Generalization of Central Place Theory Using Mathematical Programming

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Abstract

Central place theory that is the theory of location of urban settlements where goods and services are supplied was established by Walter Christaller in 1933, and expanded by August Lösch seven years later. The purpose of this thesis is to reinterpret their theories that have been forced to rely on descriptive and geometrical explanations from mathematical modelling perspective. Specifically, modelling for both of the locational principle of single good and the superimposition problem of market area networks are attempted by using mathematical programming comprising some constraints and an objective function. Moreover, the generalized model that both of the locational principle and the hierarchical structure are integrated is presented, and Lösch's and Christaller's theories are identified systematically on the generalized model.

First of all, the market area theory of Lösch as a location problem of single good was modeled, and the extended model was developed in Chapter 2. When the process of locational equilibrium of firms based on the concept of demand cone and the normal profit is premised, the market area theory of Lösch can be formulated as the total demand maximization problem. In the previous studies concerning modelling the theory of Lösch, there were some problems in the formulation to reproduce theoretical central place system for the conditions such as assuming the behavior of firm and deriving the appropriate market area. On the other hand, the model of this thesis is able to reproduce the original Lösch’s theory because of formulating an objective to maximize total demand subject to the conditions of the nearest center hypothesis and the indifference principle. According to the results of applying the model in hypothetical areas, the model will be considered as an operational model of the market area theory of Lösch that enables the derivation of a realistic central place system under the more relaxed assumptions such as non-uniform

population distribution. However, the firm obtains only the normal profit in the market area theory of Lösch that requires the free entry and the perfect competition to the market. Then, the total profit maximization problem that is antithetic to the total demand maximization problem was examined in consideration of realistic firm's behavior, and the two objectives were integrated by using multiobjective programming as the extension of the model for single good. The extended model is able to be regarded as a flexible model in handling the number of firms concerning the locational principle according to the results of applying in a hypothetical area.

Next, the method of constructing a hierarchy in Lösch’s market area theory was examined, and the mathematical formulation of the superimposition problem of market area networks was attempted in Chapter 3. On the basis of the interpretation in the previous studies concerning the hierarchical arrangement of Lösch, it has been considered that two conditions of prioritizing location in particular sectors and maximizing the number of coincident market centers are the objectives of constructing a hierarchy. Then, the model that optimizes these two objectives was formulated as a combinatorial optimization and applied to the discrete lattice network of regular equilateral triangles. As a result, the central place system by Lösch was not reproduced, and the solution of the model was optimal for the purpose of prioritization of locating in particular sectors. Namely, it has been understood that the perspective towards the rationality of the entire system was lack from Lösch's process of constructing a hierarchy. Then, this thesis attempted to reinterpret Lösch's original intention in constructing a hierarchy, and developed the extended model based on the agglomeration effects. The application of the extended model revealed that the priority location in a particular sector was not an essential condition of the agglomeration of goods and the agglomeration effect by setting neighborhood ranges was able to derive more realistic central place hierarchy.

In Chapter 4, Christaller’s central place theory was reinterpreted based on the extended models both of the locational principle of single good and the hierarchical arrangement in the market area theory by Lösch. It is the core of a subject in this thesis to identify Christaller's and Lösch's theories systematically according to the extended models. When the objective function in the extended model of single good was expanded, the problem divided into three objectives—the maximum coverage of demand, the minimization of total distance traveled, and the minimization of the number of locations. From the perspective of the locational principle by combining three objectives, Christaller's central place theory was able to be identified as a generalized median problem in which second and third objectives are integrated. Because the generalized median problem is formulated as multiobjective programming, a different number of firms can be assumed in the case of one kind of good supply. Therefore, Christaller's central place theory was identified as the superimposition problem that one kind of good was associated with multiple market area networks. The application of the extended model for hierarchical arrangement indicated that central place system based on Christaller’s marketing principle was reproduced when the agglomeration effect was prioritized.

Consequently, in this thesis, the generalized model of central place theory was presented by integrating the models both of the locational principle of single good and the hierarchical arrangement. When Lösch’s and Christaller's theories are identified using the generalized model, we can recognize that the former precedes the locational principle for single good while the latter prioritizes the agglomeration effect of goods. However, central place systems based on various locational principles and flexible hierarchical structures will be assumed in the real world. The generalized model of central place theory presented in this thesis is able to be considered as an analytical model that would be the touchstone of original theories to the real-world central place system.

Acknowledgements

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CHAPTER 1

Introduction

Central place theory—a theory for explaining the number, size and distribution of urban settlements that provide goods and services to its surrounding area—was initially established in 1933 by Walter Christaller in *Die zentralen Orte in Süddeutschland*, and later expanded in 1940 as a general schema of central place systems in the market area theory developed by August Lösch. In the history of human geography, central place theory is one of the classical location theories, and has been the subject of research that many geographers attempted to apply with quantitative analysis after the Quantitative Revolution. For instance, there are many literatures such as the empirical studies on the hierarchical structure of central place systems using multivariate statistical analysis (e.g., Berry and Barnum 1962; Beavon 1972), the theoretical studies on the geometric characteristics of systems (e.g., Dacey 1965; Arlinghaus 1985), and the mathematical approaches to dynamic model of central places (e.g., Allen and Sanglier 1981; Clarke and Wilson 1985). Of mathematical techniques, location-allocation models for solving the facility location problem 'provide a suitable tool for operationalizing the concepts of central place theory' (Beaumont 1987: 21) because of generating alternate spatial structure of central place systems by varying different assumptions (Ghosh and Rushton 1987).

Location-allocation model is stated using mathematical programming formulation to seek the optimal location subject to constraints such as consumer and producer behaviors. Mathematical programming formulation is an effective method for modelling central place theory that founded on optimality principle (Beaumont 1982). Attempts to model central place theory using mathematical programming include studies, such as those of Henderson (1972), Dökmeci (1973), Puryear (1975), and Kohsaka (1983), that have sought to develop analytical models by taking advantage of the frameworks of central place theory, as well as those that have sought to reproduce theoretical central place systems, such as those of Storbeck (1988, 1990), Kuby (1989), and Curtin and Church (2007). While these studies have incorporated mathematical programming as a means of deriving the hierarchical spatial arrangement of central places, their examination of the assumptions and locational principles of central place theory is inadequate, and they cannot necessarily be said to have faithfully replicated the original central place theory (Ishizaki 1992). If we are to consider the role of central place theory as an operational model, then we need to first of all examine the nature of the constraints and objectives that Christaller and Lösch imposed on their theory when deriving their central place systems (Beaumont 1982: 239).

However, both Christaller's and Lösch's discussion, despite their relatively rigorous discussion before deriving the schema of central place system, such as for example in their discussion of supply and demand relationship of goods and services, becomes vague when actually locating central places or firms and consequently constructing the hierarchical structure. This ambiguity has often led to misinterpretations of the theory (e.g., Berry and Garrison 1958a) and has resulted in the faithful understandings of the original theory (Saey 1973; Beavon 1977; Morikawa 1980; Hayashi 1986). It is necessary to read between the lines in the writings of Christaller and Lösch in order to supplement the inherent ambiguity and insufficient explanation in the original central place theory. Whereas their theories have previously been forced to rely on descriptive and geometric explanations, today, in an era marked by significant advances in the elaboration of mathematical models and computerized numerical analysis, I am confident that their reconceptualization from a mathematical modeling perspective can provide new insights into central place theory.

The purpose of this paper is to develop a model that reproduces central place systems using mathematical programming and to reinterpret Christaller's and Lösch's theories through the process of constructing the generalized models. Especially, this paper focuses on modeling of Lösch's theory because Ishizaki (1995) has previously attempted to develop the model of Christaller's central place theory. Furthermore, after the model of Lösch's theory is extended, the differences between Christaller's and Lösch's theories are examined according to an extended model.

Chapter 2 attempts to model Lösch's market area theory in the location of single good, and then, on the basis of the results of applying the model in a hypothetical area, it is examined whether the Lösch's system can be reproduced according to the operational model of the theory. Although Kuby (1989) has already attempted to model Lösch's market area theory, there appears to be a number of problems with its formulation in Kuby (1989) concerning the model's validity. Hence, after critically examining Kuby's model, the model of Lösch's theory is reformulated on the basis of reinterpretation of the theory. Thereafter, after establishing an objective antithetical to Lösch's theory, an extension of the model concerning the location of single good is developed by unifying the two objectives.

Chapter 3 discusses the method of constructing a hierarchy in Lösch's market area theory and attempts to model the superimposition problems of hexagonal networks. The hierarchical properties of Lösch's system were intensely debated by Beavon, Marshall, and others in the 1970s (cf. Tarrant 1973; Beavon and Mabin 1975; Haites 1976; Marshall 1977, 1978a, 1978b; Beavon 1978a, 1978b), which resulted in the clarification of a number of things, including a method for faithfully reproducing Lösch's central place systems and the geometrical and mathematical characteristics of hexagonal networks. However, there is a problem that has been overlooked in the discussions of these scholars. Namely, this problem is concerned with the rationality of Lösch's central place system from the perspective of optimal hierarchical structure. That is, there is no single method for constructing a hierarchy by superimposing multiple hexagonal networks of market areas because of the enormous possible combinations of central place systems. This rationality of the scheme presented by Lösch can be verified by comparing with the solutions of model. Furthermore, this chapter presents the reinterpretation concerning the objective of constructing a hierarchy in Lösch's theory and attempts to extend the model considering the agglomeration effect of hierarchical central place systems.

In Chapter 4, based on the extended model related to the location of single good obtained in

Chapter 2 and the extended model of hierarchical structure obtained in Chapter 3, the differences between Lösch's and Christaller's theories are clarified and both theories are generalized by the unified model. Whereas Lösch's theory is required to make a hexagonal market area network, whose size is unique to each good, correspond to that good on the basis of the concept of thresholds, Christaller's central place systems are based on the concept of the range of goods and the successively inclusive hierarchy (Schultz 1970), and while both have the same hexagonal structure, the latter differs from the former in terms of the locational principle and hierarchical structure. Therefore, this chapter attempts to reinterpret the locational principle in Lösch's and Christaller's theories on the basis of an extended model related to the location of single good, and reveals that it is possible to derive Christaller's central place systems using the extended model of hierarchical structure. On the basis of these findings, a generalized model that integrates the location problems of single good and the hierarchical structure is proposed using multiobjective programming. I am convinced that there is yet no attempt to integrate of both central place theories as viewed from the model structure using multiobjective programming.

CHAPTER 2

Modelling and extending Lösch's theory in the location problems of single good

2.1 The concept of demand cone and the location of firm

According to Lösch, the equilibrium of locations "is determined by two fundamental tendencies: the tendency as seen from the standpoint of the individual firm and hitherto alone considered, to the maximization of advantages; and, as seen from the standpoint of the economy as a whole, the tendency to maximization of the number of independent economic units" (Lösch 1954: 94). In other words, as a result of the free entry of a large number of firms in pursuit of profit, the entry of firms ends at the point where normal profit is mutually obtained, creating a state of locational equilibrium. At this point, the number of firms that have entered the market is maximized.

The concept of market area, which is based on demand cones, has been introduced to consider the process of locational equilibrium as a spatial perspective. Figure 1 shows a schematic representation of demand cones for linear market (Figure 1-a) and two-dimensional market (Figure 1-b). In Figure 1, Q represents maximum demand, with demand falling in proportion to the distance of a particular good from the firm location, such that demand falls to zero at distance S, which represents the upper limit of the range of a good. When a firm is able to monopolize a two-dimensional market, the market area of the firm in that market transforms to a circle with a radius S. However, assuming the case where a large number of competitors have entered the market, there is a possibility that firms will be unable to secure a circular market area with radius S. This is because, in the case that a profit can be obtained from a market area even if it overlaps with that of a competitor, a new firm can enter in close proximity, resulting in the reduction of the market area.

The minimum demand necessary in order for firms to remain in business is known as "threshold"

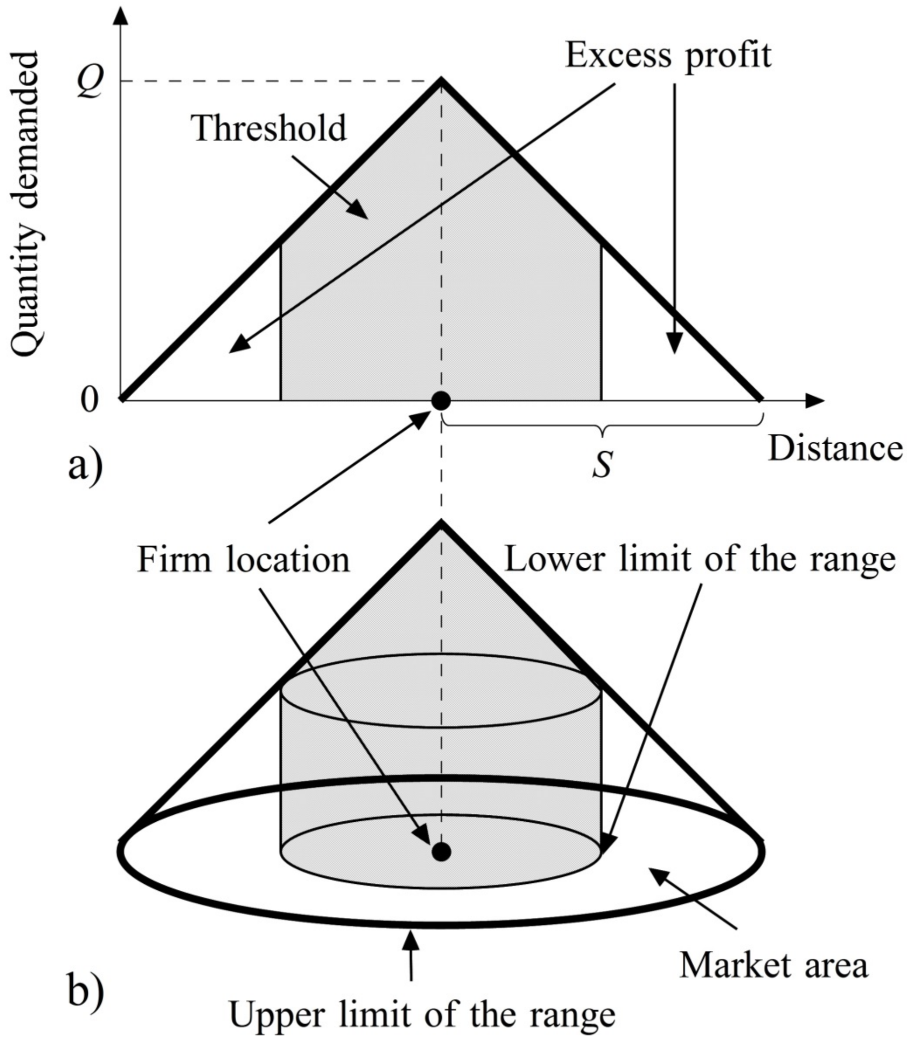

Figure 1 Concept of demand cone

(Berry and Garrison 1958b), and where there is a uniform distribution of population and households, the threshold can be expressed as the area or volume of the shaded part of Figure 1. Therefore, if a firm's threshold falls below the total demand (i.e., the whole area or whole volume of the demand cone in Figure 1), then other firms will be able to establish locations up to the point where a firm is able to secure the threshold.

Figure 2 represents linear market situations in which each firm is able to obtain maximum excess profit (Figure 2-a) or only normal profit (Figure 2-b), with the market area for each firm shrinking in

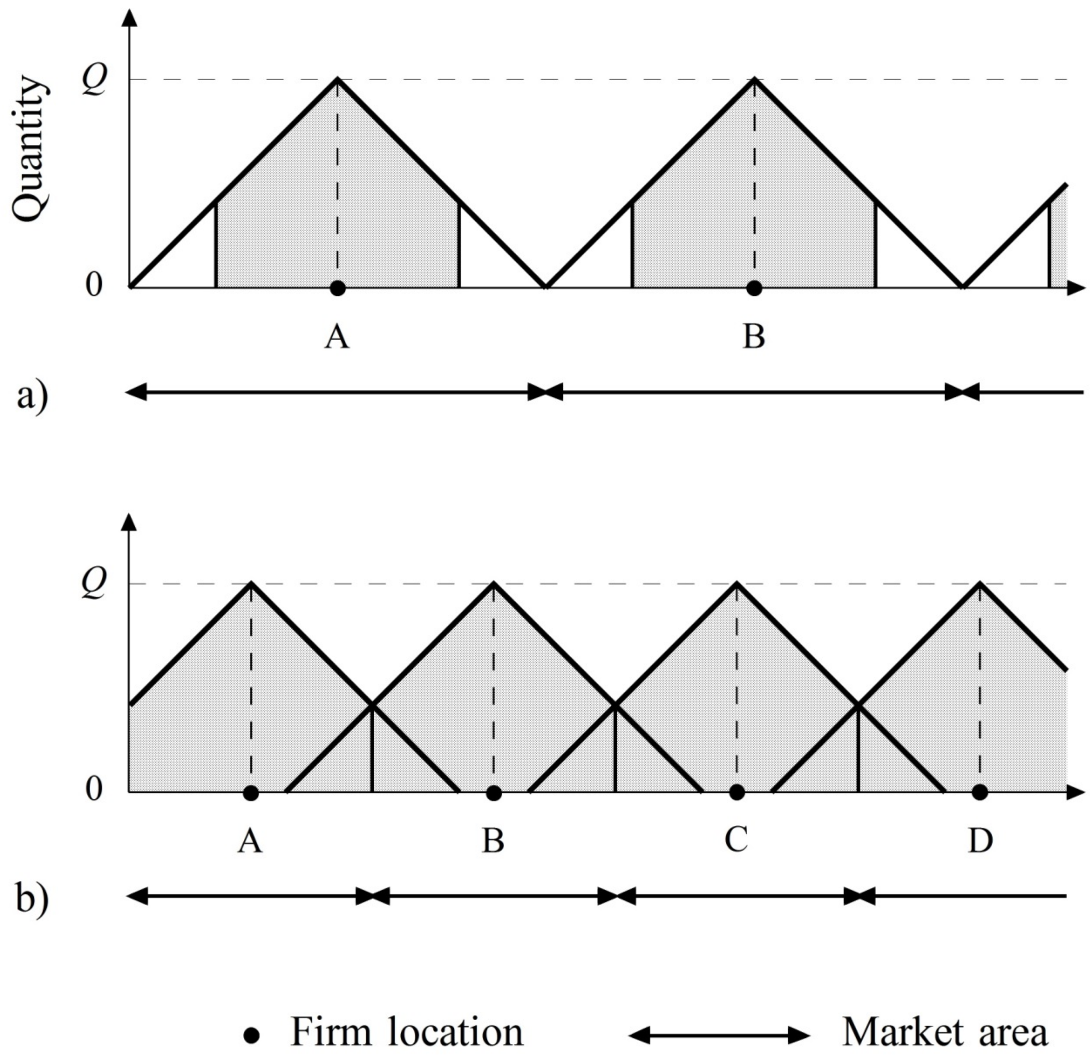

Figure 2 Löschian locational equilibrium in a linear market

The right side of a linear market is assumed to extend infinitely.

the process of shifting from the situation in Figure 2-a to the situation in Figure 2-b. Here, the situation in Figure 2-b represents the solution to the locational equilibrium process by Lösch. Assuming an infinitely spreading linear market with a uniform population distribution, firms appear at equal intervals securing market areas in which they can obtain a normal profit, which achieves the maximum number of firms in the market.

The case of a two-dimensional market is rather more complicated in that market areas overlap with one another between the locations of surrounding firms, and because the base of the demand cone is cut off, the shape of the market area is polygonal rather than circular. Of all polygons able to fill a plane, the regular hexagon is able to obtain the most efficient demand (Lösch 1954: 111), therefore, when the distribution of population is uniform, a honeycomb-like market area network comprising regular hexagons of identical size that satisfies the threshold is formed.

2.2 Problems with Kuby's formulation

Kuby (1989) attempts to model Lösch's locational equilibrium process by defining the objective of maximizing the number of firms while satisfying threshold constraints as an optimization problem. This formulation is regarded as a model that reproduces Lösch's system relatively precisely. However, there are several problems with Kuby's formulation as an operational model of the original central place theory.

First, there is a problem that the optimal solution is not always unique because of the manner in which the objective function takes an integer value. In Kuby's model, while the maximization of the number of firms is adopted as an objective function, there is a high possibility for the existence of multiple solutions that yield the same value for the objective function. For example, Figure 3 assumes a bounded linear market. When firm A and C have already been located as shown in Figure 3, a third firm B can be located between the two. However, this market is characterized by the presence of a small surplus that produces an excess profit. The demand that corresponds to this surplus will not support the establishment of a fourth firm. As a result, this surplus creates a corresponding degree of freedom for establishing the location of firm B. Specifically, firm B would be viable regardless of whether it was located adjacent to firm A (Figure 3-a) or at a position midway between firm A and C (Figure 3-b). Furthermore, there are other potential locations where the demand to secure the threshold is obtained. In the case of conditions that produce excess profit as in Figure 3, the above objective function will lead to the existence of multiple solutions for a firm's location even while the

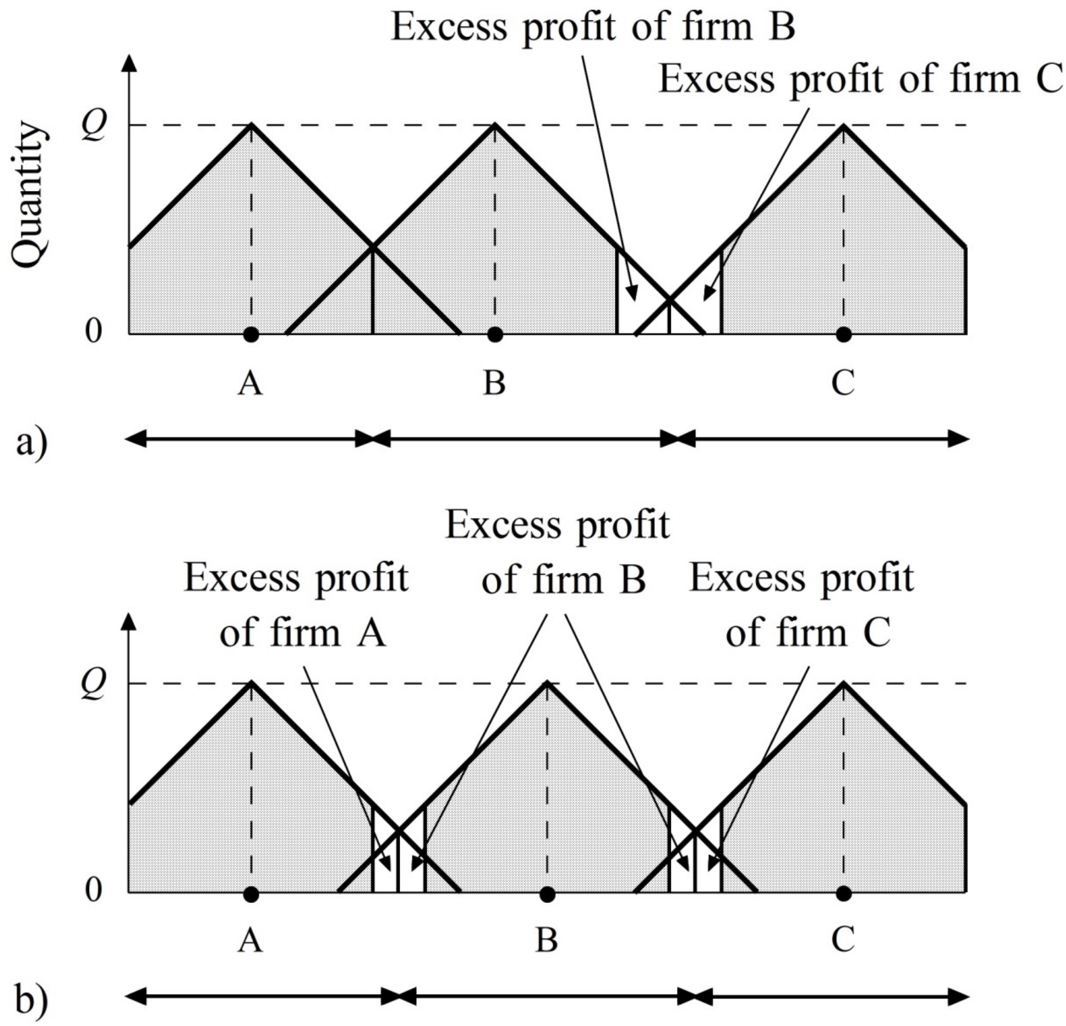

Figure 3 Potential solutions of the locational competition by three firms in a bounded market

The legend is the same as Figure 2.

maximum number of firms that can enter the market is still only three.

Of course, while the maximization of the number of firms is one of the objectives of Lösch's market area theory, an important point is that Lösch's theory assumes maximization not only of the number of firms but also of the firm's profits. Now, when we compare Figure 3-a and Figure 3-b, we see that the excess profit that firm B is able to secure is greater in Figure 3-b. Moreover, of all potential locations available to firm B, the situation in Figure 3-b, located exactly midway between firm A and C, indicates the solution that maximizes excess profit[1]. In this way, assuming that firms will act to maximize their profit and a large number of firms will appear as a result, it is desirable that they do so at points where they will be able to obtain higher profits even if the number of firms is the same. Kuby's model lacks the setting of an objective function that reflects this behavior of individual firm that leads to the maximization of the number of firms.

The second problem with Kuby's formulation is that when the model is applied in reality, the allocation of demand to the nearest firm is not guaranteed. Kuby (1989: 332-333) himself confirms this as a problem when actually applying the model in a number of hypothetical cases. This is because demand has no choice but to be allocated to firms who are not the nearest firms because of the priority given to the constraint that each firm satisfies the threshold rather than to the improvement of the objective function. Central place theory hypothetically assumes that consumers will use the nearest neighboring central place, and when this assumption is not based, the appropriate market area can no longer be defined. Accordingly, if we try to build an operational model that is faithful to at least the original central place theory, placing demand allocation on the nearest firm or central place represents an essential condition.

The final problem is regarding the condition of equal allocation toward equidistant nodes (this is the "indifference principle"). Although Kuby (1989) makes provision for a hypothetical area with a demand node arranged on a regular equilateral triangular grid when applying his model, this creates a possibility for the existence of multiple nearest firms at equal distances from the demand node. For example, when three firms are located at the vertices of an equilateral triangle of a certain size, the demand node at the center of the equilateral triangle will be equidistant from each firm. Therefore,

[1] In Figure 3, the excess profit to be obtained by firm B corresponds to the sum of the area of the blank portion of firm B's market-area. The problem of maximizing the area of blank portion can be solved analytically, with the answer being the situation in Figure 3-b located midway between adjacent firms.

theoretically, it is desirable that one third of the corresponding demand allocation should be allocated to each firm. However, when Kuby attempted to formulate the constraints to achieve equal allocation to these equidistant points, he defined the nonlinear equation that is impossible to apply the linear programming. Therefore, he adopted a linear "symmetry constraint" (Kuby 1989: 327-328) as an alternative formulation. The symmetry constraint ensures that several demand nodes equidistant from one firm become the same allocation value from that firm. Kuby is himself aware that defining this constraint is not appropriate in the formulation of an operational model for central place theory. This is because it allows for the occurrence of asymmetrical demand allocation, depending on location intervals of adjacent firms. This indifference principle problem, as described below, is critical especially when applying the model in regions where populations are not distributed uniformly.

2.3 Reformulation of Lösch's market area theory

This section attempts to improve the Kuby's model and to reformulate Lösch's market area theory. Returning to the first problem with Kuby's model mentioned above, the reproduction of the locational equilibrium process in Lösch's market area theory would ideally involve the setting of an objective function that considers both the maximization of the number of firms as well as the firm's behavior to maximize their profits. Here, the situation in which the maximum number of firms is able to enter the market at the same time, as each firm is able to achieve greater profits implies the maximization of demand for the entire market. This is because if the total demand is maximized, each firm will on average be able to obtain the maximum profit. Taking Figure 3 as an example, this is represented by a situation where the total area of the ridge formed by the overlapping of the demand cones for firm A through C is maximized (Figure 3-b)[2].

The demand maximization was pointed out as an establishing condition for central place theory

[2] However, Figure 3 assumes the case whereby the locations of firm A and C are fixed. When we hypothesize that the three firms are free to locate where they wish, total demand will be maximized when the locations of firm A and C are moved somewhat closer to the middle.

by Getis and Getis (1966) and was adopted by Kohsaka (1983) as an objective function when modelling the theory. Additionally, in locational competition by commercial establishments such as retail chain companies, which aim to increase their market share by developing multiple store locations, the maximization of acquisition demand is one of the purposes for which locations are used (Goodchild 1984; ReVelle 1986; Hua et al. 2011). Particularly, because it is possible to realize the distributed locations of stores and facilities when assuming the distance elasticity of demand expressed by the demand cone (Smithies 1941), the goal of demand maximization is also sometimes used to model the public facility location that considers accessibility for facility users (Holmes et al 1972; Calvo and Marks 1973; Wagner and Falkson 1975). Thus, in the sense that it can reflect the characteristics of distributed location and the behavior of firms in pursuit of profit, demand maximization may be reasonably interpreted as one of the purposes of implementing the ideals of central place theory. While Kuby (1989) himself proposes the total demand maximization as a potential alternative for the maximization of the number of firms, this is not applied in the actual model. Moreover, to my knowledge, there have been no studies that have explicitly modeled Lösch's theory as the total demand maximization problem. Nevertheless, the total demand maximization that is premised by the demand cone is able to reproduce Lösch's locational equilibrium process more rationally than the maximization of the number of firms.

The two remaining problems concerning Kuby's model, namely the condition of allocating demand to the nearest firms and the condition of equal allocation to equidistant points, will need to be considered together. First, as for the former, a number of constraints for encouraging the allocation of demand to the nearest facilities have been devised in studies that have dealt with the undesirable facility location problem and the capacitated facility location problem. However, because the size of problems or allocation conditions for equidistant points can differ depending on the constraints used (e.g., Gerrard and Church 1996; Espejo et al. 2012), it is also necessary to select appropriate constraints depending on the formularization of the problem at hand. On the other hand, regarding the problem of indifference principle, Gerrard and Church (1996) have demonstrated that it is possible to address the condition by adding the constraint of equal allocation to the nearest facility noted by Rojeski and ReVelle (1970). However, because this additional constraint is defined by classifying the equidistant nodes by case[3], the problem tends to be large size and complicated when there are multiple equidistant nodes. To my knowledge, there is no generalized formulation for solving the indifference principle problem using a linear programming. It might be because, to begin with, the existence of more than one equidistant node has never been assumed in facility location problems for which the model is to be applied in actual regions, although this also depends on the accuracy when measuring distances. However, it is possible to define constraints of equal allocation linearly as described below. The constraint proposed for distributing demand to the nearest facility by Wagner and Falkson (1975) is adopted because there would be no inconsistency with the indifference principle.

As a result of considering the above problems, the model of Lösch's theory in the location problem of single good is reformulated as a mixed integer programming where the total demand maximization is attempted in the following way:

$$\max Z = \sum_{i} \sum_{j \in N_i} a_i q_{ij} X_{ij} \tag{1}$$

subject to:

$$\sum_{i \in N_j} a_i \, q_{ij} X_{ij} \geq tY_j \quad \forall\, j \tag{2}$$

[3] For example, in the case of two equidistant nodes, a constraint is added such that the allocation value of 0.5, of 0.333 for three nodes, and so on. Thus, constraints are added for dividing the allocation value in accordance with the number of equidistant nodes.

$$\sum_{j \in N_i} X_{ij} \leq 1 \quad \forall\, i \tag{3}$$

$$Y_j - X_{ij} \geq 0 \quad \forall\, i \in N_j, j \tag{4}$$

$$\sum_{h \in F_{ij}} X_{ih} + Y_j \leq 1 \quad \forall\, i, j \in N_i \tag{5}$$

$$X_{ij} - X_{ik} \leq 2 - Y_j - Y_k \quad \forall\, i, j \in N_i, k \in E_{ij} \tag{6}$$

$$X_{ij} - X_{ik} \geq Y_j + Y_k - 2 \quad \forall\, i, j \in N_i, k \in E_{ij} \tag{7}$$

$$X_{ij} \geq 0 \quad \forall\, i, j \tag{8}$$

$$Y_j = 0{,}1 \quad \forall\, j \tag{9}$$

where:

a_i = population of demand node *i*;

q_{ij} = demand quantity from demand node *i* to potential firm location node *j*;

d_{ij} = distance from node *i* to node *j*;

t = threshold for a firm supplying good;

X_{ij} = fraction of demand at node *i* that is supplied by a firm at node *j*;

$Y_j = \begin{cases} 1 \text{ if a firm locates at node } j; \\ 0 \text{ otherwise;} \end{cases}$

N_i = the set of nodes *j* within radius *S*, that is $\{j | d_{ij} \leq S\}$;

N_j = the set of nodes *i* within radius *S*, that is $\{i | d_{ij} \leq S\}$;

F_{ij} = the set of potential firm location nodes *h* farther than node *j* from node *i*, that is $\{h | d_{ij} < d_{ih}\}$;

E_{ij} = the set of potential firm location nodes *k* that are equidistant to node *j* from node *i*, that is $\{k | d_{ij} = d_{ik}, k > j\}$.

Constraint (2) indicates that each firm satisfies threshold, with threshold *t* assuming a positive value. Constraint (3) prevents a demand node from over-allocating its population, and defines that demands at node *i* are equal to zero if the distance between nodes *i* and *j* is beyond the radius *S* which represents the upper limit of the range of a good. Constraint (4) states that demands at node *i* can only be assigned to a firm at node *j* if a firm is located at node *j* (this is the "self-assignment constraint"). Constraint (5) is the closest assignment constraint introduced by Wagner and Falkson (1975), and constraints (6) and (7) guarantee the indifference principle. Specifically, in the case of a firm located at node *j* (i.e., when $Y_j = 1$), all demand allocations from demand node *i* toward potential firm location node h become zero according to constraint (5), indirectly encouraging the allocation of demand to the nearest node. Then, by virtue of constraints (6) and (7), in the case that there are firms located at nodes *j* and *k*, which are equidistant from point *i* (i.e., when $Y_j = Y_k = 1$), equal allocation is realized because $X_{ik} \leq X_{ij} \leq X_{ik}$ and $X_{ij} = X_{ik}$ hold true. In the case that there is no or only one firm at nodes j and *k*, then both constraints (4) and (8) represents that either $0 \leq X_{ij} \leq 1$ or $X_{ij} = 0$ when taking X_{ij} as an example.

Here, we need to define the amount of demand q_{ij} from node *i* to node *j* on the basis of the concept of demand cone. While Kuby (1989) sets four parameters of the price of good, the transportation rate per unit of distance, the slope of the demand curve (distance elasticity), and the maximum demand, ultimately we only need consider two parameters of the maximum demand without the addition of transportation costs and the elasticity of demand per unit of distance. Accordingly, this paper assumes a simple linear relationship between distance and demand[4] as shown in Figure 1, and defines the quantity of demand as follows:

[4] It is, of course, possible to define a non-linear function, but such a function would still have the same properties in the sense that demand decreases monotonically with respect to distance.

$$q_{ij} = \begin{cases} Q - \beta d_{ij} & d_{ij} \leq S \\ 0 & d_{ij} > S \end{cases} \tag{10}$$

where Q represents the maximum demand and β the distance elasticity of demand, with $Q > 0$ and $\beta \geq 0$. Incidentally, when β is a positive, the theoretical upper limit S of the range of a good can be represented as Q/β.

2.4 Setting a hypothetical area

It is well known that Lösch assumed a hypothetical plain distributed with regular, discrete settlements as the space in which firms would be located (Lösch 1954: 114-116). Storbeck (1988, 1990) and Kuby (1989), who have attempted to model central place theory, have also used uniform lattice networks as a hypothetical area for applying the model, attempting to derive a theoretical central place system with a central places or firms located at equal intervals. Similarly, this chapter sets a hypothetical area consisting of a regular and discrete point distribution.

However, problems tend to be large size in the model detailed in the previous section, especially in constraints (4) through (7), because the number of constraint formulae increases significantly with an increase in the number of nodes. This tendency is particularly notable when using a two-dimensional market as a hypothetical area. Therefore, this chapter attempts to apply the model chiefly to the hypothetical area of a linear market in which problems can be kept comparatively small and it remains possible to obtain efficient solutions, and then expands the discussion to a two-dimensional market on the basis of the findings obtained.

The hypothetical area of the linear market used in this chapter, as shown in Figure 4, includes 61 nodes with integer values of 0 through 60 as coordinate values along the x-axis, regularly distributed in a straight line. Each node is simultaneously a demand node and a potential firm location node, and the distance between nodes is measured using Euclidean distance. The population a_i on the demand

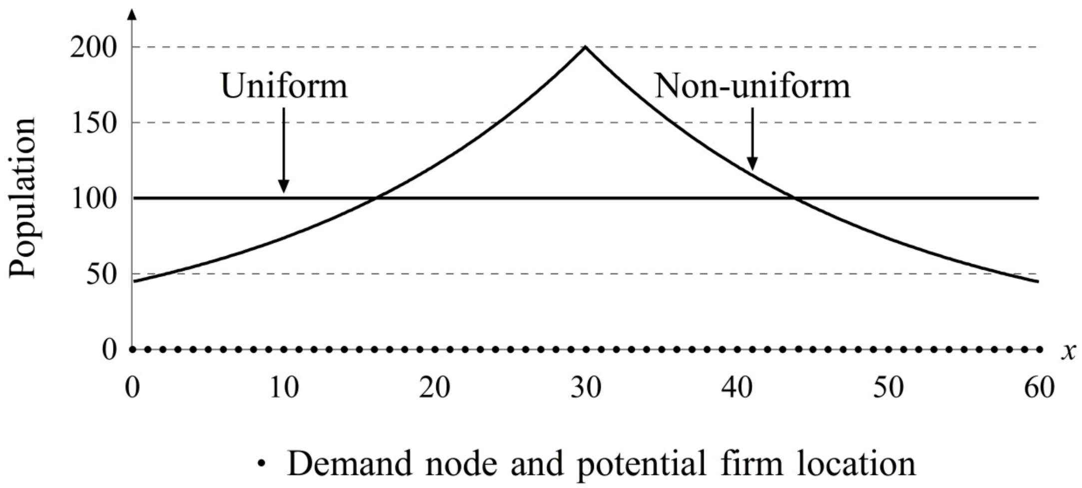

Figure 4 Hypothetical linear market and uniform and non-uniform distributions of population

node is assumed to be distributed in two patterns, namely uniformly and non-uniformly (Figure 4). In the former case, the population for all demand nodes is taken to be 100, while in the latter the population is distributed according to Clark's (1951) model presented below when the market center is a node where *x* coordinate value is 30:

$$a_i = pe^{-bd_i} \tag{11}$$

where p is a market center population, b is the decline rate of population, and d_i is the distance from the market center to node i. Specifically, $p = 200$, $b = 0.05$. Incidentally, the total population of the 61 demand nodes obtained by Equation (11) is 6,261, which is comparable with the total population of 6,100 when the population is uniformly distributed.

The reason for adopting Clark's model, known as the law of urban population density, is that Lösch's market area theory can be regarded as an alternative theory of tertiary activity on an intra-urban scale (Beavon 1977), as well as that examines to validate what the "cobweb-like" economic landscape (Parr 1973: 192) introduced by Isard (1956: 272) when their population density is as low as that of the surrounding area.

In the development of their theories, both Christaller and Lösch assumed the continuous and unbounded plain as a space for the location of their central places. However, we have no choice but to apply our model to a finite space of discrete nodes where the number of nodes is limited. The application of the model to a finite space produces distortions, especially in the regularity of the location of firms near the boundary of hypothetical area (Kuby 1989). Let us examine this problem in more detail using Figure 5, which posits a uniform population distribution.

Figure 5-a represents an example of a hypothetical area closed by boundaries, firm C, who is located near the boundary, being unable to obtain demand allocation from outside the area, is only able to obtain a smaller quantity of demand compared to firms A and B. Although theoretically the three firms locates at regular intervals, if it is estimated that the quantity of demand that firm C can acquire is less than threshold, then firm C will not enter the market or will seek elsewhere at a node where the threshold can be satisfied. Either way, the change in the demand allocation pattern will create an imbalance in acquisition demand for each firm such that the regularity of the location will be lost.

To deal with this boundary problem[5], it is necessary to consider how to assume the presence of demand allocation outside the region. Kuby (1989) enabled demand allocation from outside the region by separating his hypothetical area into an inner area comprising demand nodes and potential firm location nodes and an outer area comprising only demand nodes where firms could not be located. However, to overcome the above problem, simply setting an outer area is not sufficient.

For example, Figure 5-b indicates the presence of demand allocation to firm C from demand nodes d_5 and d_6, which are placed in the outer area. Under normal circumstances, the demand

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[5] This problem is similar to so-called “edge effects” in the point pattern analysis (Bailey and Gatrell 1995: 90).

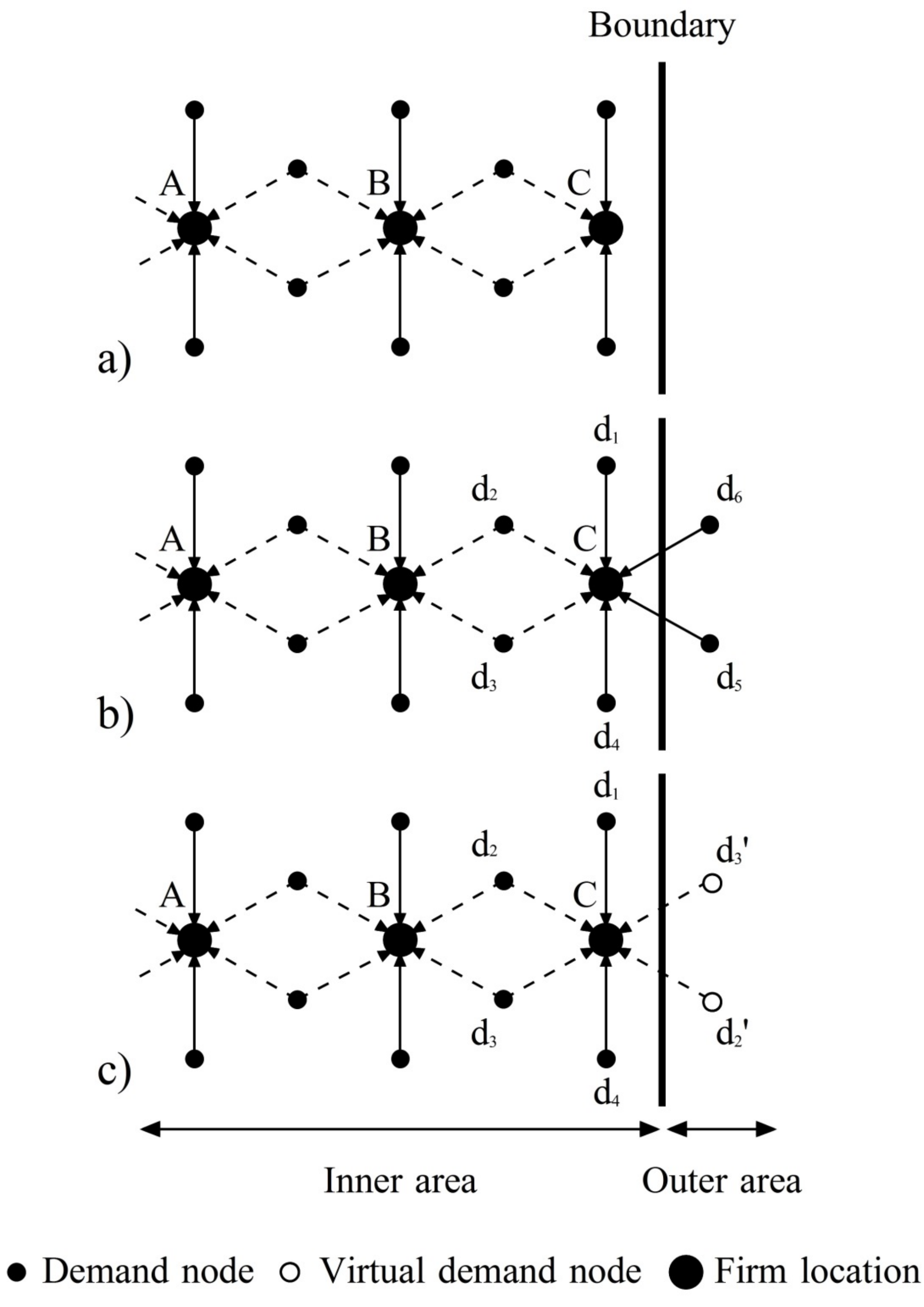

Figure 5 Demand allocation pattern near the boundary

Solid line represents to allocate all population at demand node to firm location and dashed line is half allocation.

allocation from these two nodes should be shared between firm C and the adjacent firm, as can be surmised from firm B's demand allocation pattern. However, because competing firms cannot be located in the outer area, firm C is able to have a monopoly over the demand allocation from the outer area. In this way, as the tendency to be located near boundaries with advantageous demand acquisition reinforces, this ultimately results in a number of firms different from that predicted by the theory and the occurrence of irregularities in the location pattern.

Kuby's symmetry constraints mentioned above are also a device for making a demand allocation from the outer area equivalent to that from the inner area. However, if nodes d_1 through d_6 in Figure 5-b are all equidistant from the location of firm C, this causes problems for the symmetry constraints. This is because while it is desirable for nodes d_2, d_3, d_5, and d_6 to have symmetrical allocation values of 1/2 and all demand from nodes d_1 and d_4 should be allocated to firm C. Thus, it is possible to become different allocation values even if these nodes are equidistant from the same firm.

Therefore, the following approach to the boundary problem is taken. Namely, "to add virtual demand allocation for demand nodes for which there is no point symmetry, when assuming potential firm location nodes as centers of symmetry." Using Figure 5-c as a specific example, now, for node d_1, node d_4 exists in the inner area to provide point symmetry with respect to the location of firm C. However, the locations that would provide point symmetry for nodes d_2 and d_3 correspond to the outer area and do not exist in the inner area. Thus, to account for these points d_2 and d_3, which lack point symmetry locations, we will posit the existence of virtual points $d_2^{'}$ and $d_3^{'}$. Thereafter, we will be able to add the demand allocations to firm C for an equivalent value of the demand allocations from nodes d_2 and d_3.

In fact, we are now able to deal with constraint (2) above, which relates to the threshold condition, as follows:

$$\sum_{i \in N_j} a_i\, q_{ij} X_{ij} + \sum_{i \in N_j \cap O_j} a_i\, q_{ij} X_{ij} \geq t Y_j \quad \forall\, j \tag{12}$$

where O_j is the set of demand nodes i corresponding to the potential firm location node j for which no point symmetry locations exist. That is, when point symmetry locations exist, then we include the usual demand allocation. When point symmetry locations do not exist, then we include twice the usual demand allocation.

Herein, the condition of point symmetry for demand allocation described above will be referred to for the time being as the "mirror effect." Application of the mirror effect will be limited to the threshold condition in constraint (12), and we will not introduce it into the objective function (1) that maximizes the total demand. This is because the tendency to be located near boundary areas is fostered similar to Figuire 5-b when the demand allocation from outer area is considered as an optimization variable.

2.5 Solving the total demand maximization problem

In applying the model to the linear market in Figure 4[6], we set the maximum demand Q = 10 and the distance elasticity of demand β = 0.8 in Equation (10). Additionally, the upper limit of the range of a good is set to S = 10. When β = 1, this is consistent with the theoretical range of a good (Q/β), but in that case demand would fall to 0 at a demand node ten units apart from the firm location, and demand allocation becomes meaningless. Accordingly, our parameters have been adjusted so that demand does not fall to 0 inside the demand cone.

Figure 6 shows the results of applying the model when t = 8,000 and t = 4,000. From Figure 6-a, which has a uniform population distribution and t = 8,000, we find seven firms located at equal

[6] In this paper, all problems are solved using NUOPT ver.15.1.0 by NTT DATA Mathematical Systems Inc.

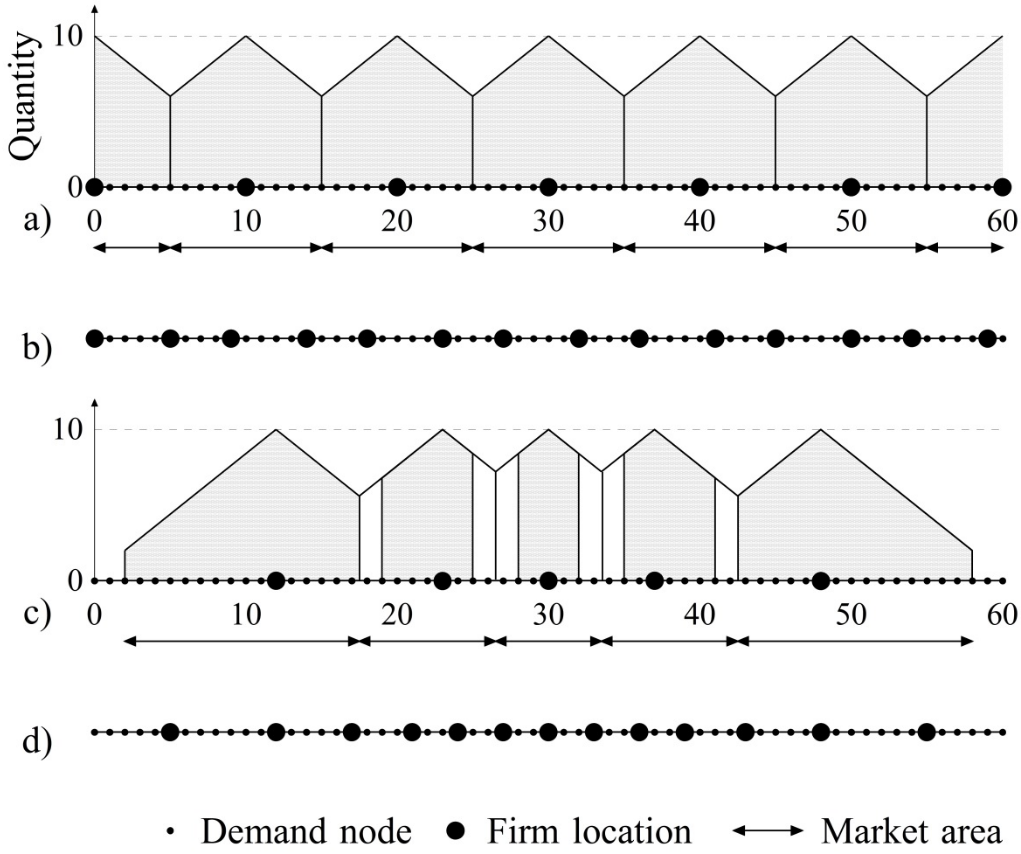

Figure 6 Solutions of the total demand maximization problem in a linear market

Shaded area corresponds to the quantity of threshold. Demand cones are omitted in b) and d).

intervals. The total demand quantity of each firm is exactly 8,000 that is sum of demand at nodes within a radius of four units from each firm location and demand at nodes divided into two among adjacent firms. Accordingly, when each firm is located next to another while maintaining a market area with a radius of five units, the number of firms able to enter the market and the total demand of the entire market is maximized with each firm able to obtain only a normal profit. In Figure 6-a, while there are firms located at both ends of the linear market at the *x*-coordinates 0 and 60, these firms are unable to meet the threshold using only demand from within the market. However, by employing the mirror effect from constraint (12), these firms will be able to secure the threshold by accepting demand allocation from a virtual outer area.

Even though the population distribution is uniform, it remains somewhat lacking in regularity for t = 4,000 (Figure 6-b). This is because, under the values we set earlier for the maximum demand Q = 10 and the distance elasticity of demand $\beta = 0.8$, there will be no solution that would be exactly equal to 4,000 for each firm, which results in the production of excess profit, leading the market area to become asymmetrical[7]. If we assume a discrete point distribution, then this presents us with a problem that we will have to somehow overcome.

This asymmetry of market area also arises in the event of a non-uniform population distribution. Figure 6-c shows the results of the case where threshold is set to 8,000 for a non-uniform population distribution. When this is compared against a case with a uniform population distribution (Figure 6-a), we see that the market area size varies significantly depending on location, producing asymmetry. In other words, while it is possible to meet the threshold within a comparatively confined market area in the vicinity of market centers with dense populations, a much wider market area is needed to acquire the demand satisfied the threshold in areas with a sparse population distribution. Because the breadth of distance between firm locations is determined in response to population distribution, drawing boundaries between market areas for adjacent firms in a linear market could cause the development of market areas characterized by left–right asymmetry. This tendency will be similar in the case where threshold is set to 4,000 for a non-uniform population distribution (Figure 6-d). As stated above, using the symmetry constraint used by Kuby (1989) implies that nodes that are equidistant from a firm location have the same demand allocation value, thus preventing the emergence of the asymmetric market areas.

From Figure 6-c, we see that in addition to excess profits occurring for firms in the vicinity of

[7] Specifically, the solutions that yield values closest to 4,000 are 3,680 and 4,100. In the latter case, which satisfies the threshold, the allocation value takes a left–right asymmetry at demand nodes at a radius of two units from the firm location.

market centers, there are also demand nodes not included in the demand cones toward both edges of the market. The former is the effect of a discrete point distribution. If the market were a continuous space, firms would likely be located even more closely together. The latter is the result of the coverage condition of a good as shown in constraint (3). As opposed to Christaller's central place theory, which covers all demand nodes inside the upper limit of the range of a good (that is "mandatory coverage"), Lösch's market area theory prioritizes firm's acquisition of profit. As a result, it is possible that uncovered area will appear in areas with low demand.

Application of the model to a non-uniform population distribution is similar to the case wherein density conversion is employed for deforming the hexagonal networks of central places in response to demand allocation; cartograms (Getis 1963) and map transformation techniques (Rushton 1972; Sugiura 1991) are a few such examples. Then, imagining a situation where we rotate the linear market, expanding it as a two-dimensional market, we can infer the bias pattern of market area networks in the case of non-uniform population distribution of Isard (1956).

Although there is a risk of large size problems occurring when the model is applied to a two-dimensional market, let us examine below the possibility of applying the model using a relatively small size two-dimensional market as a hypothetical area. The hypothetical area is a discrete space of 127 demand nodes and 43 potential firm location nodes on a regular equilateral triangular grid as shown in Figure 7. Note that not all demand nodes are potential firm location nodes for the following two reasons: (1) to reduce the size of the problem and (2) to consider the influence of blank areas where there is no demand between nodes[8].

When applying the model, because the nearest distance between nodes in Figure 7 is four units

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[8] When assuming a discrete point distribution, the fact that there is a possibility of "gaps" occurring in the supply of a good between nodes implies that our hypothetical area would not fulfill its role as a continuous virtual plane. I have addressed this problem by placing demand nodes separately from potential firm location nodes. For a closer discussion, see Ishizaki (1992). Note that there are some cases in which it is possible to examine central place systems when uncovered areas occur (Church and Bell 1990).

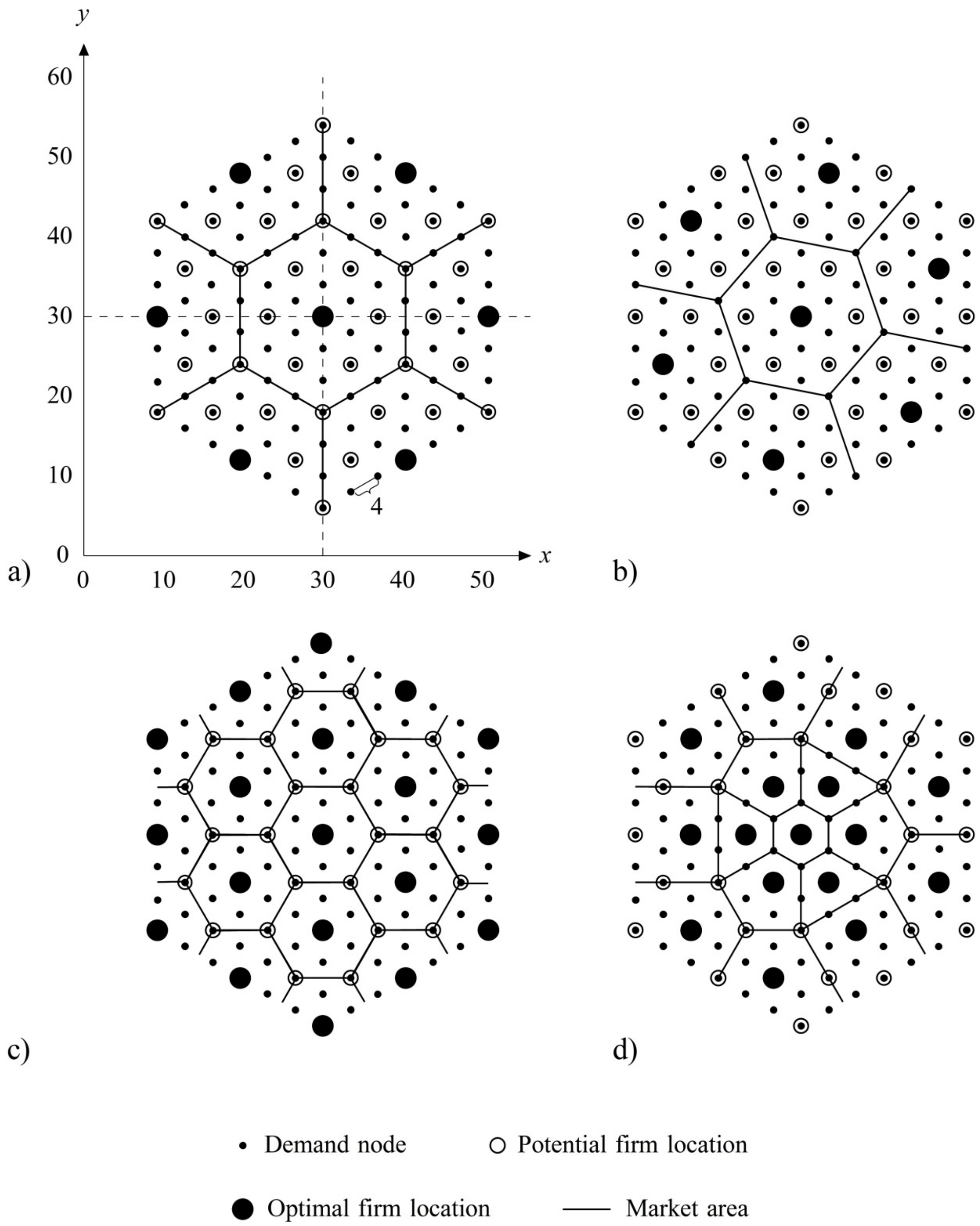


Figure 7 Solutions of the total demand maximization problem in a plane market

and considering the number of nodes, the following parameters have been set. Namely, the maximum demand $Q = 10$, the distance elasticity of demand $\beta = 0.5$, and the upper limit of the range of a good $S = 16$. As with the linear market, the population distribution is such that all nodes have a population of 100 in uniform distributions, while the non-uniform distributions of population have been calculated on the basis of Equation (11). Specifically, setting one node where *x*- and *y*-coordinate values are 30 in Figure 7 as market center, the parameters for Equation (11) have been set to $p = 300$ and $b = 0.08$. As a result, the aggregate population for 127 demand nodes is 12,013, which is approximately the equivalent of a population of 12,700 in the uniform population distribution case. In addition, to facilitate the comparison of the model's results for different population distributions and threshold conditions, the constraint that at least one firm must be located in a market center has been added to the model.

Figures 7-a through c represent the results of applying the model for uniform populations with respective thresholds of $t = 16{,}000$, $t = 14{,}000$, and $t = 6{,}000$, while Figure 7-d shows the result for a non-uniform population with a threshold of $t = 5{,}000$. Summarizing the findings obtained from these results, we can confirm the following among other things: (1) as Beavon (1977) has summarized for Lösch's theory for uniform population distributions, regular hexagonal networks of various sizes are formed because the number of demand nodes included in a market area is different according to the threshold value, and that (2) as indicated by Isard (1956), for non-uniform population distributions, market area networks emerge in which the hexagonal structure is distorted.

As can be seen by comparing Figure 7-a and Figure 7-b, the derivation of market area networks of different sizes, even for the same number of firms, is the result of the fact that an objective function of the model is not the maximization of the number of firms but the maximization of total demand. In addition, the mirror effect in Equation (12) and the condition of the equal allocation of demand in constraints (6) and (7) result in the realization of a pattern whereby firms are located at regular intervals while securing a minimum market area that yields normal profit. Furthermore, in the case of a non-uniform population distribution, we can see that it corresponds to the formation of more realistic (if more complex) market area networks, as in the case of nodes allocated equally to four equidistant firms.

From the above, we may regard the model demonstrated in this chapter as a model that reliably reproduces the locational principle and preconditions in Lösch's market area theory. Moreover, this model will be considered as an operational model that enables the derivations of theoretical location patterns even under various real-world conditions.

2.6 Developing the total profit maximization problem

As described above, in Lösch's market area theory, which considers locational equilibrium to be the result of perfect competition and free entry of firms into the market, the firm will be unable to obtain anything other than normal profit. However, in reality, according to the changes of population distribution, prices, and production costs in the market, it is possible that excess profits will begin to accrue after locating firms once or that a firm will only be able to achieve demand that falls below the threshold. To begin with, the situation where no firm is able to obtain excess profit is extremely unrealistic. Conversely, what would be the most profitable situation for a firm? This would be when a firm is able to secure all excess profits included in the market area of radius *S* within the demand cone shown in Figure 1.

Figure 2-a represents a situation in which firms A and B participate while securing maximum profit. If we assume this linear market to extend infinitely, then each firm will not only locate without overlapping a market area of radius *S*, but also will ensure that all demands are captured in the market. In contrast to "the total demand maximization problem", in which each firm is located as close to one another as possible on the condition that a normal profit can be obtained, this can be considered as "the total profit maximization problem" (Hansen and Thisse 1977). In this case, all firms participating in the market are able to obtain maximum excess profit.

In the example given in Figure 2-a, the total profit is represented by the total area of blank parts. In other words, it is simply the total area that is obtained by subtracting the total area of shaded parts as two firm's thresholds from the total area under the ridge as the total demand. Accordingly, the total profit maximization problem can be formulated by correcting the objective function (1) to the following equation and by using constraints (2) through (9) without modifying them:

$$\max Z = \sum_{i} \sum_{j \in N_i} a_i q_{ij} X_{ij} - \sum_{j} t Y_j \tag{13}$$

Here, let us apply the total profit maximization problem to the linear market of the hypothetical area in Figure 4. Our parameters, just as with Figure 6, will be maximum demand Q = 10, distance elasticity of demand β = 0.8, and the upper limit of the range of a good S = 10. Additionally, the mirror effect is applied by using constraint (12) instead of constraint (2). Figures 8-a and 8-b represent the results of applying the model to uniform populations with respective thresholds of t = 8,000 and t = 4,000, while Figure 8-c shows the result for a non-uniform population with a threshold of t = 8,000.

From Figure 8-a, we find that each firm has a monopoly over a distance of ten units that represents the maximum radius of the demand cone, with three firms participating in the market located at equal intervals. Hence, under normal circumstances, maximum excess profit will be obtained for each firm when they are located without any overlap in the demand cone. However, when threshold is set to 4,000, by increasing the number of firms participating in the market, the interval between firm locations is narrowed, necessarily leading to a subdivision of the market area (Figure 8-b).

Given infinite space, the fourth firm would enter outside area, as shown in Figure 8-a, and each firm would be able to earn even greater excess profit without any mutual market area interference. However, even considering the mirror effect expressed in constraint (12) into account, the linear

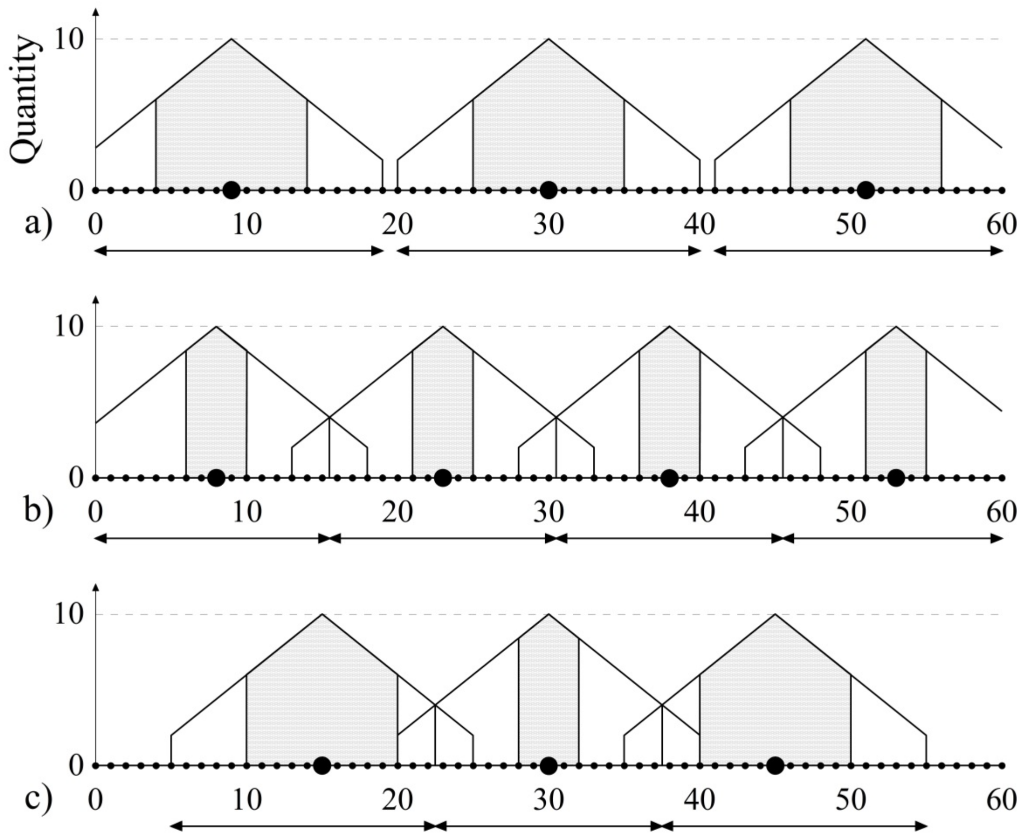


Figure 8 Solutions of the total profit maximization problem in a linear market

The legend is the same as Figure 6.

market of our hypothetical area is not infinite. Accordingly, insofar as we are considering a finite space, the addition of a firm inside a limited market as in Figure 8-b can sometimes lead to the occurrence of a rise in total overall profits depending on threshold values.

In addition, even in the case of a non-uniform population distribution, demand cone overlap can still be confirmed (Figure 8-c). This is because firms intend to be located in a densely populated area that is more advantageous for earning profit than the surrounding area with little prospective demand. As a result, even if the market areas of radius *S* are partially overlapped between firms, the total profit is maximized in the entire market. Therefore, in the case of a non-uniform population distribution, there is a conceivable possibility that demand cones will overlap regardless of whether the space is finite or infinite.

In contrast to the total demand maximization problem (Figure 6) which seeks to maximize the number of firms, there are fewer firm locations in the total profit maximization problem. The total profit maximization problem represents an antithetical model in the sense that it supplies a good by having as few locations as possible.

2.7 Extension of the model using multiobjective programming

If Lösch's market area theory is regarded as the total demand maximization problem on the unrealistic assumption of perfect and free competition, then we will be forced to admit that the total profit maximization problem is also unrealistic in the same way. This is because it is difficult to imagine that firms will make efforts together with one another to secure maximum excess profit without encroaching on each other's areas, as shown in Figure 2-a. A situation such as the one shown in Figure 2-a is limited to an extreme case, for example, when a market is monopolized by only one company (Ghosh and Harche 1993).

A solution between the two extremes of the total demand maximization and the total profit maximization is realistically conceivable. In other words, a situation in which a certain degree of excess profit is to be expected is plausible. The method for simultaneously optimizing such multiple and competing objectives is known as multiobjective programming (Cohon 1978). Multiobjective programming typically involves the use of a weighting method that combines multiple objective functions by applying a weighting to each individual objective and then adjusting the weighting minutely to derive several eclectic non-inferior solutions (Pareto optimal solutions).

Therefore, the extended model in which two objectives are integrated can be formulated as follows using multiobjective programming by substituting Z_1 for the right-hand side of objective function (1), which represents the total demand, and Z_2 for the right-hand side of objective function

(13), which represents the total profit:

$$\max Z = (1 - w)Z_1 + wZ_2 \tag{14}$$

where w is a weight that takes a value of $0 \leq w \leq 1$. Namely, when $w = 0$, the objective function (14) represents the total demand maximization problem, and when $w = 1$ it equivalent to the total profit maximization problem. Moreover, because eclectic solutions for both can be obtained when $0 < w < 1$.

Here, let us apply the extended model to the hypothetical area in Figure 4. To avoid complication, we will demonstrate only the solution for a uniform population distribution where threshold $t = 8{,}000$. Note that our parameters will be as they have been thus far, with maximum demand $Q = 10$, distance elasticity of demand $\beta = 0.8$, and the upper limit of the range of a good $S = 10$. In addition, the constraints will be according to constraints (3) through (9), with the mirror effect applied by adopting constraint (12). Weighting w is varied in increments of 0.01.

Table 1 lists five non-inferior solutions obtained by changing the weighting w. Of these solutions, Solution A corresponds to the solution to the total demand maximization problem (Figure 6-a), whereas Solution E corresponds to a solution for the total profit maximization problem (Figure 8-a). Among the objective functions, there will be cases when total profit takes a negative value, because while extrinsic demand will be added to the threshold condition by the mirror effect of constraint (12), the demand in question will not be reflected in the objective function. Accordingly, a negative total profit does not necessarily mean that the threshold is not satisfied.

By observing the transition of the two objective function values in Table 1, we can see that there is a trade-off relationship between the two objectives. In multiobjective programming, an absolutely optimal solution that is simultaneously optimal for multiple objectives generally does not exist. Hence, on the basis of the variations in each objective function value, we turn to weigh the compromise

Table 1 Non-inferior solutions of the extended model for single good

Solution	w	Objectives		Number of firms
		Total demand (Z_1)	Total profit (Z_2)	
A	0.00～0.04	49,000	-7,000	7
B	0.05～0.30	48,600	600	6
C	0.31～0.46	46,120	6,120	5
D	0.47～0.78	42,440	10,440	4
E	0.79～1.00	36,200	12,200	3

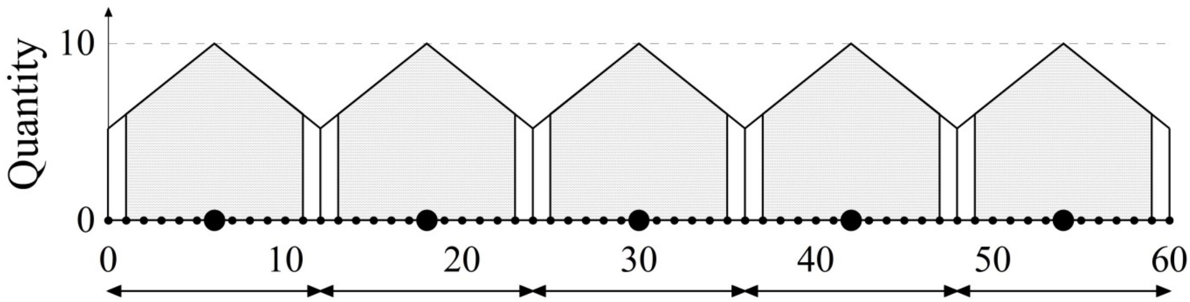


Figure 9 Locational pattern of non-inferior Solution C in Table 1

The legend is the same as Figure 6.

alternative of our non-inferior solutions. Accordingly, rather than leading us to a unique optimal solution, multiobjective programming offers a richly flexible model in the sense that it seeks out solutions interactively from among many competing objectives.

Of five non-inferior solutions in Table 1, Figure 9 shows the results of Solution C situated middle solution between the two objectives of total demand maximization and total profit maximization. All five solutions show a trend for forms to be located at equal intervals, and solution C is no exception.

We might go so far as to say that the results shown in Figure 9, which are similar to neither Figure 6-a nor Figure 8-a, illustrate the situation whereby each firm is able to earn moderate excess profit.

Ultimately, the differences between five non-inferior solutions depend on the number of established firms. The weight w of objective functions in multiobjective programming is a relative indicator for deriving multiple non-inferior solutions, and the value of the weighting itself does not necessarily have any meaning. Although each of non-inferior solutions shown in Table 1 appears for a certain range of values of weight w, the objective function values Z_1 and Z_2 remain constant within that range. In other words, the model represented by objective function (14) may be thought of as a model that variably captures the number of firm locations while extending Lösch's model ultimately to the maximum number of firms entering into the market.

CHAPTER 3

Modelling and extending Lösch's theory in hierarchical structures

3.1 Lösch's outline of constructing a hierarchy

In Lösch's market area theory, a two-dimensional market with a uniform population distribution takes shape as a market area of regular hexagons of varying sizes according to the threshold of each good. For a continuous plain, the fact that firms can be freely located on the plain implies that innumerable market area networks can exist at slightly varying intervals. Hence, Lösch fixed the places where firms can be located by assuming a plain on which the settlements that indicate demand nodes and potential firm locations are distributed regularly and discretely in a lattice network of regular equilateral triangles. As a result, the location interval between firms is defined in the distribution of the settlements, and conceptually possible market area networks are limited to a finite number. Accordingly, market area networks is determined in consideration of regular hexagonal market areas of varying sizes that regard two arbitrary settlements as the neighboring market centers.

Specifically, Lösch examined the case where the distance between market centers gradually increases, assuming as for the start from the market area network in which the distance between market centers is $\sqrt{3}a$, when distance between adjacent settlements is assumed to be a. As in Figure 10, when the distance from market center O to P_1 is taken as $\sqrt{3}a$, the six settlements that serve as the nearest market centers for O, including P_1, will be distributed along a circle of radius $\sqrt{3}a$ centered on O. The boundaries delimited between these nearest market centers represent the market area for O and yield the shape of a regular hexagon. Assigning the area Number 1 to this smallest market area A_1, market area A_2, with the next-largest distance between market centers, will be the market area delimited between six settlements along a circle of radius $2a$ from O to P_2. In this way,

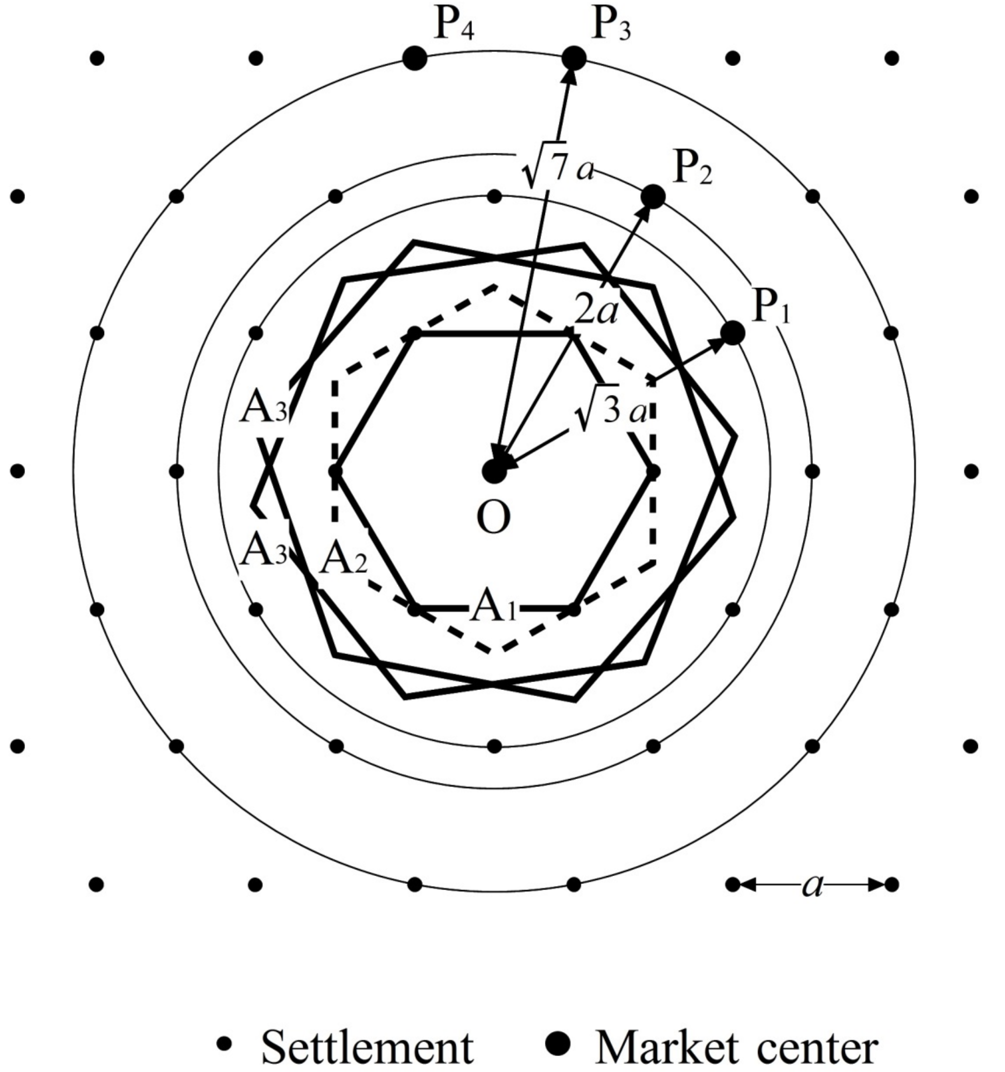


Figure 10 Derivation of the market areas of various sizes

Solid lines and dashed lines represent the boundaries of market areas assigned the market area number. The radius of circles from O is each segment OP_1, OP_2, and OP_3.

starting with O as the reference point, and following the distance from O to any settlement in ascending order, and then defining the boundary between the settlements along a circle with a radius of the distance in question, it will be possible to derive market areas of various sizes.

However, the settlements on the circumference of the circle will not necessarily be six. For example, in market area A_3, for which the distance between market centers is $\sqrt{7}a$, there are twelve settlements that exist along the circle of radius $\sqrt{7}a$ and centered on O (Figure 10). As a result, there could be two alternatives, in the hexagonal market area described by the boundary between O and P_3 and the market area described by the boundary between O and P_4. When there are only six nearest market centers, as in market areas A_1 and A_2, there can be only one market area network, but when there are twelve or more nearest market centers, as in market area A_3, then there will be multiple market area networks of identical size.

Lösch's hierarchical arrangement is able to represent market area networks of different sizes as a problem of superimposition, as shown in Figure 11. Lösch assumed that all of the market area networks "shall have at least one center in common" (Lösch 1954: 124), and he regarded this common center as a metropolis. Then, using the metropolis as a reference point, he created market area networks of varying sizes and overlaid the networks in ascending order from the smallest market area network. On the basis of the size of the market area that satisfies the threshold, each good can be associated with a single market area network[9]. Figure 11 illustrates the relation between three goods and three kinds of market area networks. Whereas the market area network corresponding to goods Numbers 1 and 2 is uniquely determined, because there are two alternative identically sized market area networks corresponding to good Number 3, it will be necessary to select one of them.

The difference between alternatives, as shown in market area A_3 for Figure 10, is the difference in the angle of the sides of the regular hexagonal market area. That is, what differs is the direction of the nearest market center, as seen from the metropolis. Lösch, when selecting one from among multiple alternatives, considered an array of market area networks, noting "we turn the nets about this center in such a way as to get six sectors with many and six with only a few production sites" (Lösch 1954: 124). The production site indicates a supply point for good or a firm location and corresponds to a

[9] However, as in Figure 11, the number of goods associated with a market area network is not necessarily one. In fact, it could be that "one and the same area will usually be the market for several goods, since there are more products than regional sizes" (Lösch 1954: 122).

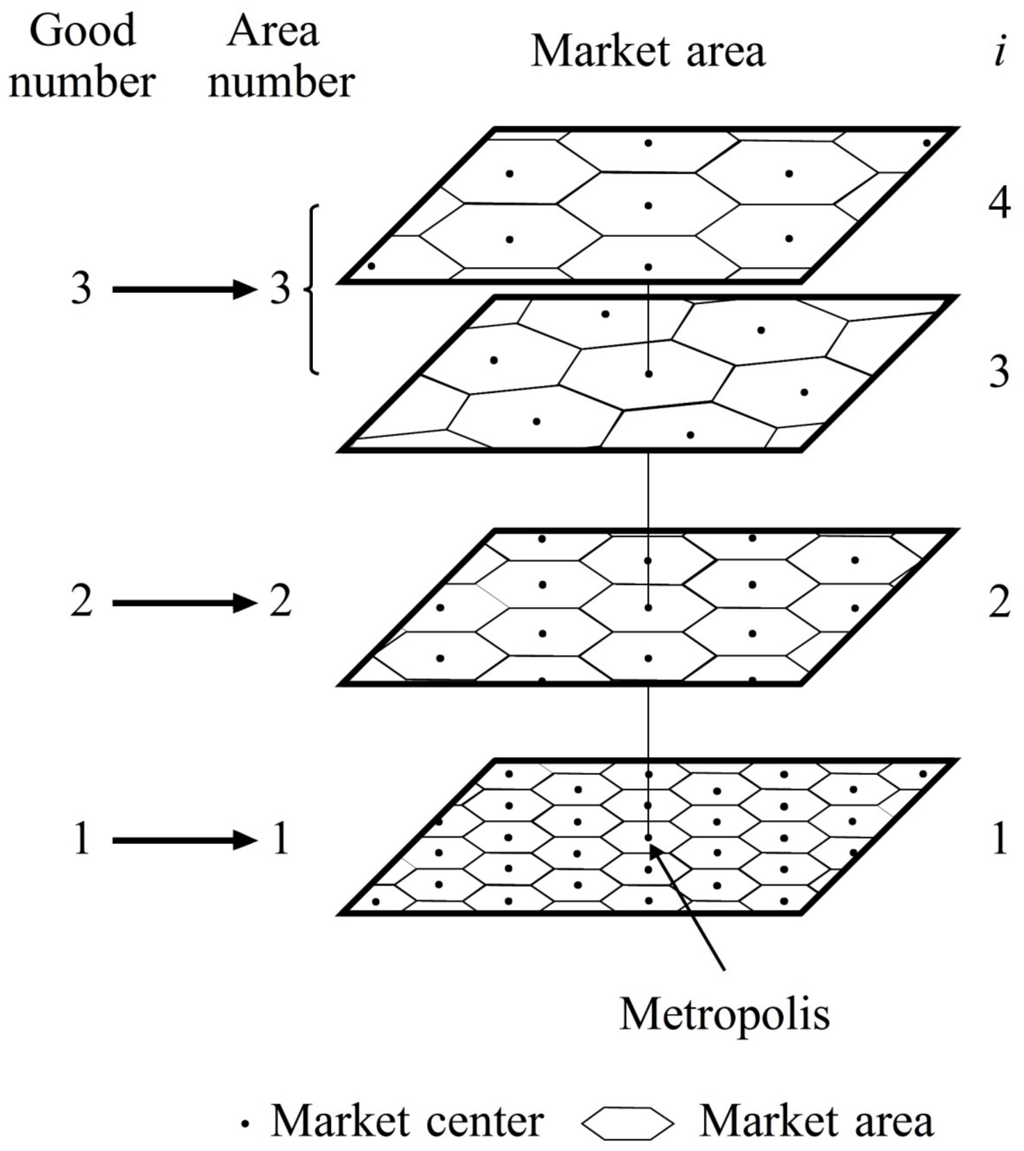


Figure 11 Superimposition of market area networks of the Lösch's system

A right-side figure is a serial number of the market area networks.

market center on the market area selected from alternatives. Thus, as a result of selecting the market area network corresponding to each good on the basic principle of concentrating supply points in a specific zone, a central place system is derived in which there is an alternate appearance of "city-rich sectors" and "city-poor sectors."

3.2 Combinational problem of the superimposition of market area networks

What becomes problematic about the hierarchical arrangement in Lösch's market area theory is the method of selecting one market area network from multiple alternatives. Beavon (1977: 89-92), after arranging market areas of various sizes, has produced a table listing 150-center market area illustrated by Lösch (1954: 127). While this table lists market area numbers and the number of settlements included in one market area, it does not specify which market areas have alternatives, or how many alternatives there are. Thus, by referring to the table provided by Beavon (1977: 92), Table 2 adds the distances between market centers for each market area and the number of alternatives. However, to avoid redundancy, to the discussion is limited to the fifty-five market area networks illustrated by Lösch (1954: 128).

Table 2 List of fifty-five market area networks

Area no.	*n*	*d*	*a*	Area no.	*n*	*d*	*a*	Area no.	*n*	*d*	*a*
1	3	1.73	1	20	52	7.21	2	39	111	10.54	2
2	4	2.00	1	21	57	7.55	2	40	112	10.58	2
3	7	2.65	2	22	61	7.81	2	41	117	10.82	2
4	9	3.00	1	23	63	7.94	2	42	121	11.00	1
5	12	3.46	1	24	64	8.00	1	43	124	11.14	2
6	13	3.61	2	25	67	8.19	2	44	127	11.27	2
7	16	4.00	1	26	73	8.54	2	45	129	11.36	2
8	19	4.36	2	27	75	8.66	1	46	133	11.53	4
9	21	4.58	2	28	76	8.72	2	47	139	11.79	2
10	25	5.00	1	29	79	8.89	2	48	144	12.00	1
11	27	5.20	1	30	81	9.00	1	49	147	12.12	3
12	28	5.29	2	31	84	9.17	2	50	148	12.17	2
13	31	5.57	2	32	91	9.54	4	51	151	12.29	2
14	36	6.00	1	33	93	9.64	2	52	156	12.49	2
15	37	6.08	2	34	97	9.85	2	53	157	12.53	2
16	39	6.24	2	35	100	10.00	1	54	163	12.77	2
17	43	6.56	2	36	103	10.15	2	55	169	13.00	3
18	48	6.93	1	37	108	10.39	1				
19	49	7.00	3	38	109	10.44	2				

Note: *n* is the number of settlements included in a single market, *d* is the distance between market centers, and *a* represents the number of alternatives.

Lösch (1954: 117) has previously referred to how the square of the distance between market centers corresponds to the number of settlements included in one market area when the distance between settlements $a = 1$. Further, depending on the value for the number of settlements, we can determine whether or not a market area will have alternatives (Marshall 1977). In fact, as previously described, the number of alternatives can be determined by measuring the distance from the settlement that serves as a reference point to any other settlement, and then counting the number of settlements distributed along a circle with the respective radiance around the metropolis and then dividing that number by six. Looking at Table 2, we find that within fifty-five market area networks, thirty-nine networks have multiple alternatives, and the maximum number of alternatives is four. Thus, the number of market area networks that should be selected from among alternatives is more than the number of networks uniquely determined. This tendency becomes even more prominent when the number of types of market area is increased.[10]

Table 2 demonstrates the fact that when all market area networks are superimposed, there will be innumerable combinations of which only one will be selected from alternatives. For example, in fifty-five market-area networks shown in Table 2, there are thirty-four market area networks with two alternatives, three with three alternatives, and two with four alternatives. As a result, the alternatives can be combined in $2^{34} \times 3^3 \times 4^2$—that is, about 7.4 trillion—different ways. If the number of market area networks to be considered increases further, the number of combinations will reach astronomical figures because the combinatorial explosion occurs[11]. The results of the superimposition of market area networks illustrated by Lösch are thus only one of the myriad numbers of such combinations.

Although the combination described above represents a number that conforms to Lösch's provision "to have at least one center in common", what might happen when this precondition is relaxed? Lösch's hierarchical arrangement has a reputation of being more flexible than Christaller's

[10] While the number and percentage of market area networks with alternatives is 39 for 55 types (70.9%), for 150 types it becomes 123 (82.0%), and 932 (93.2%) for 1,000 types.

[11] For example, the number of combinations for 150 market area networks is 5.95358×10^{42} ways!

central place theory (Dicken and Lloyd 1990: 35) because it offers a high degree of freedom as congruence of every market center is not required. In this sense, the presence of a metropolis that assimilates all market centers would be exceptional, and there is no imperative to recognize such exceptions. When the metropolis is not fixed as a reference point, a wider variety of superimposition patterns can be considered (Matsubara 2013: 46-47).

Not having a fixed reference point means that when we align market area networks with the distribution of settlements, we are able to arrange market centers offset from the metropolis. Specifically, let me explain using Figure 12 with reference to the example of market area Number 1 in Table 2. Figure 12-a shows the boundary of market area and the adjacent market centers, when a market center and the point A as a metropolis match each other. Now, how many conceivable patterns could be obtained by moving the market centers away from the point A and then drawing a different pattern than that of the market area network in Figure 12-a?

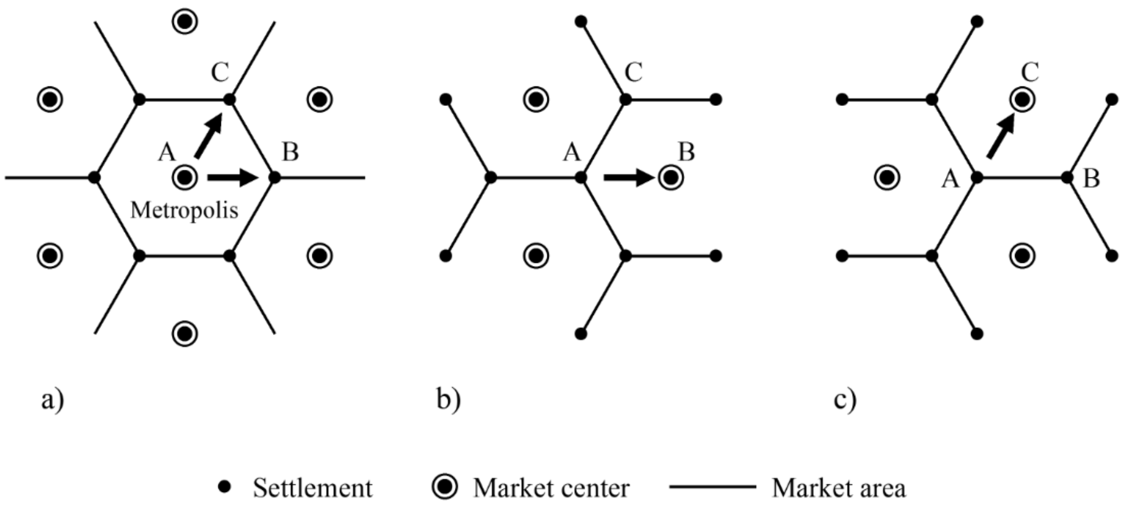


Figure 12 Market area network obtained by moving the market center to another point from a metropolis

Given that other market centers also move in tandem as a market center moves from the point A, the range of movement for a market center on the point A may be limited to be within the market area of the point A. In the example of Figure 12, the six settlements around the point A will serve as our movement point candidates. However, the market area networks that can be obtained when one actually moves from the point A, excluding the same patterns, converge to the two patterns shown in Figure 12-b and Figure 12-c. That said, when we move from the point A to the point B (Figure 12-b), two points other than the point B from among the above-mentioned six movement point candidates become market centers, and it is no longer necessary to consider these as point candidates. The same can be said to hold true when moving from the point A to the point C (Figure 12-c). In other words, we are able to form three distinct market area networks for market area Number 1, including one case where a market center is matched with the point A.

Considering other market area numbers in the same way, we find that the number of patterns of market area networks, including those that do not fix a market center to the metropolis, is equivalent to the number of settlements for each market area network in Table 2. In the case of market areas with alternatives, the existence of market area network patterns with different numbers of settlements for each alternative means that, in the end, the number of possible patterns is equal to the product of the number of settlements and the number of alternatives. In other words, this leaves us with three patterns for market area Number 1, four for market area Number 2, and fourteen for market area Number 3. Accordingly, when we perform a simple trial calculation of the combination of superimposition problems for market area networks, only ten kinds of market area networks still leave us with over 300 billion possible patterns.

The precondition, “to have at least one center in common,” might have been set by Lösch because it was impossible to examine such enormous number of combinatorial problems. However, even when we fix a market center to the metropolis for the sake of argument, there is no change in the existence of a large combination number when overlaying market area networks. Our problem, then, is how Lösch arrived at one pattern from among these many combinations.

3.3 The superimposition problem as a combinatorial optimization

When selecting a market area network with multiple alternatives, Beavon and Mabin (1975), who faithfully reproduced Lösch's process of constructing a hierarchy, have demonstrated that the pattern illustrated by Lösch can be obtained by the following three steps: 1) align the nearest market center as seen from the metropolis with specific sectors (Dacey 1965: 121), 2) when there are multiple alternatives for the location of a market center within specific sectors, choose the alternative that maximizes the number of coincident market centers, and 3) on such occasions, select a market center with comparatively higher-order goods from among points when there is an identical number of goods.

Of these, Step 1) is a condition that prioritizes locating market centers in particular sectors involved in the creation of "city-rich sectors", and Step 2) is a condition that maximizes the number of coincident market centers. The other condition 3) is a secondary condition associated with 2), and in the 150 market area networks examined by Beavon and Mabin (1975), this condition is applied to only two types of market areas A_{85} and A_{150}. Tarrant (1973) and Marshall (1977) have also mentioned the two conditions of prioritizing location in particular sectors and maximizing the number of coincident market centers, and Marshall's (1977) opinion differs from Beavon and Mabin (1975) because of preceding the latter condition (Hayashi 1986: 200).

The problem of seeking out the most desirable solution from among a finite combination number in light of a certain purpose is known as the combinatorial optimization (Korte and Vygen 2008). Lösch's process of constructing a hierarchy to derive a pattern from the superimposition of multiple market area networks can be defined as a combinatorial optimization. Here, this section attempts to model the problem of superimposing market area networks using mathematical programming. Assuming that Lösch would regard a sectoral pattern as the best spatial solution to the arrangement of market area networks (Dicken and Lloyd 1990: 31), the prioritization of locating market centers in particular sectors becomes the criteria for optimization among the steps listed by Beavon and Mabin (1975). However, because there are cases where there is no determinate solution solely for prioritizing location in particular sectors, when we define the problem by adding the maximization of the number of coincident market centers from Step 2) in the above list, then the problem of superimposing market area networks can be formulated as an integer programming problem using multiobjective programming as follows:

$$\max Z = w \sum_{j \in S} \sum_{m} Y_{jm} + (1 - w) \sum_{j} \left(\sum_{m} Y_{jm} - F_j \right) \tag{15}$$

subject to:

$$\sum_{i \in N_m} \mathrm{X}_{mi} = 1 \quad \forall\, m \tag{16}$$

$$\sum_{i \in N_m} a_{ij} \mathrm{X}_{mi} = Y_{jm} \quad \forall\, j, m \tag{17}$$

$$F_j - Y_{jm} \geq 0 \quad \forall\, j, m \tag{18}$$

$$X_{mi} = 0{,}1 \quad \forall\, m, i \tag{19}$$

$$Y_{jm} = 0{,}1 \quad \forall\, j, m \tag{20}$$

$$F_j = 0{,}1 \quad \forall\, j \tag{21}$$

where:

$$a_{ij} = \begin{cases} 1 \text{ if node } j \text{ is predefined as the center of market area network } i; \\ 0 \text{ otherwise;} \end{cases}$$

$$X_{mi} = \begin{cases} 1 \text{ if good } m \text{ is corresponding to the market area network } i; \\ 0 \text{ otherwise;} \end{cases}$$

$$Y_{jm} = \begin{cases} 1 \text{ if node } j \text{ is selected to the market center of good } m; \\ 0 \text{ otherwise;} \end{cases}$$

$$F_j = \begin{cases} 1 \text{ if node } j \text{ becomes the market center of one or more goods;} \\ 0 \text{ otherwise;} \end{cases}$$

S = the set of nodes j within particular sectors;

N_m = the set of market area networks i that can correspond to good m.

This model is applied to the discrete lattice network of regular equilateral triangles where the center of each market area is predefined according to Figure 10. The market area network i is a serial number in which all alternatives were counted. Using Figure 11 as an example, the total number of market area networks is four because there are two alternatives in the market area Number 3. Therefore, the sets of market area networks of goods Numbers 1 and 2 that correspond to only one market area network are $N_1 = \{1\}$ and $N_2 = \{2\}$, and the set of market area networks of good Number 3 with alternatives is $N_3 = \{3, 4\}$.

As for the objective function (15), the first term on the right side expresses the objective of maximizing the total number of market centers included in particular sectors, while the second term represents the objective of maximizing the coincidence of market centers. The value in parentheses in the second term is a difference between the number of goods at node j and F_j, and F_j substantially indicates the location of central place. Specifically, the number of coincident market centers would be 2 in the case that the number of goods at node j was 3, and 0 if the number of goods was only 1 or if node j is not a central place. In other words, this means that the higher the value, the more market centers there are that coincide at node j. The two objectives are combined using a weighting method within multiobjective programming. Weight w, which adjusts the level of priority given between the objectives, takes a value $0 < w < 1$, and when applying the solution in practice, minutely adjusting this weight w allows us to derive a number of eclectic non-inferior solutions.

Constraint (16) represents that either of market area network of good m from among the set N_m. Constraint (17) determines whether or not node j is the market center of good m, while constraint (18) defines the value of central place location F_j. In the objective function (15), because the endogenous variable F_j is minimized, F_j automatically becomes 0 when $F_j \geq 0$ is always satisfied.

3.4 Comparing the Lösch's system with the solution of model

Let us actually apply the model defined in previous section to the superimposition problem for the 150 market area networks presented by Lösch. If the distance between settlements is taken to be a, the distance between market centers in market area Number 150 is $\sqrt{511}a$. Therefore, the subject area is set within a circle of radius $\sqrt{511}a$ from the metropolis on a triangular lattice network to which nodes are distributed at intervals of distance a, and there are 1,867 nodes in the subject area. When the range of a 30° arc from the y-axis in clockwise with the origin as a metropolis is defined as a particular sector, the set of nodes S contains 138 nodes[12]. The total number of alternatives for 150 market area networks yields 310 options. Therefore, the superimposition problem for market area networks can be defined as the problem where correspondences between 310 market area networks i and 150 types of goods m are to be found.

Here, if we apply the model to the subject area outlined above by varying the value of weight w in the objective function (15) from 0.01 to 0.99 by an increment of 0.01, two non-inferior solutions are derived. With one solution, the first term of the objective function yields the value of 317 and the second term yields 2,525, while another solution gives the value of 314 for the first term and 2,543 for the second term. Compared with the central place system of Lösch, restored by Beavon and Mabin (1975), that consists of 150 market area networks, among the results of applying the model, the latter

[12] Nodes located on the boundary line with adjacent sectors are not included in the set of particular sectors S.

solution yields the same value for the second term of the objective function as its counterpart in Lösch. Therefore, in what follows, let us compare the latter solution with Lösch's system by focusing on the prioritizing location in particular sectors out of the two objectives as our optimization criteria.

Figure 13 represents a portion of the subject area obtained by taking the metropolis as an origin and cutting a 60° arc clockwise from the *y*-axis. As indicated by Beavon and Mabin (1975: 146), in the diagram of a central place system presented by Lösch (1954: 127), the configuration of a range consisting of a circle of radius $\sqrt{511}a$ and of a partially different area (the range of the dashed lines in Figure 13) produces excess or deficiency in some of the numbered market areas in the 150 market area networks. Here, because we want to compare all combinations of the 150 market area networks, in the same manner as Beavon and Mabin (1975), let us use the range of a circle of radius $\sqrt{511}a$.

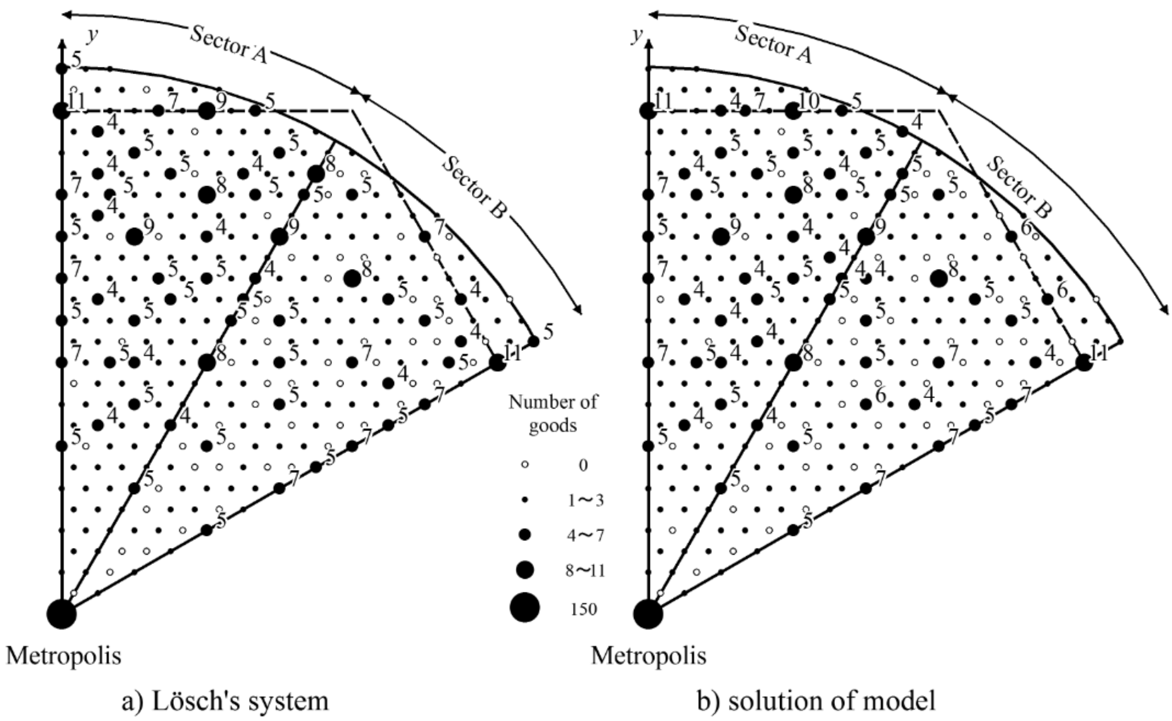


Figure 13 Central place systems of one hundred fifty market area networks

Dashed lines represent the boundaries of one hundred fifty market areas by Lösch.

From Figure 13, we see that the large numbers of goods are concentrated within the Sector A, a particular sector of Figure 13, both in the results of Lösch's system (Figure 13-a) and the solution of model (Figure 13-b). This Sector A is a "city-rich sector" within Lösch's central place system, while Sector B is a "city-poor sector." However, when we observe in detail, slight differences exist between Lösch's system and the solution of model in the number of goods and the location of central place.

In Beavon and Mabin (1975: 148), the relative supply of goods in "city-rich sectors" and "city-poor sectors" is compared in the number of central places by number of goods in each sector. However, to accurately evaluate differences between two sectors, one should consider except the nodes on the boundary line demarcating the two sectors (Hayashi 1986: 195). Furthermore, focus should be placed on differences not only in the number of central places but also in terms of the actual number of market centers where each good is supplied. Table 3 compares the solution of model with Lösch's system by the product of the number of goods and the number of places divided among Sector A, Sector B, and the boundary between both sectors, as shown in Figure 13.

Table 3 Comparison of the sectors in the Lösch's system and the solution of model

Number of goods	Lösch's system						Solution of model					
	Nunber of places			Number of market centers			Nunber of places			Number of market centers		
	Sector A	Sector B	Boundary	Sector A	Sector B	Boundary	Sector A	Sector B	Boundary	Sector A	Sector B	Boundary
1	52	31	5	52	31	5	46	30	7	46	30	7
2	27	32	5	54	64	10	34	32	3	68	64	6
3	25	11	7	75	33	21	21	12	10	63	36	30
4	8	3	2	32	12	8	12	3	2	48	12	8
5	12	8	8	60	40	40	11	6	6	55	30	30
6	0	0	0	0	0	0	0	3	0	0	18	0
7	1	2	3	7	14	21	1	1	3	7	7	21
8	1	1	2	8	8	16	1	1	1	8	8	8
9	2	0	1	18	0	9	1	0	1	9	0	9
10	0	0	0	0	0	0	1	0	0	10	0	0
11	0	0	1	0	0	11	0	0	1	0	0	11
Total	128	88	34	306	202	141	128	88	34	314	205	130

Comparison of the two results shows that there is no change in the total number of places in each sector. However, when we compare the number of places that possess multiple goods, i.e., the number of places where the number of goods is two or more, then in contrast to Lösch's system, which found a total of 76 such places in Sector A, the solution of model yield 82. In addition, when we look at the total number of market centers, whereas Lösch's system found 306 in Sector A, the model finds 314. In other words, the solution of model show a higher degree of prioritization of locating in particular sectors, and more "city-rich sectors" are created than in the results of Lösch's superimposition[13].

3.5 Testing the rationality of constructing a hierarchy on Lösch's system

Why might something like this occur? As mentioned above, Lösch aligned the nearest market center from the metropolis with particular sectors on the creation of "city-rich sectors." However, it is not necessarily the case that this choice actually results in the greatest number of market centers in particular sectors. For example, Figure 14 shows only the first quadrant of the range of a circle of radius $\sqrt{511}a$ for the market centers of a market area Number 6 from Table 2. Market area Number 6 has two alternatives, and Figure 14 shows the results of aligning the nearest market center from the metropolis with the particular Sector A. Excluding the boundary line, in contrast to eleven centers that are located inside Sector A, there are twelve centers located inside Sector B. Namely, at least in regard to market area Number 6, in the case of the basis of Lösch's method, we can say that there will be more market centers in "city-poor sectors" than in "city-rich sectors."

In the area of the 150 central place system shown by Lösch (the range of the dashed lines in Figure 14), the exclusion of one of the market centers in Sector B makes it look like the number of

[13] Even when we share the number of market centers on the boundary in Table 3 in Sector A and Sector B, the total number of market centers for Sector A is greater in the case of the model, with 376.5 in the Lösch's system, and 379 in the solution of model. In addition, when the model is applied by including the nodes on boundary in the set of particular sectors *S*, the total number of market centers in Sector A becomes 447 in the Lösch's system and 449 in the solution of model.

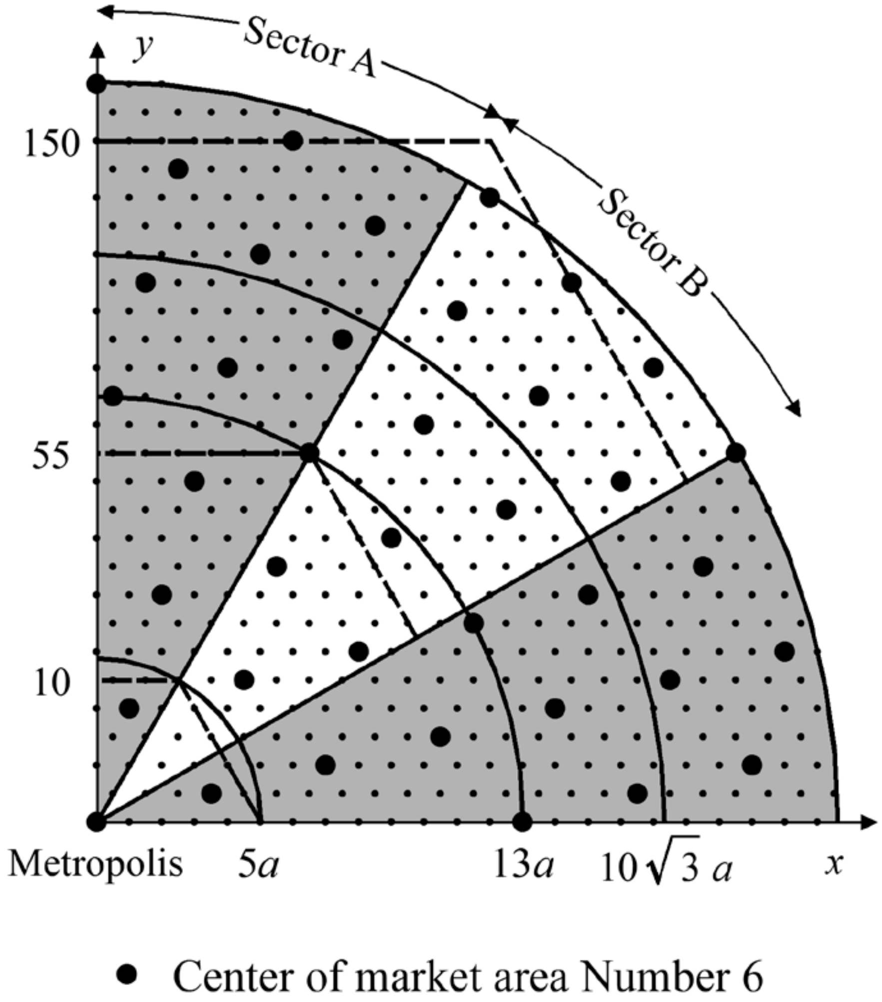


Figure 14 Distribution of market centers for market area Number 6

Shaded areas are particular sectors. Dashed lines represent ranges of each market area network of the number on the *y*-axis illustrated by Lösch. Circles from the metropolis with the radiuses of the number on the *x*-axis correspond to ranges in all combinations of the market area networks of the number on the *y*-axis.

centers does not change from Sector A to Sector B. While it remains unclear why Lösch chose this range for his illustration, it is a fact that this resulted in the obscuration of an “inversion problem” like that described above. Whatever the case, to ensure the location of more market centers in “city-rich sectors”, we will need to take note of the fact that in some cases it is better to align the nearest market center from the metropolis with “city-poor sectors.”[14] Actually, such a solution has already been derived as the solution of model.

However, whether such “inversion problem” actually occurs will depend on the size of the subject area. The range of the central place system for fifty-five market area networks shown by Lösch (1954: 128) and the area of a circle with radius of 13*a*, which is equal to the distance between market centers in market area Number 55, are shown in Figure 14. Even so, the number of market centers in this range remains unchanged in both Sector A and Sector B. However, in the interior of a circle of radius $10\sqrt{3}a$ which lies more or less mid-way between the market areas Numbers 55 and 150, Sector A contains the locations of seven market centers, in contrast to Sector B which has six (Figure 14). Furthermore, turning to the range of ten market area networks shown by Lösch (1954: 118), the fact that there are no locations of interest other than the six nearest market centers from the metropolis means that it is necessarily divided into sectors in which market centers exist and sectors in which they do not.

Ultimately, as Marshall (1977: 12) has pointed out, the nearest market center from the metropolis is no more than the “tip of iceberg”. For example, market areas Numbers 56 through 150 in the 150 market area networks contain only six market centers inside the subject area of a circle of radius $\sqrt{511}a$, excluding the metropolis. In other words, in the same way as with the range of market area Number 10 in Figure 14, only the locations of the nearest market centers from the metropolis contribute to relative amounts between sectors in these market area networks. However, as we can also infer from Figure 14, when we configure a subject area broader than a circle of radius $\sqrt{511}a$, then even for market areas Numbers 56 through 150, either the differences between sectors will disappear or there will be the possibility of a “inversion problem.”

Importantly, the solution of the objective of prioritization of location in a particular sector is the existence of the possibility of change depending on the range of the subject area. In other words, in

[14] A similar “inversion problem” can be seen in market area Number 25.

order to know which alternative to choose in a market area network where multiple alternatives exist, it is necessary to survey the whole range of the subject area, that is to say the entirety of the central place system, and to pursue its rationality. Missing from Lösch's process of constructing a hierarchy is a perspective towards the rationality of the entire system.

Nevertheless, of course, this is only natural given the period when Lösch was writing. At the time, the methodologies and techniques for solving the combinatorial optimization formulated in this chapter had not yet been developed[15]. What Lösch was able to do, as reproduced by Beavon and Mabin (1975), was to apply the geometric processing of aligning nearest market centers from the metropolis with particular sectors and to adopt a sequential method of searching for the place with the largest number of coincident market centers during processing. In other words, he had to resort to heuristic algorithms in which solution is sought on a trial and error basis. However, the application of heuristic algorithms does not necessarily guarantee a global optimal solution for the entire system.

What we should question when trying to understand Lösch's theory as a superimposition problem for market area networks is the nature of Lösch's original intention in constructing a hierarchy. In order to explore the true meaning, it will be necessary to step away from the interpretation by Beavon and Mabin (1975) for once and grapple with the task of a reinterpretation that will decipher the objective of constructing a hierarchy that Lösch was attempting to pursue.

3.6 Hierarchical arrangement based on the agglomeration effect

Lösch, after having superimposed his market area networks to achieve "six sectors with many production sites," continues by saying that "with this arrangement the greatest number of production locations coincide, the number of local purchases is maximized, the sum of the minimum distances

[15] The simplex method for solving linear programming and the integer programming formulated in this paper was developed in 1947 by Dantzig (Kubo et al. 2002). In addition, the branch and bound method that allows the exact solution of combinatorial optimization was developed later, and depending on the size of the problem, the use of a computer can be essential for solving problems such as these.

between industrial locations is least, and in consequence not only transportation amount but also transport lines are reduced to a minimum" (Lösch 1954: 124). From this description, it is clear that Lösch had taken into account the notion of agglomeration economies in the supply of goods (Webber 1971: 17). If the advantage of agglomeration was Lösch's original objective, then the prioritization of locations in particular sectors would simply be a means of achieving that end. Therefore, let us then consider the agglomeration effect of goods brought about by the economies of agglomeration and attempt to reformulate the model in the previous section.

The objective pertaining to the agglomeration effect of goods is also expressed in the maximization of the number of coincident market centers, which is addressed by the second term on the right-hand side of objective function (15). Here, when the second term on right-hand side of objective function (15) is expanded, the second objective divides into the maximization of the total number of market centers and the minimization of the total number of central places. Assuming that market centers are determined in advance for each market area network and that the subject area is a circle of a certain radius from the metropolis, there will be no difference in the number of market centers between alternatives that keep the same distance between market centers. In other words, the number of market centers will not change no matter which alternative is chosen from the market area networks corresponding to each good. Thus, because the total number of market centers in the subject area is a constant value, the objective of maximizing of the number of coincident market centers is synonymous with the minimization of the total number of central places.

The minimization of the total number of central places is introduced by Parr and Denike (1970: 572), Parr (1973: 187), and Mulligan (1984: 10), among others, as a condition that represents the agglomeration effects in Lösch's hierarchical arrangement. However, for my own part, I do not believe that the agglomeration effect of goods can be properly represented by means of the minimization of the total number of central places. Let us consider this effect concretely, assuming an agglomeration of four kinds of goods ranging from a lower-order good Number 1 to a higher-order good Number 4, as listed in Table 4.

The circle mark in Table 4 shows that each good is supplied in a place and there are 16 different conceivable patterns according to whether or not four kinds of goods are supplied in each place. These patterns can be divided roughly into an agglomerated locational pattern that possesses two or more goods and other non-agglomerated locational pattern. The non-agglomerated locational pattern is classified into “nothing” when none of goods are supplied, and “single good” when any one of a good is supplied. Furthermore, the agglomerated locational pattern is classified into “complete agglomeration” when all goods, from the hierarchical marginal good (Berry and Garrison 1958a) down to the lowest-order good Number 1, are supplied, and “partial agglomeration” when goods can be seen to be missing lower-order goods than the hierarchical marginal good.

Table 4 Relationship between types and effects of agglomeration in the case of four kinds of goods

		Non-agglomerated location					Agglomerated location										
		Nothing	Single good				Partial agglomeration								Complete agglomeration		
Order of goods	4					○			○	○	○	○	○	○			○
	3				○		○	○			○		○	○		○	○
	2			○				○		○		○	○		○	○	○
	1		○				○		○			○		○	○	○	○
Central place location		0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Number of goods		0	1	1	1	1	2	2	2	2	2	3	3	3	2	3	4
Number of coincident market centers		0	0	0	0	0	1	1	1	1	1	2	2	2	1	2	3
Number of missing goods		0	0	1	2	3	1	1	2	2	2	1	1	1	0	0	0

"Central place location" in Table 4 is the same as F_j in objective function (15) and "number of goods" refers to the number of goods supplied at a respective place. Therefore, "number of coincident market centers" corresponds to the number in which "central place location" is subtracted from "number of goods". Looking at the number of coincident market centers, while the values are both 0 in cases of "nothing" and "single good", both take a positive value in cases of "partial agglomeration" and "complete agglomeration", in accordance with the degree of agglomeration. While it is possible to distinguish the agglomerated locational pattern by the number of coincident market centers, the problem will be whether or not the agglomeration effect of goods is accurately expressed by the relevant indicators.

For example, for the pattern whereby three kinds of goods (goods Numbers 1, 2, and 3) are supplied in "complete agglomeration" and the pattern whereby good Number 1 is missing from among goods Numbers 1 through 4 in "partial agglomeration", the number of coincident market centers is two as both of which are the same value, but can they both truly be said to have the same agglomeration effect? In general, lower-order goods have higher convenience than higher-order goods, with a greater number of supply points. Accordingly, regarding the goods supplied at places, the case of higher-order good Number 4 missing from Table 4 would have a different significance than the case of when lowest-order good Number 1 is missing despite the fact that higher-order goods than good Number 1 are supplied. In addition, in situations where the total number of market centers is constant, such as the one described above, anything other than a "nothing" of 0 will be regarded as having the same degree of agglomeration because the "central place location" ends up being the only indicator for determining the agglomeration effect.

When considered in this manner, neither the number of coincident market centers nor the number of central places can be regarded as an adequate indicator for capturing the agglomeration effect of goods. Therefore, in this paper, I focus particularly on the number of missing goods as an alternative indicator of the agglomeration effect of goods. The number of missing goods is the number of goods missed lower-order goods than the hierarchical marginal good. It is possible to distinguish clearly between complete agglomeration and partial agglomeration, because the number of missing goods becomes to zero in the former but takes a positive value corresponding to the degree of agglomeration in the latter. Moreover, the difference can be identified by using the number of missing goods even if a part of "nothing" and "partial agglomeration" is considered to be the same agglomeration effect when measuring the pattern by the number of coincident market centers.

If we use the number of missing goods as an index of the agglomeration effect, then we can realize the economies of agglomeration by minimizing the total number of missing goods. Here, reformulating the model for the problem of superimposing market area networks, we may revise the objective function (15) with the objective that seeks to minimize of the total number of missing goods, as in following formula (22), and add the following constraints to constraints (16), (17), (19), and (20):

$$\min \mathrm{Z} = \sum_{j} A_j \tag{22}$$

subject to:

$$A_j + \sum_{k}^{m-1} Y_{jk} - (m-1)Y_{jm} \geq 0 \quad \forall\, j, m \geq 2 \tag{23}$$

$$A_j \geq 0 \quad \forall\, j \tag{24}$$

where:

A_j = the number of missing goods at node j.

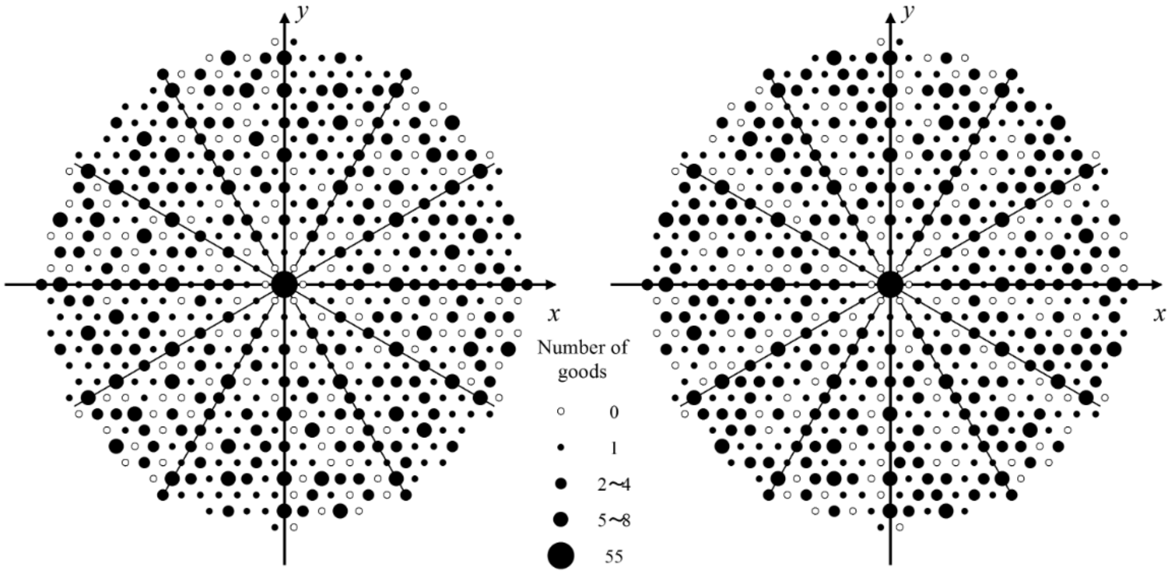
Constraints (23) and (24) specify the number of missing goods. Specifically, if node j is assumed a place where possess two kinds of goods (goods Numbers 2 and 4) as "partial agglomeration" in Table 4, then because $A_j \geq 2$ when goods m = 4, $A_j \geq 0$ when m = 3, and $A_j \geq 1$ when m = 2, we will end up with $A_j \geq 2$. Because A_j is minimized by the objective function (22), A_j will take a value of 2.

3.7 Testing the agglomeration effect

Let us apply our model for minimizing the total number of missing goods to the superimposition problem for market area networks. Because the size of the problem has grown with the addition of our new constraints, our discussion limits to the 55 market area networks shown in Table 2. Assuming the distance between settlements to be *a*, the circle of radius 13*a*, which is the distance between market centers for market area Number 55, contains 613 nodes on the discrete lattice network of regular equilateral triangles. Solving the problem of dealing with the 101 market area networks that integrate all alternatives for the 55 market area networks, we derive the result shown in Figure 15-a and an objective function value of 8,640, which represents the total number of missing goods. The difference in the number of market centers between sectors is less than that shown in the diagram given by Lösch (1954: 128) for a central place system of 55 market areas[16]. However, from the fact that calculating the total number of missing goods in the figure presented by Lösch yields a result of 9,240, the solution of model show a higher agglomeration effect of goods. That is, if our objective adequately represents the notion of agglomeration economies in the supply of goods, then we can say that the prioritization of location in particular sectors is not a necessary requirement.

That said, because one center of each market area network is fixed as the metropolis and market area networks of different sizes in relation to each good are superimposed, the metropolis as center point of the subject area is the only location where complete agglomeration can be achieved while

[16] In terms of the relative numbers of "city-rich sectors" and "city-poor sectors", comparing the total number of market centers, Lösch's results show 75 and 44 respectively, while the solution of model yield 63 and 56.



a) no neighborhood effect b) neighborhood effect

Figure 15 Solutions of the model based on the agglomeration effect

supplying multiple goods. By applying the model, it is possible to derive the central place system that results in the maximum agglomeration effect of goods. However, this effect will still be limited, even when the agglomeration effect is determined only for the combination of goods supplied at each point.

When we determine the agglomeration effect without limiting the combination of goods at a single place, including goods supplied at neighboring places, we can apply the model more flexibly. For example, as in Table 4, let us assume the supply of four kinds of goods at the five places A through E shown in Figure 16. Of the five places, only place D satisfies complete agglomeration on its own, whereas at the other places either only good Number 1 is supplied or there is some missing good. Here, if the goods in the possession of neighboring places were able to complement the missing goods from the place in question, then we could expect an improved agglomeration effect. Now, focusing on place C, we see that point C is determined to have partial agglomeration, with the supply

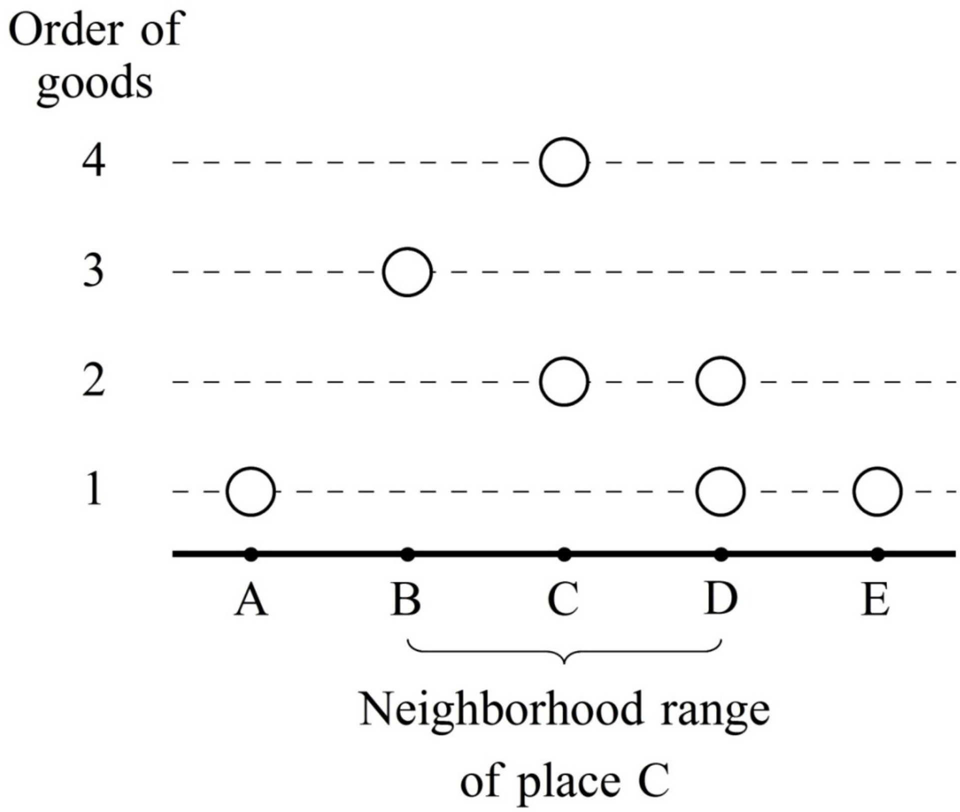


Figure 16 Agglomeration by the neighborhood effect

Round marks indicate the goods supplied at each place.

of two kinds of goods (goods Numbers 2 and 4). However, goods Numbers 1 and 3, which are not supplied at place C, are present at the neighboring points B and D. This distance to the adjacent places will be regarded as the “neighborhood range” of place C, and if we evaluate the agglomeration of goods inside the neighborhood range, then it will be determined that place C offers complete agglomeration.

However, when we evaluate place D using the same method, it is necessary not to consider good Number 4 which is supplied at place C. This is because when our evaluation includes good Number 4, point D—which satisfies complete agglomeration on its own—will be regarded as showing partial agglomeration that lacks good Number 3. Accordingly, the object for complementary goods in a neighborhood range should be limited to the lower-order missing goods as seen from the hierarchical marginal good occurring at the respective place. As a result, a pattern of complete agglomeration will be determined for place B up to good Number 3, for place C up to good Number 4, and for place D up to good Number 2.

Denoting locations with localized clusters of firms as integrated units seems to be a viable idea, although it also depends on the configuration of the distance between places and neighborhood range. What Lösch refers to as "the sum of the minimum distances between industrial locations is least" and "the maximum number of purchases can be made locally" is not what can be achieved solely at a specific place but places at which the agglomeration effect is considered including neighboring firms, as a result of which multi-purpose shopping behavior is facilitated for the consumers (Eaton and Lipsey 1982: 58).

The problem of minimizing the total number of missing goods for a configured neighborhood range can be expressed by replacing constraint (23) with the addition of the following:

subject to:

$$A_j + \sum_{k}^{m-1} B_{jk} - (m-1)Y_{jm} \geq 0 \quad \forall\, j, m \geq 2 \tag{25}$$

$$B_{jm} - \sum_{l \in C_j} Y_{lm} \leq 0 \quad \forall\, j, m \tag{26}$$

$$B_{jm} - Y_{lm} \geq 0 \quad \forall\, j, m, l \in C_j \tag{27}$$

$$B_{jm} = 0{,}1 \quad \forall\, j, m \tag{28}$$

where:

$$B_{jm} = \begin{cases} 1 \text{ if good } m \text{ is supplied inside the neighborhood range of node } j; \\ 0 \text{ otherwise;} \end{cases}$$

C_j = the set of nodes inside the neighborhood range of node j.

Constraints (26) through (28) hold that when good m is supplied in the neighborhood range of node j, $B_{jm} = 1$, and that when there is no supply at any node, then $B_{jm} = 0$. As well as constraint (23), constraint (25) is a condition that determines the number of missing goods, and when good m is not supplied at node j, and then there will be no count of missing goods for the good in question. Accordingly, goods whose order are higher than that of the hierarchical marginal good will have no relation to the calculation of the number of missing goods. While we can determine the configuration of neighborhood range for each node, for the sake of convenience, we may configure this as the nodes included inside a circle of a specified radius R as $C_j = \{l | d_{jl} \leq R\}$.

Applying the model to the superimposition problem of 55 market area networks with a radius R as the distance a between settlements, the central place system shown in Figure 15-b is drawn as the solution of model. Calculating the number of missing goods, including adjacent nodes, yields an objective function value of 6,120, which can be confirmed as an improved agglomeration effect when compared with the result of Figure 15-a, in which the neighborhood range is not configured. In addition, from the fact that the total number of missing goods is 6,648 when considering the complementing of goods from neighboring nodes in the diagram presented by Lösch, we can say that the solution of model yield a higher agglomeration effect even when a neighborhood range is set.

The complementary effect of goods from the neighborhood range allows a complete agglomeration pattern to be derived even from nodes other than the metropolis. Specifically, whereas Lösch finds 30 nodes with complete agglomeration up to good Number 2, the model yields 36 nodes with complete agglomeration up to good Number 2, and six nodes with complete agglomeration up to good Number 3. However, as with Figure 15-a, the difference of the number of market centers between 30-degree sectors is not clear (Figure 15-b). Because goods are rather seen to agglomerate in a narrow range moving counter clockwise from the boundary between each sector, the same kind of dense pattern of agglomeration of goods appears in every sector. Of the 55 market area networks shown in Table 2, the market centers for sixteen market area networks with no multiple alternatives will necessarily be located on the boundary line between 30-degree sectors. Therefore, the complementary effect with these market centers on the boundary lines might have caused tendency of agglomeration of goods around the boundary lines.

We can easily imagine that the agglomeration effect changes depending on the setting of the neighborhood range. In this chapter, while the neighborhood range has defined as a circle of a specified radius at each node, as the radius increases in size, the complementary effect of missing goods will rise, and the number of missing goods will tend to decrease overall. On the other hand, by variably configuring the radius of the neighborhood range for different nodes and by setting neighborhood ranges for only specific nodes, we could consider a regional difference of agglomeration effect or a regional planning of agglomeration. If we assume that Lösch's objective in hierarchical arrangement as the realization of the economies of agglomeration, by using the reinterpretation that we have given to the agglomeration effect of goods in this chapter, we could bring Lösch's intention to fruition. In addition, the model to minimize the total number of missing goods can be considered as the extended model in which more flexible central place systems can be derived concerning the hierarchical structure.

CHAPTER 4

Generalization of central place theory

Thus far, through the process of modelling Lösch's market area theory, we have attempted the extension of the model to the location of single good and to hierarchical arrangement as seen from the agglomeration effect of goods. In general, as against Lösch's market area theory, which seeks to maximize the number of firms on the basis of the concept of threshold, Christaller's central place theory has been regarded as a model for the supply of goods from the minimum number of firms on the basis of the upper limit of the range of goods (Saey 1973; Matsubara 2006). Comparing both theories from the perspective of hierarchical structure, as opposed to Lösch's central place systems, in which partial agglomeration of goods is inevitably caused, Christaller's central place theory, in which central places hold all lower order goods than the hierarchical marginal good, posits a complete agglomeration pattern for every central place. In this chapter, I would like to summarize the differences between Lösch's and Christaller's theories in order to generalize both theories by a unified model.

4.1 Reinterpreting central place theory from the perspective of the locational principle

Given the results of our discussion in Chapter 2, as with Figure 2-a, some similarities can be pointed out between Christaller's theory and the problem of maximizing total profit, which realizes fewer locations while covering nodes to the upper limit of the for a good *S*[17]. However, for clarifying the differences in Lösch's and Christaller's theories and for positioning both theories in the framework

[17] In this regard, Ishikawa and Toda (2000) discussed the systematic change of Christaller's central place systems when firms take action to maximize profit.

of the locational principle, it will be necessary to examine carefully the locational process as seen from the structure of model, rather than focusing solely on the number and pattern of locations as results of the model. Therefore, in what follows, taking a cue from the model in objective function (14), which extends Lösch's theory with regard to the location of single good, I would like to reinterpret Lösch's and Christaller's theories, particularly from the perspective of the locational principle.

First, substituting the right-hand side of objective functions (1) and (13) for the two objectives Z_1 and Z_2, respectively, in objective function (14), the objective function (14) can be summarized as follows:

$$\max Z = \sum_{i} \sum_{j \in N_i} a_i q_{ij} X_{ij} - w \sum_{j} t Y_j \tag{29}$$

Moreover, when the quantity of demand q_{ij} is defined as in Equation (10), which expresses the demand cone, the objective function is expressed as the following equation composed of three terms[18]:

$$\max Z = \sum_{i} \sum_{j \in N_i} Q a_i X_{ij} - \sum_{i} \sum_{j \in N_i} \beta a_i d_{ij} X_{ij} - w \sum_{j} t Y_j \tag{30}$$

Here, let us speculate on the locational principles of the three terms on the right-hand side of objective function (30). The first term seeks to maximize the value obtained by multiplying population a_i by maximum demand Q for the set of points N_i inside radius S. Because maximum

[18] Erlenkotter (1977), for example, has developed three terms in a similar manner to this formulation for the maximization of total profit problem.

demand Q is a constant value, the first term ultimately signifies the maximization of the population covered by the upper limit of the range of a good S. This is none other than the maximum covering location problem (Church and ReVelle 1974), which seeks maximum coverage for a population within a specified coverage distance. Interpreted likewise, the second term corresponds to the median problem for minimizing the total distance travelled (ReVelle and Swain 1970) because it maximizes the negative value of a population-weighted distance; whereas the third term corresponds to the so-called location set covering problem (Toregas et al. 1971) in the sense that it minimizes the number of locations multiplied by the threshold constant t when all demand nodes are covered by radius S.

In other words, the extended model in the location of single good that was presented in Chapter 2, among the basic location-allocation models (Ishizaki 2003), may be considered to combine the three models, with the exception that it excludes the center problem. However, depending on conditions, the three objectives are in some cases not subject to optimization. First of all, the first term becomes meaningless when all demand nodes are covered by radius S. In other words, if the coverage condition shown in constraint (3) expresses equality rather than inequality, the first term becomes a constant and thus excluded from optimization. Next, cases where the second term is satisfied are limited to those that assume the demand cone where the distance elasticity of demand β takes a positive value. When $\beta = 0$, the second term will not be subject to optimization, and the quantity of demand will take a discrete value of either Q or 0. Further, for the third term, when $w = 0$, i.e., when the extended model is expressed as a total demand maximization problem, it will be excluded from the object for optimization.

Given this way, it is possible to classify objective function (30) into the combination of several models according to the three objects. Table 5 shows the well-known location-allocation models and other models that become relevant for combinations of these three objectives—the maximum coverage of demand for the first term, the minimization of total distance traveled for the second term, and the minimization of the number of locations for the third term. As mentioned above, although

Table 5 Classification of the extended model based on locational principles

Coverage condition	Elasticity of demand	Weight	
		$w = 0$	$0 < w \leq 1$
$\sum_{j \in N_i} X_{ij} = 1$	$\beta = 0$	——	Set covering problem
	$\beta > 0$	Median problem	Generalized median problem (Christaller's model)
$\sum_{j \in N_i} X_{ij} \leq 1$	$\beta = 0$	Maximal covering problem	Fixed charge maximal covering problem
	$\beta > 0$	Total demand maximization problem (Lösch's model)	Flexible central place model

objective function (30) corresponds to each of the three basic models when only one of its terms is satisfied[19], when none of the three terms is the object to be optimized, it cannot be established as an optimization problem. On the other hand, there are also models that consist of combinations of two objectives. For example, a model combining the minimization of total distance traveled with the minimization of the number of locations will have the same form as a generalized median problem (Mavrides 1979)[20]. In addition, the fixed charge maximal covering location problem (Church and

19 Under a strict definition, because the constraints of the upper limit of the range of a good are taken into account, the median problem is also with maximum distance constraints problem (Khumawala 1973). Moreover, if we apply the threshold condition presented in constraint (2), we get a median problem with threshold requirements (Carreras and Serra 1999), which resolves as a problem of maximum coverage with threshold constraints (Balakrishnan and Storbeck 1991).

20 While not specifically formulated as a multiobjective programming solution by Mavrides (1979), Church and Davis (1992) extended the generalized median problem with multiobjective programming. In addition, the generalized median problem takes the same form as the plant

Davis 1992), formulated as a multiobjective programming solution for a maximal coverage problem and fixed charge minimization problem, can be regarded as a composite model of the first and third terms.

Here, how are Lösch's market area theory and Christaller's central place theory defined in Table 5? As described earlier in Chapter 2, because we can formulate Lösch's market area theory as the total demand maximization problem, it may be understood, from objective function (30), as a composite model of the first and second terms. However, because the third term—which adjusts the number of locations—is not included, we will need a constraint to determine the number of firms. This is also why constraints that fix the number of locations are necessary in usual median problem and maximal covering location problem. In Lösch's market area theory, the number of firms is decided by the threshold condition shown in constraint (2). This is because, without constraint (2), all firms would be located in all potential firm location nodes.

Additionally, because the range of a circle whose radius is the upper limit of the range of a good is typically greater than that of a market area that meets the threshold; consequently, it is possible that all demand nodes can be covered under a radius *S*. In particular, this is inevitable where a population's distribution is regular and uniform, and because this results in the first term of objective function (30) becoming a constant, it will therefore be excluded as the object to be optimized.[21] Therefore, to establish the total demand maximization problem as an optimization problem, the second term becomes a prerequisite. In other words, the distance elasticity of demand must always be $\beta > 0$. From this, the concept of the demand cone and the threshold condition for determining the number of firms are essential for Lösch's market area theory.

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location problem, which seeks to minimize transportation costs and fixed costs (Efroymson and Ray 1966).

[21] In other words, it turns into a median problem. ReVelle et al. (1975) have indicated a similar case with comment to a model resembling the total demand maximization problem presented by Holmes et al. (1972). However, in the case of a non-uniform population distribution, as shown in Figure 6-c, the possibility of the occurrence of demand nodes that lack supply implies that first term may be the object to be optimized.

On the other hand, with regard to Christaller's central place theory, as Saey (1973) and Beaumont (1987) have pointed out, one possible interpretation is that of set covering problem that seeks fewest locations while ensuring coverage over a region within the upper limit of the range of a good. Because the set covering problem implies that only the third term in objective function (30) will be optimized, there will always be the same number of locations, regardless of the value of the constants weight w and threshold t. Basically, the upper limit of the range of a good S affects the number of locations[22]. However, some questions remain about whether Christaller's theory can be reinterpreted as the set covering problem alone. As clarified by Ishizaki (1992, 1995), this is because the marketing principle underlying Christaller's central place theory does not necessarily assume the coverage of all goods with the minimum number of firms.

According to Christaller's discussion, when the seven central places that were called B-places are required to be distributed equally, some unsupplied areas appear at the supply of good Number 20. Then, because "it is more reasonable to suppose that the places which should regularly supply the unsupplied ring with central good Number 20", "other central places, which we call K, must lie at those points farthest distant from the neighboring B-places" (Christaller 1966: 61-62). However, as a result, there will no longer be the minimum number of firms to supply a good Number 20. In the case of good Number 21, which is supplied from the seven B-places, the overlapping of the circles of which the radius is equal to the upper limit of the range of the good is kept to a minimum, and the number of firms to supply the relevant good is also minimized. However, good Number 20 whose order is lower by one is now supplied by a total of 13 places at B-places and K-places, and the overlap also becomes excessive.

The reason for a lack of consistency between the number of firms and the extent of coverage by the difference of good is that in Christaller's central place theory, it is the central place, rather than

[22] In a set covering problem, the coverage distance will determine the number of locations covered overall. However, in the case of a non-uniform population distribution, the number of locations may be considered to vary depending on the threshold condition in constraint (2).

each individual firm, that is subject to location. Central places possess all goods of lower order than the hierarchical marginal good (i.e. successively inclusive). Accordingly, when considering the location of central places, it is necessary to consider all goods held by the central place, rather than to pursue optimality solely for the supply of specific goods. In this regard, Ishizaki (1995) interprets the locational principle of the marketing principle as a problem for arranging the aggregate range of goods from a higher to lower order.

Thus, is there any reason why we should not be able to specify some kind of locational principle in Christalller's theory in relation to the location of single good? In the process of deriving central place systems on the basis of the marketing principle, Christaller relies exclusively on the upper limit of the range of a good in his explanation. Such an explanation seems to give the impression only the coverage by firms is assumed to be an issue[23]. However, when Christaller discussed the concept of the range of goods, we should pay attention to the fact that he has considered the same notion of demand cone by using the example of a relationship between the doctor established at the center and the distance decay effect of the number of consultations (Morikawa 1980: 38).

My own thoughts are as follows. Namely, although the reason why central places to be added are needed is certainly due to the coverage of unsupplied area, the optimality based on the notion of demand cone might be implicitly considered about the problem of where a firm that supply a good establish themselves. This idea implies that the distance elasticity of demand in the extended model is $\beta > 0$, and, as a result, the second term of objective function (30) is added to the set covering problem. In other words, the locational principle for single good in Christaller's central place theory, while assuming the set covering problem as its basic model, might also be interpreted as the generalized median problem in Table 5, which adds the minimization of total weighted-distance to its objectives[24].

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[23] Especially, in Beavon (1977: 18-27), the coverage of the circle whose radius is the upper limit of the range of a good is given particular emphasis in the supplementary explanation of Christaller's central place theory.

[24] In this sense, the interpretation given in Berry and Garrison (1958a), which apprehends the

Importantly, the generalized median problem, which corresponds to a composite model of the second and third terms from objective function (30), has been formulated as a multiobjective programming that can grasp the number of firms variably depending on the weight *w*. In Christaller's theory, as mentioned earlier, while there are cases that lack the minimum number of firms depending on the good, this can be explained if interpreted as the result of seeking a compromise solution between the two objectives of minimizing the number of firms and minimizing total weighted-distance.

For example, because the non-inferior solution shown in Table 1 ends up with the coverage of all demand nodes within radius *S*, the first term in objective function (30) will be a constant, and the non-inferior solution, in fact, becomes synonymous with a generalized median problem. Thus, if we consider that for each good it is possible to select the appropriate alternative from among the various non-inferior solutions for number of firms, from the minimum number to the maximum number, then Christaller's central place theory may be said to envision an extremely flexible model, at least with respect to the locational principle for single good.

As described above, when we try clarifying both Lösch's and Christaller's theories from the perspective of the locational principle for single good, we are able to interpret the former as a composite model comprising first and second terms from objective function (30), and the latter as a composite model comprising the second and third terms. Then, as in Table 5, it is possible to say "flexible central place model" in which all three terms are the objects to be optimized, in the sense that it acts as a bridge between Lösch's market area theory and Christaller's central place theory. The flexible central place model practically becomes Christaller's central place theory when all demand nodes end up being covered, and Lösch's market area theory when the weight *w* is as close as

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location of central places from the perspective of the minimization of the total distance traveled, could not necessarily be said to be mistaken. However, because they emphasized the threshold condition in order to determine the number of firms as well as Lösch's theory, Saey (1973) points out the confusion of Christaller's theory with that of Lösch.

possible to 0.

4.2 Christaller's central place theory as a superimposition problem for market area networks

If we accept the interpretation in the previous section as correct, then, because the number of firms—as opposed to in Lösch's theory, wherein it is uniquely determined on the basis of the threshold—in Christaller's central place theory is understood as variable, it will be possible to derive multiple market areas of different sizes even for a single good.

As in Chapter 3, when we assume Christaller's central place theory as a superimposition problem for market area networks, we need to take note of the relationship between each good and its corresponding market area network. Figure 17 shows that multiple market area networks are compatible with a single good. Each good is associated with one of the market area networks from the upper limit of the range of the good to the threshold (the lower limit of the range of the good). The market area network of the former is shaped with least number of market centers while the latter case reaches the maximum number of market centers. As a result, the number of combinations that associate each good with each market area network is even greater than that in the superimposition problem of Figure 11, which illustrates Lösch's hierarchical arrangement.

Here, let us try to reproduce Christaller's central place system on the basis of the agglomeration effect of goods shown in objective function (22). When multiple market area networks are accessible, as in Figure 17, the agglomeration effect of goods alone will be inadequate as an objective function for solving the superimposition problem for market area networks. This is because there can be multiple solutions comprising various combinations of market area networks even for the same agglomeration effect, by increasing the degree of freedom for potential market area networks. As discussed in the previous section, if the locational principle for single good in Christaller's central place theory is based on the set covering problem, then we can add the objective of minimizing the

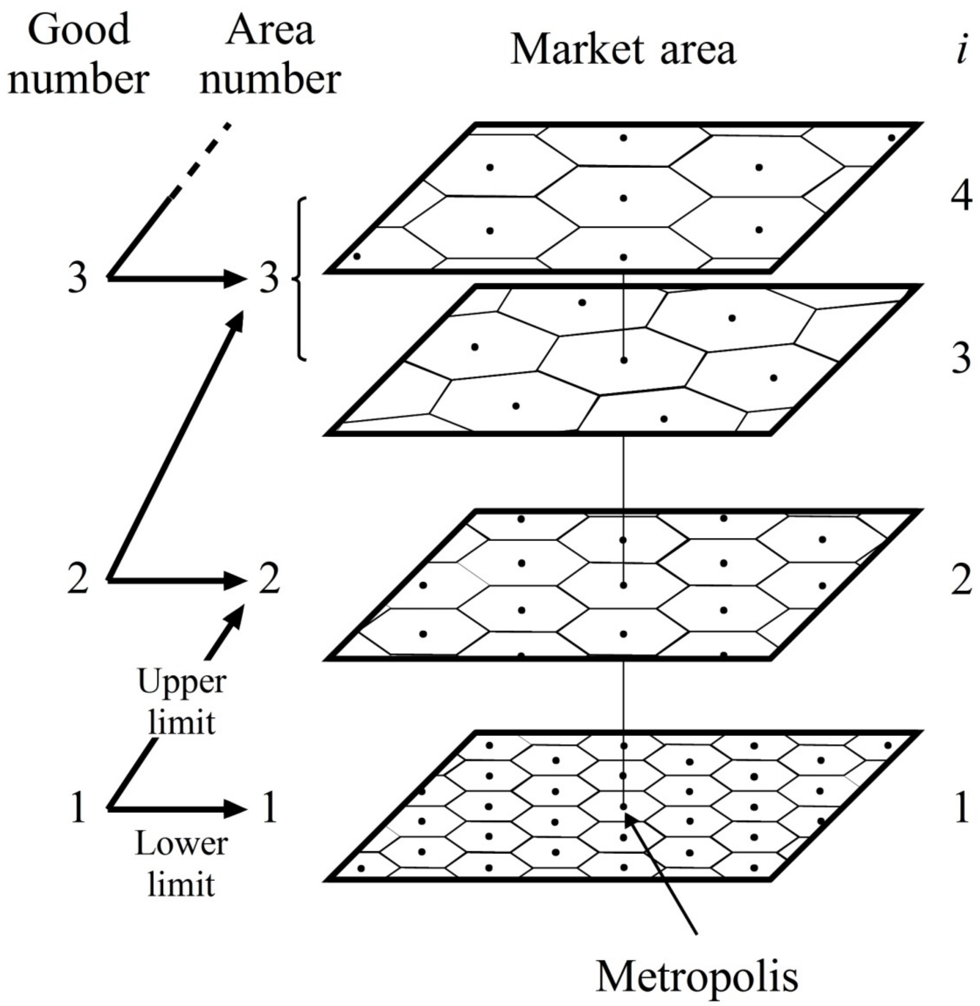

Figure 17 Superimposition of market area networks of the Christaller's system

The legend is the same as Figure 11. The arrows from a good number to area numbers indicate that one good corresponds to the alternatives of two or more market area networks from the upper limit of the range of a good to its lower limit.

number of market centers in order to specify the solution.

However, in the finite space that takes as its subject area a circle of a specified radius as configured in Chapter 3, the number of market centers cannot be distinguished because there are cases where market area networks of different sizes have the same number of market centers. In that case, we can use the distance between market centers for each market area network as an alternative index for expressing the number of market centers in a certain range. Because location density decreases as this distance grows, we will be able to minimize the number of market centers by maximizing the distance between market centers.

Thus, when formulating these two objectives of minimizing the total number of missing goods, which expresses the agglomeration effect of goods, and maximizing the total distance between market centers, which is standing in for the minimization of the total number of market centers, we can express them in the following formula:

$$\min Z = w \sum_{j} A_j - (1-w) \sum_{m} \sum_{i \in N_m} d_i X_{mi} \tag{31}$$

where:

d_i = distance between market centers for market area network i.

The second term in objective function (31) is synonymous with a maximization problem because of minimizing the negative value of the total distance between market centers. The weight w takes a value $0 < w < 1$.

The model is applied to the subject area as the range of a circle with the distance between market centers for market area Number 55 as the radius to address the problem of optimizing objective function (31) under the constraints (16) , (17), (19), and (20), with the addition of constraints (23) and (24). It is well known that Christaller considered goods with varying ranges, starting from good Number 21 that had an upper limit of the range of 21 km, and then proceeding in 1 km increments. According to Christaller (1966: 60-62), a total of 21 kinds of goods originating in B-place up to M-place are for consideration: from good Number 21 down to good Number 4, as well as goods

Numbers 22 through 24, which are of a higher order than good Number 21.

Here, let us assume that the entire plane market is covered with some circles of the upper limit of the range of a good without overlapping as much as possible. In that case, the distance between market centers for each good will correspond to $\sqrt{3}$ times the upper limit of the range of the good because the market centers are distributed in a lattice network of regular equilateral triangles. Therefore, the market area network that the distance between market centers match with a value $\sqrt{3}$ times the good number can be considered to be the market area network associated with the upper limit of the range of each good. Now, if we assume that a distance between settlements is 4 km[25], then the market area network that corresponds to the upper limit of the range of good Number 24 is market area Number 37 as listed in Table 2. In addition, all market area networks will be able to cope down to market area Number 1, which corresponds to the lowest-order good, because "the lower limit of the range will definitely be more elastic" (Christaller 1966: 61). Consequently, when integrating the number of alternatives in Table 2, we find 63 market area networks that correspond to the good Number 24. The set of market area networks that can correspond to each good N_m is defined in a similar way.

When the model is applied by adjusting the weight w in increments of 0.001 from 0.001 to 0.999, the twenty non-inferior solutions shown in Figure 18-d are derived. On the right-hand side of objective function (31), there is a trade-off relationship between the first term (the total number of missing goods) and the second term (the total distance between market centers). Among the non-inferior solutions, the total number of missing goods for Solution A—which gives highest priority to the first term—is 0, implying that complete agglomeration is achieved for all central places. However, because the solutions other than Solution A yield a positive value for the total number of missing goods, there is a mixture of central places with a single good and of those where partial

[25] Christaller used 4 km, the area radius of M-place, as the basis for specific distances in the central place system (Hayashi 1986: 117). If the distance between settlements is set to 4 km, then the market area of M-places, which are the lowest place of central place hierarchy, can exist.

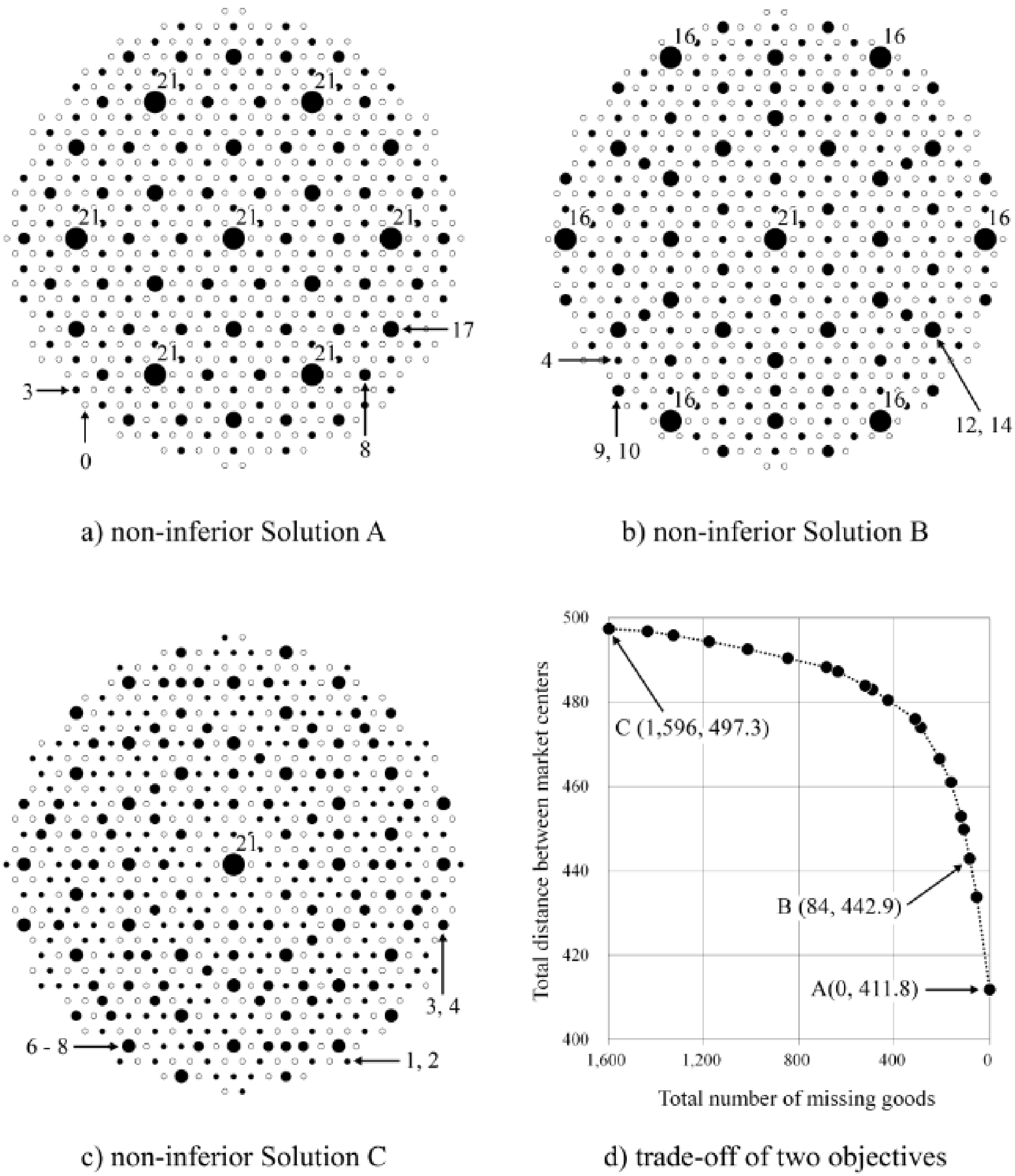

Figure 18 Non-inferior solutions of the generalized model using multiobjective programming

The numerals in figure a) through c) indicate the number of goods. Filled circles in figure d) show obtained non-inferior solutions; in parentheses of Solution A through C among non-inferior solutions, the left numerical value indicates the total number of missing goods and the right numerical value indicates the total distance of market centers.

agglomeration means that some goods are missing. This is because when optimality for the first term is lost, the distance between market centers increases, achieving a minimization of the number of market centers.

Figure 18-a shows the results of Solution A, we find that the central places that supply all 21 kinds of goods are located not only in the center of the subject area but that there are also six locations at equal intervals along a 9*a* ring (i.e. 36km) away from the center. Furthermore, at the centers of the regular equilateral triangles, created by the higher-order central places, are located lower-order central places that supply the 17 kinds of goods comprising goods Numbers 4 through 20. Goods Numbers 4 through 6 correspond to the market area Number 1, goods Numbers 7 through 11 in the market area Number 4, goods Numbers 12 through 20 in the market area Number 11, and goods Numbers 21 through 24 in the market area Number 30, and the hierarchical differentiation that emerges as the result of these respective correspondences, shown in Figure 18-a, illustrates that this central place system is undeniably based on Christaller's marketing principle. In Figure 18-a, in addition to central places corresponding to B-places and K-places above, A-place, which supplies eight kinds of goods (Numbers 4 through 11), and M-place, which in terms of the marketing principle belongs to the lowest-order of the hierarchy, are located at the center of regular equilateral triangles formed by the higher-order central places in the same manner as K-place. Namely, assuming a variable number of market centers for each good, the most rational solution when superimposing market area networks such that there are as few locations as possible while keeping complete agglomeration in play is the schema given by the marketing principle.

On the other hand, when the objective of minimizing the total number of market centers is given priority over the agglomeration effect, the maximum value for the distance between market centers is 497.3, as in Solution C show in Figure 18-d. In this case, the market area networks corresponding to each good are all consistent with the market area numbers corresponding to the upper limit of the range of the good. From Figure 18-c, which yields the results of Solution C, beyond the center of a subject area that has been fixed as a metropolis, we find central place systems of partial agglomeration or single good in which hierarchy has become unclear. This is the same pattern as in Figure 15-a, and despite differences in the upper or lower limit—in the sense that a market area network of a unique size is associated with each good—it is the same as the hierarchical arrangement presented by Lösch.

In practice, commercial districts are not always provided with every good, starting from the highest- to lowest-order goods, supplied at a given place. Simultaneously, cities that possess higher-order central functions without any lower-order functions are probably rare. While what are referred to as "central places" depends on the sort of areal units or spatial scales in which they are understood, at the very least, it would be difficult to imagine that only the central place systems in Figures 18-a and 18-c reproduce actual commercial districts or urban hierarchy. For example, in Figure 18-b, which shows Solution B from Figure 18-d, when some central places lose goods and the situation moves from that shown in Figure 18-a, in which all central places have complete agglomeration, to one of partial agglomeration, a central place system forms that exhibits a location pattern and hierarchy distinct from the marketing principle. By generalizing with multiobjective programming, it is possible to seek flexible solutions that lie between the two objectives. It could be that central place systems that could actually exist in reality might be found in a schema with these diverse hierarchies, wherein the agglomeration effect of goods works, but it does so gradually.

4.3 Generalized model of central place theory

If objective function (30) is a model that situates the theories of both Christaller and Lösch within the locational principle for single good, and objective function (22) is a model that reproduces the hierarchy of central place systems on the basis of the agglomeration effect of goods, then by integrating both models it will be possible to build a model that can generalize the central place theory.

Now, defining the objective function (30) for a good *m* and assuming the right-hand side to be L_m, we can extend the locational principle for single good to the problem of multiple goods as in the following formula:

$$L_m = \sum_i \sum_{j \in N_{im}} Q_m a_i X_{ijm} - \sum_i \sum_{j \in N_{im}} \beta_m a_i d_{ij} X_{ijm} - w_m \sum_j t_m Y_{jm} \tag{32}$$

Here, as with objective function (31), when we formulate L_m as the locational principle for each good and the agglomeration effect of goods using multiobjective programming, a model integrating the hierarchical arrangement and the locational principle of central place theory can be defined as follows:

$$\min Z = W \sum_j A_j - (1 - W) \sum_m L_m \tag{33}$$

On the basis of objective function (33), let us try to reinterpret Christaller's and Lösch's theories. If L_m expresses a generalized median problem, the objective of minimizing the total number of firms is included in the second term of objective function (33). When the weight W takes a relatively large value, giving priority to the first term, the objective function value of the first term becomes 0, deriving Christaller's central place system, which fulfills complete agglomeration. Conversely, when L_m is synonymous with the total demand maximization problem and the weight W assumes a relatively small value that gives priority to the second term, then the market area for each good will be limited to a minimum size according to the threshold, and the maximum number of firms will be achieved. As per our reinterpretation in Chapter 3, if we assume that the Lösch's objective in constructing a hierarchy was the realization of the economies of agglomeration, then the central place system in Figure 15 is one such solution. In other words, what can be understood from the model in objective function (33) is that the difference between Christaller's and Lösch's theories in terms of hierarchical arrangement is nothing other than the decision of whether to prioritize the hierarchy of the overall system or to prioritize the behavior of individual firms.

In Lösch's market area theory, which lay emphasis on the behavior of individual firms, complete agglomeration is never realized outside of a metropolis. The agglomeration effect of goods is considered only secondarily. However, the maximum agglomeration effect is not achieved by prioritizing location in particular sectors as attempted by Lösch. Even where prioritizing location itself to be an objective, it is possible to create more "city-rich sectors" than Lösch's results, as revealed in Chapter 3. If Lösch believed that prioritizing location in particular sectors would lead to the most rational solution for the superimposition problem for market area networks, he was only chasing phantoms.

On the other hand, Christaller's central place theory, which is premised on complete agglomeration, encourages adaptive behavior on the supply of each good by firm in order to prefer the agglomeration effect. As a result, goods of differing orders are associated with market area networks of the same size. This naturally produces differences in the profits that can be achieved by each good. The goods with different size of the upper limit of the range are usually thought to differ also in terms of the threshold and the market area needed by existence of firm, but the economic rationality of individual firms is unquestioned solely on the basis that they are "more elastic."

In conclusion, it may be said that both Christaller and Lösch envisioned a model that specialized on one or the other of the two objectives in objective function (33). However, in practice, as indicated in the previous section, there exists a plurality of central place systems made up of many different types of hierarchical structure between the two inversely related objectives. Moreover, while the superimposition problem for market area networks assumes the uniform distribution of settlements and potential firm locations, population distributions and traffic networks are not uniform in reality, resulting in the formation of market area networks of different shapes and sizes. Therefore, when applying the model to the real world, it follows that a model that achieves hierarchical arrangement without resorting to geometric processing is essential.

The first term of objective function (33), which minimizes the total number of missing goods, determines whether firms will be located at a node j through the endogenous variable Y_{jm}. Because Y_{jm} is also included in Equation (32), which corresponds to the second term of objective function (33), it can also be applied in the case of a non-uniform population distribution such as was attempted in Chapter 2 or a problem that assumes a point distribution with even greater irregularity. While it depends on how the locational principle for single good is defined, the model of objective function (33), which captures flexible hierarchies based on the agglomeration effect of goods, can be defined as a generalized model of central place theory that subsumes the theories of both Christaller and Lösch and can be applied to real-world problems.

CHAPTER 5

Conclusion

In this paper, I have formulated the model that reproduces theoretical central place systems using a mathematical programming comprising some constraints and an objective function. Further, by placing particular focus on the structure of the model, I elaborated an attempt to reinterpret systematically both theories of Christaller and Lösch.

What can be understood from modelling Lösch's theory, first of all, is that Lösch's market area theory for the location of single good can be reproduced as a total demand maximization problem. Unlike the model developed in Kuby (1989), the model presented in this paper offers a corrected formulation of Lösch's objective in terms of the process of locational equilibrium and conditions such as demand allocation. Accordingly, it may be considered to stand as an operational model of Lösch's market area theory that enables the derivation of realistic central place systems, taking into account a more relaxed set of assumptions. Meanwhile, if Lösch's hierarchical arrangement is modelled as a superimposition problem for market area networks, Lösch's central place systems are not reproduced, with the model deriving its own rational solution with regard to prioritizing location in particular sectors. The fact that the optimality of Lösch's system is not supported, even when the Lösch's objective in constructing a hierarchy is reinterpreted as leveraging the agglomeration effect of goods, indicates that Lösch's method in constructing a hierarchy lacks a perspective on the rationality of the entire system.

When we reinterpret Christaller's central place theory by extending the model of Lösch's market area theory, the former can be considered as a generalized median problem as opposed to the latter, which is a total demand maximization problem. Furthermore, when Christaller's central place theory is regarded as a superimposition problem for market area networks, assuming the variable number of market centers envisioned in the generalized median problem, each good can be associated with multiple market area networks of different sizes. Thus, as a result of applying the extended model of Lösch's hierarchical arrangement, we succeeded in reproducing a central place system based on Christaller's marketing principle when prioritizing the agglomeration effect of goods. That is, in the sense of deriving the optimal solution in terms of hierarchical structure, Christaller's method may be considered more successful at deriving a rational solution than that of Lösch.

However, both theories are the same in the meaning of premising on special kinds of central place systems. In this paper, I have presented a generalized model that places the agglomeration effect of goods in opposition to the locational principle for single good, and identified Lösch's and Christaller's theories systematically. Consequently, we can recognize the former precedes the locational principle for single good while the latter prioritizes the agglomeration effect of goods.

It should be obvious why these had to be respectively conceived as antithetical objectives. Both Christaller and Lösch thought that there were an infinite number of kinds of goods to be considered originally. If it is assumed that the range and the threshold are slightly different depending on each good, then the distance between market centers will also change only a little. It would be almost impossible to solve a superimposition problem for an infinite number of market area networks whose interval of market centers are spaced at slightly different from one another. Hence, Lösch, assuming a point distribution of discrete settlements, replaced the superimposition problem with a combinatorial problem on finite set by specifying one of the market area networks in accordance with a threshold for each good. On the other hand, Christaller, by assuming a pattern of complete agglomeration in which market centers of higher-order goods must necessarily integrate all lower-order centers, succeeded in fixing the market centers on a continuous plane.

While Lösch regarded Christaller's central place systems as a special case of his own "complete systems" (Lösch 1954: 130-132)[26], it is apparent from the discussion in this paper that Christaller's central place system was no such thing. Certainly, Christaller's central place system comprised a part of the market area networks of various sizes that Lösch considered. Nevertheless, this was a product of the fact that these market area networks were aggregated by complete agglomeration and a varying number of market centers. As we can imagine from a comparison of Figures 11 and 17, Christaller captured the correspondence between goods and market area networks more flexibly than Lösch. The rational central place systems that realized complete agglomeration, obtained as a result of this flexibility, are incompatible with those of Lösch, which lacked rationality from the perspective of a whole system. To begin with, there is no containment relationship between the two central place systems, which are formed with different objectives and processes. Further, while these findings have been partially indicated in previous studies (e.g., Morikawa 1980; Hayashi 1986), it seems that the locus of the problem and differences between the two theories can be clarified by revisiting the question anew from a model-building perspective.

Incidentally, the generalized model in objective function (33), presented when reinterpreting Christaller's and Lösch's theories, embodies the potential to bring central place theory closer to reality in the sense that it can derive the various central place systems that are present in both theories. Real-world central place systems are not limited to those that satisfy the description of regular hexagonal market area networks, even when populations are uniformly distributed. Depending on the location pattern for each good, there could be cases where market area networks can take shape in various other ways, such as triangular and quadrilateral. When deriving central place systems based on the superimposition problem for market area networks elaborated in Chapter 3 and Chapter 4, the addition of options for market areas to correspond with goods in ways other than through these regular hexagons makes it possible to verify their consistency with real-world central place systems

[26] By "complete" (vollständig) Lösch meant that he had taken into account market area networks of varying sizes. However, he also points out that it is "not all possible market areas need occur in reality" (Lösch 1954: 120).

such as those of Parr (1978, 1980, 1981), which assume a general hierarchical model. In addition, the model for objective function (33) can be regarded as a generalized model for the problem of hierarchical facility location problems. The second term on the right-hand side of objective function (33), as classified in Table 5, is associated with the existing location-allocation models often invoked in the facility location problems. Conventionally, in the hierarchical facility location problems, a facility hierarchy is assumed to have either successively inclusive or successively exclusive of the goods and services being supplied[27], which are defined in a model by constraints (Narula 1984; Şahin and Süral 2007). In objective function (33), which employs multiobjective programming, because the condition of facility hierarchy is incorporated into the first term of the right-hand side, it will be possible to construct this hierarchy gradually in the move from successively inclusive of goods to successively exclusive of goods. From the standpoint of application in terms of theorizing the planning of central place theory (Sugiura 2013), it is possible to say that the model presented in this paper, in addition to being applicable in proposing plans for the layout of real-world facilities and cities, can also be invoked as an analytical model for understanding the locational principle and processes of hierarchy when locating central place systems and events in the real world.

In the context of the re-examination of central place theory that has been taking place in recent years with a focus on spatial economics, the significance and limits of central place theory are being questioned anew (e.g., Mulligan et al. 2012). For the purpose of realizing this re-examination of the theory, the aims of this paper are consistent with these research trajectory. However, before debating the validity of central place theory, I believe that there is still some work needed to understand the implications of classical central place theory, which does not seem to have been thoroughly discussed. This paper is one such attempt. As mentioned at the beginning, the significance of using mathematical

[27] Successively inclusive refers to higher-order facilities being in possession of the goods and services provided by lower-order facilities, while successively exclusive denotes a situation where facilities at each level of a hierarchy are in possession of their respective specialized goods and services (Ishizaki 2003). The former corresponds to hierarchy in Christaller’s central place systems, while the latter corresponds to that in Lösch’s central place systems.

techniques to reinterpret theories of Christaller and Lösch, who had no choice but to rely on descriptive and geometric explanations, has not decreased in the slightest.

References

Allen, P. M. and Sanglier, M. 1981. A dynamic model of a central place system – II. *Geographical Analysis* 13: 149-164.

Arlinghaus, S. L. 1985. Fractals take a central place. *Geografiska Annalers Ser. B* 67: 83-88.

Bailey, T. C. and Gatrell, A. C. 1995. *Interactive spatial data analysis*. Harlow: Longman.

Balakrishnan, P. V. and Storbeck, J. E. 1991. McTHRESH: Modeling maximum coverage with threshold constraints. *Environment and Planning B* 18: 459–472.

Beaumont, J. R. 1982. Mathematical programming in human geography. *Mathematical Social Sciences* 2: 213-243.

Beaumont, J. R. 1987. Location-allocation models and central place theory. In *Spatial analysis and location-allocation models,* ed. A. Ghosh and G. Rushton, 21-54. New York: Van Nostrand Reinhold.

Beavon, K. S. O. 1972. The intra-urban comtinuum of shopping centers in Cape Town. *South African Geographical Journal* 54: 58-71.

Beavon, K. S. O. 1977. *Central place theory: A reinterpretation.* New York: Longman.

Beavon, K. S. O. 1978a. City-rich and city-poor sectors: A comment on the construction of the Löschian landscape. *Geographical Analysis* 10: 77-82.

Beavon, K. S. O. 1978b. A comment on the procedure for determining the general structure of a Löschian hierarchy. *Journal of Regional Science* 18: 127-132.

Beavon, K. S. O. and Mabin, A. S. 1975. The Lösch system of market areas: Derivation and extension. *Geographical Analysis* 7: 131-151.

Berry, B. J. L. and Barnum, H. G. 1962. Aggregate relations and components of central place systems. *Journal of Regional Science* 4: 35-68.

Berry, B. J. L. and Garrison, W. L. 1958a. Recent developments of central place theory. *Papers and*

Proceedings of the Regional Science Association 4: 107-120.

Berry, B. J. L. and Garrison, W. L. 1958b. A note on central place theory and the range of a good. *Economic Geography* 34: 304-311.

Calvo, A. B. and Marks, D. H. 1973. Location of health care facilities: An analytical approach. *Socio-Economic Planning Sciences* 7: 407-422.

Carreras, M. and Serra, D. 1999. On optimal location with threshold requirements. *Socio-Economic Planning Sciences* 33: 91-103.

Christaller, W. 1966. *Central places in southern Germany* (translated by C. W. Baskin). Englewood Cliffs: Prentice-Hall.

Church, R. L. and Bell, T. L. 1990. Unpacking central place geometry I: Single level theoretical k systems. *Geographical Analysis* 22: 95-115.

Church, R. L. and Davis, R. R. 1992. The fixed charge maximal covering location problem. *Papers in Regional Science* 71: 199-215.

Church, R. and ReVelle, C. 1974. The maximal covering location problem. *Papers of the Regional Science Association* 32: 101-118.

Clark, C. 1951. Urban population densities. *Journal of the Royal Statistical Society A* 114: 490-496.

Clarke, M. and Wilson, A. G. 1985. The dynamics of urban spatial structure: The progress of a research programme. *Transaction, Institute of British Geographers* NS10: 427-451.

Cohon, J. L. 1978. *Multiobjective programming and planning*. New York : Academic Press.

Curtin, K. M. and Church, R. L. 2007. Optimal dispersion and central places. *Journal of Geographical Systems* 9: 167-187.

Dacey, M. F. 1965. The geometry of central place theory. *Geografiska Annaler* 47B: 111-124.

Dicken, P. and Lloyd, P. E. 1990. *Location in space,* 3rd ed. London: Harper & Row.

Dökmeci, V. F. 1973. An optimization model for a hierarchical spatial system. *Journal of Regional Science* 13: 439-451.

Eaton, B. C. and Lipsey, R. G. 1982. An economic theory of central places. *The Economic Journal* 92: 56-72.

Efroymson, M. A. and Ray, T. L. 1966. A branch-bound algorithm for plant location. *Operations Research* 14: 361-368.

Erlenkotter, D. 1977. Facility location with price-sensitive demands: Private, public and quasi-public. *Management Science* 24: 378-386.

Espejo, I., Marín, A. and Rodríguez-Chía, A. M. 2012. Closest assignment constraints in discrete location problems. *European Journal of Operational Research* 219: 49-58.

Gerrard, R. A. and Church, R. L. 1996. Closest assignment constraints and location models: Properties and structure. *Location Science* 4: 251-270.

Getis, A. 1963. The determination of the location of retail activities with the use of a map transformation. *Economic Geography* 39: 14-22.

Getis, A. and Getis, J. 1966. Christaller's central place theory. *Journal of Geography* 65: 220-226.

Ghosh, A. and Harche, F. 1993. Location-allocation models in the private sector: Progress, problems and prospects. *Location Science* 1: 81-106.

Ghosh, A. and Rusuton, G. 1987. Progress in location-allocation modeling. In *Spatial analysis and location-allocation models*, ed. A. Ghosh and G. Rushton, 1-18. New York: Van Nostrand Reinhold.

Goodchild, M. F. 1984. ILACS: Location-allocation model for retail site selection. *Journal of Retailing* 60: 84-100.

Haites, E. F. 1976. A note on the general structure of a Löschian hierarchy. *Journal of Regional Science* 16: 257-260.

Hansen, P. and Thisse, J-F. 1977. Multiplant location for profit maximisation. *Environment and Planning A* 9: 63-73.

Hayashi, N. 1986. *Chushinchi riron kenkyu* (The study of central place theory). Taimeido (in

Japanese).

Henderson, J. V. 1972. Hierarchy models of city size: an economic evaluation. *Journal of Regional Science* 12: 435-441.

Holmes, J., Williams, F. B. and Brown, L. A. 1972. Facility location under a maximum travel restriction: An example using day care facilities. *Geographical Analysis* 4: 258-266.

Hua, G., Chenga, T. C. E. and Wang, S. 2011. The maximum capture per unit cost location problem. *International Journal of Production Economics* 131: 568-574.

Isard, W. 1956. *Location and space-economy: A general theory relating to industrial location, market areas, land use, trade and urban structure*. New York: John Wiley.

Ishikawa, T. and Toda, M. 2000. Some economic extensions of central-place theory involving profit maximisation. *Urban Studies* 37: 481-495.

Ishizaki, K.1992. Ritchi - haibun moderu ni yoru Christaller chuushinchi riron no teishikika no kokoromi (Formulating Christaller's central place theory by location-allocation model). *Chirigaku Hyoron* (Geographical Review of Japan) 65A: 747-768 (in Japanese with English abstract).

Ishizaki, K.1995. Christaller no chuushinchi riron no haichi genri ni kansuru ichikousatsu: kyokyu genri kara ippanka saidai kaba mondai e (A reinterpretation of Christaller's central place theory: From marketing principle to generalized maximal covering location problem). *Chirigaku Hyoron* (Geographical Review of Japan) 68A: 579-602 (in Japanese with English abstract).

Ishizaki, K. 2003. Ritchi haibun moderu (Location-allocation model). In *Chiri kukan bunseki* (Geographical and spatial analysis), ed. Y. Sugiura, 61-83. Asakura Shoten (in Japanese).

Khumawala, B. M. 1973. An efficient algorithm for the p-median problem with maximum distance constraints. *Geographical Analysis* 5: 309-321.

Kohsaka, H. 1983. A central-place model as a two-level location-allocation system. *Environment and Planning A* 15: 5-14.

Korte, B. and Vygen, J. 2008. *Combinatorial optimization: Theory and algorithms*. Berlin

Heidelberg: Springer.

Kubo, M., Tamura, A. and Matsui, T. eds. 2002. *Oyo suri keikaku handobukku* (The handbook of applied mathematical programming). Asakura Shoten (in Japanese).

Kuby, M. 1989. A location-allocation model of Lösch's central place theory: Testing on a uniform lattice network. *Geographical Analysis* 21: 316-337.

Lösch, A. 1954. *The economics of location* (translated by W. H. Woglom 1954). New Haven: Yale University Press.

Marshall, J. U. 1977. The construction of the Löschian landscape. *Geographical Analysis* 9: 1-13.

Marshall, J. U. 1978a. The truncated Löschian landscape: A reply to Beavon. *Geographical Analysis* 10: 84-86.

Marshall, J. U. 1978b. On the structure of the Löschian landscape. *Journal of Regional Science* 18: 121-125.

Matsubara, H. 2006. *Keizai chirigaku: Ritchi chiiki toshi no riron* (Economic geography: Theories relating to industrial location, regional economy, and urban structure). Tokyo-daigaku shuppankai (University of Tokyo Press) (in Japanese).

Matsubara, H. 2013. Chushinchi riron no kiso to oyo (Fundamentals and applications of central place theory). In *Gendai no ritchi ron* (Contemporary location theory), ed. H. Matsubara, 38-53. Kokon shoin (in Japanese).

Mavrides, L. P. 1979. An indirect method for the generalized k-median problem applied to lock-box location. *Management Science* 25: 990-996.

Morikawa, H. 1980. *Chushinchi ron- I, II* (Central place theory vol. I, II). Taimeido (in Japanese).

Mulligan, G. F. 1984. Agglomeration and central place theory: A review of the literature. *International Regional Science Review* 9: 1-42.

Mulligan, G. F., Partridge, M. D. and Carruthers, J. I. 2012. Central place theory and its reemergence in regional science. *Annals of Regional Science* 48: 405-431.

Narula, S. C. 1984. Hierarchical location-allocation problems: A classification scheme. *European Journal of Operational Research* 15: 93-99.

Parr, J. B. 1973. Structure and size in the urban system of Lösch. *Economic Geography* 49: 185-212.

Parr, J. B. 1978. Models of the central place system: a more general approach. *Urban Studies* 15: 35-49.

Parr, J. B. 1980. Frequency distributions of central places in southern Germany: A further analysis. *Economic Geography* 56: 141-154.

Parr, J. B. 1981. Temporal change in a central-place system. *Environment and Planning A* 13: 97-118.

Parr, J. B. and Denike, K. G. 1970. Theoretical problems in central place analysis. *Economic Geography* 46: 568-586.

Puryear, D. 1975. A programming model of central place theory. *Journal of Regional Science* 15: 307-316.

ReVelle, C. 1986. The maximum capture or "sphere of influence" location problem: Hotelling revisited on a network. *Journal of Regional Science* 26: 343-358.

ReVelle, C. S. and Swain, R. W. 1970. Central facilities location. *Geographical Analysis* 2: 30-42.

ReVelle, C., Church, R. and Schilling, D. 1975. A note on the location model of Holmes, Williams and Brown. *Geographical Analysis* 7: 457-459.

Rojeski, P. and ReVelle, C. 1970. Central facilities location under an investment constraint. *Geographical Analysis* 2: 343-360.

Rushton, G. 1972. Map transformations of point patterns: Central place patterns in areas of variable population density. *Papers of the Regional Science Association* 28: 111-129.

Saey, P. 1973. Three fallacies in the literature on central place theory. *Tijdschrift voor Economische en Sociale Geografie* 64: 184-194.

Şahin, G. and Süral, H. 2007. A review of hierarchical facility location models. *Computers & Operations Research* 34: 2310-2331.

Schultz, G. P. 1970. The logic of health care facility planning. *Socio-Economic Planning Sciences* 4: 383-393.

Smithies, A. 1941. Optimum location in spatial competition. *Journal of Political Economy* 49: 423-439.

Storbeck, J. E. 1988. The spatial structuring of central places. *Geographical Analysis* 20: 93-110.

Storbeck, J. E. 1990. Classical central places as protected thresholds. *Geographical Analysis* 22: 4-21.

Sugiura, Y. 1991. A map-transformation approach to the location of central places: The case of the central Kanto region in preindustrial Japan. *Environment and Planning A* 23: 969-985.

Sugiura, Y. 2013. Chushinchi riron (Central place theory). In *Jimbun chirigaku jiten* (The dictionary of human geography), ed. Jimmonchiri gakkai (The Human Geographical Society of Japan), 32-35. Maruzen shuppan (in Japanese).

Tarrant, J. R. 1973. Comments on the Lösch central place system. *Geographical Analysis* 5: 113-121.

Toregas, C., Swain, R., ReVelle, C. and Bergman, L. 1971. The location of emergency service facilities. *Operations Research* 19: 1363-1373.

Wagner, J. L. and Falkson, L. M. 1975. The optimal nodal location of public facilities with price-sensitive demand. *Geographical Analysis* 7: 69-83.

Webber, M. J. 1971. Empirical verifiability of classical central place theory. *Geographical Analysis* 3: 15-28.