American Scientific Research Journal for Engineering, Technology, and Sciences (ASRJETS)

ISSN (Print) 2313-4410, ISSN (Online) 2313-4402

http://asrjetsjournal.org/

Application of HPM to Solve Unsteady Squeezing Flow of a Second-Grade Fluid between Circular Plates

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Abstract

In this article, Homotopy Perturbation Method (HPM) is used to provide two approximate solutions to the nonlinear differential equation that describes the behaviour for the unsteady squeezing flow of a second grade fluid between circular plates. Comparing results between approximate and numerical solutions shows that our results are capable to provide an accurate solution and are extremely efficient.

Keywords: boundary layer; fluid mechanics; kinematics viscosity; density; homotopy perturbation method.

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1. Introduction

According to the classification of Prandtl, the fluid motion is divided into two regions. The first, study the region near the object where the effect of friction is important and is known as the boundary layer; while for the second type, these effects can be neglected [1,2,3]. It is common to define the boundary layer as the region where the fluid velocity parallel to the surface is less than 99% of the free stream velocity [1].

The boundary layer thickness δ increases from the edge along the surface on which fluid moves. Even in the case of a laminar flow, the exact solution of equations describing the laminar boundary layer is very difficult and only few simple problems can be analysed easily [1,3].

An interesting case study is that of a squeezing flow between parallel plates. Although the studies of squeezing flows have its origins in the 19 th century, at present it is an issue of considerable importance, due to the practical applications in different areas such as physical, biophysical, and food industry among many others. Of particular interest for fluid mechanics, are the polymer extrusion process modelled using squeezing flow of viscous fluids [61]. Also the squeezing flow between parallel plates when the confining walls have a transverse motion is of a great importance in hydrodynamic lubrication theory [62]. Despite the importance of these processes, getting analytical approximate solutions is complicated, thus numerical solutions of squeezing flow between parallel plates has been conducted by Verma [63] and later by Singh and his colleagues [64]. Therefore in this article, two approximate solutions for the case of the squeezing flow of a fluid between circular plates are obtained. In [4], may be found a detailed discussion of this topic.

Ji Huan He [5,6]; proposed the standard Homotopy Perturbation Method (HPM); it was introduced as a powerful tool to approach various kinds of nonlinear problems. The HPM can be considered as combination of the classical perturbation technique and the homotopy (whose origin is in the topology), but not restricted to the limitations found in traditional perturbation methods. For instance, HPM method does need neither small parameter nor linearization, just few iterations to obtain accurate results [5,6,20-31,33-38,41,47,48,50-55,58].

There are other modern alternatives to find approximate solutions to the differential equations that describe some nonlinear problems such as those based on: variational approaches [7-9, 31], tanh method [10], exp-function [11,12], Adomian's decomposition method [13-18,42], parameter expansion [19], homotopy analysis method [4,32,49], and perturbation method [56] among many others.

This paper is arranged as follows. Sections 2 provide the basis of HPM method. In Section 3, we introduce governing equations. We present two approximate solutions of the fluid's equation in Section 4. Comparisons between the two methods are presented in Section 5. Finally, the conclusions will be presented in Section 6.

2. Basic concepts of HPM

To figure out how HPM method works, consider a general nonlinear equation in the form

$$A(u) - f(r) = 0, \qquad r \in \Omega, \tag{1}$$

with the following boundary conditions

$$B(u, \partial u / \partial n) = 0, \quad r \in \Gamma, \tag{2}$$

where A is a general differential operator, B is a boundary operator, $f(r)$ a known analytical function and Γ is the domain boundary for Ω.

Also, A can be divided into two parts L and N, where L is linear and N nonlinear; from this last statement, (1) can be rewritten as

$$L(u) + N(u) - f(r) = 0, \tag{3}$$

In a broad sense, a homotopy can be constructed in the form [5, 6]

$$H(v, p) = (1-p)[L(v) - L(u_0)] + p[L(v) + N(v) - f(r)] = 0, \ p \in [0,1], \ r \in \Omega. \tag{4}$$

or

$$H(v, p) = L(v) - L(u_0) + p[L(u_0) + N(v) - f(r)] = 0, \quad p \in [0,1], \ r \in \Omega. \tag{5}$$

where p is a homotopy parameter, whose values are within range of 0 and 1, u_0 is the first approximation to the solution of (3) that satisfies the boundary conditions.

Assuming that solution for (4) or (5) can be written as a power series of p.

$$v = v_0 + v_1 p + v_2 p^2 + \cdots \tag{6}$$

Substituting (6) into (5) and equating identical powers of p terms, it is possible to obtain the values for the sequence $u_0, u_1, u_2, \ldots$

When $p \to 1$, it yields in the approximate solution for (1) in the form

$$v = v_0 + v_1 + v_2 + v_3 \ldots \tag{7}$$

Another way to build a homotopy, which is relevant for this paper, is by considering the following general equation

$$L(v) + N(v) = 0, \tag{8}$$

where $L(v)$ and $N(v)$ are the linear and no linear operators respectively. It is desired that solution for $L(v)=0$ describes, accurately, the original nonlinear system.

By the homotopy technique, a homotopy is constructed as follows [28]

$$(1-p)L(v)+p[L(v)+N(v)]=0. \tag{9}$$

Again, it is assumed that solution for (9) can be written in the form (6); thus taking the limit when $p\to 1$ results in the approximate solution of (8).

3. Mathematical Formulation

Consider a two dimensional squeezing flow of nonconducting, incompressible second grade fluid between two circular plates. The instantaneous distance between the plates at any time t is $2a(t)$. The central axis of the channel is taken as the r-axis and z-axis is normal to it (see Fig. 1). The velocity components along the radial and axial directions are $u(r,z,t)$ and $v(r,z,t)$ respectively. The fluid is assumed to have density ρ, kinematics viscosity ν, coefficient of viscosity μ, and material constants α_1 and α_2.

The relevant equations of motion for a homogeneous and incompressible second grade two dimensional unsteady fluid, neglecting the thermal effects are [4].

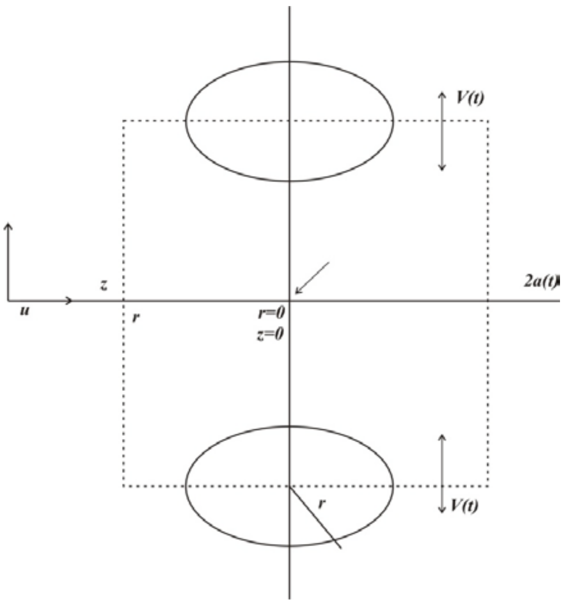


Figure 1: Geometry of the problem

$$\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial w}{\partial z}=0, \tag{10}$$

$$\frac{\partial h}{\partial r}+\rho\left(\frac{\partial u}{\partial t}-w\Omega\right)=-\left(\mu+\alpha_1\frac{\partial}{\partial t}\right)\frac{\partial\Omega}{\partial t}-\alpha_1 w\left(\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}\right)\Omega-\frac{\Omega}{r^2}\right)+\left(\alpha_1+\alpha_2\right)\left(\left(\frac{2}{r}\frac{\partial(\Omega u)}{\partial z}\right)+\left(\frac{\Omega^2}{r}\right)\right),$$

(11)

$$\frac{\partial h}{\partial z}+\rho\left(\frac{\partial w}{\partial t}+u\Omega\right)=\left(\mu+\alpha_1\frac{\partial}{\partial t}\right)\left(\frac{\partial\Omega}{\partial r}+\frac{\Omega}{r}\right)+\alpha_1 u\left(\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}\right)\Omega-\frac{\Omega}{r^2}\right)-\left(\alpha_1+\alpha_2\right)\left(\left(\frac{2}{r}\frac{\partial(\Omega u)}{\partial z}\right)\right). \quad (12)$$

Where

$$\Omega=\frac{\partial w}{\partial r}-\frac{\partial u}{\partial z}, \quad (13)$$

$$h=\frac{\rho}{2}\left(u^2+w^2\right)+p-\alpha_1\left(w\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)-u\frac{\partial}{\partial z}\right)\Omega-\frac{1}{4}\left(3\alpha_1+2\alpha_2\right)A_1^2, \quad (14)$$

and to simplify, we have defined

$$A_1^2=\left[4\left(\frac{\partial u}{\partial r}\right)^2+4\left(\frac{\partial w}{\partial z}\right)^2+4\left(\frac{u}{r}\right)^2+2\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)^2\right]. \quad (15)$$

Eliminating h between (11) and (12) we obtain

$$\rho\left(\frac{\partial\Omega}{\partial t}+\left(\left(u\frac{\partial}{\partial r}+w\frac{\partial}{\partial z}-\frac{u}{r}\right)\right)\Omega\right)=$$

$$\left(\mu+\alpha_1\frac{\partial}{\partial t}\right)\left(\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}\right)\Omega-\frac{\Omega}{r^2}\right)+ \quad (16)$$

$$\alpha_1\left(u\frac{\partial}{\partial r}+w\frac{\partial}{\partial z}-\frac{u}{r}\right)\left(\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}\right)\Omega-\frac{\Omega}{r^2}\right)-$$

$$2\left(\frac{\alpha_1+\alpha_2}{r}\right)\left(\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}\right)u\Omega+\frac{1}{2}\frac{\partial\Omega^2}{\partial z}\right).$$

The boundary conditions on $u(r,z,t)$ and $w(r,z,t)$ are given by

$u(r,z,t)=0$, $w(r,z,t)=V(t)$, $z=a$,

$$w(r,z,t)=0\,,\frac{\partial u(r,z,t)}{\partial z}=0,\;\; z=a\,, \tag{17}$$

where $V=da(t)/dt$ denotes the velocity of the plates (see Fig. 1).

Introducing dimensionless variable $\eta = z/a(t)$, equations (13), (10), and (16) adopt the form

$$\frac{\partial w}{\partial r}-\frac{\partial u}{a\partial \eta}=\Omega, \tag{18}$$

$$\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial w}{a\partial \eta}=0, \tag{19}$$

$$\rho\left(\frac{\partial \Omega}{\partial t}+\left(\left(u\frac{\partial}{\partial r}+w\frac{\partial}{a\partial \eta}-\frac{u}{r}\right)\right)\Omega\right)=$$

$$\left(\mu+\alpha_1\frac{\partial}{\partial t}\right)\left(\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}\right)\Omega-\frac{\Omega}{r^2}\right)+ \tag{20}$$

$$\alpha_1\left(u\frac{\partial}{\partial r}+w\frac{\partial}{a\partial \eta}-\frac{u}{r}\right)\left(\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}\right)\Omega-\frac{\Omega}{r^2}\right)-$$

$$2\left(\frac{\alpha_1+\alpha_2}{r}\right)\left(\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}\right)u\Omega+\frac{1}{2a}\frac{\partial \Omega^2}{\partial \eta}\right).$$

The boundary conditions on $u(r,\eta,t)$ and $w(r,\eta,t)$ are given by

$$u(r,\eta,t)=0\,, w(r,\eta,t)=V(t)\;\;,\; \eta=1\;, \tag{21}$$

$$w(r,\eta,t)=0\,,\frac{\partial u(r,\eta,t)}{\partial \eta}=0,\; \eta=0\,. \tag{22}$$

Defining the velocity components as [4], [64]:

$$u=\frac{-r}{2a(t)}V(t)y'(\eta)\,, w=V(t)y(\eta)\,, \tag{23}$$

it follows that (18) takes the following form

$$\Omega = \frac{r}{2a(t)} V(t) y''(\eta). \tag{24}$$

The substitution of (22) and (23) into (19) and (20) yield

$$\frac{aV}{\nu}\left(yy''' - 2y'' - \eta y''' \right) + \frac{a(dV/dt)}{\nu V} y'' =$$

$$y^{iv} + \frac{(\alpha_1 + \alpha_2)V}{\mu a}\left(2y''y''' + y'y^{iv} \right) + \frac{\alpha_1 V}{\mu a}\left(yy^{v} - \eta y^{v} - 4y^{iv} + \frac{a(dV/dt)}{\nu V^2} y^{iv} \right) \tag{25}$$

where prime denotes differentiation with respect to η.

The boundary conditions on $y(\eta)$ are deduced from (21), (22), and (23) so that

$$y(1) = 1,\ y'(1) = 0,$$

$$y(0) = 0,\ y''(0) = 0 \tag{26}$$

In order to obtain a similarity solution, we define

$$\frac{aV}{\nu} = R_e, \tag{27}$$

$$\frac{a^2(dV/dt)}{\nu V} = R_e Q, \tag{28}$$

$$\frac{\alpha_1 V}{\mu a} = W_{e1}, \tag{29}$$

$$\frac{\alpha_2 V}{\mu a} = W_{e2}, \tag{30}$$

for a similarity solution, R_e, Q, W_{e1} and W_{e2} are considered constants.

After integrating (27), we obtain

$$a(t) = (Kt + a_0)^{1/2}, \tag{31}$$

where K and a_0 are constants; in particular $2a_0$ denotes the initial separation between the plates.

From (27)-(31), it follows that $Q = -1$, and (25) becomes

$$y^{iv} + K^2 y'' = L(y-\eta)y''' + M(\eta - y)y^{v} - N(2y''y''' + y'y^{iv}), \quad (32)$$

in the above equation we have defined

$$K = \sqrt{\frac{3R_e}{1-5W_{e1}}},$$

$$L = \frac{R_e}{1-5W_{e1}}, \quad (33)$$

$$M = \frac{W_{e1}}{1-5W_{e1}},$$

$$N = \frac{W_{e1} + W_{e2}}{1-5W_{e1}}.$$

In this work, we will consider the following case for constants: K = 1.732, L = 1, M = 0, N = 0.05 as reported in [4].

4. Approximate solution for the unsteady squeezing flow of a second-grade fluid equation by using HPM.

In this section, we will employ two different variants of HPM formulation, to find two accurate approximate solutions of (32), by using the first order approximation.

4.1 First HPM Approximation

Identifying the linear part as

$$l = y^{iv}(\eta), \quad (34)$$

and the nonlinear

$$n = K^2 y''(\eta) - L(y(\eta) - \eta)y'''(\eta) + N(2y''(\eta)y'''(\eta) + y'(\eta)y^{iv}(\eta)), \quad (35)$$

we construct a homotopy starting from (9), in the form

$$\begin{aligned}&(1-p)y^{iv}(\eta)+p\big(y^{iv}(\eta)+K^2y''(\eta)-L(y(\eta)-\eta)y'''(\eta)\\&+N\big(2y''(\eta)y'''(\eta)+y'(\eta)y^{iv}(\eta)\big)\big)\end{aligned} \quad (36)$$

Then substituting (6) into (36), and equating terms having identical powers of p we obtain

$$\begin{aligned}&p^0: y_0^{iv}(\eta)=0,\\&p^1: -Ly_0(\eta)\big(y_0'''(\eta)\big)+y_1^{iv}(\eta)-M\eta(y_0^{v}(n))+L\eta(y_0'''(\eta))+My_0(\eta)(y_0^{v}(\eta))\\&+2N(y_0''(\eta))\big(y_0'''(\eta)\big)+N(y_0'(\eta))(y_0^{iv}(\eta))+K^2(y_0''(\eta))\end{aligned} \quad (37)$$

In order to fulfill the boundary conditions from (26), it follows that $y_0(0)=0, y_0''(0)=0, y_0(1)=1, y_0'(1)=0,$ and $y_1(0)=0, y_1''(0)=0, y_1(1)=0, y_1'(1)=0.$

Thus, the results obtained from the above equations are

$$\begin{aligned}&y_0=\frac{1}{2}\eta^3+\frac{3}{2}\eta\\&y_1=\frac{1}{560}L\eta^7+\left(\frac{1}{40}K^2-\frac{1}{80}L-\frac{3}{20}N\right)\eta^5+\left(-\frac{1}{20}K^2+\frac{11}{560}L+\frac{3}{10}N\right)\eta^3\\&+\left(\frac{1}{40}K^2-\frac{1}{112}L-\frac{3}{20}N\right)\eta\end{aligned} \quad (38)$$

By substituting solutions (38) into the first to terms of (6) and calculating the limit when $p\to 1$, results in a first order approximation.

$$y(\eta)=y_0(\eta)+y_1(\eta)=-0.615348\eta^3+1.558567\eta+0.0017857\eta^7+0.055\eta^5 \quad (39)$$

4.2 Second HPM Approximation

Identifying the linear part as

$$l=y^{iv}(\eta)+K^2y''(\eta)\,, \quad (40)$$

and the nonlinear

$$n=-L(y(\eta)-\eta)y'''(\eta)-M(\eta-y(\eta))y^{v}(\eta)+N(2y''(\eta)y'''(\eta)+y'(\eta)y^{iv}(\eta))\,, \quad (41)$$

we construct a homotopy starting from (9), in the form

$$(1-p)(y^{iv}(\eta)+K^2y''(\eta))+p(y^{iv}(\eta)+K^2y''(\eta)-L(y(\eta)-\eta)y'''(\eta)+N(2y''(\eta)y'''(\eta)+y'(\eta)y^{iv}(\eta)) \quad (42)$$

Substituting (6) into (42), and equating terms having identical powers of p we obtain

$$\begin{aligned}
&p^0 : y_0^{iv}(\eta) + K^2 y_0'' = 0, \\
&p^1 : K^2\left(y_1''(\eta)\right) + 2N\left(y_0''(\eta)\right)\left(y_0'''(\eta)\right) + N\left(y_0'(\eta)\right)\left(y_0^{iv}(\eta)\right) \\
&+ y_1^{iv}(\eta) + L\eta\left(y_0'''(\eta)\right) - Ly_0(\eta)\left(y_0'''(\eta)\right) \\
&\vdots
\end{aligned} \tag{43}$$

In order to fulfill the boundary conditions from (26), it follows that $y_0(0) = 0, y_0''(0) = 0, y_0(1) = 1, y_0'(1) = 0,$ and $y_1(0) = 0, y_1''(0) = 0, y_1(1) = 0, y_1'(1) = 0.$ Then, solving (43), results

$$\begin{aligned}
&y_0(\eta) = c_1\eta + c_2\sin(K\eta), \\
&y_1(\eta) = \frac{1}{24}\frac{1}{K^4(\cos(K)K - \sin(K))}\left[-c_3(\cos(K)K - \sin(K))\sin(2K\eta)\right. \\
&\quad + \left(2Kc_3\cos(2K) - c_3\sin(2K) + 6\left(\left(-Lc_2(\eta-2)(\eta+2)(c_1-1)K^2\right.\right.\right. \\
&\quad \left.\left.\left. - 2c_4\right)\sin(K) + c_2K^3L\cos(K)(\eta-1)(\eta+1)(c_1-1)\right)K^2\right)\sin(K\eta) \\
&\quad + 30\left[\left(\cos(K)K - \sin(K)\right)\left(Lc_2(c_1-1)K^2 - \tfrac{2}{5}c_4\right)\cos(K\eta)\right. \\
&\quad - \tfrac{1}{15}c_3\cos(2K)\sin(K) + \tfrac{1}{30}c_3\cos(K)\sin(2K) \\
&\quad - \tfrac{3}{5}\left(Lc_2(c_1-1)K^2 - \tfrac{2}{3}c_4\right)K\sin(K)^2 + \cos(K)\left(Lc_2(c_1-1)K^2 - \tfrac{2}{5}c_4\right)\sin(K) \\
&\quad \left.\left. - \cos(K)^2\left(Lc_2(c_1-1)K^2 - \tfrac{2}{5}c_4\right)K\right]\eta K\right],
\end{aligned} \tag{44}$$

$$c_1 = \frac{\cos(K)K}{\cos(K)K - \sin(K)}, \quad c_2 = \frac{1}{\cos(K)K - \sin(K)},$$

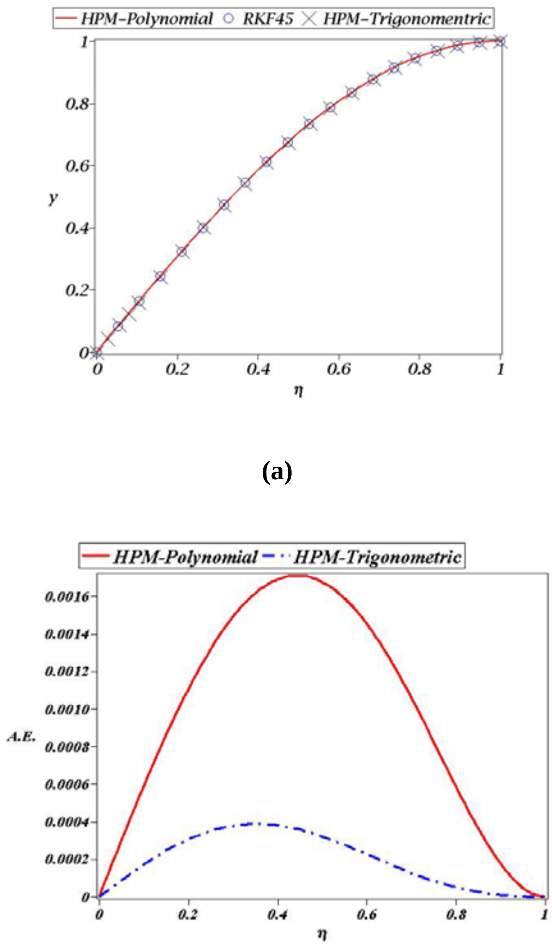
$$c_3 = Lc_2^2K^3 + 3Nc_2^2K^5, \quad c_4 = Nc_2K^4c_1,$$

By substituting solutions (44) into (42) and calculating the limit when $p \to 1$, results in a trigonometric first order approximation.

$$\begin{aligned}
y(\eta) = y_0(\eta) + y_1(\eta) = &-0.27657665\eta + 1.367\sin(1.732\eta) - 0.0218\sin(3.464\eta) \\
&- 0.4526567\eta\cos(1.732\eta) - 0.15419473\eta^2\sin(1.732\eta)
\end{aligned}. \tag{45}$$

5. Discussion

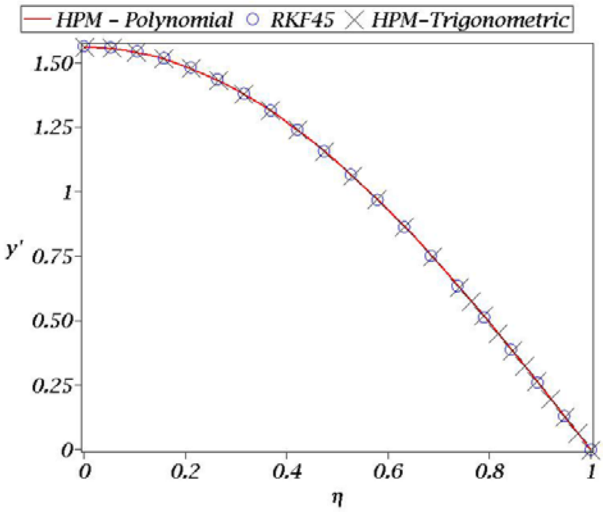
For comparison purposes we use the Fehlberg fourth-fifth order Runge-Kutta method with degree four interpolant (RKF45) [65, 66] solution (built-in function of Maple software). In order to obtain a good numerical reference the accuracy of RKF45 was set to an absolute error of 10^{-7} and relative error of 10^{-6}. Fig. 2(a) shows the comparison between RKF45 numerical solution of (32) and approximations given by (39) and (45), obtained by using HPM formulations (36) and (42), respectively. From Fig. 2(b) for absolute error (A.E), it is clear that proposed approximations are in good agreement with RKF45. In fact, the bigger A.E. for (39) is 0.0016 and for (45) is only 0.0004. On the other hand, Fig. 3(a) compares RKF45 numerical solution for $y'(\eta)$ and the corresponding approximations obtained differentiating (39) and (45). Finally, from Fig. 3(b), we conclude that A.E. is low for both cases, being it 0.006 for the derivative of (39) and less than 0.002 for the derivative of (45). Although (32) was originally solved by HAM with good accuracy [4], the obtained results correspond to a 20-th order approximation and therefore requiring more computational resources for practical use. In contrast (39) and (45) are both, highly accurate and handy.



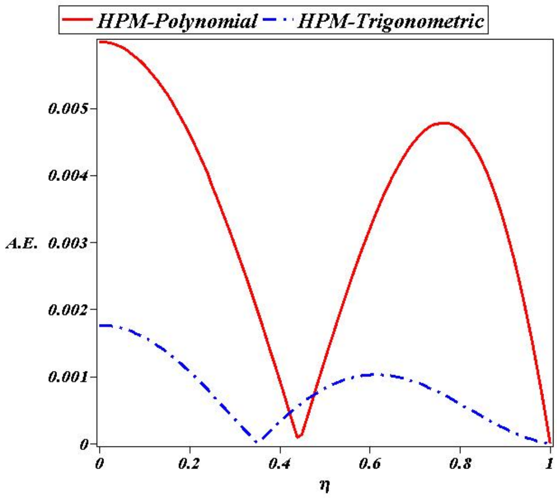
(a)

(b)

Figure 2: Approximate HPM solutions: polynomial (39) and trigonometric (45) and its error with respect to RKF45



(a)



(b)

Figure 3: Approximate derivative of: polynomial (39) and trigonometric (45) and its error with respect to RKF45.

6. Conclusion

An important task is to find analytic expressions that provide a good description of the solution to the nonlinear differential equations like (32). For instance, the case of an unsteady squeezing flow between circular plates is adequately described by our approximations given by (39) and (45) (see Fig. 2 and Fig. 3). A relevant fact of HPM method is that even utilizing the first order approximation, we obtained highly precise solutions for (32). Moreover, Figs. 2(b) and 3(b) for the absolute error show that the proposed solutions are highly accurate. In contrast to RKF45 numerical solutions, HPM methods allow both quantitatively and qualitatively analyse the solution. Therefore is expected that other problems in the field of fluid mechanics, described by nonlinear differential equations can be solved in a similar way, following the techniques employed in this work.

Acknowledgements

We gratefully acknowledge the financial support from the National Council for Science and Technology of Mexico (CONACyT) through grant CB-2010-01 #157024. The authors would like to thank Rogelio-Alejandro Callejas-Molina, and Roberto Ruiz-Gomez for their contribution to this project.

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