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Application of Kuhn-Tucker Optimality Criteria in the Selection of Fertilizer Combination for Crop Optimal Yield

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Abstract

The application of response surface methodology in agricultural context, especially in agronomical research for years now, has been of great interest to many statisticians. Most of their earlier works were to a large extent on ordinary polynomials which exhibited undesirable problem of unboundedness, symmetry about the optimum, false location of optimum and nonsensical extrapolation. In this work, Kuhn-Tucker optimality criteria have proved to be more efficient when compared with methods like Berry and Mitscherlich. In fact, the initial problem of unboundedness, symmetry about the optimum, etc, are removed. Numerical application using different types of fertilizer combination to compare crop yield confirmed this assertion.

Keywords*:* Complementary; hessian matrix; optimality; response surface; reciprocal polynomial.

1. Introduction

The introduction of response surface methodology in agricultural context was done by [1] in Germany, though most of the earlier works on it by the author were to a large extent on ordinary polynomial. It also focuses on experimental models. Reference [2] also expressed the hope that the method will be of immense value on other fields where experimentation is sequential and the error, fairly small.

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Therefore, the introduction of factorial experiment in agricultural context by [3] may have sided the fulfillment of their hope in the application of response surface methodology beyond the walls of chemical industries, particularly in agronomic research. Reference [4] also opined that the fundamental applications of response surface methodology in any experimental design, is in approximating the true models discriminating between surfaces and the exploration of response surface.

Reference [5] has also demonstrated in his work entitled "Agricultural Response Surface Experiment based on Four Level Factorial Design" that the ordinary polynomials, particularly the second order, have been very reliable in exploring response surfaces; this is because of the conceptual and computational simplicity and easy location of the response surface. It also shows that the experimental models, on the other hand, exhibit apparent difficulties in the estimation of parameters of interest by the usual least square method; and the model transformation may be difficult in the case of intrinsic non-linear model in the parameter. On the other hand, [6] also shows that reciprocal polynomials have the desirable properties of boundedness, asymptotic distribution free, invariance and speedy convergence. Awareness of the uses of reciprocal polynomial increases following its use in multifactor experiment on Bermuda grass by [7] for parameter estimation. The main use of reciprocal polynomials has been in agronomy – the research area from which they were derived. Its various formulations are mainly as growth studies in plant-yield relationship and inverse linear regression method of calibration.

2. Materials and Methods

In the study of crop-yield fertilizer relationship, [1] proposed the model

$$y = A\left(1 - e^{-C(x+D)}\right) \qquad (1)$$

y is the crop yield, x is the rate of fertilizer application, and A, C, D are parameters which measure, respectively: maximum yield which could not be exceeded by the use of the fertilizer, the efficiency of the fertilizer [assumed constant], and the soil content of the fertilizer in the control plots. The model was popularly used; and though most suitable in all biometric research. However, this was tried by various experimenters and later found not always adequate ([8]; [9] ; [10] ; [11]). This was because C was found not to be constant but varies with the kind of plant, form of nutrient, fertility and planting rate. Thus, much mathematical complexity was involved in estimating C which has less appeal to experimenters, [12]

For purposes of parameter estimation, [13] suggested a transformation of the model into the form

$$y = \alpha + \beta\rho_*^{x}, \quad 0 < \rho < 1, \quad x = 0,1,\ldots,(n-1) \qquad (2)$$

ρ_* represents the factor by which the deviation from its asymptotic value is reduced for a unit increase in x. The model was found not to give a good fit for some plants, and its convergence, sometimes very slow, [13]. [14] therefore proposed the first use of inverse model

$$w^{-1} = \alpha + \beta\rho \qquad (3)$$

[where ρ is the density defined as the number of plants per unit area, and α, β are parameters] on the assumptions that the growth curve is logistic and yield per area, w, is ultimately independent of density. Therefore, the model is adequate for vegetable plants [that is, those plants cultivated from any part(s) of a plant other than the seeds, e.g. Bryophium from leaves, yam, potato from tuber, cassava, sugar cane from stem-cutting] but inadequate for reproductive plants [that is, limit of yield obtainable from a particular area with the particular crop]; and reproductive plants have parabolic relationship [that is, higher densities resulting in low yields]. Therefore, modified the model to allow for both relationships. That is,

$$w^{-\theta} = \alpha + \beta\rho, \quad 0 < \theta \leq 1 \qquad (4)$$

[where θ, which depends on any part of the plant like leaves, shoots, stem, etc, is called Critical Parameter; where $\theta = 1$ refers to vegetative plant and $\theta < 1$ refers to reproduction plants].

To allow for symmetric designs for spacing experiments, Nelder and Bermuda (1966) modified it to

$$w^{-\theta} = \alpha + \beta\rho^{\Phi} \qquad (5)$$

[where Φ defines the nature of the two curves]. [7] generalized them to the family of inverse polynomials by proposing

$$\frac{\prod_{i=1}^{k} x_i}{y} = \text{polynomial in } x_1, x_2, \ldots, x_k \qquad (6)$$

[where $x_1's$ are levels of k experimental factors]. For a single factor, [7] has given it as

$$\frac{x}{y} = \alpha x + \beta \qquad (7)$$

and for a 2×2 design, it is

$$\frac{x_1 x_2}{y} = \beta_{00} + \beta_{10} x_1 + \beta_{11} x_2 + \beta_{12} x_1 x_2 \qquad (8)$$

which is the inverse analogue of 2^2 factorial experiment. In order to allow for the effects of density to be separated into within row and between row spacing [x_1 and x_2], [13] suggested the model

$$w^{-\theta} = \alpha + \beta_1 x_1^{-1} + \beta_1 x_2^{-1} + \beta_{12} (x_1 x_2)^{-1}, \quad 0 < \theta \leq 1 \qquad (9)$$

particularly for regularly spaced crops,the theoretical properties [form] of (θ, Φ), (w, ρ) are compared and such relationship for the response curves.

$$w^{-\theta} = \alpha + \beta\rho^{\Phi}$$

$\theta = \Phi$ implies asymptotic yield [vegetable plant]

$\theta < \Phi$ implies parabolic yield [reproductive plants]

$\theta > \Phi$ gives $w^{-\theta}$ as unbounded which is biologically unrealistic

An agriculturist will always be interested in obtaining the maximum yield. Our interest is to determine the maximum yield. Let our maximum yield be defined by

$$w' = w_{\max}$$

$$w = x_1^{-1}x_2^{-1}\left(\alpha + \beta_1 x_1^{-1} + \beta_2 x_2^{-1} + \beta_{12} x_1^{-1} x_2^{-1}\right)^{-\frac{1}{\theta}} \qquad (10)$$

was obtained for a one-factor experiment disregarding θ as variable. However, since θ depends on some parts of the plant and, in certain plants change with maturity [7] , it can be considered a variable. Hence,

$$w = f\left(x_1, x_2, \theta\right) \qquad (11)$$

On finding x_1, x_2 and θ which maximize w, we obtain

$$w_{\max} = \frac{1 \pm \beta - \alpha\beta \pm \sqrt{1 + \beta - \alpha\beta}}{\beta}$$

Therefore, the value of x_1, x_2 and θ which yield $w_{\max}$ are as given in [1]. We estimate the various values of $\hat{\theta}$ and test for statistical significance between the various obtained by [13] and the analytical values so far obtained. The parameter estimates of $w_{\max}$ have been obtained using Gaussian-Newton method. Consequently, the various values of $\hat{\theta}$ are obtained. In certain circumstances, an agriculturist may be forced to seek some proportion, $\lambda\left(0 < \lambda < 1\right)$ of this maximum yield. This may be a situation where full yield is unattainable or simply unavailable. This may be due to some natural disasters which may affect the survival of these plants at full harvest time.

3. Results

There are four fertilizers: Nitrate base, Phosphate base and Potassium base, to be involved in this combination and applied to four crops: rice, yam, cassava and cocoyam. Let nc_q be different ways n fertilizer combination could be obtained from q ways. This implies that there are different groups of fertilizers, each one consisting of n fertilizers that can be selected from q combinations.

Then there are $\sum nc_q$ possible groups of fertilizers in all form which selection can be made. This approach requires us to solve $\sum_{n=2}^{q} nc_q$ different quadratic programming problem from which an optimal group of crops can be selected. This will help us feed the parameter of each combination into the model and solving the resulting problems one by one to obtain the optimal solution. The combination of group of fertilizer with the maximum objective function value gives the optimal solution.

Let

ML = Model

FC = Fertilizer combination

$MOFV$ = Mitscherlich objective function values

$BOFV$ = Berry objective function values

$KUOFV$ = Kuhn-Tucker objective function values

MCT = Mitscherlich computer time

MNI = Mitscherlich number of iterates

$KUCT$ = Kuhn-Tucker computer time

$KUNI$ = Kuhn-Tucker number of iterates

BCT = Berry computer time

BNI = Berry number of iterates

Manual Illustration Using the Kuhn-Tucker Conditions

Consider the quadratic programming problem

$$\begin{aligned}
&P : Max\, Z = 3x_2^3 - 3x_1x_2 + 2x_2^2 - 26x_1 - 8x_2 \\
&S.t.: \quad I : x_1 + 2x_2 \leq 6 \\
&\qquad\quad II : x_1 - x_2 \geq 1 \\
&\qquad\quad x_1, x_2 \geq 0
\end{aligned}$$

Here,

$n = 2$ [two variables, x_1 and x_2]

$m = 2$ [two constraints]

TABLE 1: Quadratic Program Results for Three Methods

MC	FC	MOFV	MNI	BOFV	BNI	KUOFV	KUNI
QP_1	1, 2	178.3	1	176.1	1	175.8	1
QP_2	1, 3	197.8	2	196.3	2	195.3	1
QP_3	1, 4	163.4	2	166.2	1	164.9	1
QP_4	2, 3	176.8	1	176.1	1	175.2	1
QP_5	2, 4	218.6	3	214.1	3	213.1	2
QP_6	3, 4	24.2	2	21.1	3	20.4	2
QP_7	1, 2, 3	73.1	2	74.3	2	74.1	1
QP_8	1, 2, 4	37.6	1	38.2	1	36.4	1
QP_9	1, 3, 5	48.4	1	45.2	2	43.1	2
QP_{10}	2, 3, 5	46.8	3	45.1	1	44.4	1
QP_{11}	1, 2, 3, 4	50.1	3	50.4	4	49.1	3

TABLE 2: Computer Time for the Three Methods

NANO SECONDS		
MCT	**BCT**	**KUCT**
0.1314	0.1261	0.1258
0.1148	0.1142	0.1137
0.1210	0.1163	0.1136
0.1145	0.1142	0.1128
0.2291	0.2194	0.2187
0.2165	0.2168	0.2169
0.1258	0.1160	0.1109
0.1138	0.1139	0.1140
0.1140	0.1140	0.1140
0.1143	0.1142	0.1141
0.4013	0.7942	0.4032

Then the Kuhn-Tucker conditions are:

1. $\frac{\partial f}{\partial x_j} - \sum_{i=1}^{m} u_i \frac{\partial g_i}{\partial x_j} \le 0$

2. $x_j^*\left(\frac{\partial f}{\partial x_j}-\sum_{i=1}^{m}u_i\frac{\partial g_i}{\partial x_j}\right)=0$

3. $g_i\left(x^*\right)-b_i\le 0$

4. $u_i\left(g_i\left(x^*\right)-b_i\right)=0$

5. $x_j^*\ge 0$

6. $u_i\ge 0 \quad x=x^*=x_j,\quad i=1,\ldots,m,\quad j=1,\ldots,n$

The applications of the Kuhn-Tucker conditions are as follows:

Given that $g_1(x)=x_1+2x_2$ and $g_2(x)=-x_1+x_2$

1. $(j=1)\Rightarrow\frac{\partial Z}{\partial x_1}-\sum_{i=1}^{2}u_i\frac{\partial g_i}{\partial x_1}\le 0$

$$\Rightarrow\frac{\partial Z}{\partial x_1}-\left(u_1\frac{\partial g_1}{\partial x_1}+u_2\frac{\partial g_2}{\partial x_1}\right)\le 0$$

$$\Rightarrow(6x_1-2x_2-26)-(u_1-u_2)\le 0$$

2. $(j=1)\Rightarrow x_1\left(\frac{\partial Z}{\partial x_1}-\sum_{i=1}^{2}u_i\frac{\partial g_i}{\partial x_1}\right)=0$

$$\Rightarrow x_1\left(\frac{\partial Z}{\partial x_1}-\left(u_1\frac{\partial g_1}{\partial x_1}+u_2\frac{\partial g_2}{\partial x_1}\right)\right)=0$$

$$\Rightarrow x_1\left((6x_1-2x_2-26)-(u_1-u_2)\right)=0$$

1. $(j=2)\Rightarrow\frac{\partial Z}{\partial x_2}-\sum_{i=1}^{2}u_i\frac{\partial g_i}{\partial x_2}\le 0$

$$\Rightarrow\frac{\partial Z}{\partial x_2}-\left(u_1\frac{\partial g_1}{\partial x_2}+u_2\frac{\partial g_2}{\partial x_2}\right)\le 0$$

$$\Rightarrow(-2x_1+4x_2-8)-(2u_1+u_2)\le 0$$

2. $\left(j=2\right) \Rightarrow x_2\left(\frac{\partial Z}{\partial x_2}-\sum_{i=1}^{2} u_i \frac{\partial g_i}{\partial x_2}\right)=0$

$$\Rightarrow x_2\left(\frac{\partial Z}{\partial x_2}-\left(u_1 \frac{\partial g_1}{\partial x_2}+u_2 \frac{\partial g_2}{\partial x_2}\right)\right)=0$$

$$\Rightarrow x_2\left(\left(-2x_1+4x_2-8\right)-\left(2u_1+u_2\right)\right)=0$$

3. $\left(i=1\right) \Rightarrow g_1\left(x^*\right)-b_1 \leq 0$

$$\Rightarrow x_1+2x_2-6 \leq 0$$

4. $\left(i=1\right) \Rightarrow u_1\left(g_1\left(x^*\right)-b_1\right)=0$

$$\Rightarrow u_1\left(x_1+2x_2-6\right)=0$$

3. $\left(i=2\right) \Rightarrow g_2\left(x^*\right)-b_2 \leq 0$

$$\Rightarrow -x_1+x_2+1 \leq 0$$

4. $\left(i=2\right) \Rightarrow u_2\left(g_2\left(x^*\right)-b_2\right)=0$

$$\Rightarrow u_2\left(-x_1+x_2+1\right)=0$$

5. $x_1 \geq 0, \quad x_2 \geq 0$

4. Discussion

First and foremost, we note the similarities in the result from the objective function value for all the fertilizer combinations using all the methods, including Kuhn-Tucker. We noticed that they are about 98 percent the same in all cases; but the number of iterates before this as obtained differ. We also noticed that in the Kuhn-Tucker, the number of iterates before the optimal solution as obtained in all cases were smaller. Also, the computer run-time to obtain the optimal solution from QP_1 to QP_{11} differs significantly. In QP_1, the computer run-time to obtain an optimal solution was 0.4032 Nano seconds, while the average of others stood at 0.7942 about, twice that of Kuhn-Tucker. On the whole, from the tables we noticed that the Kuhn-Tucker optimality criteria are better than any other method discussed in this work. Also, in terms of optimal yield, the fertilizer combination QP_5 gives the greatest yield which stands at 214kg as compared with others.

5. Conclusion

The introduction of Kuhn-Tucker optimality criteria in the selection of fertilizer combinations for crop optimal yield has been found to be more reliable in the presence of computing facilities. It has been maximally used in obtaining maximum yield of its proportion in the absence of full yield. A balance has also been struck over which fertilizer will give maximum yield under different crop formation. This has been done by establishing optimum design points for optimum responses in different steps of the experiment. The significant difference established between Kuhn-Tucker optimality criteria and other methods in the eleven quadratic programs [see Table 1] shows that the objective function values, the computation time and the number of iterates were fewer in all cases. This significant difference between Kuhn-Tucker and other methods underscores the need for a non-empirical approach in the determination of maximum yield.

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