Riesz potential estimates for elliptic equations with drift terms

Takanobu Hara

Department of Mathematics and Information Sciences,
Tokyo Metropolitan University

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Abstract. In this thesis, we consider stationary drift-diffusion equations using energy methods. More precisely, we consider divergence form elliptic equations with drift terms $-\mathrm{div}\,(A\nabla u) + \mathbf{b}\cdot\nabla u = \mu$ in a domain $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$. First, we give Harnack type inequalities. Next, we give global and local weak-type $L^1 - L^{n/(n-2),\infty}$ estimates and also give pointwise potential estimate iterating local version of $L^1 - L^{n/(n-2),\infty}$ estimates. Moreover, we derive a pointwise lower bound of non-negative supersolutions. These estimates have many applications. For example, the pointwise estimate immediately gives a necessary and sufficient condition for continuity of solutions, and also, we can prove Wiener's criterion using these estimate.

Contents

CHAPTER 1

Introduction

In this thesis, we consider pointwise behavior of weak solutions to divergence form stationary linear drift-diffusion equations with force terms

$$\mathcal{L}u = -\mathrm{div}\,(A(x)\nabla u) + \mathbf{b}(x) \cdot \nabla u = \mu \quad \text{in } \Omega, \tag{1}$$

where Ω is an open set in $\mathbb{R}^n$ with $n \geq 3$. Our assumption on A is standard. The matrix valued function $A = A(x)$ belongs to $(L^\infty(\Omega))^{n\times n}$ and there is a positive constant $\nu > 0$ such that

$$(A(x)\xi) \cdot \xi \geq \nu |\xi|^2 \quad \forall \xi \in \mathbb{R}^n,\ x \in \Omega.$$

We assume that the right-hand side μ is expressed as $\mu = \mu_+ - \mu_-$ each of which is a finite non-negative Radon measure in $H^{-1}(\Omega) = (H^1_0(\Omega))^*$. If $\mu \in L^{2n/(n+2)}(\Omega)$, then μ is immediately decomposed as above. The aim of this thesis is to give quantitative regularity estimates for weak solutions to (1) under appropriate conditions on $\mathbf{b}$. Throughout the thesis, we assume that vector field $\mathbf{b}$ belongs to $(L^2_{\mathrm{loc}}(\Omega))^n$, but we give stronger conditions for $\mathbf{b}$ depending on situations. Roughly speaking, we will assume that $|\mathbf{b}|(x) = O(|x|^{-1})$ in the sense of integral average and it satisfies a geometric condition or a smallness condition. We assume that $\mathbf{b}$ is expressed as

$$\mathbf{b} = \mathbf{b}_0 + \mathbf{b}_1$$

and

$$\mathrm{div}\,\mathbf{b}_0 = 0$$

in the sense of distributions and $\mathbf{b}_1$ is sufficiently 'small' with respect to the minimum eigenvalue of A. Under these assumptions (see assumptions (21), (38) and (56) for the precise meaning of this smallness conditions), we will derive the two-sided pointwise Riesz potential type estimate, which is one of the main results of this thesis,

$$\frac{1}{C}\mathbf{I}^\mu_2(x_0, R) \leq u(x_0) \leq C\left(\operatorname*{ess\,inf}_{B(x_0,R)} u + \mathbf{I}^\mu_2(x_0, 2R)\right) \tag{2}$$

for nonnegative solution to $\mathcal{L}u = \mu \geq 0$ in $B(x_0, 2R)$, where $\mathbf{I}^\mu_2(x_0, R)$ is a truncated version of Riesz potential of μ. Later, we will see that its meaning is different respectively when deriving estimates type of (4), (2). Once these estimates have been obtained, we get (2) immediately by combining these estimates. If $\mathbf{b} = 0$, more generally, if $\mathrm{div}\,\mathbf{b} = 0$, then $\mathbf{b}_1 = 0$ and it is small in any senses, thus, we can prove the two-sided pointwise potential bounds (2). However, it is necessary again to treat equations directly when we prove each estimate. Each of one is not automatically derived from the previous one. In this thesis, we derive (2) using a method of nonlinear potential theory. This method has two advantages. First, by using energy methods, we can use the divergence-free structure of drifts naturally. Secondly, this method does not rely on the existence of Green's function, so, our method can apply to elliptic equation with strongly singular drifts directly. In the proof, we give new global and local weak-type $L^1 - L^{n/(n-2),\infty}$ estimates. See, Theorem 50, Theorem 53 and Theorem 64. The local weak-type estimate (Theorem 53) seems new even in the case of $\mathbf{b} = 0$ whenever $A \neq I$ (see, [**31**] and [**8**]). Also, using one of these estimates, we give a shorter proof of the estimate in [**5, 36**].

To think about why the divergence-free condition is effective, we shall recall the following flow independent energy estimate for solutions to $\mathcal{L}u = \mu$ with homogeneous Dirichlet boundary data:

$$\|\nabla u\|_{L^2(\Omega)} \leq C\|\mu\|_{H^{-1}(\Omega)}.$$

If $\operatorname{div} \mathbf{b} = 0$, then we can take $C = \nu$. Indeed, testing the equation by u, we have

$$\int_\Omega A\nabla u \cdot \nabla u \, \mathrm{d}x + \int_\Omega (\mathbf{b} \cdot \nabla u) u \, \mathrm{d}x = \langle \mu, u \rangle_{(H^{-1}(\Omega), H_0^1(\Omega))}.$$

However, the second term in the left-hand side vanishes by integration by parts, so, we can get the desired flow-independent estimate. Such a cancellation is often used in analysis of equations related with fluid dynamics. In recent years, Berestycki, Hamel, and Nadirashvili [**4**] proved a uniform flow-independent lower bound of the first Dirichlet eigenvalue. In recent years, Berestycki, Kiselev, Novikov and Ryzhik proved flow-independent global $L^p - L^\infty$ $(p > n/2)$ bounds, and applied them to semilinear problems [**5, 36**]. For other usages of the divergence-free condition, see also [**71, 4, 59, 44, 42**]. There are related results for parabolic equations. For example, Carlen and Loss [**9**] gave an on-diagonal heat kernel estimate with the optimal constant, and applied them to two dimension Navier-Stokes equations. Osada [**65**] introduced consideration of generalized divergence form and gave an Aronson type estimate for parabolic equation with divergence-free drifts. He rewrote the parabolic equations with divergence free drifts $\partial_t u + \mathbf{b} \cdot \nabla u - \triangle u = 0$ as

$$\partial_t u - \operatorname{div}((I + V)\nabla u) = 0$$

using an anti-symmetric matrix valued function $V(x) = (V_{ij}(x)) \in (L^\infty(\Omega))^{n \times n}$ which satisfies $\mathbf{b}(x) = (b_i(x)) = \left(\sum_{j=1}^n \partial_j V_{ij}(x)\right)$. Note that the same idea is often found in the different context by many authors. After years, Liskevich and Zhang [**52**] gave an Aronson-type estimates for weak fundamental solutions to parabolic equations with singular drifts. Their assumptions on drifts are closely related to our assumption (38). See also [**43, 67**]. Nazarov and Ural′tseva [**64**] gave parabolic Harnack inequality for equations with divergence-free space-time singular drifts using Morrey spaces. Friedlander and Vicol [**23**] and Seregin, Silvestre, Šverák and Zlatoš [**68**] gave parabolic Harnack inequality for equations with divergence-free $L^\infty(BMO^{-1})$ drifts.

We briefly discuss the history and the background of the subject on quantitative properties of weak solutions to divergence form elliptic equations. First of all, we recall the basics of weak solutions of divergence form linear elliptic equations with bounded measurable coefficients

$$-\operatorname{div}(A(x)\nabla u) = 0. \tag{3}$$

From theory of functional analysis and calculus of variation, existence theorems of weak solutions of these equations are not difficult, but its regularity estimates had been an important problem in the first half of last century since Hilbert's 19th problem. In 1957, De Giorgi [**17**] proved Hölder continuity of weak solutions. Moser

[**62, 63**] gave a new proof of De Giorgi's theorem using Harnack's inequality. He proved the following: If u is a nonnegative weak solution to (3) in Ω, then

$$\operatorname*{ess\,sup}_{B(x_0,R/2)} u \le C \operatorname*{ess\,inf}_{B(x_0,R/2)} u \tag{4}$$

whenever $B(x_0, 2R) \subset \Omega$, where C is a constant depending only on n, ν and $\|A\|_{L^\infty(\Omega)}$. De Giorgi's theorem follows from this inequality and an iteration argument directly. Note that their proofs do not depend on the modulus of continuity of $A(x)$ neither.

After few years, Littman, Stampacchia and Weinberger [**53**] considered Green's function of linear elliptic equations (3) with the homogeneous Dirichlet boundary condition. In other words, they construct a solution $G(x, y)$ to the Dirichlet problem with measure data $-\mathrm{div}\,(A\nabla G(x, y)) = \delta_y$, where δ_y is Dirac's delta measure centered at $y \in \Omega$. They used De Giorgi-Moser's uniform estimates and solutions of perturbed equations. They also proved that pointwise behavior of $G(x, y)$ is equivalent to that of the Laplace equation. From the representation formula of Green's function of the Laplace equation, they established the Riesz potential estimates

$$G(x, y) \le C|x - y|^{2-n} \tag{5}$$

and

$$G(x, y) \ge \frac{1}{C}|x - y|^{2-n} \quad \text{if } |x - y| \le \frac{1}{2}\mathrm{dist}(y, \partial\Omega). \tag{6}$$

They also gave Wiener's boundary regularity criterion using the equivalence of the Green functions. Their proof of Wiener's criterion was not quantitative, but, Maz′ya [**56**] gave a modulus of continuity of solutions of near boundary at about the same time. Grüter and Widman [**28**] introduced a mollified version of Green's function and gave another definition of Green's function. More precisely, they considered a sequence of functions $\{G_{\rho,y}\}_{\rho>0} \subset H^1_0(\Omega)$ ($y \in \Omega$, $\rho > 0$) such that each of which satisfies

$$-\mathrm{div}\,(A\nabla G_{\rho,y}) = \frac{1}{|B(y, \rho)|}\mathbf{1}_{B(y,\rho)} \quad \text{in } \Omega,$$

moreover, defined Green's function as the limit $G(x, y) = \lim_{\rho\to 0} G_{\rho,y}(x)$ using uniform estimates

$$\|G_{\rho,y}\|_{L^{n/(n-2),\infty}(\Omega)}, \quad \|\nabla G_{\rho,y}\|_{L^{n/(n-1),\infty}(\Omega)} \le C(n) \tag{7}$$

and

$$\int_{\Omega\setminus B(y,R)} |\nabla G_{\rho,y}|^2 \,\mathrm{d}x \le C(n)R^{n-2} \quad \forall R > 0.$$

Their construction methods of Green's function are frequently used at present. They also gave estimates (5) and (6) directly. Moreover, they gave a Maz′ya-type estimate using Green's function. For further results about Wiener's critrion, see also [**19, 21, 22, 15, 61**].

De Girogi and Moser's arguments do not depend on the linearity of equations, so, their estimates were immediately extended to solutions to quasilinear equations

$$-\mathrm{div}\,\mathcal{A}(x, \nabla u) = 0, \tag{8}$$

where $\mathcal{A} : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function which satisfies

$$|\mathcal{A}(x, z)| \le L|z|^{p-1},$$
$$\mathcal{A}(x, z) \cdot z \ge \nu |z|^p,$$
$$(\mathcal{A}(x, z_2) - \mathcal{A}(x, z_1)) \cdot (z_2 - z_1) > 0$$
$$\forall x \in \Omega,\ \forall z, z_1 \neq z_2 \in \mathbb{R}^n$$

for some constants $1 < p < \infty$ and $0 < \nu \le L < \infty$. For example, if $\mathcal{A}(x, z) = |z|^{p-2}z$, then these conditions are fulfilled, and (8) becomes the p-Laplacian equation. Existence theorems of weak solutions to these equations follows from theory of monotone operators. For such weak solutions, we can show Harnack's inequality and Hölder continuity: see standard textbooks [**51, 41, 47, 25, 26, 33, 55**] and the references therein.

Since these equations are nonlinear, the concept of Green's function is not available. However, we can get an analog of (7) using truncated test functions. By using these estimates and weak convergence methods, we reach the concept of equation with measure data such as $-\operatorname{div} \mathcal{A}(x, \nabla u) = \delta_y$; see e.g. [**6, 37, 3, 7, 16**]. Unfortunately, uniqueness of such generalized solutions is not clear in general. Thus, an appropriate definition of a class of very weak solutions to quasilinear equations has been treated by many authors. On the other hand, Maz′ya [**57**] gave a sufficient condition of the Wiener boundary regularity for quasilinear equations (8). Necessity of this condition was considered by Lindqvist and Martio [**50**]. After years, Kilpeläinen and Malý [**37, 38**] proved the following two-sided pointwise estimate: if u is a non-negative solution to the equation $-\operatorname{div} \mathcal{A}(x, \nabla u) = \mu \ge 0$ in Ω, then,

$$\frac{1}{C} \mathbf{W}_p^\mu(x_0, R) \le u(x_0) \le C \left(\inf_{B(x_0,R)} u + \mathbf{W}_p^\mu(x_0, 2R) \right) \tag{9}$$

whenever $B(x_0, 2R) \subset \Omega$, where C is a constant depending only on n, ν and L, and $\mathbf{W}_p^\mu(x_0, R)$ is the Wolff potential of μ which defined by

$$\mathbf{W}_p^\mu(x_0, R) = \int_0^R \left(s^{p-n} \mu(B(x_0, s)) \right)^{1/(p-1)} \frac{\mathrm{d}s}{s}.$$

Note that if $p = 2$, then the Wolff potential is a truncated version of Riesz potential. They proved necessity of Maz′ya's condition using the second inequality of (9). Conversely, sufficiency follows from the first inequality. Trudinger and Wang [**70**] gave another proof of (9) for more general equations. For other proofs of this pointwise estimate, see also [**45, 31**]. For related results and topics of this estimate, see textbooks of nonlinear potential theory [**33, 55**].

Another extension of equations (3) is an equation with lower order terms. This problem is taken up by quite many authors including Morrey [**60**]. Quasilinear equations with lower order terms can also be considered, in fact, the above references treated such equations. We will focus on linear equations with drift terms. From standard results, if $\mathbf{b} \in (L^p(\Omega))^n$ with $p > n$ then solutions to the equation (1) are Hölder continuous and Harnack's inequality holds for nonnegative solutions. Note that it is not necessary to prove Harnack-type estimates. Indeed, Stampacchia [**69**] proved Harnack's inequality for elliptic equation with small $L^n(\Omega)$ drifts. Recently, Nazarov and Ural′tseva [**64**] gave a similar estimate for arbitrary $L^n(\Omega)$ drifts using Safonov's technique [**66**]. For equations with more general drifts, see also [**68, 64,**

29]. Stampacchia also gave existence of Green's function for elliptic equations with $L^p(\Omega)$ $(p > n)$ drifts. For Green's function for elliptic equation with drift terms, see also [**14, 35, 40, 42**].

Organization of the thesis In Chapter 3, we derive basic properties of supersolutions and subsolutions to the homogeneous equation $\mathcal{L}u = 0$. We also give Harnack's inequality for nonnegative solutions. See Theorem 47. In Chapter 4, we prove a global flow-independent estimate. Moreover, we prove the local $L^1 - L^{n/(n-2),\infty}$ estimate and the potential upper bound (2). See Theorem 53 and Theorem 55. In Chapter 5, we prove the potential bound of (2). See Theorem 71. Moreover, we discuss the continuity of solutions and Wiener's criterion as application of the potential bounds.

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CHAPTER 2

Preliminaries

1. Notation

We use the following notation in this thesis. Let U and U' be open sets in $\mathbb{R}^n$. For a Banach space X, we denote by X^* the dual of X. Here, $\operatorname{ess\,sup}_A f$ and $\operatorname{ess\,inf}_A f$ are the essential supremum and essential infimum of f on A.

- $B(x_0, R) := \{x \in \mathbb{R}^n;\ |x - x_0| < R\}$.
- $\complement A := \mathbb{R}^n \setminus A$.
- $\operatorname{dist}(x, A) := \inf\{|x-y|\ :\ y \in A\}$.
- $\operatorname{diam} A := \sup\{|x-y|\ :\ x, y \in A\}$.
- $|A|$:= the Lebesgue measure of a measurable set A.
- $⨍_A f\ \mathrm{d}x := \frac{1}{|A|}\int_A f\ \mathrm{d}x$.
- $\mathbf{1}_A(x)$:= the indicator function of A.
- $f_+ := \max\{f, 0\}$, $f_- := \max\{-f, 0\}$
- $\operatorname{osc}_A f := \operatorname{ess\,sup}_A f - \operatorname{ess\,inf}_A f$.
- $U' \Subset U :\Leftrightarrow \overline{U'} \subset U$ and $\overline{U'}$ is compact.
- $C_c^\infty(U)$:= the set of all infinitely-differentiable functions with compact support in U.
- $\nabla f = (\frac{\partial f}{\partial x_1}, \dots \frac{\partial f}{\partial x_n})^T$, $\operatorname{div} F := \sum_{i=1}^n \frac{\partial F_i}{\partial x_i}$, $\triangle f := \operatorname{div}(\nabla f) = \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2}$.

The Sobolev space $H^1(\Omega)$ is the set of all weakly differentiable functions f such that $\|f\|_{H^1(\Omega)}$ is finite, where

$$\|f\|_{H^1(\Omega)}^2 := \|f\|_{L^2(\Omega)}^2 + \|\nabla f\|_{L^2(\Omega)}^2.$$

The space $H_0^1(\Omega)$ is the closure of $C_c^\infty(\Omega)$ in $H^1(\Omega)$. We say that a function f belongs to $H_{\mathrm{loc}}^1(\Omega)$ if $\|f\|_{H^1(D)} < \infty$ for all $D \Subset \Omega$. We write $(H_0^1(\Omega))^*$ as $H^{-1}(\Omega)$. Moreover, we introduce the Dirichlet space $\mathcal{D}^{1,2}(\Omega)$ as follows:

$$\mathcal{D}^{1,2}(\Omega) = \{u \in H_{\mathrm{loc}}^1(\Omega);\ \nabla u \in (L^2(\Omega))^n\}.$$

The space $\mathcal{D}_0^{1,2}(\Omega)$ is the completion of $C_c^\infty(\Omega)$ with respect to the norm $\|\nabla \cdot\|_{L^2(\Omega)}$. From the Poincaré inequality, if Ω is bounded, then $\mathcal{D}_0^{1,2}(\Omega) = H_0^1(\Omega)$. Below, when Ω is bounded, we write $\mathcal{D}_0^{1,2}(\Omega)$ as $H_0^1(\Omega)$. We write the duality pairing on $(\mathcal{D}_0^{1,2}(\Omega))^* \times \mathcal{D}_0^{1,2}(\Omega)$ as $\langle \cdot, \cdot \rangle_\Omega$. Throughout this article, the letters C denote positive constants whose values may be different at different instances. When the value of a constant in significant, it will be clearly stated.

2. Sobolev spaces

First, we recall some properties of Sobolev spaces:

Lemma 1 ([**33**, p.18]). *Suppose that $\varphi \in C^1(\mathbb{R})$, φ' is bounded, and $u \in H^1(\Omega)$. If $\varphi \circ u \in L^2(\Omega)$, then $\varphi \circ u \in H^1(\Omega)$.*

LEMMA 2 ([**33**, p.20]). *If u and v belong to $H^1(\Omega)$, then* $\max\{u,v\}$ *and* $\min\{u,v\}$ *belong to $H^1(\Omega)$. Moreover,*

$$\nabla \max\{u,v\}(x) = \begin{cases} \nabla u(x) & \text{if } u(x) \geq v(x) \\ \nabla v(x) & \text{if } u(x) \leq v(x) \end{cases}$$

and

$$\nabla \min\{u,v\}(x) = \begin{cases} \nabla u(x) & \text{if } u(x) \leq v(x) \\ \nabla v(x) & \text{if } u(x) \geq v(x). \end{cases}$$

In particular, if $u \in H^1_{\mathrm{loc}}(\Omega)$ and $k \in \mathbb{R}$, then

$$\nabla u = 0 \quad \text{a.e. on } \{x \in \Omega;\ u(x) = k\}.$$

LEMMA 3 ([**33**, p.21]). *Suppose that u and v belong to $H^1(\Omega) \cap L^\infty(\Omega)$. Then*

(1) *$uv \in H^1(\Omega) \cap L^\infty(\Omega)$ and*

$$\nabla(uv) = v\nabla u + u\nabla v.$$

(2) *If, in addition, $u \in H^1_0(\Omega) \cap L^\infty(\Omega)$, then $uv \in H^1_0(\Omega) \cap L^\infty(\Omega)$.*

3. Riesz potentials and the capacity

Next, we recall properties of Riesz potentials. See also [**1, 25, 33, 58, 46**].

DEFINITION 4. Let μ be a non-negative Radon measure in Ω. For $x_0 \in \Omega$ and $0 < R \leq \mathrm{dist}(x_0, \partial\Omega)$, we define

$$\mathbf{I}_2^\mu(x_0, R) = \int_0^R s^{2-n}\mu(B(x_0,s))\frac{\mathrm{d}s}{s}.$$

Since $\int_{|x_0-x|}^R s^{1-n}\,\mathrm{d}s = (n-2)^{-1}(|x_0-x|^{2-n} - R^{2-n})$, by Fubini's thereom, we have

$$\begin{aligned} \mathbf{I}_2^\mu(x_0,R) &= \int_0^R s^{1-n}\left(\int_{B(x_0,R)} \mathbf{1}_{\{|x_0-x|<s\}}\,\mathrm{d}\mu(x)\right)\mathrm{d}s \\ &= \int_{B(x_0,R)}\left(\int_0^R s^{1-n}\mathbf{1}_{\{|x_0-x|<s\}}\,\mathrm{d}s\right)\mathrm{d}\mu(x) \\ &= \frac{1}{n-2}\int_{B(x_0,R)} (|x_0-x|^{2-n} - R^{2-n})\,\mathrm{d}\mu(x). \end{aligned}$$

Thus, if u is the solution to the Dirichlet problem

$$\begin{cases} -\triangle u = \mu & \text{in } B(x_0,R) \\ u = 0 & \text{on } \partial B(x_0,R), \end{cases}$$

then $u(x_0) = (n-2)^{-1}\mathbf{I}_2^\mu(x_0,R)$ (see e.g. [**34**, p.19]). In particular, for any $0 < R \leq \infty$,

$$\mathbf{I}_2^\mu(x_0,R) \leq \mathbf{I}_2^\mu(x_0,\infty) = \frac{1}{n-2}\int_{B(x_0,R)} \frac{\mathrm{d}\mu(x)}{|x_0-x|^{n-2}}.$$

LEMMA 5. *Let $R_m = 2^{-m}R$ for $m = 0, 1, \ldots$. Then there is a constant C, depending only on n and p, such that*

$$\begin{aligned}\frac{1}{C}\mathbf{I}_2^{\mu}(x_0, R) &\leq \sum_{m=0}^{\infty} R_m^{2-n}\mu(B(x_0, R_m)) \\ &\leq \sum_{m=0}^{\infty} R_m^{2-n}\mu(\overline{B(x_0, R_m)}) \leq C\mathbf{I}_2^{\mu}(x_0, 2R).\end{aligned}$$

PROOF. We only prove the latter inequality. Since μ is a non-negative measure, we have

$$\begin{aligned}\int_0^{2R} s^{2-n}\mu(B(x_0, s))\frac{\mathrm{d}s}{s} &= \sum_{m=0}^{\infty}\int_{R_m}^{2R_m} s^{2-n}\mu(B(x_0, s))\frac{\mathrm{d}s}{s} \\ &\geq \sum_{m=0}^{\infty}\int_{R_m}^{2R_m} s^{2-n}\mu(\overline{B(x_0, R_m)})\frac{\mathrm{d}s}{s} \\ &= C(n)\sum_{m=0}^{\infty} R_m^{2-n}\mu(\overline{B(x_0, R_m)}).\end{aligned}$$

By a similar calculation, we can get the first inequality. □

Next, we recall the definition of capacity:

DEFINITION 6. Let Ω be an open set in $\mathbb{R}^n$. For a compact set $K \subset \Omega$, we take

$$_*\operatorname{cap}(K, \Omega) := \inf\left\{\int_\Omega |\nabla\varphi|^2\,\mathrm{d}x;\ \varphi \in C_c^\infty(\Omega),\ \varphi \geq 1 \text{ on } K\right\}.$$

Moreover, for $E \subset \Omega$, we define

$$\operatorname{cap}(E, \Omega) := \inf_{\substack{E\subset U\subset\Omega \\ U;\,\text{open}}}\ \sup_{K\Subset U}\ {}_*\operatorname{cap}(K, \Omega).$$

The number $\operatorname{cap}(E, \Omega)$ is called the capacity of the condenser (E, Ω).

LEMMA 7. *The set function $E \mapsto \operatorname{cap}(E, \Omega)$, $E \subset \Omega$, satisfies the following properties:*

(1) *If $E_1 \subset E_2$, then*

$$\operatorname{cap}(E_1, \Omega) \leq \operatorname{cap}(E_2, \Omega).$$

(2) *If $\Omega_1 \subset \Omega_2$ and $E \subset \Omega_1$, then*

$$\operatorname{cap}(E, \Omega_2) \leq \operatorname{cap}(E, \Omega_1).$$

(3) *If $E = \bigcup_{i=0}^{\infty} E_i \subset \Omega$, then*

$$\operatorname{cap}(E, \Omega) \leq \sum_{i=0}^{\infty}\operatorname{cap}(E_i, \Omega).$$

DEFINITION 8. We say that a property holds *quasieverywhere*, abbreviated q.e., if it holds except on a set of capacity zero.

Definition 9. Let Ω be an open set in $\mathbb{R}^n$. A function $u : \Omega \to [-\infty, \infty]$ is a quasicontinuous function in Ω if for every $\epsilon > 0$, there is an open set V such that $C_2(V) < \epsilon$ and the restriction of u to $\Omega \setminus V$ is finite and continuous, where $C_2(V)$ is the Sobolev capacity of V which defined by

$$C_2(V) = \inf \left\{ \int_{\mathbb{R}^n} |u|^2 + |\nabla u|^2 \, \mathrm{d}x; \ u \in H^1(\mathbb{R}^n), \ u \geq 1 \text{ a.e. in } V \right\}.$$

If $u \in H^1(\Omega)$, then the limit

$$\lim_{R \to 0} \fint_{B(x_0,R)} u(x) \, \mathrm{d}x = u(x_0)$$

exists and defines u quasieverywhere in Ω. Moreover, we have the following:

Lemma 10 ([**33**, pp.89-90]). *Suppose that $u \in H^1(\Omega)$. Then there exists a quasicontinuous function v such that $u = v$ a.e. in Ω. Moreover, a function $u \in H^1(\Omega)$ belongs to $H_0^1(\Omega)$ if and only if there is a quasicontinuous function v in $\mathbb{R}^n$ such that $u = v$ a.e. in Ω and $v = 0$ q.e. in $\complement\Omega$.*

4. Lorentz spaces and embedding theorems

Next, we recall the definition of the Lorentz spaces $L^{p,q}(\Omega)$.

Definition 11. For $0 < p \leq \infty$ and $0 < r \leq \infty$, we take

$$L^{p,r}(\Omega) := \left\{ f : \Omega \to \mathbb{R} \text{ measurable}; \ \|f\|_{L^{p,r}(\Omega)} < \infty \right\},$$

where

$$\|f\|_{L^{p,r}(\Omega)} = \left(p \int_0^\infty \left(t |\{x \in \Omega; \ |f(x)| \geq t\}|^{1/p} \right)^r \frac{\mathrm{d}t}{t} \right)^{1/r}$$

and

$$\|f\|_{L^{p,\infty}(\Omega)} := \sup_{t>0} t \, |\{x \in \Omega; \ |f(x)| \geq t\}|^{1/p}.$$

The space $L^{p,\infty}$ is also called the weak-L^p space. By definition,

$$\| |f|^q \|_{L^{p,r}(\Omega)} = \| |f| \|^q_{L^{pq,rq}(\Omega)}.$$

The quantity $\| \cdot \|_{L^{p,\infty}(\Omega)}$ does not satisfy the triangle inequality, in general. However, it satisfies the quasi triangle inequality

$$\|f + g\|_{L^{p,\infty}(\Omega)} \leq C_p \left(\|f\|_{L^{p,\infty}(\Omega)} + \|g\|_{L^{p,\infty}(\Omega)} \right).$$

Moreover, it satisfies the following Fatou-type property (see [**27**, p.14]); for any measurable function sequence $\{f_j\}_{j=1}^\infty$, we have

$$\| \liminf_{j \to \infty} |f_j| \, \|_{L^{p,\infty}(\Omega)} \leq C_p \liminf_{j \to \infty} \|f_j\|_{L^{p,\infty}(\Omega)}. \tag{10}$$

The following Hölder-type inequality is well-known:

Lemma 12 ([**27**, p.52]). *Let $1 < p, p' < \infty$ and $1/p + 1/p' = 1$. Then*

$$\left| \int_\Omega fg \, \mathrm{d}x \right| \leq \|f\|_{L^{p,1}(\Omega)} \|g\|_{L^{p',\infty}(\Omega)}. \tag{11}$$

By using Lorentz spaces, the usual Sobolev's inequality

$$\|f\|_{L^{2n/(n-2),2}(\mathbb{R}^n)} \leq S(n) \|\nabla f\|_{L^2(\mathbb{R}^n)}, \quad \forall f \in \mathcal{D}_0^{1,2}(\mathbb{R}^n)$$

is improved as follows:

LEMMA 13 ([**2**]). *Let $n > 2$. Then, the embedding into Lorentz space*

$$\|f\|_{L^{2n/(n-2),2}(\mathbb{R}^n)} \le S_2 \|\nabla f\|_{L^2(\mathbb{R}^n)}, \quad \forall f \in \mathcal{D}_0^{1,2}(\mathbb{R}^n)$$

holds, where

$$S_2 = S_2(n) = |B(0,1)|^{-1/n} \frac{2}{n-2}.$$

Here, $|B(0,1)|$ is the Lebesgue measure of the unit ball in $\mathbb{R}^n$.

Recently, the best constant of the following embedding was obtained:

LEMMA 14 ([**11**]). *Let $n > 2$. Then, the embedding into Lorentz space*

$$\|f\|_{L^{2n/(n-2),\infty}(\mathbb{R}^n)} \le S_\infty \|\nabla f\|_{L^2(\mathbb{R}^n)}, \quad \forall f \in \mathcal{D}_0^{1,2}(\mathbb{R}^n)$$

holds, where

$$S_{p,\infty} = n^{-1/2} |B(0,1)|^{-1/n} \left(\frac{1}{n-2} \right)^{1/2}.$$

REMARK 15. Since $L^p \subset L^{p,\infty}$ (see [**27**]), from the usual Sobolev inequality, the embedding $\mathcal{D}_0^{1,2}(\mathbb{R}^n) \hookrightarrow L^{2n/(n-2),\infty}(\mathbb{R}^n)$ is well-known. Nevertheless, we cite Lemma 14, because this sharp embedding is useful for our argument. For the extremal functions of this inequality, see Theorem 2 in [**11**]. For details, see Theorem 64. On the other hand, as arguments in [**54**], if Lemma 14 holds, then we can show the embedding $\mathcal{D}_0^{1,2}(\mathbb{R}^n) \hookrightarrow L^{2n/(n-2),2}(\mathbb{R}^n)$.

Let us recall the following Marcinkiewicz interpolation theorem:

LEMMA 16 ([**27**, p.56]). *Let $0 < r \le \infty$, $0 < p_0 < p_1 \le \infty$ and $0 < q_0 < q_1 \le \infty$. Let T be a linear operator defined on the set of simple functions on Ω. Assume that for $M_0, M_1 < \infty$ the following restricted weak type estimates hold:*

$$\|T(\mathbf{1}_A)\|_{L^{q_0,\infty}(\Omega)} \le M_0 |A|^{1/p_0},$$

$$\|T(\mathbf{1}_A)\|_{L^{q_1,\infty}(\Omega)} \le M_1 |A|^{1/p_1},$$

for all $A \subset \Omega$ with $|A| < \infty$. Fix $0 < \theta < 1$ and let

$$\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1} \quad \text{and} \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Then there exists a constant M, which depends on K, p_0, p_1, q_0, q_1, M_0, M_1, r and θ, such that for all functions f in the domain of T and in $L^{p,r}(\Omega)$ we have

$$\|T(f)\|_{L^{q,r}(\Omega)} \le M \|f\|_{L^{p,r}(\Omega)}.$$

LEMMA 17 ([**27**, p.63]). *Let $1 < p, q, r < \infty$ satisfy*

$$\frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{r},$$

and let $0 < s \le \infty$. Then for all $f \in L^{q,s}(\mathbb{R}^n)$ and $g \in L^{r,\infty}(\mathbb{R}^n)$,

$$\|f * g\|_{L^{q,s}(\mathbb{R}^n)} \le C(p,q,r,s) \|g\|_{L^{r,\infty}(\mathbb{R}^n)} \|f\|_{L^{q,s}(\mathbb{R}^n)},$$

where $(f * g)(x) = \int_{\mathbb{R}^n} f(x-y) g(y)\, \mathrm{d}y$.

5. Miscellaneous facts

LEMMA 18 ([**25**, p.166]). *Let* Ω *be a convex domain, and let* $f \in W^{1,1}(\Omega)$*. Suppose that there is a constant* M *such that*

$$\int_{\Omega\cap B(y,r)} |\nabla f| \,\mathrm{d}x \le M r^{n-1}$$

for all balls $B(y,r) \subset \Omega$*. Then, there exist positive constants* σ_0 *and* C *depending only on* n *such that*

$$\int_\Omega \exp\Big(\frac{\sigma}{M}|u - u_\Omega|\Big) \,\mathrm{d}x \le C(\operatorname{diam}\Omega)^n$$

whenever $\sigma < \sigma_0 |\Omega| (\operatorname{diam}\Omega)^{-n}$*.*

LEMMA 19 ([**26**, p.220]). *Let* $\{U_m\}_{m=0}^{\infty}$ *be a sequence of non-negative numbers. Assume that*

$$U_{m+1} \le C b^m U_m^{1+\alpha}$$

for all $m \ge 0$*, where* $C > 0$*,* $b > 1$ *and* $\alpha > 0$*. Assume also that*

$$U_0 \le C^{-1/\alpha} b^{-1/\alpha^2}.$$

Then $U_m \to 0$ *as* $m \to \infty$*.*

LEMMA 20 ([**26**, p.191]). *Let* $0 < \rho < R$*. Let* $Z(t)$ *be a bounded non-negative function in the interval* $[\rho, R]$*. Assume that for* $\rho \le t < s \le R$ *we have*

$$Z(t) \le A(s-t)^{-\alpha} + \theta Z(s)$$

with $A \ge 0$*,* $\alpha > 0$ *and* $\theta \in (0,1)$*. Then,*

$$Z(\rho) \le C(\alpha,\theta) A (R-\rho)^{-\alpha}.$$

CHAPTER 3

Energy estimates and related results

In this chapter, we introduce weak solutions to the equations $\mathcal{L}u = \mu$ in Ω using the divergence structure of equations:

$$\langle \mathcal{L}u, \varphi \rangle = \int_\Omega A\nabla u \cdot \nabla \varphi + (\mathbf{b} \cdot \nabla u)\varphi \, \mathrm{d}x = \langle \mu, \varphi \rangle_\Omega,$$

where A and $\mathbf{b}$ satisfy

$$A(x) \in (L^\infty(\Omega))^{n\times n}, \quad (A(x)\xi)\cdot \xi \geq \nu|\xi|^2 \quad \forall \xi \in \mathbb{R}^n, \ x \in \Omega \tag{12}$$

and

$$\mathbf{b} = \mathbf{b}_0 + \mathbf{b}_1, \quad \operatorname{div} \mathbf{b}_0 = 0.$$

If the bilinear form $\langle \mathcal{L}u, v\rangle$ is bounded and coercive, i.e. if

$$|\langle \mathcal{L}u, v\rangle| \leq C\|\nabla u\|_{L^2}\|\nabla v\|_{L^2},$$
$$\frac{1}{C}\|\nabla u\|_{L^2}^2 \leq \langle \mathcal{L}u, u\rangle,$$

then, from the Lax-Milgram theorem, this operator gives a one-to-one relation between $H_0^1(\Omega)$ solutions and $H^{-1}(\Omega)$ data. Therefore, we can get uniqueness and existence of weak solutions to Dirichlet problems. Moreover, in the same condition, we can show Caccioppoli type estimates. Consequently, we obtain some estimates from De Giorgi-Moser theory. In particular we will prove a Hölder estimate of solutions (Theorem 44) and a Harnack estimate for nonnegetive solutions (Theorem 47). Our framework allows that $\mathbf{b} = O(|x|^{-1})$. It include equations

$$\mathcal{L}u = -\triangle u + \frac{\beta x}{|x|^2}\cdot \nabla u = 0 \quad \text{in } \Omega = B(0,1), \tag{13}$$

where $\beta \in \mathbb{R}$. From Hardy's inequality

$$\int_{\mathbb{R}^n} \frac{|u|^2}{|x|^2}\,\mathrm{d}x \leq \left(\frac{2}{n-2}\right)^2 \int_{\mathbb{R}^n} |\nabla u|^2 \,\mathrm{d}x$$

the bilinear form $\langle \mathcal{L}u, v\rangle$ is bounded on $H_0^1(\Omega)$. Moreover, if $\beta < (n-2)/2$, then by using the best constant of Hardy's inequality and integrating by parts, we can show the coercivity of $\langle \mathcal{L}u, v\rangle$. Thus, uniqueness of weak solutions to Dirichlet problems follows. On the other hand, this equation has a classical solution

$$u(x) = \begin{cases} c|x|^{2-n+\beta} & \beta \neq n-2 \\ c\log|x| & \beta = n-2 \end{cases} \tag{14}$$

in $\mathbb{R}^n\backslash\{0\}$. If $\beta > (n-2)/2$, then this solution is a weak solution because $u \in H^1(\Omega)$. Therefore, uniqueness of weak solutions to the Dirichlet problem does not hold. Thus, an appropriate smallness assumption is necessary. For further properties of this operator, see also [**48**]. Throughout this chapter, we assume the conditions

(17) and (21) on the drift $\mathbf{b}$, which will be explained precisely in Section 1 and Section 2.

1. Definition of weak solutions

Let us state a first assumption on drift $\mathbf{b}$. Let $\mathbf{b} \in (L^2_{\text{loc}}(\Omega))^n$. We say that $|\mathbf{b}|^2$ belongs to the class of admissible measures $\mathfrak{M}^{1,2}_+$ if there is a constant $C > 0$ which satisfies

$$\int_\Omega |\mathbf{b}|^2 |\varphi|^2 \,\mathrm{d}x \le C^2 \int_\Omega |\nabla \varphi|^2 \,\mathrm{d}x$$

for all $\varphi \in C^\infty_c(\Omega)$. For $\mathbf{b} \in (L^2_{\text{loc}}(\Omega))^n$ with $|\mathbf{b}|^2 \in \mathfrak{M}^{1,2}_+$, we define

$$|||\mathbf{b}|||_\Omega := \inf \left\{ C > 0;\ \frac{\int_\Omega |\mathbf{b}|^2 \varphi^2 \,\mathrm{d}x}{\int_\Omega |\nabla \varphi|^2 \,\mathrm{d}x} \le C^2,\ \forall \varphi \in C^\infty_c(\Omega)\ \varphi \ne 0 \right\}. \tag{15}$$

From Theorem 1 in [58, p.189], there is a constant C such that

$$\frac{1}{C} |||\mathbf{b}|||_\Omega \le \sup_{K;\ \text{compact} \subset \Omega} \left(\frac{1}{\text{cap}(K, \Omega)} \int_K |\mathbf{b}|^2 \,\mathrm{d}x \right)^{1/2} \le C |||\mathbf{b}|||_\Omega.$$

From this characterization (or by using a bump function), we have

$$\sup_{B(y,2r) \subset \Omega} \left(\frac{1}{r^{n-2}} \int_{B(y,r)} |\mathbf{b}|^2 \,\mathrm{d}x \right)^{1/2} \le C |||\mathbf{b}|||_\Omega.$$

On the other hand, from a result in [20] (see also [13] and [12]), for any $\epsilon > 0$, there is a constant $C = C(n, \epsilon)$ such that

$$|||\mathbf{b}|||_{\mathbb{R}^n} \le C \sup_{B(y,r) \subset \mathbb{R}^n} \left(\frac{1}{r^{n-2(1+\epsilon)}} \int_{B(y,r)} |\mathbf{b}|^{2(1+\epsilon)} \,\mathrm{d}x \right)^{1/2(1+\epsilon)}.$$

Other sufficient conditions are as follows: According to Lemma 13, we have

$$\begin{aligned} \int_\Omega |\mathbf{b}|^2 \phi^2 \,\mathrm{d}x &\le \| |\mathbf{b}|^2 \|_{L^{n/2,\infty}(\Omega)} \| \varphi^2 \|_{L^{n/(n-2),1}(\Omega)} \\ &= \|\mathbf{b}\|^2_{L^{n,\infty}(\Omega)} \|\varphi\|^2_{L^{2n/(n-2),2}(\Omega)} \le S_2^2 \|\mathbf{b}\|^2_{L^{n,\infty}(\Omega)} \|\nabla \varphi\|^2_{L^2(\Omega)}. \end{aligned}$$

Thus, if $\mathbf{b} \in (L^{n,\infty}(\Omega))^n$, then the quantity $|||\mathbf{b}|||_\Omega$ is finite:

$$|||\mathbf{b}|||_\Omega \le S_2 \|\mathbf{b}\|_{L^{n,\infty}(\Omega)}.$$

In particular, if $|\mathbf{b}(x)| \le C/|x - x_0|$, then $|||\mathbf{b}|||_\Omega < \infty$. On the other hand, if Ω is a Lipschitz domain, then we have Hardy's inequality

$$\int_\Omega \frac{|\varphi|^2}{\text{dist}(x, \partial\Omega)^2} \,\mathrm{d}x \le C \int_\Omega |\nabla \varphi|^2 \,\mathrm{d}x.$$

Therefore, if $|\mathbf{b}(x)| \le C \text{dist}(x, \partial\Omega)^{-1}$, then $|||\mathbf{b}|||_\Omega < \infty$. This vector field $\mathbf{b}$ need not belong to $(L^{n,\infty}(\Omega))^n$ in general, because it may be strongly singular near the boundary.

If $|||\mathbf{b}|||_\Omega$ is finite, then, by using Cauchy-Schwarz inequality, we have

$$\begin{aligned} \left| \int_\Omega \mathbf{b} \cdot \nabla u v \,\mathrm{d}x \right| &\le \left(\int_\Omega |\mathbf{b}|^2 v^2 \,\mathrm{d}x \right)^{1/2} \left(\int_\Omega |\nabla u|^2 \,\mathrm{d}x \right)^{1/2} \\ &\le |||\mathbf{b}|||_\Omega \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \end{aligned}$$

for all $u \in \mathcal{D}^{1,2}(\Omega)$ and $v \in \mathcal{D}^{1,2}_0(\Omega)$. Thus, the bilinear form

$$\langle \mathcal{L}u, v\rangle = \int_\Omega A\nabla u \cdot \nabla v + (\mathbf{b}\cdot\nabla u)v \,\mathrm{d}x \tag{16}$$

is bounded on $\mathcal{D}^{1,2}_0(\Omega)$:

$$|\langle \mathcal{L}u, v\rangle| \le (\|A\|_{L^\infty(\Omega)} + |||\mathbf{b}|||_\Omega)\|\nabla u\|_{L^2(\Omega)}\|\nabla\|_{L^2(\Omega)}.$$

For more sharp sufficient (and necessary) conditions for boundedness of (16), see [**59**].

Hereafter, for simplicity of notation, we write

$$\mathcal{B} = \|A\|_{L^\infty(\Omega)} + |||\mathbf{b}_0|||_\Omega, \quad \mathcal{B}^* = \mathcal{B} + |||\mathbf{b}_1|||_\Omega \tag{17}$$

and assume that $\mathcal{B}^*$ is finite. Under these boundedness conditions, let us define weak solutions to $\mathcal{L}u = \mu$ as follows:

Definition 21. Let $\mu \in \mathcal{D}^{-1,2}(\Omega)$. We say that a function $u \in H^1_{\mathrm{loc}}(\Omega)$ is a weak solution to the equation $\mathcal{L}u = \mu$ in Ω if

$$\int_\Omega A\nabla u\cdot\nabla\varphi + (\mathbf{b}\cdot\nabla u)\varphi\,\mathrm{d}x = \langle \mu,\varphi\rangle_\Omega \tag{18}$$

for all $\varphi \in C^\infty_c(\Omega)$. We say that a function $u \in H^1_{\mathrm{loc}}(\Omega)$ is a weak supersolution to the equation $\mathcal{L}u = \mu$ in Ω if

$$\int_\Omega A\nabla u\cdot\nabla\varphi + (\mathbf{b}\cdot\nabla u)\varphi\,\mathrm{d}x \ge \langle \mu,\varphi\rangle_\Omega \tag{19}$$

for all $\varphi \in C^\infty_c(\Omega)$, $\varphi \ge 0$. Moreover, we say that a function $u \in H^1_{\mathrm{loc}}(\Omega)$ is a weak subsolution to the equation $\mathcal{L}u = \mu$ in Ω if $-u$ is a weak supersolution to the equation $\mathcal{L}u = \mu$ in Ω.

When $u \in \mathcal{D}^{1,2}(\Omega)$, from density of $C^\infty_c(\Omega)$ in $\mathcal{D}^{1,2}_0(\Omega)$, (19) holds for all $\varphi \in \mathcal{D}^{1,2}_0(\Omega)$, $\varphi \ge 0$. Similarly, (19) holds if $\varphi \in \mathcal{D}^{1,2}_0(\Omega)$ has a compact support. From this fact, if u is a supersolution and a subsolution to the equation $\mathcal{L}u = \mu$ in Ω, then u is a solution to the same equation because

$$\begin{aligned}\int_\Omega A\nabla u\cdot\nabla\varphi + (\mathbf{b}\cdot\nabla u)\varphi\,\mathrm{d}x &= \int_\Omega A\nabla u\cdot\nabla\varphi_+ + (\mathbf{b}\cdot\nabla u)\varphi_+\,\mathrm{d}x\\ &\quad - \int_\Omega A\nabla u\cdot\nabla\varphi_- + (\mathbf{b}\cdot\nabla u)\varphi_-\,\mathrm{d}x\\ &= \langle\mu,\varphi_+\rangle - \langle\mu,\varphi_-\rangle = \langle\mu,\varphi\rangle.\end{aligned}$$

Conversely, if u is a solution to $\mathcal{L}u = \mu$ in Ω, then u is a supersolution and a subsolution to the same equation.

If $u \in H^1_{\mathrm{loc}}(\Omega)$ is a supersolution to $\mathcal{L}u = 0$ in Ω, then the distribution

$$C^\infty_c(\Omega) \ni \varphi \mapsto \int_\Omega A\nabla u\cdot\nabla\varphi + (\mathbf{b}\cdot\nabla u)\varphi\,\mathrm{d}x$$

is non-negative. Thus, there is a unique non-negative Radon measure μ such that

$$\int_\Omega \varphi\,\mathrm{d}\mu = \int_\Omega A\nabla u\cdot\nabla\varphi + (\mathbf{b}\cdot\nabla u)\varphi\,\mathrm{d}x \quad \forall\varphi\in C^\infty_c(\Omega). \tag{20}$$

The measure $\mu = \mu[u]$ is called the Riesz measure (or Riesz mass) of u. From the definition and boundedness of $\langle \mathcal{L}u, v\rangle$, for any supsersolution u, $\mu[u] \in \mathcal{D}^{-1,2}(D)$ whenever $D \Subset \Omega$.

DEFINITION 22. Suppose that $u \in H^1_{\mathrm{loc}}(\Omega)$ is a supersolution to $\mathcal{L}u = 0$ in Ω. We say that a non-negative Radon measure $\mu = \mu[u]$ is the Riesz measure of u if (20) holds.

From the assumption on A, we have the following Kato type inequality. Note that this lemma holds without the coercivity of the bilinear form $\langle \mathcal{L}u, v\rangle$.

LEMMA 23. *Suppose that u is a subsolution to $\mathcal{L}u = \mu$ in Ω, where μ is the measure in $H^{-1}(\Omega)$. then u_+ is a subsolution to the equation $\mathcal{L}u_+ = \mu\lfloor_{\{u>0\}}$ in Ω. In particular,*

1. *If u is a subsolution to the equation $\mathcal{L}u = 0$ in Ω, then for any $k \in \mathbb{R}$, $(u-k)_+$ is a subsolution to the equation $\mathcal{L}u = 0$ in Ω.*
2. *If u is a supersolution to the equation $\mathcal{L}u = 0$ in Ω, then for any $k \in \mathbb{R}$, $(u-k)_-$ is a supersolution to the equation $\mathcal{L}u = 0$ in Ω.*

PROOF. For $k > 0$, we take $H_k(t) = \frac{1}{k}T_k(t)$, where $T_k(t) = \min\{\max\{t, -k\}, k\}$. Note that for any $t > 0$, $H_k(t) \to 1$ as $k \to 0$. Fix any non-negative function $\varphi \in C_c^\infty(\Omega)$. Testing the equation by $H_k(u_+)\varphi$, we have

$$\int_\Omega A\nabla u \cdot \nabla(H_k(u_+)\varphi) + (\mathbf{b}\cdot\nabla u)\,(H_k(u_+)\varphi)\,\mathrm{d}x \le \int_\Omega (H_k(u_+)\varphi)\,\mathrm{d}\mu.$$

Therefore,

$$\int_\Omega A\nabla u \cdot \nabla H_k(u_+)\varphi\,\mathrm{d}x + \int_\Omega \{A\nabla u_+ \cdot \nabla\varphi + (\mathbf{b}\cdot\nabla u_+)\varphi\}\,H_k(u_+)\,\mathrm{d}x \le \int_\Omega (H_k(u_+)\varphi)\,\mathrm{d}\mu.$$

Let $k \to 0$. Since the first term in the left-hand side is non-negative, the Lebesgue dominated convergence theorem yields

$$\int_\Omega A\nabla u_+ \cdot \nabla\varphi + (\mathbf{b}\cdot\nabla u_+)\,\varphi\,\mathrm{d}x \le \int_\Omega \mathbf{1}_{\{u>0\}}\varphi\,\mathrm{d}\mu.$$

This implies the desired assertion. □

The following logarithmic Caccioppoli inequality also holds without the coercivity of $\langle \mathcal{L}u, v\rangle$:

LEMMA 24. *Let $u \ge \epsilon > 0$ be a positive weak supersolution to $\mathcal{L}u = 0$ in Ω. Then, there exists a constant C depending only on $\mathcal{B}^*/\nu$ such that for any $\eta \in C_c^\infty(\Omega)$,*

$$\int_\Omega |\nabla \log u|^2\eta^2\,\mathrm{d}x \le C\int_\Omega |\nabla\eta|^2\,\mathrm{d}x.$$

PROOF. Let us choose a test function $u^{-1}\eta^2$. Then we have

$$0 \le \int_\Omega A\nabla u\cdot\nabla(u^{-1}\eta^2)\,\mathrm{d}x + \int_\Omega (\mathbf{b}\cdot\nabla u)\,(u^{-1}\eta^2)\,\mathrm{d}x.$$

Therefore,

$$\int_\Omega A\nabla u\cdot\nabla u u^{-2}\eta^2\,\mathrm{d}x \le 2\int_\Omega A\nabla u\cdot\nabla\eta u^{-1}\eta\,\mathrm{d}x + \int_\Omega (\mathbf{b}\cdot\nabla u)\,u^{-1}\eta^2\,\mathrm{d}x.$$

Since $\nabla\log u = \nabla u u^{-1}$, we have

$$\int_\Omega A\nabla\log u\cdot\nabla\log u\eta^2\,\mathrm{d}x \le 2\int_\Omega A\nabla\log u\cdot\nabla\eta\eta\,\mathrm{d}x + \int_\Omega (\mathbf{b}\cdot\nabla\log u)\,\eta^2\,\mathrm{d}x.$$

From the Cauchy-Schwarz inequality,

$$\nu \int_\Omega |\nabla \log u|^2 \eta^2 \, \mathrm{d}x \le 2\|A\|_{L^\infty(\Omega)} \left(\int_\Omega |\nabla \eta|^2 \, \mathrm{d}x \right)^{1/2} \left(\int_\Omega |\nabla \log u|^2 \eta^2 \, \mathrm{d}x \right)^{1/2} + \left(\int_\Omega |\mathbf{b}|^2 \eta^2 \, \mathrm{d}x \right)^{1/2} \left(\int_\Omega |\nabla \log u|^2 \eta^2 \, \mathrm{d}x \right)^{1/2}.$$

This implies that

$$\int_\Omega |\nabla \log u|^2 \eta^2 \, \mathrm{d}x \le 4 \frac{(\mathcal{B}^*)^2}{\nu^2} \int_\Omega |\nabla \eta|^2 \, \mathrm{d}x.$$

We arrived at the desired inequality. □

2. The comparison principle and existence theorems

In general, even if $|||\mathbf{b}|||_\Omega$ is finite, the bilinear form (16) need not be coercive on $\mathcal{D}_0^{1,2}(\Omega)$. However, if $|||\mathbf{b}_1|||_\Omega < \nu$, then the bilinear form (16) is coercive on $\mathcal{D}_0^{1,2}(\Omega)$ since

$$\begin{aligned} \langle \mathcal{L}u, u \rangle &= \int_\Omega A\nabla u \cdot \nabla u + (\mathbf{b} \cdot \nabla u) u \, \mathrm{d}x \\ &= \int_\Omega A\nabla u \cdot \nabla u \, \mathrm{d}x + \int_\Omega (\mathbf{b}_0 \cdot \nabla u) u \, \mathrm{d}x + \int_\Omega (\mathbf{b}_1 \cdot \nabla u) u \, \mathrm{d}x \\ &\ge (\nu - |||\mathbf{b}_1|||_\Omega) \|\nabla u\|^2_{L^2(\Omega)}, \end{aligned}$$

In particular, when

$$|||\mathbf{b}_1|||_\Omega \le \frac{\nu}{2}, \tag{21}$$

the bilinear form (16) is coercive on $\mathcal{D}_0^{1,2}(\Omega)$:

$$\langle \mathcal{L}u, u \rangle \ge \frac{\nu}{2} \|\nabla u\|^2_{L^2(\Omega)}.$$

For simplicity, hereafter we assume (21).

LEMMA 25. *Let Ω be a bounded open set. Let $u, v \in H^1(\Omega)$. Suppose that $\mathcal{L}v - \mathcal{L}u$ is a non-negative measure in $H^{-1}(\Omega)$ and $(u-v)_+ \in H_0^1(\Omega)$. Then*

$$u(x) \le v(x) \quad \text{for a.e. } x \in \Omega. \tag{22}$$

PROOF. Testing the equation by $(u-v)_+ \in H_0^1(\Omega)$, we have

$$\begin{aligned} 0 &\le \langle (\mathcal{L}v - \mathcal{L}u), (u-v)_+ \rangle_\Omega \\ &= \int_\Omega A\nabla v \cdot \nabla (u-v)_+ + (\mathbf{b} \cdot \nabla v)(u-v)_+ \, \mathrm{d}x \\ &\quad - \int_\Omega A\nabla u \cdot \nabla (u-v)_+ + (\mathbf{b} \cdot \nabla u)(u-v)_+ \, \mathrm{d}x, \end{aligned}$$

hence

$$\int_\Omega A\nabla (u-v) \cdot \nabla (u-v)_+ \, \mathrm{d}x \le -\int_\Omega (\mathbf{b} \cdot \nabla (u-v))(u-v)_+ \, \mathrm{d}x.$$

If $(u-v)_+(x) \neq 0$, then $(u-v)(x) = (u-v)_+(x)$. Therefore, from assumption (21), this implies that

$$\left(\nu - \frac{\nu}{2} \right) \int_\Omega |\nabla (u-v)_+|^2 \, \mathrm{d}x \le 0.$$

Since $(u-v)_+ \in H_0^1(\Omega)$, we have $u \le v$ a.e. in Ω as required. □

From the theory of variational inequality ([**41**]), we have the following existence theorem for obstacle problems:

LEMMA 26. *Let Ω be a open set. For any measurable function $g : \Omega \to [-\infty, \infty]$ and $\theta \in \mathcal{D}^{1,2}(\Omega)$, we define*

$$\mathcal{K}_{g,\theta}(\Omega) := \left\{ u \in \mathcal{D}^{1,2}(\Omega);\ u \ge g \text{ a.e.},\ u - \theta \in H_0^1(\Omega) \right\}. \tag{23}$$

Let $\mu \in \mathcal{D}^{-1,2}(\Omega)$. Then, the variational inequality

$$\int_\Omega A\nabla u \cdot \nabla(v-u) + (\mathbf{b} \cdot \nabla u)(v-u)\,\mathrm{d}x \ge \langle \mu, (v-u) \rangle_\Omega \quad \forall v \in \mathcal{K}_{g,\theta}(\Omega)$$

has a unique solution $u \in \mathcal{K}_{g,\theta}(\Omega)$ whenever $\mathcal{K}_{g,\theta}(\Omega) \ne \emptyset$.

PROOF. This theorem follows from a general theorem in [**41**, pp24-26,32]. However, we give a (concrete) full proof for completeness.

Step 1. First of all, we reduce the problem to a simple variational inequality with parameter. Take

$$\mathcal{K} = (-\theta) + \mathcal{K}_{g,\theta} \subset \mathcal{D}_0^{1,2}(\Omega)$$

and

$$f = \mu - \mathcal{L}\theta.$$

From the assumption on $\mathcal{K}_{g,\theta}$, the set $\mathcal{K}$ is closed and convex. Moreover, we decompose bilinear form $\langle \mathcal{L}u, v \rangle$ as follows:

$$a_0(u,v) = \frac{1}{2}\left(\langle \mathcal{L}u, v \rangle + \langle \mathcal{L}v, u \rangle \right),$$

$$V(u,v) = \frac{1}{2}\left(\langle \mathcal{L}u, v \rangle - \langle \mathcal{L}v, u \rangle \right).$$

For $t \in [0,1]$, we take

$$a_t(u,v) = a_0(u,v) + tV(u,v).$$

Since $\langle \mathcal{L}u, u \rangle \ge \alpha \|\nabla u\|^2_{L^2(\Omega)}$ with $\alpha = \nu/2$,

$$a_t(u,u) \ge \alpha \|\nabla u\|^2_{L^2(\Omega)}$$

for all t. Fix $t \in [0,1]$. Let us show existence of a function $u \in \mathcal{K}$ such that

$$a_t(u,(v-u)) \ge \langle f, (v-u) \rangle \quad \forall v \in \mathcal{K}, \tag{24}$$

where f is any functional in $\mathcal{D}^{-1,2}(\Omega)$. If it is proved that (24) has a solution with $t=1$, then the proof of lemma is complete.

Step 2. We first prove the case of $t=0$. Let us consider the minimizing problem

$$(d :=) \inf_{u \in \mathcal{K}} I(u), \tag{25}$$

where

$$I(u) = \frac{1}{2} a_0(u,u) - \langle f, u \rangle.$$

Since

$$\begin{aligned} I(u) &\ge \frac{\alpha}{2}\|\nabla u\|^2_{L^2(\Omega)} - \|f\|_{(\mathcal{D}^{1,2}(\Omega))^*}\|\nabla u\|_{L^2(\Omega)} \\ &\ge -\frac{1}{2\alpha}\|f\|^2_{(\mathcal{D}^{1,2}(\Omega))^*}, \end{aligned}$$

we have d is bounded from below. Choose $\{u_j\}_{j=1}^{\infty} \subset \mathcal{K}$ so that

$$d \le I(u_j) \le d + j^{-1}.$$

Then from the parallelogram law $\|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 - \|x+y\|^2$,

$$\begin{aligned}\alpha\|\nabla(u_j - u_i)\|^2_{L^2(\Omega)} &\le a_0(u_j - u_i, u_j - u_i)\\ &= 2a_0(u_j, u_j) + 2a_0(u_i, u_i) - 4a_0(\frac{1}{2}(u_j + u_i), \frac{1}{2}(u_j + u_i))\\ &= 4I(u_j) + 4I(u_i) - 8I(\frac{1}{2}(u_j + u_i))\\ &\le 4(j^{-1} + i^{-1}).\end{aligned}$$

Thus, $\{u_j\}_{j=1}^{\infty}$ is a Cauchy sequence in $\mathcal{K}$. Let $u = \lim_{j\to\infty} u_j$. Then $u \in \mathcal{K}$ and $I(u) = d$. For any $v \in \mathcal{K}$ and $\epsilon \in [0,1]$, we have $u + \epsilon(v-u) \in \mathcal{K}$, Since

$$\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} I(u + \epsilon(v-u)) \ge 0,$$

it follows that u satisfies (24). Thus, the minimizing problem (25) had a solution in $\mathcal{K}$. Next we show the uniqueness. Let u_1 and u_2 be solutions to the same data f. Then, since

$$a_t(u_1, (u_1 - u_2)) \le \langle f, (u_1 - u_2)\rangle$$

and

$$a_t(u_2, (u_1 - u_2)) \ge \langle f, (u_1 - u_2)\rangle,$$

we have

$$\alpha\|\nabla(u_1 - u_2)\|^2_{L^2(\Omega)} \le a_t((u_1 - u_2), (u_1 - u_2)) \le 0.$$

Thus, the solution is unique.

Step 3. To treat general cases, we use a method of continuity. Assume that (24) is solvable with $t = \tau_1$. Let

$$M = \sup\{|V(u,v)|;\ u, v \in \mathcal{D}_0^{1,2}(\Omega),\ \|\nabla u\|_{L^2(\Omega)}, \|\nabla v\|_{L^2(\Omega)} \le 1\}$$

and fix $\tau_2 > \tau_1$ such that $0 < (\tau_2 - \tau_1) < \alpha/M$. Let us define the mapping $T : \mathcal{D}_0^{1,2}(\Omega) \to \mathcal{K}$ by $u = Tw$ if

$$a_{\tau_1}(u, v-u) \ge \langle F(w), (v-u)\rangle \quad \forall v \in \mathcal{K},$$

where $F(w)$ is a bounded linear functional on $\mathcal{D}_0^{1,2}(\Omega)$ defined by

$$\langle F(w), \varphi\rangle = \langle f, \varphi\rangle - (\tau_2 - \tau_1)V(w, \varphi) \quad \forall \varphi \in \mathcal{D}_0^{1,2}(\Omega).$$

Let $w_i \in \mathcal{D}_0^{1,2}(\Omega)$ $(i = 1,2)$ and $u_i = T(w_i)$. Then, since

$$a_t(u_1, (u_1 - u_2)) \le \langle F(w_1), (u_1 - u_2)\rangle$$

and

$$a_t(u_2, (u_1 - u_2)) \ge \langle F(w_2), (u_1 - u_2)\rangle,$$

we have

$$\|\nabla u_1 - \nabla u_2\|_{L^2(\Omega)} \le \frac{1}{\alpha}(\tau_2 - \tau_1)M\|\nabla w_1 - \nabla w_2\|_{L^2(\Omega)}$$

with $\alpha^{-1}(t-\tau)M < 1$. Thus, T is a contraction mapping on $\mathcal{K}$. Therefore, there is a unique function u such that

$$a_{\tau_1}(u, v-u) \ge \langle f, (v-u)\rangle - (\tau_2 - \tau_1)V(u, v) \quad \forall v \in \mathcal{K}.$$

Therefore, (24) is solvable with $t = \tau_2$. Iterating this argument, we arrive at the desired assertion. □

REMARK 27. If A is symmetric and $\operatorname{div} \mathbf{b} = 0$, then $a_0(u,v) = \int_\Omega A\nabla u \cdot \nabla v \, \mathrm{d}x$ and $V(u,v) = \int_\Omega (\mathbf{b} \cdot \nabla u) v \, \mathrm{d}x$.

COROLLARY 28. *Suppose that Ω is a bounded open set. Let $\mu \in H^{-1}(\Omega)$, and let $\theta \in H^1(\Omega)$. Then the Dirichlet problem*

$$\begin{cases} \mathcal{L}u = \mu & \text{in } \Omega \\ u = \theta & \text{on } \partial\Omega. \end{cases}$$

has a unique weak solution $u \in \theta + H_0^1(\Omega)$.

DEFINITION 29. Assume that $\mathcal{K}_{g,\theta}(\Omega) \neq \emptyset$, where $\mathcal{K}_{g,\theta}(\Omega)$ is the set defined by (23). A function $u \in \theta + H_0^1(\Omega)$ is called a solution to the obstacle problem in $\mathcal{K}_{g,\theta}(\Omega)$ if

$$\int_\Omega A\nabla u \cdot \nabla(v-u) + (\mathbf{b} \cdot \nabla u)(v-u) \, \mathrm{d}x \geq 0 \quad \forall v \in \mathcal{K}_{g,\theta}(\Omega).$$

If u is a solution to the obstacle problem in $\mathcal{K}_{g,\theta}(\Omega)$, then u is a supersolution to the equation $\mathcal{L}u = 0$ in Ω. Conversely, if u is a supersolution to the equation $\mathcal{L}u = 0$ in Ω, then u is a solution to the obstacle problem in $\mathcal{K}_{u,u}(D)$ for any $D \Subset \Omega$.

LEMMA 30. *Suppose that u is a solution to the obstacle problem in $\mathcal{K}_{g,\theta}(\Omega)$. Let $v \in H^1(\Omega)$ be a supersolution to the equation $\mathcal{L}v = 0$ in Ω such that* $\min\{u,v\} \in \mathcal{K}_{g,\theta}(\Omega)$. *Then $v \geq u$ a.e. in Ω.*

PROOF. Note that $u - \min\{u,v\} = (u-v)_+ \geq 0$. From the assumptions, we have

$$\begin{aligned} 0 \leq &\int_\Omega (A\nabla v) \cdot \nabla(u - \min\{u,v\}) + (\mathbf{b} \cdot \nabla v)(u - \min\{u,v\}) \, \mathrm{d}x \\ &- \int_\Omega (A\nabla u) \cdot \nabla(u - \min\{u,v\}) + (\mathbf{b} \cdot \nabla u)(u - \min\{u,v\}) \, \mathrm{d}x, \end{aligned}$$

hence

$$0 \geq \int_\Omega A\nabla(u-v) \cdot \nabla(u-v)_+ \, \mathrm{d}x + \int_\Omega (\mathbf{b} \cdot \nabla(u-v))(u-v)_+ \, \mathrm{d}x.$$

This implies that

$$\begin{aligned} 0 &\geq \int_\Omega A\nabla(u-v)_+ \cdot \nabla(u-v)_+ \, \mathrm{d}x + \int_\Omega (\mathbf{b} \cdot \nabla(u-v)_+)(u-v)_+ \, \mathrm{d}x \\ &\geq \frac{\nu}{2} \int_\Omega |\nabla(u-v)_+|^2 \, \mathrm{d}x. \end{aligned}$$

Therefore, $|\{x \in \Omega;\ (u-v)(x) > 0\}| = 0$. □

LEMMA 31. *A function $u \in H^1(\Omega)$ is a solution to the obstacle problem in $\mathcal{K}_{g,u}(\Omega)$ if and only if u is a solution to the obstacle problem in $\mathcal{K}_{g,u}(D)$ whenever $D \subset \Omega$ is open.*

PROOF. Assume that u is a solution to the solution to the obstacle problem in $\mathcal{K}_{g,u}(\Omega)$. For any $v \in \mathcal{K}_{g,u}(D)$, the function

$$\tilde{v} = \begin{cases} v & \text{in } D, \\ u & \text{otherwise} \end{cases}$$

belongs to $\mathcal{K}_{g,u}(\Omega)$. Thus, we have

$$\begin{aligned} &\int_D A\nabla u \cdot \nabla(v-u) + (\mathbf{b}\cdot\nabla u)(v-u)\,\mathrm{d}x \\ &= \int_\Omega A\nabla u \cdot \nabla(\tilde{v}-u) + (\mathbf{b}\cdot\nabla u)(\tilde{v}-u)\,\mathrm{d}x \\ &\geq \langle \mu, (\tilde{v}-u)\rangle_\Omega = \langle \mu, (v-u)\rangle_D. \end{aligned}$$

Therefore, u is a solution to the solution to the obstacle problem in $\mathcal{K}_{g,u}(D)$. The converse follows by taking $D = \Omega$. □

LEMMA 32. *Suppose that $u \in H^1(\Omega)$ is a solution to the obstacle problem in $\mathcal{K}_{g,\theta}(\Omega)$, and that $D \subset \Omega$ is open. If there is a subsolution v to the equation $\mathcal{L}v = 0$ in D with $g \leq v \leq u$ a.e. in D, then u is a solution to the equation $\mathcal{L}u = 0$ in D. In particular, if there is a constant c such that $g \leq c \leq u$ in D, then then u is a solution to the equation $\mathcal{L}u = 0$ in D.*

PROOF. Let $h \in u + H^1_0(D)$ be the solution to the equation $\mathcal{L}h = 0$ in D. From the comparison principle, $g \leq v \leq h \leq u$ in D. From Lemma 31, u is a solution to the obstacle problem in $\mathcal{K}_{g,u}(D)$. Since $h \in \mathcal{K}_{g,u}(D)$, it follows from Lemma 30 that $u \leq h$ in D. Therefore, $u = h$ a.e. in D. □

Next, we introduce the Poisson modification of supersolutions and $\mathcal{L}$-equilibrium potentials.

DEFINITION 33. Suppose that u is a supersolution to the equation $\mathcal{L}u = 0$ in Ω. For $D \Subset \Omega$, we define

$$P(u, D) = \begin{cases} u_D & \text{in } D, \\ u & \text{otherwise}, \end{cases}$$

where $u_D \in u + H^1_0(D)$ is the solution to the Dirichlet problem $\mathcal{L}u = 0$ in D.

Take $D \Subset D' \Subset \Omega$. Then, from Lemma 32, v is the solution to the obstacle problem in $\mathcal{K}_{g,u}(D')$, where

$$g = \begin{cases} -\infty & \text{in } D, \\ u & \text{otherwise}. \end{cases}$$

Therefore, we have the following:

LEMMA 34. *Suppose that u is a supersolution to the equation $\mathcal{L}u = 0$ and $v = P(u, D)$ is the Poisson modification of u in D. Then,*

(1) *v is a solution to the equation $\mathcal{L}u = 0$ in D.*
(2) *v is a supersolution to the equation $\mathcal{L}u = 0$ in Ω.*
(3) *$v \leq u$ in Ω.*

DEFINITION 35. Let Ω be a bounded open set, and let E be a closed set (in $\mathbb{R}^n$) which is contained in Ω. We say that a function u is the $\mathcal{L}$-equilibrium potential of (E,Ω) if u is the solution to the obstacle problem in $\mathcal{K}_{\mathbf{1}_E,0}(\Omega)$ with respect to $\mathcal{L}$. Moreover, we denote

$$u = \mathfrak{R}(E,\Omega) = \mathfrak{R}(E,\Omega;\mathcal{L}).$$

If $u = \mathfrak{R}(E,\Omega)$, then $u = \min\{u,1\} = 1$ q.e. on E. Hence, u satisfies the boundary condition $u = 1$ on ∂E in Sobolev's sense. However, in general, it does not satisfy the boundary condition in the classical sense.

3. Caccioppoli's inequality and related results

Under (21), we also get the following local version of energy estimate. This estimate is also called Caccioppoli's inequality.

LEMMA 36. *Let u be a weak subsolution to $\mathcal{L}u = 0$ in Ω. Then, there exists a constant C_E depending only on $\mathcal{B}/\nu$ such that for any $\eta \in C_c^\infty(\Omega)$,*

$$\int_\Omega |\nabla u_+|^2 \eta^2 \, \mathrm{d}x \le C_E \int_\Omega u_+^2 |\nabla \eta|^2 \, \mathrm{d}x.$$

PROOF. For $k > 0$, we take $\bar{u} = \min\{u,k\}$. Let us choose the test function $\bar{u}_+\eta^2$. Then we have

$$0 \ge \int_\Omega A\nabla u \cdot \nabla(\bar{u}_+\eta^2) \, \mathrm{d}x + \int_\Omega (\mathbf{b}\cdot\nabla u)\,(\bar{u}_+\eta^2)\, \mathrm{d}x.$$

Note that if $\bar{u}_+(x) \neq 0$, then $\nabla u(x) = \nabla u_+(x)$. Therefore,

$$0 \ge \int_\Omega A\nabla u_+ \cdot \nabla(\bar{u}_+\eta^2) \, \mathrm{d}x + \int_\Omega (\mathbf{b}\cdot\nabla u_+)\,(\bar{u}_+\eta^2)\, \mathrm{d}x,$$

and hence

$$\begin{aligned}\int_\Omega A\nabla u_+ \cdot \nabla \bar{u}_+ \eta^2 \, \mathrm{d}x \le -2\int_\Omega A\nabla u_+ \cdot \nabla\eta\, \bar{u}_+\eta \, \mathrm{d}x - \int_\Omega (\mathbf{b}_0\cdot\nabla u_+)\,\bar{u}_+\eta^2\, \mathrm{d}x \\ - \int_\Omega (\mathbf{b}_1\cdot\nabla u_+)\,\bar{u}_+\eta^2\, \mathrm{d}x.\end{aligned} \tag{26}$$

We shall estimate the third term of the right-hand side in (26). By Young's inequality $ab \le \frac{\epsilon}{2}a^2 + \frac{1}{2\epsilon}b^2$, for any $\epsilon_1 > 0$,

$$\begin{aligned}\left|\int_\Omega (\mathbf{b}_1\cdot\nabla u_+)\,\bar{u}_+\eta^2\, \mathrm{d}x\right| &\le \frac{\epsilon_1}{2}\int_\Omega |\nabla u_+|^2\eta^2 \, \mathrm{d}x + \frac{1}{2\epsilon_1}\int_\Omega |\mathbf{b}_1|^2(\bar{u}_+\eta)^2 \, \mathrm{d}x \\ &\le \frac{\epsilon_1}{2}\int_\Omega |\nabla u_+|^2\eta^2 \, \mathrm{d}x + \frac{1}{2\epsilon_1}|||\mathbf{b}_1|||_\Omega^2 \int_\Omega |\nabla(\bar{u}_+\eta)|^2 \, \mathrm{d}x.\end{aligned}$$

From Jensen's inequality, for any $a, b \ge 0$ and $\theta \in (0,1)$, we have

$$(a+b)^2 = \left(\int_0^\theta \frac{a}{\theta} + \int_\theta^1 \frac{b}{1-\theta}\right)^2 \le \int_0^\theta \left(\frac{a}{\theta}\right)^2 + \int_\theta^1 \left(\frac{b}{1-\theta}\right)^2 = \frac{a^2}{\theta} + \frac{b^2}{1-\theta}.$$

Therefore, taking $\theta = (1+\epsilon_2)^{-2}$ with $\epsilon_2 > 0$, we get

$$\int_\Omega |\nabla(\bar{u}_+\eta)|^2 \, \mathrm{d}x \le (1+\epsilon_2)^2 \int_\Omega |\nabla \bar{u}_+|^2\eta^2 \, \mathrm{d}x + \frac{(1+\epsilon_2)^2}{(1+\epsilon_2)^2 - 1}\int_\Omega \bar{u}_+^2 |\nabla\eta|^2 \, \mathrm{d}x.$$

Taking $\epsilon_1 = (1+\epsilon_2)^{-1}|||\mathbf{b}_1|||_\Omega$ and combining these inequalities, we obtain

$$\left|\int_\Omega (\mathbf{b}_1 \cdot \nabla u_+)\, \bar{u}_+ \eta^2 \,\mathrm{d}x\right| \le (1+\epsilon_2)|||\mathbf{b}_1|||_\Omega \int_\Omega |\nabla u_+|^2 \eta^2 \,\mathrm{d}x + \frac{(1+\epsilon_2)}{2\{(1+\epsilon_2)^2 - 1\}} |||\mathbf{b}_1|||_\Omega \int_\Omega u_+^2 |\nabla \eta|^2 \,\mathrm{d}x. \tag{27}$$

Next we shall estimate the second term of the right-hand side in (26). We have

$$\int_\Omega (\mathbf{b}_0 \cdot \nabla u_+)\bar{u}_+\eta^2 \,\mathrm{d}x = \int_\Omega (\mathbf{b}_0 \cdot \nabla \bar{u}_+)\bar{u}_+\eta^2 \,\mathrm{d}x + \int_\Omega (\mathbf{b}_0 \cdot \nabla (u-k)_+)\bar{u}_+\eta^2 \,\mathrm{d}x.$$

Note that

$$\begin{aligned}
&\left|\int_\Omega (\mathbf{b}_0 \cdot \nabla (u-k)_+)\, \bar{u}_+\eta^2 \,\mathrm{d}x\right| \\
&\le \left(\int_\Omega |\mathbf{b}_0|^2 (\bar{u}_+\eta)^2 \,\mathrm{d}x\right)^{1/2} \left(\int_\Omega |\nabla (u-k)_+|^2 \eta^2 \,\mathrm{d}x\right)^{1/2} \\
&\le |||\mathbf{b}_0|||_\Omega \left(\int_\Omega |\nabla (\bar{u}_+\eta)|^2 \,\mathrm{d}x\right)^{1/2} \left(\int_\Omega |\nabla (u-k)_+|^2 \eta^2 \,\mathrm{d}x\right)^{1/2}.
\end{aligned}$$

Thus, since $\operatorname{div} \mathbf{b}_0 = 0$, by integrating by parts, we have

$$\int_\Omega (\mathbf{b}_0 \cdot \nabla u_+)\, \bar{u}_+\eta^2 \,\mathrm{d}x = -\int_\Omega (\mathbf{b}_0 \cdot \nabla \eta)\, \bar{u}_+^2 \eta \,\mathrm{d}x + o(1).$$

On the other hand, for any $\epsilon_3 > 0$, we have

$$\begin{aligned}
\left|\int_\Omega (\mathbf{b}_0 \cdot \nabla \eta)\, \bar{u}_+^2 \eta \,\mathrm{d}x\right| &\le \frac{\epsilon_3}{2} \int_\Omega |\mathbf{b}_0|^2 (\bar{u}_+\eta)^2 \,\mathrm{d}x + \frac{1}{2\epsilon_3} \int_\Omega \bar{u}_+^2 |\nabla \eta|^2 \,\mathrm{d}x \\
&\le \frac{\epsilon_3}{2} |||\mathbf{b}_0|||_\Omega^2 \int_\Omega |\nabla (\bar{u}_+\eta)|^2 \,\mathrm{d}x + \frac{1}{2\epsilon_3} \int_\Omega \bar{u}_+^2 |\nabla \eta|^2 \,\mathrm{d}x.
\end{aligned}$$

Therefore, taking $\epsilon_3 = \epsilon_4 / |||\mathbf{b}_0|||_\Omega^2$, we get

$$\left|\int_\Omega (\mathbf{b}_0 \cdot \nabla u_+)\, \bar{u}_+\eta^2 \,\mathrm{d}x\right| \le \epsilon_4 \int_\Omega |\nabla u_+|^2 \eta^2 \,\mathrm{d}x + \left(\epsilon_4 + \frac{|||\mathbf{b}_0|||_\Omega^2}{2\epsilon_4}\right) \int_\Omega u_+^2 |\nabla \eta|^2 \,\mathrm{d}x + o(1). \tag{28}$$

Finally, we estimate the first term of the right-hand side in (26). By Young's inequality, for any $\epsilon_5 > 0$, we have

$$-2\int_\Omega A\nabla u_+ \cdot \nabla \eta\, \bar{u}_+ \eta \,\mathrm{d}x \le \epsilon_5 \int_\Omega |\nabla u_+|^2 \eta^2 \,\mathrm{d}x + \frac{1}{\epsilon_5} \|A\|_{L^\infty(\Omega)}^2 \int_\Omega u_+^2 |\nabla \eta|^2 \,\mathrm{d}x. \tag{29}$$

Let $\epsilon_2 = 1/2$, $\epsilon_4 = \nu/8$ and $\epsilon_5 = \nu/16$. Combining (26)-(29) and taking the limit $k \to +\infty$, we obtain

$$\int_\Omega |\nabla u_+|^2 \eta^2 \,\mathrm{d}x \le C\left(\frac{\mathcal{B}^2}{\nu^2} + 1\right) \int_\Omega u_+^2 |\nabla \eta|^2 \,\mathrm{d}x. \tag{30}$$

We arrived at the desired inequality. □

The following two lemmas are direct consequences of Caccioppoli's inequality.

LEMMA 37. *Let E be a closed subset of Ω, and let $u = \mathfrak{R}(E, \Omega)$. Suppose that K is a compact subset of Ω. Then,*

$$\mu[u](K) \le \mathcal{B}^*(C_E)^{1/2} \operatorname{cap}(E \cap K, \Omega).$$

where C_E is the constant as in Lemma 36.

PROOF. Since u is a solution to the equation $\mathcal{L}u = 0$ in $\Omega \setminus E$, it follows that

$$\mu[u](K \setminus E) \le \mu[u](\Omega \setminus E) = 0,$$

so

$$\mu[u](K) = \mu[u](E \cap K).$$

Take $\eta \in C_c^\infty(\Omega)$ such that $\eta = 1$ on $E \cap K$. Then we have

$$\begin{aligned}
\mu[u](E \cap K) &\le \int_\Omega \eta^2 \, \mathrm{d}\mu[u] \le \int_\Omega A\nabla u \cdot \nabla \eta^2 + (\mathbf{b} \cdot \nabla u)\eta^2 \, \mathrm{d}x \\
&\le \|A\|_{L^\infty(\Omega)} \left(\int_\Omega |\nabla \eta|^2 \, \mathrm{d}x \right)^{1/2} \left(\int_\Omega |\nabla u|^2 \eta^2 \, \mathrm{d}x \right)^{1/2} \\
&+ \left(\int_\Omega |\mathbf{b}|^2 \eta^2 \, \mathrm{d}x \right)^{1/2} \left(\int_\Omega |\nabla u|^2 \eta^2 \, \mathrm{d}x \right)^{1/2} \\
&\le \mathcal{B}^* \left(\int_\Omega |\nabla \eta|^2 \, \mathrm{d}x \right)^{1/2} \left(\int_\Omega |\nabla u|^2 \eta^2 \, \mathrm{d}x \right)^{1/2}.
\end{aligned}$$

Since $0 \le u \le 1$, it follows from Lemma 36 that

$$\mu[u](K) \le \mathcal{B}^*(C_E)^{1/2} \int_\Omega |\nabla \eta|^2 \, \mathrm{d}x.$$

Taking the infimum with respect to η, we arrive at the desired inequality. □

LEMMA 38. *Let $\{u_j\}_{j=1}^\infty$ be a non-decreasing sequence of supersolutions to $\mathcal{L}u = 0$ in Ω which converges to a function u almost everywhere. Suppose that one of the following conditions is fulfilled:*

(1) $u \in L^\infty_{\mathrm{loc}}(\Omega)$.
(2) $u \in H^1_{\mathrm{loc}}(\Omega)$.

Then, $u \in H^1_{\mathrm{loc}}(\Omega)$ and u is a supersolution to $\mathcal{L}u = 0$ in Ω. Moreover, there exists a sequence of bounded supersolutions $\{v_j\}_{j=1}^\infty$ such that $\mu[v_j] \to \mu[u]$ weakly.

PROOF. (1). Fix $D' \Subset D \Subset \Omega$ and choose $\eta \in C_c^\infty(D)$ such that $\eta \equiv 1$ in D'. Let $k = \operatorname{ess\,sup}_D u < \infty$, and let $v_j = (k - u_j)$. Then $\{v_j\}_{j=1}^\infty$ is a nonincreasing sequence of subsolutions. According to Caccioppoli's inequality, we have a constant C such that

$$\int_{D'} |\nabla v_j|^2 \, \mathrm{d}x \le C_E \int_\Omega v_j^2 |\nabla \eta|^2 \, \mathrm{d}x \le C.$$

Therefore, taking a subsequence, we may assume that $\{\nabla u_j\}_{j=1}^\infty$ converges weakly in $L^2(D')$. Moreover, from the monotonicity of $\{u_j\}_{j=1}^\infty$, this subsequence converges weakly to ∇u in $L^2(D')$. Using the diagonal argument with respect to D' and D, we can choose a subsequence $\{\nabla u_j\}_{j=1}^\infty$ such that $\nabla u_j \to \nabla u$ weakly in $L^2_{\mathrm{loc}}(\Omega)$. Therefore, $u \in H^1_{\mathrm{loc}}(\Omega)$. Moreover, since $\mathbf{b} \in (L^2_{\mathrm{loc}}(\Omega))^n$, we have

$$\int_\Omega A\nabla u \cdot \nabla \varphi + (\mathbf{b} \cdot \nabla u)\varphi \, \mathrm{d}x = \lim_{j \to \infty} \int_\Omega A\nabla u_j \cdot \nabla \varphi + (\mathbf{b} \cdot \nabla u_j)\varphi \, \mathrm{d}x \ge 0$$

for any $\varphi \in C_c^\infty(\Omega)$ with $\varphi \geq 0$. This implies that u is a supersolution to $\mathcal{L}u = 0$ in Ω and $\mu[u_j] \to \mu[u]$ weakly.

(2). For any fixed $l \in \mathbb{R}$, $\{\min\{u_j, l\}\}_{j=1}^\infty$ is an increasing sequence of supersolutions. Therefore, from the previous result, $\min\{u, l\}$ is a supersolution. By using Lebesgue's dominated convergence theorem, we have

$$\begin{aligned}&\int_\Omega A\nabla u \cdot \nabla\varphi + (\mathbf{b} \cdot \nabla u)\varphi \, \mathrm{d}x \\ &= \lim_{l \to \infty} \int_\Omega A\nabla \min\{u, l\} \cdot \nabla\varphi + (\mathbf{b} \cdot \nabla \min\{u, l\})\varphi \, \mathrm{d}x \geq 0\end{aligned}$$

for any $\varphi \in C_c^\infty(\Omega)$, $\varphi \geq 0$. Thus, u is a supersolution in Ω and $\mu[\min\{u, l\}] \to \mu[u]$ weakly. □

Remark 39. In the proof, $\{\nabla u_j\}_{j=1}^\infty$ converges weakly in $L^2_{\mathrm{loc}}(\Omega)$. However, in general, it need not converges weakly in $L^2(\Omega)$. For instance, consider the functions

$$u_j = \min\{j(1 - |x|^2), 1\} \in H_0^1(B(0,1)).$$

Then, $\{u_j\}_{j=1}^\infty$ is a monotone increasing sequence of supersolutions to $-\triangle u = 0$ and $u_j \to 1$ as $j \to \infty$. However, $1 \notin H_0^1(B(0,1))$. In this case, the Riesz measures $\{\mu[u_j]\}_{j=1}^\infty$ concentrate on $\partial B(0,1)$.

4. Harnack-type inequalities and Hölder estimates

In this section, we establish Harnack-type inequalities. Moreover, we give interior and boundary Hölder estimates using a weak Harnack inequality. First, we give a local $L^2 - L^\infty$ (or $L^p - L^\infty$) estimate using De Giorgi's iteration methods. For another proof of this estimate (Moser's methods), see also [**33**, pp.63-66].

Theorem 40. *Let u be a subsolution to $\mathcal{L}u = 0$ in Ω. Then, for any $\gamma > 0$, there exists a constant C_B depending only on n, $\mathcal{B}/\nu$ and γ such that*

$$\operatorname*{ess\,sup}_{B(x_0,\lambda R)} u_+ \leq \frac{C_B}{(1-\lambda)^{n/\gamma}} \left(⨍_{B(x_0,R)} u_+^\gamma \, \mathrm{d}x \right)^{1/\gamma},$$

whenever $B(x_0, R) \subset \Omega$.

Proof. First, we prove the case of $\gamma = 2$. Let $0 \leq l < h < \infty$. From Lemma 12, we have

$$\begin{aligned}\int_\Omega ((u-h)_+\eta)^2 \, \mathrm{d}x \leq &\|((u-h)_+\eta)^2\|_{L^{n/(n-2),\infty}(\Omega)} \\ &\times |\{x \in \Omega;\, u(x) \geq h\} \cap \operatorname{supp}\eta|^{2/n}.\end{aligned}$$

By using Lemma 14 and Lemma 36, we have

$$\begin{aligned}\|((u-h)_+\eta)\|^2_{L(2n/(n-2),\infty)(\Omega)} &\leq S_\infty^2 \int_\Omega |\nabla((u-h)_+\eta)|^2 \, \mathrm{d}x \\ &\leq S_\infty^2 (C_E + 1) \int_\Omega (u-h)_+^2 |\nabla\eta|^2 \, \mathrm{d}x.\end{aligned}$$

Here, S_∞ is the constant as in Lemma 14. On the other hand, from Chebyshev's inequality, we have

$$|\{x \in \Omega;\, u(x) \geq h\} \cap \operatorname{supp}\eta| \leq \frac{1}{(h-l)^2} \int_{\operatorname{supp}\eta} (u-l)_+^2 \, \mathrm{d}x.$$

Combining the two inequalities, we get

$$\int_\Omega ((u-h)_+\eta)^2\,\mathrm{d}x \le \frac{C\|\nabla\eta\|_{L^\infty}^2}{(h-l)^{4/n}}\left(\int_{\operatorname{supp}\eta}(u-l)_+^2\,\mathrm{d}x\right)^{1+2/n}.$$

For $m = 0, 1, \ldots$, set

$$B_m = B(x_0, (\lambda + \frac{1}{2^m}(1-\lambda))R)$$

and take $\eta_m \in C_c^\infty(B_m)$ such that

$$\eta_{m+1} \equiv 1 \text{ on } \overline{B_{m+1}}, \quad |\nabla\eta_m| \le \frac{2^{m+1}}{(1-\lambda)R}.$$

Let $k > 0$ be a constant to be chosen later. Taking $k_m = (1-\frac{1}{2^m})k$ and substituting $h = k_{m+1}$, $l = k_m$ and $\eta = \eta_m$, we obtain

$$\frac{1}{k^2}\fint_{B_{m+1}}(u-k_{m+1})_+^2\,\mathrm{d}x \le \frac{C}{(1-\lambda)^2}(4^{1+2/n})^m\left(\frac{1}{k^2}\fint_{B_m}(u-k_m)_+^2\,\mathrm{d}x\right)^{1+2/n}.$$

Let $\alpha = 2/n$, $b = 4^{1+2/n}$ and

$$U_m = \frac{1}{k^2}\fint_{B_m}(u-k_m)_+^2\,\mathrm{d}x.$$

Then this inequality is rewritten as $U_{m+1} \le C(1-\lambda)^{-2}b^m U_m^{1+\alpha}$. Thus, from Lemma 19, choosing

$$k^2 = \left(\frac{C}{(1-\lambda)^2}\right)^{1/\alpha} b^{1/\alpha^2}\fint_{B_0} u_+^2\,\mathrm{d}x,$$

we have $U_m \to 0$ as $m \to \infty$. This implies that

$$\operatorname*{ess\,sup}_{B(x_0,\lambda R)} u^2 \le k^2.$$

If $\gamma > 2$, then the assertion follows from the Hölder's inequality. For $\gamma < 2$, we use a rescaling argument. Let $0 < \lambda < 1$ and $0 < r < R$. Take $\beta = 2/\gamma$.

$$\begin{aligned}\operatorname*{ess\,sup}_{B(x_0,\lambda r)} u &\le \left(\frac{C_B^2}{(1-\lambda)^n}\fint_{B(x_0,r)}(\operatorname*{ess\,sup}_{B(x_0,r)} u_+)^{2-\gamma}u_+^\gamma\,\mathrm{d}x\right)^{1/2}\\ &= (\operatorname*{ess\,sup}_{B(x_0,r)} u_+)^{1-1/\beta}\left\{\left(\frac{C_B^2}{(1-\lambda)^n}\fint_{B(x_0,r)}u_+^\gamma\,\mathrm{d}x\right)^{1/\gamma}\right\}^{1/\beta}.\end{aligned}$$

From Young's inequality $ab \le \epsilon a^p + C(\epsilon)b^{p'}$, we get

$$\operatorname*{ess\,sup}_{B(x_0,\lambda r)} u \le \frac{1}{2}(\operatorname*{ess\,sup}_{B(x_0,r)} u_+) + \left(\frac{C(\gamma)C_B^2}{(1-\lambda)^n}\fint_{B(x_0,r)}u_+^\gamma\,\mathrm{d}x\right)^{1/\gamma}.$$

Let $\rho = \lambda r$, $h(\rho) = \operatorname{ess\,sup}_{B(x_0,\rho)} u$ and

$$A = \sup_{0<r<R}\left(C(\gamma)C_B^2\fint_{B(x_0,r)}u^\gamma\,\mathrm{d}x\right)^{1/\gamma}.$$

Then this inequality is rewritten as

$$h(\rho) \le \frac{1}{2}h(r) + A(r-\rho)^{-n/\gamma},$$

so, it follows from Lemma 20 that $h(\rho) \le CA(R-\rho)^{-n/\gamma}$. This implies that

$$\operatorname*{ess\,sup}_{B(x_0,\lambda R)} u_+ \le \frac{C}{(1-\lambda)^{n/\gamma}} \left(\fint_{B(x_0,R)} u_+^\gamma \, \mathrm{d}x \right)^{1/\gamma}.$$

We arrived at the desired estimate. □

Lemma 41. *Let u be a weak supersolution to $\mathcal{L}u = 0$ in Ω. Suppose that $x_0 \in \Omega$ is a Lebesgue point of u. Then*

$$u(x_0) = \lim_{R\to 0} \fint_{B(x_0,R)} u \, \mathrm{d}x = \lim_{R\to 0} \operatorname*{ess\,inf}_{B(x_0,R)} u.$$

In particular, u is lower semicontinuous after redefinition in a set of measure zero.

Proof. We follow the method in [**49**, pp.82-83]. Since $u(x_0) - u$ is a weak subsolution to $\mathcal{L}u = 0$ in Ω, by using Theorem 40, we have

$$\begin{aligned} \operatorname*{ess\,sup}_{B(x_0,R)} (u(x_0) - u) &\le C_B \fint_{B(x_0,2R)} (u(x_0) - u)_+ \, \mathrm{d}x \\ &\le C_B \fint_{B(x_0,2R)} |u(x_0) - u| \, \mathrm{d}x \end{aligned}$$

for any $0 < R < \operatorname{dist}(x_0, \partial\Omega)/2$. The right-hand side goes to 0 as $R \to 0$ since x_0 is a Lebesgue point of u. Therefore,

$$u(x_0) \le \lim_{R\to 0} \operatorname*{ess\,inf}_{B(x_0,R)} u \le \lim_{R\to 0} \fint_{B(x_0,R)} u \, \mathrm{d}x = u(x_0).$$

This completes the proof. □

Combining Theorem 40, Lemma 24 and Lemma 18, we obtain the following weak Harnack inequality. For another proof of this inequality, see also [**63, 26, 64, 18**].

Theorem 42. *Let u be a non-negative weak supersolution to $\mathcal{L}u = 0$ in Ω. Then, there exist constants $\gamma > 0$ and C_W depending only on n and $\mathcal{B}/\nu$ such that*

$$\left(\fint_{B(x_0,R)} u^\gamma \, \mathrm{d}x \right)^{1/\gamma} \le C_W \operatorname*{ess\,inf}_{B(x_0,R/2)} u,$$

whenever $B(x_0, 2R) \subset \Omega$.

Proof. Let $\epsilon > 0$ and $v = \log(u + \epsilon)$. Then from Lemma 24 we have

$$\left(\fint_{B(y,r)} |\nabla v| \, \mathrm{d}x \right)^2 \le \fint_{B(y,r)} |\nabla v|^2 \, \mathrm{d}x \le Cr^{-2}$$

for any $y \in B(x_0, R)$ and $0 < r \le R$. Fix any positive constant $0 < \gamma < \sigma(n)/C^{1/2}$, where $\sigma(n)$ is the constant as in Lemma 18. Then, it follows from Lemma 18 that

$$\int_{B(x_0,R)} \exp(\gamma |v - c|) \, \mathrm{d}x \le CR^n,$$

where $c = \fint_{B(x_0,R)} v\,\mathrm{d}x$. Since

$$\begin{aligned}&\int_{B(x_0,R)} (u+\epsilon)^{\gamma}\,\mathrm{d}x \cdot \int_{B(x_0,R)} (u+\epsilon)^{-\gamma}\,\mathrm{d}x\\ &= \int_{B(x_0,R)} \exp(\gamma v)\,\mathrm{d}x \cdot \int_{B(x_0,R)} \exp(-\gamma v)\,\mathrm{d}x\\ &= \int_{B(x_0,R)} \exp(\gamma(v-c))\,\mathrm{d}x \cdot \int_{B(x_0,R)} \exp(-\gamma(v-c))\,\mathrm{d}x,\end{aligned}$$

we have

$$\left(\fint_{B(x_0,R)} (u+\epsilon)^{\gamma}\,\mathrm{d}x\right)^{1/\gamma} \le C\left(\fint_{B(x_0,R)} (u+\epsilon)^{-\gamma}\,\mathrm{d}x\right)^{-1/\gamma}. \tag{31}$$

On the other hand, testing the equation by $(u+\epsilon)^{-\gamma/2-1}\varphi$ with non-negative $\varphi \in C_c^{\infty}(\Omega)$, we get

$$\int_{\Omega} A\nabla(u+\epsilon)\cdot\nabla((u+\epsilon)^{-\gamma/2-1}\varphi) + (\mathbf{b}\cdot\nabla(u+\epsilon))((u+\epsilon)^{-\gamma/2-1}\varphi)\,\mathrm{d}x \ge 0.$$

Therefore,

$$\begin{aligned}&\frac{2}{-\gamma}\left(\int_{\Omega} A\nabla(u+\epsilon)^{-\gamma/2}\cdot\nabla\varphi + (\mathbf{b}\cdot\nabla(u+\epsilon)^{-\gamma/2})\varphi\,\mathrm{d}x\right)\\ &\ge (1+\frac{\gamma}{2})\int_{\Omega} A\nabla(u+\epsilon)\cdot\nabla(u+\epsilon)\,(u+\epsilon)^{\gamma/2-2}\varphi\,\mathrm{d}x \ge 0.\end{aligned}$$

Since $(u+\epsilon)^{-\gamma/2} \in H^1_{\mathrm{loc}}(\Omega)$, this implies that $(u+\epsilon)^{-\gamma/2}$ is a subsolution to $\mathcal{L}u = 0$. Therefore, from Theorem 40, we get

$$\operatorname*{ess\,sup}_{B(x_0,R/2)} (u+\epsilon)^{-\gamma/2} \le C_B\left(\fint_{B(x_0,R)} (u+\epsilon)^{-\gamma}\,\mathrm{d}x\right)^{1/2},$$

hence

$$\operatorname*{ess\,inf}_{B(x_0,R/2)} (u+\epsilon) \ge \frac{1}{C_B^{2/\gamma}}\left(\fint_{B(x_0,R)} (u+\epsilon)^{-\gamma}\,\mathrm{d}x\right)^{-1/\gamma}. \tag{32}$$

Combining (31) and (32), we arrive at

$$\left(\fint_{B(x_0,R)} (u+\epsilon)^{\gamma}\,\mathrm{d}x\right)^{1/\gamma} \le C_W \operatorname*{ess\,inf}_{B(x_0,R/2)} (u+\epsilon).$$

Taking the limit $\epsilon \to 0$, we complete the proof. □

COROLLARY 43. *Let Ω be a connected open set, and let u be a lower semicontinuous weak supersolution to $\mathcal{L}u = 0$ in Ω. If u has an interior minimum point, then u is constant in Ω*

PROOF. Let $\bar{u} = u - \operatorname{ess\,inf}_{\Omega} u$. From the lower semicontinuity of $\bar{u}$, $E = \{x \in \Omega;\ \bar{u}(x) = 0\}$ is a closed subset of Ω. On the other hand, if $x_0 \in E$, then for sufficiently small $R > 0$, we have

$$\left(\fint_{B(x_0,R)} \bar{u}^{\gamma}\,\mathrm{d}x\right)^{1/\gamma} \le C_W \operatorname*{ess\,inf}_{B(x_0,R/2)} \bar{u} = 0.$$

Therefore, E is a open subset of Ω. Since Ω is connected, this implies $E = \Omega$. □

THEOREM 44. *Let u be a (bounded) weak solution to $\mathcal{L}u = 0$ in Ω. Then, u is locally Hölder continuous in Ω. Moreover, there exist constants C and β depending only on n and $\mathcal{B}/\nu$ such that*

$$\operatorname*{osc}_{B(x_0,\rho)} u \le C \left(\frac{\rho}{R}\right)^{\beta} \operatorname*{osc}_{B(x_0,2R)} u,$$

whenever $B(x_0, 2R) \subset \Omega$. Moreover, for any $\gamma > 0$, there exist a constant C depending only on n, $\mathcal{B}/\nu$ and γ such that

$$\operatorname*{osc}_{B(x_0,\rho)} u \le C \left(\frac{\rho}{R}\right)^{\beta} \left(\fint_{B(x_0,4R)} |u|^{\gamma}\,\mathrm{d}x\right)^{1/\gamma},$$

whenever $B(x_0, 4R) \subset \Omega$.

PROOF. Let

$$M(R) = \operatorname*{ess\,sup}_{B(x_0,R)} u, \quad m(R) = \operatorname*{ess\,inf}_{B(x_0,R)} u$$

and $\omega(R) = M(R) - m(R)$. Since u is a solution to $\mathcal{L}u = 0$ in $B(x_0, 2R)$, applying Theorem 42 for $u - m(2R)$ and $M(2R) - u$, we get

$$\left(\fint_{B(x_0,R)} (u - m(2R))^{\gamma}\,\mathrm{d}x\right)^{1/\gamma} \le C_W(m(R/2) - m(2R))$$

and

$$\left(\fint_{B(x_0,R)} (M(2R) - u)^{\gamma}\,\mathrm{d}x\right)^{1/\gamma} \le C_W(M(2R) - m(R/2)).$$

On the other hand, from the quasi-triangle inequality, we have

$$\omega(2R) \le \frac{C(\gamma)}{|B(x_0,R)|^{1/\gamma}} \left(\|M(2R) - u\|_{L^{\gamma}(B(x_0,R))} + \|u - m(2R)\|_{L^{\gamma}(B(x_0,R))}\right).$$

Therefore, combining these inequalities, we obtain

$$\omega(2R) \le C(\omega(2R) - \omega(R/2)),$$

hence

$$\omega(R/2) \le \frac{C-1}{C}\omega(2R).$$

Iterating this estimate, we arrive at

$$\operatorname*{osc}_{B(x_0,\rho)} u \le C \left(\frac{\rho}{R}\right)^{\beta} \operatorname*{osc}_{B(x_0,2R)} u,$$

where $\beta = -\log_4(\frac{C-1}{C})$. This completes the proof. □

THEOREM 45. *Let $D \Subset \Omega$, and let $\theta \in H^1(D) \cap C(\overline{D})$. Let $u \in \theta + H^1_0(D)$ be the weak solution to $\mathcal{L}u = 0$ in D. Assume that D satisfies the following volume density condition at $x_0 \in \partial D$: There exist positive constants $\alpha \in (0,1)$ and $R_0 > 0$ such that*

$$|B(x_0,R) \setminus D| \ge \alpha |B(x_0,R)| \tag{33}$$

for all $0 < R < R_0$. *Then* u *is continuous at* x_0. *Moreover, there exist constants* C *and* $\beta \in (0,1)$ *depending only on* n, $\mathcal{B}/\nu$ *and* α *such that*

$$\omega(\rho) \le C \left(\frac{\rho}{R}\right)^{\beta} \omega(2R) + \omega_\theta(2R)$$

for any $0 < R < \min\{R_0, \operatorname{dist}(x_0, \partial\Omega)/2\}$, *where*

$$\omega(R) = \operatorname*{osc}_{D\cap B(x_0,R)} u, \quad \omega_\theta(R) = \operatorname*{osc}_{\partial D\cap B(x_0,R)} \theta.$$

Proof. For simplicity of notation, we let

$$M(R) = \operatorname*{ess\,sup}_{D\cap B(x_0,R)} u, \quad m(R) = \operatorname*{ess\,inf}_{D\cap B(x_0,R)} u, \quad \omega(R) = M(R) - m(R)$$

and

$$M_\theta(R) = \sup_{\partial D\cap B(x_0,R)} \theta, \quad m_\theta(R) = \inf_{\partial D\cap B(x_0,R)} \theta, \quad \omega_\theta(R) = M_\theta(R) - m_\theta(R).$$

Since ∂D is compact, from the maximum principle, these quantities are finite. For fixed $R > 0$, we consider the function

$$v = \begin{cases} \min\{u, m_\theta(2R)\} & \text{in } D, \\ m_\theta(2R) & \text{in } B(x_0, 2R) \setminus D. \end{cases}$$

Let $\tilde{v}$ be the solution to the obstacle problem in $\mathcal{K}_{v,v}(B(x_0, 2R))$. Since $\tilde{v}$ is a solution to the obstacle problem in $\mathcal{K}_{v,v}(D \cap B(x_0, 2R))$ (Lemma 31) and v is a supersolution in $D \cap B(x_0, 2R)$, we see that $\tilde{v} = v$ a.e. in $D \cap B(x_0, 2R)$ (Lemma 32). Applying Theorem 42 for $\tilde{v} - m(2R)$ in $B(x_0, 2R)$, we get

$$\left(\fint_{B(x_0,R)} (\tilde{v} - m(2R))^\gamma \, \mathrm{d}x \right)^{1/\gamma} \le C_W \operatorname*{ess\,inf}_{B(x_0,R/2)} (\tilde{v} - m(2R)).$$

It follows from (33) and the definition of v that

$$\begin{aligned} (m_\theta(2R) - m(2R))\, \alpha^{1/\gamma} &\le (m_\theta(2R) - m(2R)) \left(\frac{|B(x_0,R) \setminus D|}{|B(x_0,R)|} \right)^{1/\gamma} \\ &\le \left(\fint_{B(x_0,R)} (\tilde{v} - m(2R))^\gamma \, \mathrm{d}x \right)^{1/\gamma}. \end{aligned}$$

Therefore, we get

$$m_\theta(2R) - m(2R) \le C(m(R/2) - m(2R)),$$

where $C = \alpha^{-1/\gamma} C_W$. From a similar argument, we can show that

$$M(2R) - M_\theta(2R) \le C(M(2R) - M(R/2)).$$

Combining the two inequalities, we obtain

$$\omega(2R) - \omega_\theta(2R) \le C(\omega(2R) - \omega(R/2)),$$

hence

$$\omega(R/2) \le \frac{C-1}{C} \omega(2R) + \frac{1}{C} \omega_\theta(2R).$$

Iterating this estimate, we arrive at the assertion. □

REMARK 46. We say that a bounded open set D has an (exterior) corkscrew at $x_0 \in \partial D$ if there are constants $\lambda \in (0,1)$ and $R_0 > 0$ such that the ball $B(x_0, R)$ contains a ball $B(y, \lambda R) \subset \complement D$ whenever $0 < R \leq R_0$. Also, we say that D has an exterior cone at $x_0 \in \partial D$, if there is a truncated cone in $\complement D$ with vertex at x_0. From the definition,

$$x_0 \text{ has an exterior cone} \implies x_0 \text{ has a corkscrew} \implies x_0 \text{ satisfies (33).}$$

In particular, if D has a Lipschitz boundary, then (33) holds for any $x_0 \in \partial D$.

Combining Theorem 40 and Theorem 42, we get the following Harnack's inequality.

THEOREM 47. *Let u be a non-negative weak solution to $\mathcal{L}u = 0$ in Ω. Then, there exists a constant C_H depending only on n and $\mathcal{B}/\nu$ such that*

$$\operatorname*{ess\,sup}_{B(x_0,R/2)} u \leq C_H \operatorname*{ess\,inf}_{B(x_0,R/2)} u,$$

whenever $B(x_0, 2R) \subset \Omega$. Moreover, if $D \Subset \Omega$ is connected, then, there exists a constant C depending only on n, $\mathcal{B}/\nu$, D and Ω such that

$$\operatorname*{ess\,sup}_D u \leq C \operatorname*{ess\,inf}_D u.$$

PROOF. The first assertion immediately follows from Theorem 40 and Theorem 42. Let $R = \operatorname{dist}(\overline{D}, \partial\Omega)/2$. Since D is connected and $\overline{D}$ is compact, we can cover $\overline{D}$ by a chain of balls $\{B_m\}_{m=1}^M$ such that each radius is $R/2$ and $B_m \cap B_{m+1} \neq \emptyset$ for $m = 1, \dots M-1$. Then, it follows from Theorem 47 that

$$\operatorname*{ess\,sup}_D u \leq (C_H)^M \operatorname*{ess\,inf}_D u.$$

This completes the proof. □

The following Harnack's convergence theorem and a one-sided Liouville-type theorem are standard consequences of Harnack's inequality.

COROLLARY 48. *Let $\{u_j\}_{j=1}^\infty$ be a non-decreasing sequence of continuous weak solutions to $\mathcal{L}u = 0$ in Ω. Assume that there is a point $x_0 \in \Omega$ such that $\{u_j(x_0)\}_{j=1}^\infty$ converges a finite value. Then, u is a continuous function belonging to $H^1_{\mathrm{loc}}(\Omega)$ and u is a weak solution to $\mathcal{L}u = 0$ in Ω.*

PROOF. Fix $D \Subset \Omega$ such that $x_0 \in D$. Since $\{u_j\}_{j=1}^\infty$ is a non-decreasing sequence, we have $u_i - u_j \geq 0$ for any $i \geq j$. By Theorem 47,

$$\operatorname*{ess\,sup}_D (u_i - u_j) \leq C \operatorname*{ess\,inf}_D (u_i - u_j).$$

From the assumption, the right-hand side is a Cauchy sequence. Hence, $\{u_j\}_{j=1}^\infty$ converges locally uniformly and $u \in C(\Omega) \subset L^\infty_{\mathrm{loc}}(\Omega)$. From Lemma 38, u belongs to $H^1_{\mathrm{loc}}(\Omega)$ and satisfies the equation $\mathcal{L}u = 0$ in Ω. □

COROLLARY 49. *Let u be a weak solution to $\mathcal{L}u = 0$ in $\mathbb{R}^n$. If u is bounded from below (or above) in $\mathbb{R}^n$, then u is a constant.*

PROOF. Applying Harnack's inequality for $v = u - \operatorname{ess\,inf}_{\mathbb{R}^n} u$, we get

$$\operatorname*{ess\,sup}_{B(x_0,R/2)} v \leq C_H \operatorname*{ess\,inf}_{B(x_0,R/2)} v \to 0 \quad \text{as } R \to 0.$$

Therefore, $u \equiv \operatorname{ess\,inf}_{\mathbb{R}} u$ in $\mathbb{R}^n$. □

CHAPTER 4

Potential upper bounds

If u is a superharmonic function in $B(x_0,R)=\Omega$, then, from the Riesz decomposition theorem (see, [**34**, p.159]),

$$u(x)=\int_\Omega G_\Omega(x,y)\,\mathrm{d}\mu(y)+h(x),$$

where $G_\Omega(\cdot,\cdot)$ is the Green function for Ω and h is the greatest harmonic minorant of u on Ω. If $h=0$, in other words, if $u=0$ on $\partial\Omega$, then, from the upper pointwise estimate of Green's function

$$G_\Omega(x,y)\le\frac{1}{(n-2)n|B(0,1)|}|x-y|^{2-n}, \tag{34}$$

we have the weak-type estimate

$$\|u\|_{L^{n/(n-2),\infty}(\Omega)}\le C(n)\mu(\Omega). \tag{35}$$

On the other hand, for any $x\in B(x_0,R/2)$, we have the upper bound

$$\begin{aligned}h(x)&\le\sup_{B(x_0,3R/4)}h=\sup_{\partial B(x_0,3R/4)}h\le\sup_{y\in\partial B(x_0,3R/4)}\fint_{B(y,R/4)}h\,\mathrm{d}x\\&\le C(n)\fint_{B(x_0,R)\setminus\overline{B(x_0,R/2)}}h\,\mathrm{d}x\le C(n)\fint_{B(x_0,R)\setminus\overline{B(x_0,R/2)}}u_+\,\mathrm{d}x\end{aligned}$$

from the comparison principle and the mean value property. Thus, generally, we have the local pointwise bound

$$u_+(x_0)\le C\left(\fint_{B(x_0,R)\setminus\overline{B(x_0,R/2)}}u_+\,\mathrm{d}x+\mathbf{I}_2^\mu(x_0,2R)\right) \tag{36}$$

and the local weak-type estimate

$$\begin{aligned}&R^{2-n}\|u_+\|_{L^{n/(n-2),\infty}(B(x_0,R/2))}\\&\qquad\le C\left(\fint_{B(x_0,R)\setminus\overline{B(x_0,R/2)}}u_+\,\mathrm{d}x+R^{2-n}\mu(B(x_0,R))\right).\end{aligned} \tag{37}$$

From the results in [**53**] and [**28**], the existence of Green's function and the pointwise estimate of Green's function (34) also hold for equations (3). Therefore, a similar decomposition argument also works for equations

$$-\mathrm{div}\,(A\nabla u)=\mu\ge 0.$$

Moreover, using the De Giorgi and Moser local boundedness estimate (Theorem 40) we can obtain local estimates. Unfortunately, due to the effects of drift, we can not use this argument for (1). Under condition (21), the estimate (34) does not holds in general. The condition (21) is a sufficient condition to relate solutions and

H^{-1} data, However, this condition is not sufficient to relate solutions and measure data. Moreover, even if (38) holds, the existence of Green's function is not clear.

In this chapter, considering about the relation between the upper bounds of solutions and the measure data, we establish the global weak-type estimate (35), the local weak-type estimate (36) and the potential upper bound (37) under the assumption (21) and the additional assumption (38). Our method differs from the Green function method. We first prove (35) using a truncation function and energy method. Next, we consider its local version (37). Finally, iterating (37), we get (36). These arguments are based on the nonlinear potential theory.

1. Global weak-type estimates

Hereafter, we assume that

$$\|(\operatorname{div}\mathbf{b})_+\|_{L^{n/2,1}(\Omega)} \le \frac{\nu}{4S_\infty^2} \tag{38}$$

in addition to (21). Under these conditions, we have the following global $L^1 - L^{n/(n-2),\infty}$ estimate:

THEOREM 50. *Let μ be a finite Radon measure in $(\mathcal{D}_0^{1,2}(\Omega))^*$, and let $u \in \mathcal{D}_0^{1,2}(\Omega)$ be a weak solution to $\mathcal{L}u = \mu$ in Ω. Then*

$$\|u\|_{L^{n/(n-2),\infty}(\Omega)} \le \frac{4S_\infty^2}{\nu}|\mu|(\Omega), \tag{39}$$

$$\|\nabla u\|_{L^{n/(n-1),\infty}(\Omega)} \le \frac{4S_\infty}{\nu}|\mu|(\Omega) \tag{40}$$

and

$$\|\nabla T_k(u)\|_{L^2(\Omega)}^2 \le \frac{4k}{\nu}|\mu|(\Omega) \tag{41}$$

for all $k > 0$.

PROOF. First, we prove (39). Taking the test function $T_k(u)$, we get

$$\begin{aligned}\int_\Omega T_k(u)\,\mathrm{d}\mu &= \int_\Omega A\nabla u\cdot\nabla T_k(u) + (\mathbf{b}\cdot\nabla u)T_k(u)\,\mathrm{d}x\\ &= \int_\Omega A\nabla u\cdot\nabla T_k(u) - (\mathbf{b}\cdot\nabla T_k(u))\,u - \operatorname{div}\mathbf{b}\,u\,T_k(u)\,\mathrm{d}x.\end{aligned}$$

If $\nabla T_k(u)(x) \neq 0$, then $T_k(u)(x) = u(x)$. Therefore,

$$\int_\Omega T_k(u)\,\mathrm{d}\mu = \int_\Omega A\nabla T_k(u)\cdot\nabla T_k(u) - (\mathbf{b}\cdot\nabla T_k(u))\,T_k(u) - \operatorname{div}\mathbf{b}\,u\,T_k(u)\,\mathrm{d}x.$$

This implies that

$$\begin{aligned}\nu\int_\Omega |\nabla T_k(u)|^2\,\mathrm{d}x &\le k\left(|\mu|(\Omega) + \int_\Omega (\operatorname{div}\mathbf{b})_+|u|\,\mathrm{d}x\right) + \int_\Omega \mathbf{b}_0\cdot\nabla(\frac{1}{2}T_k(u)^2)\,\mathrm{d}x\\ &\quad + \left(\int_\Omega |\mathbf{b}_1|^2T_k(u)^2\,\mathrm{d}x\right)^{1/2}\left(\int_\Omega |\nabla T_k(u)|^2\,\mathrm{d}x\right)^{1/2}.\end{aligned}$$

By using integration by parts, the second term in the right-hand side is zero. Thus, from (21), we have

$$\frac{\nu}{2}\int_\Omega |\nabla T_k(u)|^2\,\mathrm{d}x \le k\left(|\mu|(\Omega) + \int_\Omega (\operatorname{div}\mathbf{b})_+|u|\,\mathrm{d}x\right).$$

From Lemma 14, we get

$$k^2|\{x \in \Omega;\ |u(x)| \geq k\}|^{(n-2)/n} \leq \frac{2S_\infty^2}{\nu} k \left(|\mu|(\Omega) + \int_\Omega (\operatorname{div} \mathbf{b})_+ |u| \,\mathrm{d}x \right).$$

Dividing by k and taking the supremum over $k > 0$, we get

$$\|u\|_{L^{n/(n-2),\infty}(\Omega)} \leq \frac{2S_\infty^2}{\nu} \left(|\mu|(\Omega) + \|(\operatorname{div} \mathbf{b})_+\|_{L^{n/2,1+}(\Omega)} \|u\|_{L^{n/(n-2),\infty}(\Omega)} \right).$$

From (38), this implies (39). Since

$$\int_{\{x \in \Omega;\ |u(x)| < k\}} |\nabla u|^2 \,\mathrm{d}x = \int_\Omega |\nabla T_k(u)|^2 \,\mathrm{d}x \leq \frac{4k}{\nu} |\mu|(\Omega)$$

for any $k > 0$, we have (41). By Chebyshev's inequality we have

$$l^2 |\{x \in \Omega;\ |\nabla u(x)| \geq l \text{ and } |u(x)| < k\}| \leq \frac{4k}{\nu} |\mu|(\Omega).$$

Since

$$|\{x \in \Omega;\ |u(x)| \geq k\}| \leq \left(\frac{4S_\infty^2 |\mu|(\Omega)}{\nu k} \right)^{n/(n-2)},$$

we have

$$|\{x \in \Omega;\ |\nabla u(x)| \geq l\}| \leq \frac{4k}{\nu l^2} |\mu|(\Omega) + \left(\frac{4S_\infty^2 |\mu|(\Omega)}{\nu k} \right)^{n/(n-2)}.$$

Therefore, choosing

$$k = S_\infty^{n/(n-1)} \left(\frac{4|\mu|(\Omega)}{\nu} \right)^{1/(n-1)} l^{(n-2)/(n-1)},$$

we obtain

$$|\{x \in \Omega;\ |\nabla u(x)| \geq l\}| \leq S_\infty^{n/(n-1)} \left(\frac{4|\mu|(\Omega)}{\nu} \right)^{n/(n-1)} l^{-n/(n-1)}.$$

This implies that

$$\|\nabla u\|_{L^{n/(n-1),\infty}(\Omega)} = \sup_{l>0} l |\{x \in \Omega;\ |\nabla u(x)| \geq l\}|^{(n-1)/n} \leq \frac{4S_\infty}{\nu} |\mu|(\Omega).$$

We arrived at the desired estimate (40). □

Remark 51. It is necessary to add the condition (38) to lead Theorem 50. Consider the case of $A = I$, $\mathbf{b} = \epsilon x/|x|^2$ and $\Omega = B(0, R)$. For sufficiently small positive $\epsilon > 0$, (21) holds; however (38) does not hold. Let us consider functions

$$u_r(x) = \min\{|x|^{2-n+\epsilon}, r^{2-n+\epsilon}\} - 1.$$

Then u_r are supersolutions to $\mathcal{L}u = 0$, moreover, their Riesz measures are represented by

$$\int_\Omega \varphi \,\mathrm{d}\mu[u_r] = \frac{(n-2-\epsilon)}{r^{n-1-\epsilon}} \int_{\partial B(0,r)} \varphi \,\mathrm{d}\mathcal{H}^{n-1}$$

for all $\varphi \in C_c^\infty(\Omega)$, $\varphi \geq 0$, where $\mathcal{H}^{n-1}$ is the $(n-1)$ dimension Hausdorff measure. Then $u_r \to |x|^{2-n+\epsilon} - 1$ and $\mu[u_r] \to 0$ as $r \to 0$. Therefore, under (21), the estimates (39), (40) and (41) do not hold in general.

REMARK 52. More generally, if $u \in H_0^1(\Omega)$ is the weak solution to the equation

$$\mathcal{L}u + \mathbf{c}\cdot\nabla u = -\mathrm{div}\,(A\nabla u) + (\mathbf{b}+\mathbf{c})\cdot\nabla u = \mu$$

with

$$\|\mathbf{c}\|_{L^{n,1}(\Omega)} \le \frac{\nu}{8S_\infty},$$

then we have

$$\|u\|_{L^{n/(n-2),\infty}(\Omega)} \le 8S_\infty^2|\mu|(\Omega)$$

and

$$\|\nabla u\|_{L^{n/(n-1),\infty}(\Omega)} \le 8S_\infty|\mu|(\Omega).$$

Indeed, from (40), we have

$$\begin{aligned}\|\nabla u\|_{L^{n/(n-1),\infty}(\Omega)} &\le \frac{4S_\infty}{\nu}\big(\|\mathbf{c}\cdot\nabla u\|_{L^1(\Omega)} + |\mu|(\Omega)\big)\\ &\le \frac{4S_\infty}{\nu}\|\mathbf{c}\|_{L^{n,1}(\Omega)}\|\nabla u\|_{L^{n/(n-1),\infty}(\Omega)} + \frac{4S_\infty}{\nu}|\mu|(\Omega).\end{aligned}$$

From the assumption on $\mathbf{c}$, this implies that

$$\|\nabla u\|_{L^{n/(n-1),\infty}(\Omega)} \le \frac{8S_\infty}{\nu}|\mu|(\Omega).$$

Consequently, we have

$$\begin{aligned}\|u\|_{L^{n/(n-2),\infty}(\Omega)} &\le \frac{4S_\infty^2}{\nu}\big(\|\mathbf{c}\cdot\nabla u\|_{L^1(\Omega)} + |\mu|(\Omega)\big)\\ &\le \frac{4S_\infty^2}{\nu}\|\mathbf{c}\|_{L^{n,1}(\Omega)}\|\nabla u\|_{L^{n/(n-1),\infty}(\Omega)} + \frac{4S_\infty^2}{\nu}|\mu|(\Omega) \le \frac{8S_\infty^2}{\nu}|\mu|(\Omega).\end{aligned}$$

2. Local $L^1 - L^{n/(n-2),\infty}$ estimates and potential upper bounds

When $\mathbf{b} = 0$, the previous global weak-type $L^1 - L^{n/(n-2),\infty}$ estimate is well-known. However, its local version is not known valid even if $\mathbf{b} = 0$. Below, we give a local weak-type estimate using the Poisson modification technique based on the method in Trudinger and Wang [**70**]. For an analog for p-Laplacian equations, see [**31**].

THEOREM 53. *Let u be a weak solution to $\mathcal{L}u = \mu_+ - \mu_-$ in Ω. Then, for any $\gamma > 0$, there exist constants C_1 depending only on n, $\mathcal{B}/\nu$ and γ and C_2 depending only on n such that*

$$\begin{aligned}R^{2-n}\|u_\pm\|_{L^{n/(n-2),\infty}(B(x_0,R/2))} &\le C_1\left(\fint_{B(x_0,R)\setminus\overline{B(x_0,R/2)}} u_\pm^\gamma\,\mathrm{d}x\right)^{1/\gamma}\\ &\quad + \frac{C_2}{\nu}R^{2-n}\mu_\pm(\overline{B(x_0,R)}),\end{aligned}$$

whenever $B(x_0,R) \Subset \Omega$.

PROOF. We prove only the estimate on u_+. Set $D = B(x_0,R)\setminus\overline{B(x_0,R/2)}$ and $B_* = B(x_0,3R/4)$. By definition, $\partial B_* \Subset D$. We divide the proof into several steps:

Step 1. First, we take a comparison function $v \in H^1_{\mathrm{loc}}(\Omega)$ as follows:

$$\begin{cases}\mathcal{L}v = 0 - \mu_- & \text{in } D,\\ v = u & \text{in } \Omega\setminus D.\end{cases}$$

By Lemma 28, this function v can be defined. From the comparison principle, we have

$$v(x) \leq u(x) \quad \text{for a.e. } x \in D.$$

Step 2. Next, we shall estimate the size of $\int_{\overline{B_*}} \mathrm{d}(\mathcal{L}v)_+$ (see (42)). For $k > 0$, we take $H_k(t) = \frac{1}{k}T_k(t)$, where $T_k(t) = \min\{\max\{t, -k\}, k\}$. Note that for any $t > 0$, $H_k(t) \to 1$ as $k \to 0$. Choose $\psi \in H_0^1(B_*)$ such that $0 \leq \psi \leq 1$, and consider the non-negative function $\varphi = \psi H_k(u - v)$. This function belongs to $H_0^1(D) \cap L^\infty(D)$, because $H_k(u - v) \in H_0^1(D) \cap L^\infty(D)$ and $\psi \in H^1(D) \cap L^\infty(D)$. By using this function, we have

$$\begin{aligned}
0 &\leq \langle \mu_+, \varphi \rangle_D = \langle (\mathcal{L}u - \mathcal{L}v), \varphi \rangle_D \\
&= \int_D A\nabla u \cdot \nabla(\psi H_k(u - v)) + (\mathbf{b} \cdot \nabla u)\psi H_k(u - v) \, \mathrm{d}x \\
&- \int_D A\nabla v \cdot \nabla(\psi H_k(u - v)) + (\mathbf{b} \cdot \nabla v)\psi H_k(u - v) \, \mathrm{d}x \\
&= \int_D \{(A\nabla u \cdot \nabla\psi + (\mathbf{b} \cdot \nabla u)\psi) - (A\nabla v \cdot \nabla\psi + (\mathbf{b} \cdot \nabla v)\psi)\} \, H_k(u - v) \, \mathrm{d}x \\
&+ \int_D (A\nabla u - A\nabla v) \cdot (\nabla H_k(u - v))\psi \, \mathrm{d}x.
\end{aligned}$$

From positivity of ψ and uniform ellipticity of A, the second term of the right-hand side can be estimated by

$$\begin{aligned}
&\int_D (A\nabla u - A\nabla v) \cdot (\nabla H_k(u - v))\psi \, \mathrm{d}x \\
&= \frac{1}{k} \int_{\{x \in D;\ (u-v)(x) < k\}} (A\nabla u - A\nabla v) \cdot \nabla(u - v)\psi \, \mathrm{d}x \\
&\leq \frac{1}{k} \int_{\{x \in D;\ (u-v)(x) < k\}} (A\nabla u - A\nabla v) \cdot \nabla(u - v) \, \mathrm{d}x.
\end{aligned}$$

Since $\langle (\mathcal{L}u - \mathcal{L}v), \varphi \rangle_D = \langle \mu_+, \varphi \rangle_D$, taking $\varphi = T_k(u - v)$, we get

$$\begin{aligned}
&\int_{\{x \in D;\ (u-v)(x) < k\}} (A\nabla u - A\nabla v) \cdot \nabla(u - v) \, \mathrm{d}x \\
&\leq k \left(\mu_+(D) + \int_\Omega (\operatorname{div} \mathbf{b})_+ (u - v) \, \mathrm{d}x \right) + \int_D \mathbf{b}_0 \cdot \nabla(\frac{1}{2}T_k(u - v)^2) \, \mathrm{d}x \\
&+ \left(\int_D |\mathbf{b}_1|^2 T_k(u - v)^2 \, \mathrm{d}x \right)^{1/2} \left(\int_D |\nabla T_k(u - v)|^2 \, \mathrm{d}x \right)^{1/2}.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\frac{1}{k} \int_{\{x \in D;\ (u-v)(x) < k\}} (A\nabla u - A\nabla v) \cdot \nabla(u - v) \, \mathrm{d}x \\
&\leq \mu_+(D) + \|(\operatorname{div} \mathbf{b})_+\|_{L^{n/2,1}(D)} \|u - v\|_{L^{n/(n-2),\infty}(D)} \\
&\quad + \frac{1}{k} |||\mathbf{b}_1|||_D \|\nabla T_k(u - v)\|_{L^2(D)}^2.
\end{aligned}$$

It follows from Theorem 50 ((39) and (41)) that

$$\frac{1}{k} \int_{\{x \in D;\ (u-v)(x) < k\}} (A\nabla u - A\nabla v) \cdot \nabla(u - v) \, \mathrm{d}x \leq 4\mu_+(D).$$

Therefore, we get

$$\int_D \{(A\nabla v\cdot\nabla\psi+(\mathbf{b}\cdot\nabla v)\psi)-(A\nabla u\cdot\nabla\psi+(\mathbf{b}\cdot\nabla u)\psi)\}\,H_k(u-v)\,\mathrm{d}x\le 4\mu_+(D).$$

Let $k\to 0$. By the Lebesgue dominated convergence theorem, we then obtain

$$\int_A \{(A\nabla v\cdot\nabla\psi+(\mathbf{b}\cdot\nabla v)\psi)-(A\nabla u\cdot\nabla\psi+(\mathbf{b}\cdot\nabla u)\psi)\}\,\mathrm{d}x\le 4\mu_+(D).$$

Since $v=u$ in $\Omega\setminus D$ and $D\subset\overline{B(x_0,R)}$, it follows that

$$\int_{B(x_0,R)} \{(A\nabla v\cdot\nabla\psi+(\mathbf{b}\cdot\nabla v)\psi)-(A\nabla u\cdot\nabla\psi+(\mathbf{b}\cdot\nabla u)\psi)\}\,\mathrm{d}x\le 4\mu_+(\overline{B(x_0,R)}).$$

On the other hand, since $B_*\subset\overline{B(x_0,R)}$, we have

$$\int_{B(x_0,R)} A\nabla u\cdot\nabla\psi+(\mathbf{b}\cdot\nabla u)\psi\,\mathrm{d}x=\langle\mu,\psi\rangle_{B_*}\le\mu_+(\overline{B(x_0,R)}).$$

Consequently, we obtain

$$\sup_{\substack{\psi\in H_0^1(B_*)\\ 0\le\psi\le 1}}\int_{B(x_0,R)} A\nabla v\cdot\nabla\psi+(\mathbf{b}\cdot\nabla v)\psi\,\mathrm{d}x\le 5\mu_+(\overline{B(x_0,R)}). \tag{42}$$

Step 3. Next, we shall estimate the upper part of u, or the effect of external force. Note that $v=u$ in $B(x_0,R/2)$. Let

$$l=\sup_{\partial B_*} v_+ := \inf\{l\in\mathbb{R};\ (v_+-l)_+\in H_0^1(B_*)\}.$$

Since v is a subsolution to $\mathcal{L}v=0$ in D, it follows from Theorem 40 that

$$0\le l\le C(n)C_B\left(\fint_D v_+^\gamma\,\mathrm{d}x\right)^{1/\gamma}<\infty. \tag{43}$$

Using this l, we consider the function

$$\psi=H_k((v_+-l)_+)\quad(k>0).$$

Then, it follows from (42) that

$$\int_{B_*} A\nabla v\cdot\nabla\psi-(\mathbf{b}\cdot\nabla\psi)\,v\,\mathrm{d}x-\int_{B_*}\operatorname{div}\mathbf{b}\,v\,\psi\,\mathrm{d}x\le 5\mu_+(\overline{B(x_0,R)}).$$

Therefore, from (21),

$$\begin{aligned}&\frac{\nu}{2k}\int_{B_*}|\nabla\min\{(v_+-l)_+,k\}|^2\,\mathrm{d}x\\ &\le 5\mu_+(\overline{B(x_0,R)})+\|(\operatorname{div}\mathbf{b})_+\|_{L^{n/2,1}(B_*)}\|v_+\|_{L^{n/(n-2),\infty}(B_*)}.\end{aligned}$$

From Lemma 14, this implies that

$$\begin{aligned}\|(v_+-l)_+\|_{L^{n/(n-2),\infty}(B_*)}\le{}&\frac{10S_\infty^2}{\nu}\mu_+(\overline{B(x_0,R)})\\ &+\frac{2S_\infty^2}{\nu}\|(\operatorname{div}\mathbf{b})_+\|_{L^{n/2,1}(B_*)}\|v_+\|_{L^{n/(n-2),\infty}(B_*)}.\end{aligned} \tag{44}$$

Step 4. Finally, we estimate the lower part of u, or the effect of value around. Since $0 \le v_+ \le (v_+ - l)_+ + l$, we have

$$\begin{aligned}\|v_+\|_{L^{n/(n-2),\infty}(B_*)} &\le \|(v_+ - l)_+ + l\|_{L^{n/(n-2),\infty}(B_*)}\\ &\le \|(v_+ - l)_+\|_{L^{n/(n-2),\infty}(B_*)} + l|B_*|^{(n-2)/n}.\end{aligned}$$

Therefore, combining this inequality, (43) and (44), we obtain

$$\begin{aligned}\|v_+\|_{L^{n/(n-2),\infty}(B_*)} &\le C(n)R^{n-2}C_B\left(\fint_D v_+^\gamma\,\mathrm{d}x\right)^{1/\gamma}\\ &\quad+\frac{10S_\infty^2}{\nu}\mu_+(\overline{B(x_0,R)})\\ &\quad+\frac{2S_\infty^2}{\nu}\|(\operatorname{div}\mathbf{b})_+\|_{L^{n/2,1}(B_*)}\|v_+\|_{L^{n/(n-2),\infty}(B_*)}.\end{aligned}$$

It follows from (38) that

$$\begin{aligned}\|v_+\|_{L^{n/(n-2),\infty}(B(x_0,R/2))} &\le 2C(n)R^{n-2}C_B\left(\fint_D v_+^\gamma\,\mathrm{d}x\right)^{1/\gamma}\\ &\quad+\frac{20S_\infty^2}{\nu}\mu_+(\overline{B(x_0,R)}).\end{aligned}$$

Since $v = u$ in $B(x_0, R/2)$ and $v \le u$ in D, replacing v by u, we arrive at the desired estimate. □

REMARK 54. If $\mu \ge 0$, then we can give a simpler proof. Indeed, the following claim holds: Suppose that u is a supersolution to $\mathcal{L}u = 0$ in Ω and $D \Subset \Omega$. Then the Poisson modification $v = P(u, D)$ of u in D satisfies

$$\mu[v](\overline{D}) \le 2\mu[u](\overline{D}).$$

To prove this, let us choose a function $\eta \in C_c^\infty(\Omega)$ such that $\eta \equiv 1$ on $\overline{D}$ and $0 \le \eta \le 1$. Since $u = v$ on $\operatorname{supp}\nabla\eta$, we have

$$\begin{aligned}\int_\Omega \eta\,\mathrm{d}\mu[u] &= \int_\Omega A\nabla u\cdot\nabla\eta - (\mathbf{b}\cdot\nabla\eta)u - \operatorname{div}\mathbf{b}\,u\,\eta\,\mathrm{d}x\\ &= \int_\Omega A\nabla v\cdot\nabla\eta - (\mathbf{b}\cdot\nabla\eta)v - \operatorname{div}\mathbf{b}\,v\,\eta\,\mathrm{d}x - \int_\Omega \operatorname{div}\mathbf{b}\,(u-v)\,\eta\,\mathrm{d}x\\ &= \int_\Omega \eta\,\mathrm{d}\mu[v] - \int_\Omega \operatorname{div}\mathbf{b}\,(u-v)\,\mathrm{d}x.\end{aligned}$$

Note that $v \le u$ from the comparison principle. Therefore, it follows from Theorem 50 that

$$\begin{aligned}\mu[v](\overline{D}) &\le \mu[u](\overline{D}) + \|(\operatorname{div}\mathbf{b})_+\|_{L^{n/2,1}(D)}\|u-v\|_{L^{n/(n-2),\infty}(D)}\\ &\le 2\mu[u](\overline{D}).\end{aligned}$$

Therefore, replacing Step 2 by this estimate, we can get the same estimate.

Using Theorem 53 and an iteration argument, we reach the truncated Riesz potential estimates. The concept of our iteration method is due to Kilpeläinen and Malý [**38**]. The following modified version of iteration method is in [**30, 31**].

THEOREM 55. *Let u be a weak solution to $\mathcal{L}u = \mu_+ - \mu_-$ in Ω. Suppose that x_0 is a Lebesgue point of u. Then there exists a constant C_U depending only on n, $\mathcal{B}/\nu$ such that*

$$u_\pm(x_0) \le C_U \left\{ \left(\fint_{B(x_0,R)\setminus\overline{B(x_0,R/2)}} u_\pm^\gamma \, \mathrm{d}x \right)^{1/\gamma} + \frac{1}{\nu}\mathbf{I}_2^{\mu_\pm}(x_0, 2R) \right\},$$

whenever $B(x_0, 2R) \subset \Omega$

PROOF. We prove the estimate only of u_+. Let $\theta \in (0,1)$ be a sufficiently small constant to be chosen later. For $m = 0, 1, \ldots$, we take $R_m = 2^{-m}R$, $B_m = B(x_0, R_m)$ and

$$l_0 = 0, \quad l_{m+1} = l_m + \frac{1}{(\theta|B_{m+1}|)^{(n-2)/n}} \|(u - l_m)_+\|_{L^{n/(n-2),\infty}(B_{m+1})}.$$

From the definition of l_m, for any $m \ge 1$,

$$\|(u - l_{m-1})_+\|_{L^{n/(n-2),\infty}(B_m)} = (l_m - l_{m-1})(\theta|B_m|)^{(n-2)/n} \tag{45}$$

holds. Assume that $l_m > l_{m-1}$. Then we have

$$|\{x \in B_m;\, u(x) \ge l_m\}| = |\{x \in B_m;\, (u(x) - l_{m-1})_+ \ge l_m - l_{m-1}\}| \le \theta|B_m|.$$

Therefore, by using Lemma 12, we get

$$\begin{aligned}\int_{B_m} (u - l_m)_+ \, \mathrm{d}x &\le C(n)|\{x \in B_m;\, u(x) \ge l_m\}|^{2/n} \|(u - l_m)_+\|_{L^{n/(n-2),\infty}(B_m)} \\ &\le C(n)(\theta|B_m|)^{2/n} \|(u - l_m)_+\|_{L^{n/(n-2),\infty}(B_m)},\end{aligned}$$

This implies that

$$\begin{aligned}\fint_{B_m} (u - l_m)_+ \, \mathrm{d}x &\le C(n)\theta^{2/n} \frac{1}{|B_m|^{(n-2)/n}} \|(u - l_{m-1})_+\|_{L^{n/(n-2),\infty}(B_m)} \\ &= C(n)\theta^{2/n} \frac{1}{|B_m|^{(n-2)/n}} \|(u - l_{m-1})_+\|_{L^{n/(n-2),\infty}(B_m)}.\end{aligned}$$

On the other hand, applying Theorem 53 for $u - l_m$ with $\gamma = 1$, we have

$$\begin{aligned}\frac{1}{|B_{m+1}|^{(n-2)/n}} \|(u - l_m)_+\|_{L^{n/(n-2),\infty}(B_{m+1})} &\le C_1 \fint_{B_m\setminus\overline{B_{m+1}}} (u - l_m)_+ \, \mathrm{d}x \\ &\quad + \frac{C_2}{\nu} R_m^{2-n} \mu_+(\overline{B_m}).\end{aligned}$$

Combining the two inequalities, we get

$$(l_{m+1} - l_m) \le C(n)C_1\theta^{2/n}(l_m - l_{m-1}) + \frac{C(n)}{\theta^{(n-2)/n}\nu} R_m^{2-n} \mu_+(\overline{B_m}).$$

On the other hand, if $l_m = l_{m-1}$, then $l_{m+1} = l_m$ by (45). Therefore, this inequality holds for any $m \ge 1$. Choose $\theta > 0$ such that $C(n)C_1\theta^{2/n} \le 1/2$. Summing over $m = 1, 2, \ldots M$, we get

$$l_{M+1} - l_1 \le \frac{1}{2} l_M + \frac{C(n,\theta)}{\nu} \sum_{m=1}^{M} R_m^{2-n} \mu_+(\overline{B_m}).$$

Taking the limit $M \to \infty$, we then obtain

$$\frac{1}{2} l_\infty := \frac{1}{2} \lim_{M\to\infty} l_M \leq l_1 + \frac{C(n,\theta)}{\nu} \sum_{m=1}^{\infty} R_m^{2-n} \mu_+(\overline{B_m}).$$

On the other hand, by Theorem 53, we have

$$\begin{aligned} l_1 &= \frac{1}{(\theta|B_1|)^{(n-2)/n}} \|u_+\|_{L^{n/(n-2),\infty}(B_1)} \\ &\leq \frac{1}{\theta^{(n-2)/n}} \left\{ C_1 \left(\fint_{B_0\setminus\overline{B_1}} u_+^\gamma \,\mathrm{d}x \right)^{1/\gamma} + \frac{C_2}{\nu} R_0^{2-n} \mu_+(\overline{B_0}) \right\}. \end{aligned}$$

Therefore, we get

$$l_\infty \leq C(n,\theta) \left\{ \left(\fint_{B_0\setminus\overline{B_1}} u_+^\gamma \,\mathrm{d}x \right)^{1/\gamma} + \frac{1}{\nu} \sum_{m=0}^{\infty} R_m^{2-n} \mu_+(\overline{B_m}) \right\}.$$

By Lemma 5, the right-hand side may be assumed to be finite; therefore, $(l_m - l_{m-1}) \to 0$ as $m \to \infty$. Thus, it follows from Lemma 12 that

$$\fint_{B_m} (u - l_m)_+ \,\mathrm{d}x \leq C(n)\theta(l_m - l_{m-1}) \to 0 \quad \text{as } m \to \infty.$$

Hence

$$\lim_{m\to\infty} \fint_{B_m} (u - l_\infty) \,\mathrm{d}x \leq \lim_{m\to\infty} \fint_{B_m} (u - l_\infty)_+ \,\mathrm{d}x \leq \lim_{m\to\infty} \fint_{B_m} (u - l_m)_+ \,\mathrm{d}x = 0.$$

Therefore,

$$u(x_0) = \lim_{m\to\infty} \fint_{B_m} u \,\mathrm{d}x \leq l_\infty.$$

Consequently, we arrive at

$$u(x_0) \leq C(n,\theta) \left\{ \left(\fint_{B(x_0,R)\setminus\overline{B(x_0,R/2)}} u_+^\gamma \,\mathrm{d}x \right)^{1/\gamma} + \frac{1}{\nu} \mathbf{I}_2^{\mu_+}(x_0, 2R) \right\}.$$

This completes the proof. □

Corollary 56. *Let u be a weak solution to $\mathcal{L}u = \mu \geq 0$ in Ω. Let x_0 be a Lebesgue point of u. Then there exists a constant $C_{U'}$ depending only on n and $\mathcal{B}/\nu$ such that*

$$u(x_0) \leq C_{U'} \left(\operatorname*{ess\,inf}_{B(x_0,R)} u + \frac{1}{\nu} \mathbf{I}_2^{\mu}(x_0, 2R) \right),$$

whenever $B(x_0, 2R) \subset \Omega$.

Proof. Combine Theorem 42 and Theorem 55. □

3. Applications of upper bounds

Using Theorem 55, we can give some sufficient conditions of interior regularity.

COROLLARY 57. *Let u be a weak solution to $\mathcal{L}u = \mu_+ - \mu_-$ in Ω.*

(1) *Assume that*

$$\lim_{R\to 0} \sup_{x\in B(x_0,R)} \mathbf{I}_2^{|\mu|}(x_0, R) = 0$$

for some $x_0 \in \Omega$. Then u is continuous at x_0. Moreover, if

$$\lim_{R\to 0} \sup_{x_0\in D} \mathbf{I}_2^{|\mu|}(x_0, R) = 0$$

for $D \Subset \Omega$, then u is continuous in D.

(2) *If $\mu \in L^{n/2,1}_{\mathrm{loc}}(\Omega)$, then $u \in C(\Omega)$.*

(3) *If $\mu \in L^{p,r}_{\mathrm{loc}}(\Omega)$ with*

$$\frac{1}{q} = \frac{1}{p} - \frac{2}{n}$$

and $0 < r \le \infty$, then $u \in L^{q,r}_{\mathrm{loc}}(\Omega)$.

PROOF. Without loss of generality, we may assume that x_0 is a Lebesgue point of u. From Theorem 55,

$$|u(x) - u(x_0)| \le C \left(\fint_{B(x,R)} |u - u(x_0)| \,\mathrm{d}x + \frac{1}{\nu}\mathbf{I}_2^{|\mu|}(x, R) \right).$$

Taking the supremum for $x \in B(x_0, R)$, we get

$$\operatorname*{ess\,sup}_{B(x_0,R)} |u - u(x_0)| \le C \left(\fint_{B(x_0,2R)} |u - u(x_0)| \,\mathrm{d}x + \sup_{x\in B(x_0,R)} \frac{1}{\nu}\mathbf{I}_2^{|\mu|}(x, R) \right).$$

The first term of the right-hand side goes to 0 as $R \to 0$. From the assumption on μ, we obtain

$$\lim_{R\to 0} \operatorname*{ess\,sup}_{B(x_0,R)} |u - u(x_0)| = 0.$$

The second assertion follows from the first assertion. The third assertion is a consequence of Lemma 17. □

COROLLARY 58. *Let u be a weak solution to $\mathcal{L}u = \mu \ge 0$ in Ω. If there are constants K and $\epsilon > 0$ such that*

$$\mu(B(x, R)) \le KR^{n-2+\epsilon}$$

for all $x \in \Omega$, Then there exists constants C_1, C_2 and $\beta > 0$ depending only on n and $\mathcal{B}/\nu$ such that

$$\operatorname*{osc}_{B(x_0,\rho)} u \le C_1 \left(\frac{\rho}{R}\right)^{\beta} \operatorname*{osc}_{B(x_0,5R)} u + \frac{C_2}{\nu} KR^{\epsilon},$$

whenever $B(x_0, 5R) \subset \Omega$.

PROOF. Let

$$M(R) = \operatorname*{ess\,sup}_{B(x_0,R)} u, \quad m(R) = \operatorname*{ess\,inf}_{B(x_0,R)} u, \quad \omega(R) = M(R) - m(R).$$

From Corollary 56 and the assumption on μ, we have

$$\begin{aligned}
M(R) - m(5R) &= \sup_{B(x_0,R)} (u - m(5R)) \\
&\le \sup_{x\in B(x_0,R)} C_{U'} \left(\inf_{B(x,2R)} (u - m(5R)) + \frac{1}{\nu}\mathbf{I}_2^{\mu}(x,4R) \right) \\
&\le C_{U'} \left(\inf_{B(x_0,R)} (u - m(5R)) + \frac{1}{\nu} \sup_{x\in B(x_0,R)} \mathbf{I}_2^{\mu}(x,4R) \right) \\
&\le C_{U'}(m(R) - m(5R)) + C_{U'}\frac{C(n)}{\nu}KR^{\epsilon}.
\end{aligned}$$

Since $M(5R) - M(R) \le C_{U'}(M(5R) - M(R))$, combining the two inequality, we obtain

$$\omega(5R) \le C_{U'}(\omega(5R) - \omega(R)) + C_{U'}\frac{C(n)}{\nu}KR^{\epsilon},$$

hence

$$\omega(R) \le \frac{C_{U'} - 1}{C_{U'}}\omega(5R) + \frac{C(n)}{\nu}KR^{\epsilon}.$$

Iterating this estimate, we get the desired inequality. □

From the method in [**39**], we can estimate the growth order of non-negative subsolutions:

Corollary 59. *Let u be a non-negative solution to $\mathcal{L}u = \mu \le 0$ in Ω. Then there exist constants C and $0 < \lambda < 1$ depending only on n, ν and $\mathcal{B}$ such that*

$$\operatorname*{ess\,sup}_{B(x_0,\lambda R)} u \le 2u(x_0) + \frac{C}{\nu}\left(\mathbf{I}_2^{\mu}(x_0,2R) + R^{2-n}\mu(B(x_0,2R))\right),$$

whenever $B(x_0,2R) \subset \Omega$.

Proof. Choose a natural number M such that

$$\left(\frac{2C_U}{2C_U - 1}\right)^{M-1} \le 2C_H \le \left(\frac{2C_U}{2C_U - 1}\right)^{M},$$

and take $\lambda = 2^{-M}$. We divide the proof two cases.

Case 1. Assume that for all $m = 1, \dots M$,

$$\sup_{B(x_0,2^{-m}R)} u \le \left(\frac{2C_U - 1}{2C_U}\right) \sup_{B(x_0,2^{1-m}R)} u.$$

Then,

$$\sup_{B(x_0,2^{-M}R)} u \le \frac{1}{2C_H} \sup_{B(x_0,R)} u.$$

Let h be the solution to the Dirichlet problem

$$\begin{cases} \mathcal{L}h = 0 & \text{in } B(x_0,2R) \\ h = u & \text{on } \partial B(x_0,2R), \end{cases}$$

and let $k = (2C_H)^{-1}\sup_{B(x_0,R)} u$. By the comparison principle, $h \ge 0$ in $B(x_0,2R)$. It follows from Theorem 47,

$$\inf_{B(x_0,R)} h \ge \frac{1}{C_H}\sup_{B(x_0,R)} h = \frac{1}{C_H}\sup_{B(x_0,R)} u = 2k.$$

Thus, from the assumption on u,

$$\inf_{B(x_0,\lambda R)}(h-u) \geq 2k - k = k.$$

Consider the function $w = \min\{h-u, k\}$. Then from Lemma 14,

$$\begin{aligned} k^2 C(n)(\lambda R)^{n-2} &\leq k^2 |\{x \in B(x_0, 2R);\ w(x) \geq k\}|^{(n-2)/n} \\ &\leq S_\infty^2 \int_{B(x_0,2R)} |\nabla w|^2 \, \mathrm{d}x. \end{aligned}$$

On the other hand, from Theorem 50 we have

$$\int_{B(x_0,2R)} |\nabla w|^2 \, \mathrm{d}x \leq \frac{4k}{\nu} \mu(B(x_0, 2R)).$$

Therefore,

$$k \leq \frac{C(n)}{\nu} (\lambda R)^{2-n} \mu(B(x_0, 2R)).$$

This implies that

$$\sup_{B(x_0,R)} u \leq \left(\frac{C(n) C_H 2^{M(n-2)}}{\nu}\right) R^{2-n} \mu(B(x_0, 2R)). \tag{46}$$

Case 2. Otherwise, there is a natural number $m \in [1, M]$ such that

$$\sup_{B(x_0,2^{-m}R)} u > \left(\frac{2C_U - 1}{2C_U}\right) \sup_{B(x_0,2^{1-m}R)} u.$$

Then we have

$$\left(C_U - \frac{1}{2}\right) \sup_{B(x_0,2^{1-m}R)} u \leq C_U \sup_{B(x_0,2^{-m}R)} u.$$

Let us consider the function $\sup_{B(x_0,2^{1-m}R)} u - u$. Then, we have

$$\begin{aligned} C_U \inf_{B(x_0,2^{-m}R)} \left(\sup_{B(x_0,2^{1-m}R)} u - u \right) &= C_U \left(\sup_{B(x_0,2^{1-m}R)} u - \sup_{B(x_0,2^{-m}R)} u \right) \\ &\leq \frac{1}{2} \sup_{B(x_0,2^{1-m}R)} u. \end{aligned}$$

Thus, applying Corollary 56 for $\sup_{B(x_0,2^{1-m}R)} u - u$, we get

$$\sup_{B(x_0,2^{1-m}R)} u - u(x_0) \leq \frac{1}{2} \sup_{B(x_0,2^{1-m}R)} u + \frac{C_U}{\nu} \mathbf{I}_2^\mu(x_0, 2^{1-m}R).$$

Therefore,

$$\sup_{B(x_0,2^{1-m}R)} u \leq 2u(x_0) + 2\frac{C_U}{\nu} \mathbf{I}_2^\mu(x_0, 2^{1-m}R). \tag{47}$$

Thus, combining (46) and (47), for any case, we have

$$\sup_{B(x_0,2^{-M}R)} u \leq 2u(x_0) + \frac{C}{\nu} \left(\mathbf{I}_2^\mu(x_0, 2R) + R^{2-n} \mu(B(x_0, 2R)) \right).$$

We arrived at the desired inequality. □

Next, we give a necessary condition of for the solvability of the Dirichlet problem.

DEFINITION 60. Let Ω be a open set, and let E be a closed subset of Ω. We say that E is thin at x_0 with respect to $\mathcal{L}$ if x_0 is not a limit point of E or there is a quasicontinuous supersolution to $\mathcal{L}u = 0$ in a neighborhood of x_0 such that

$$\liminf_{\substack{x \to x_0 \\ x \in E}} u > u(x_0) \left(= \lim_{R \to 0} \fint_{B(x_0,R)} u \,\mathrm{d}x \right).$$

THEOREM 61. *Let E be a closed subset of Ω, and let x_0 be a limit point of E. Suppose that*

$$\int_0^{R_0} s^{2-n} \operatorname{cap}(E \cap \overline{B(x_0,s)}, B(x_0,2s)) \frac{\mathrm{d}s}{s} < \infty \tag{48}$$

for some $R_0 > 0$. Then E is thin at x_0 with respect to $\mathcal{L}$.

PROOF. Let $0 < R < \min\{R_0/2, \operatorname{dist}(x_0, \partial\Omega)\}$ be a constant to be chosen later. Let

$$u = \mathfrak{R}(E \cap \overline{B(x_0,R)}, B(x_0,2R)).$$

From Theorem 50, we have

$$\begin{aligned} \fint_{B(x_0,R)} u \,\mathrm{d}x &\le C(n) R^{2-n} \|u\|_{L^{n/(n-2),\infty}(B(x,R))} \\ &\le \frac{C(n)}{\nu} R^{2-n} \mu[u](B(x_0,2R)) \\ &= \frac{C(n)}{\nu} R^{2-n} \mu[u](\overline{B(x_0,R)}). \end{aligned}$$

Combining this inequality and Theorem 55, we get

$$\begin{aligned} u(x_0) &\le C_U \left(\fint_{B(x_0,R)} u \,\mathrm{d}x + \frac{1}{\nu} \mathbf{I}_2^{\mu[u]}(x_0,2R) \right) \\ &\le \frac{C_U(C(n)+1)}{\nu} \int_0^{2R} s^{2-n} \mu[u](B(x_0,s)) \frac{\mathrm{d}s}{s}. \end{aligned}$$

From Lemma 37, we have

$$\begin{aligned} \mu[u](\overline{B(x_0,s)}) &\le \mathcal{B}^*(C_E)^{1/2} \operatorname{cap}(E \cap \overline{B(x_0,s)}, B(x_0,2R)) \\ &\le \mathcal{B}^*(C_E)^{1/2} \operatorname{cap}(E \cap \overline{B(x_0,s)}, B(x_0,2s)). \end{aligned}$$

Therefore,

$$\begin{aligned} &\frac{C}{\nu} \int_0^{2R} s^{2-n} \mu[u](B(x_0,s)) \frac{\mathrm{d}s}{s} \\ &\le C(n, \frac{\mathcal{B}}{\nu}) \int_0^{2R} s^{2-n} \operatorname{cap}(E \cap \overline{B(x_0,s)}, B(x_0,2s)) \frac{\mathrm{d}s}{s}. \end{aligned}$$

From assumption, we can take the right-hand side less than $1/2$ by choosing R sufficiently small. On the other hand, from definition, $u = 1$ q.e. on E. Therefore,

$$\liminf_{\substack{x \to x_0 \\ x \in E}} u = 1 > \frac{1}{2} \ge \lim_{R \to 0} \fint_{B(x_0,R)} u \,\mathrm{d}x.$$

Since u is a supersolution to $\mathcal{L}u = 0$ in $B(x_0,2R)$, E is thin at x_0 with respect to $\mathcal{L}$. □

EXAMPLE 62. Let us recall Lebesgue's spine:

$$D = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3;\ |x| < 1 \quad \text{and} \quad u(x) < 2\},$$

where

$$u(x) = \int_0^1 \frac{s\,\mathrm{d}s}{\sqrt{(x_1 - s)^2 + x_2^2 + x_3^2}}.$$

Then, $\complement D$ is thin at $x = 0$ with respect to $-\triangle$, and (48) holds at $x_0 = 0$ (see [**34**, pp.144-145] and Corollary 77). Thus, $\complement D$ is thin at $x = 0$ with respect to $\mathcal{L}$.

4. Divergence-free drifts

In this section, we give further remarks for equations with divergence-free drifts. Since we do not need the perturbation arguments, we can obtain the optimal constants for some estimates (see, (49), (50), (52) and the first term in (54)).

THEOREM 63. *Assume that* $\operatorname{div}\mathbf{b} = 0$. *Let* u *be a supersolution to* $\mathcal{L}u = 0$ *in* Ω, *and let* $D \Subset \Omega$. *Then the Poisson modification* $v = P(u, D)$ *of* u *in* D *satisfies the charge conservation*

$$\mu[v](\overline{D}) = \mu[u](\overline{D}). \tag{49}$$

Conversely, assume that

$$A(x) \in (C^1(\Omega))^{n\times n}, \quad \mathbf{b} \in (L^n(\Omega))^n$$

and (49) *holds for any Poisson modification. Then* $\operatorname{div}\mathbf{b} = 0$ *in the sense of distributions.*

PROOF. Choose a function $\eta \in C_c^\infty(\Omega)$ such that $\eta \equiv 1$ on $\overline{D}$ and $0 \le \eta \le 1$. Since $u = v$ on $\operatorname{supp}\nabla\eta$, we have

$$\begin{aligned}\int_\Omega \eta\,\mathrm{d}\mu[u] &= \int_\Omega A\nabla u\cdot\nabla\eta - (\mathbf{b}\cdot\nabla\eta)u\,\mathrm{d}x\\ &= \int_\Omega A\nabla v\cdot\nabla\eta - (\mathbf{b}\cdot\nabla\eta)v\,\mathrm{d}x = \int_\Omega \eta\,\mathrm{d}\mu[v].\end{aligned}$$

Hence (49) holds We show the converse. Since $\|\mathbf{b}\|_{L^n(\Omega')} \to 0$ as $|\Omega'| \to 0$, replacing Ω by a sufficiently small subdomain Ω', we can assume that $|||\mathbf{b}|||_\Omega$ is sufficiently small. Thus, without loss of generality, we may assume that $\langle \mathcal{L}u, v\rangle$ is coercive. Fix any $\varphi \in C_c^\infty(\Omega)$ and take an open set $D \Subset \Omega$ such that $\operatorname{supp}\varphi \subset D$. Then, since

$$f = \mathcal{L}\varphi = -\operatorname{div}(A\nabla\varphi) + \mathbf{b}\cdot\nabla\varphi \in L^{2n/(n+2)}(D),$$

we can take functions $w_p, w_m \in H_0^1(D)$ such that

$$\mathcal{L}w_p = f_+ \quad \text{and} \quad \mathcal{L}w_m = f_- \quad \text{in } D.$$

Then $\varphi = w_p - w_m$. Let $u \in H_0^1(\Omega)$ be the solution to the Dirichlet problem $\mathcal{L}u = f_+$ in Ω and let $v = P(u, D)$. Choose a sequence of smooth functions $\{\eta_j\}_{j=1}^\infty$ such that each of which satisfies $\eta_j \equiv 1$ on $\overline{D}$, $0 \le \eta_j \le 1$ and $\eta_j \downarrow \mathbf{1}_{\overline{D}}$ as $j \to \infty$.

Then, for each j,

$$\begin{aligned}&\int_\Omega \eta_j \,\mathrm{d}\mu[u] - \int_\Omega \eta_j \,\mathrm{d}\mu[v]\\ &= \int_\Omega A\nabla u\cdot\nabla\eta_j - (\mathbf{b}\cdot\nabla\eta_j)u\,\mathrm{d}x - \int_\Omega A\nabla v\cdot\nabla\eta_j - (\mathbf{b}\cdot\nabla\eta_j)v\,\mathrm{d}x\\ &+ \int_\Omega \mathbf{b}\cdot\nabla((u-v)\eta_j)\,\mathrm{d}x\\ &= \int_\Omega \mathbf{b}\cdot\nabla(u-v)\,\mathrm{d}x.\end{aligned}$$

Therefore, from assumption,

$$0 = \lim_{j\to\infty}\left(\int_\Omega \eta_j\,\mathrm{d}\mu[u] - \int_\Omega \eta_j\,\mathrm{d}\mu[v]\right) = \int_\Omega \mathbf{b}\cdot\nabla(u-v)\,\mathrm{d}x = \int_\Omega \mathbf{b}\cdot\nabla w_p\,\mathrm{d}x.$$

Applying the same argument for w_m, we can show that

$$\int_\Omega \mathbf{b}\cdot\nabla\varphi\,\mathrm{d}x = \int_\Omega \mathbf{b}\cdot\nabla w_p\,\mathrm{d}x - \int_\Omega \mathbf{b}\cdot\nabla w_m\,\mathrm{d}x = 0-0=0.$$

Therefore, $\operatorname{div}\mathbf{b}=0$ in the sense of distributions. □

THEOREM 64. *Assume that* $\operatorname{div}\mathbf{b} = 0$. *Let* μ *be a finite Radon measure in* $H^{-1}(\Omega)$, *and let* $u \in H_0^1(\Omega)$ *be a weak solution to* $\mathcal{L}u=\mu$ *in* Ω. *Then*

$$\|u\|_{L^{n/(n-2),\infty}(\Omega)} \le \frac{S_\infty^2}{\nu}|\mu|(\Omega), \tag{50}$$

$$\|\nabla u\|_{L^{n/(n-1),\infty}(\Omega)} \le \frac{S_\infty}{\nu}|\mu|(\Omega) \tag{51}$$

and

$$\|\nabla T_k(u)\|_{L^2(\Omega)}^2 \le \frac{k}{\nu}|\mu|(\Omega) \tag{52}$$

for all $k>0$. *Here,* S_∞ *is the constant as in Lemma 14.*

REMARK 65. The constant S_∞^2/ν in (50) is sharp. We note that the sharpness of (50) has been mentioned in [**10**] in the case of $\mathcal{L} = -\triangle$. For $0<r<R$, consider the superharmonic functions

$$u_{r,R}(x) = \min\{|x|^{2-n}, r^{2-n}\} - R^{2-n} \in H_0^1(B(0,R)).$$

Then, its Riesz measure $\mu_{r,R} = \mu[u_{r,R}]$ satisfies

$$\mu_{r,R}(B(0,R)) = (n-2)n|B(0,1)|.$$

Since

$$\|u_{r,R}\|_{L^{n/(n-2),\infty}(\mathbb{R}^n)} = \sup_{r^{2-n}-R^{2-n}>k>0} \frac{k}{(k+R^{2-n})}|B(0,1)|^{(n-2)/n},$$

the functions $u_{r,R}$ are archive the sharp constant when $R\to\infty$ or $r\to 0$. Let us consider an example of equations with nontrivial drift. For smooth function $\phi:\mathbb{R}\to\mathbb{R}$, we take

$$\mathbf{b}_\phi(x) = \left(\frac{\partial\phi(|x|)}{\partial x_2}, -\frac{\partial\phi(|x|)}{\partial x_1}, 0,\ldots,0\right)^T.$$

By definition, $\operatorname{div} \mathbf{b}_\phi = 0$. Moreover, since

$$\mathbf{b}_\phi \cdot \nabla u_{r,R} = 0.$$

we have

$$-\triangle u_{r,R} + \mathbf{b}_\phi \cdot \nabla u_{r,R} = -\triangle u_{r,R} = \mu_{r,R}.$$

Remark 66. By using a parabolic equation, we can give another proof of (50). Let us consider the parabolic problem

$$\partial_t u + \mathbf{b} \cdot \nabla u - \operatorname{div}(A \nabla u) = 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

Then, from a method in [**9**], we have

$$\|u(\cdot, T)\|_{L^\infty(\Omega)} \le \frac{1}{(4\pi T)^{n/2}} \|u(\cdot, 0)\|_{L^1(\Omega)}. \tag{53}$$

Moreover, testing the equation by $H_k(u)$ and taking the limit $k \to 0$, we can show the maximum principle

$$\|u(\cdot, T)\|_{L^1(\Omega)} \le \|u(\cdot, 0)\|_{L^1(\Omega)}.$$

Since $U(x) = \int_0^\infty u(x, T)\, \mathrm{d}T$ satisfies the equation $\mathcal{L}U = u_0$, by using an interpolation argument (see e.g [**24**]), we can reach the desired estimate.

By using (50), (51) and a duality argument, we can give another proof of the estimate in [**5**]; see also [**36**].

Corollary 67. *Assume that* $\operatorname{div} \mathbf{b} = 0$. *Let* $f \in L^{n/2,1}(\Omega)$ *and* $F \in (L^{n,1}(\Omega))^n$. *Let* $u \in H_0^1(\Omega)$ *be a weak solution to*

$$\mathcal{L}u = f + \operatorname{div} F \quad \text{in } \Omega.$$

Then u *is bounded. Moreover, we have the estimate*

$$\|u\|_{L^\infty(\Omega)} \le \frac{S_\infty^2}{\nu} \|f\|_{L^{n/2,1}(\Omega)} + \frac{S_\infty}{\nu} \|F\|_{L^{n,1}(\Omega)}. \tag{54}$$

Proof. Let $\mathcal{L}^*$ be the formal adjoint operator of $\mathcal{L}$:

$$\mathcal{L}^* v = -\operatorname{div}(A^T \nabla v + \mathbf{b} v) = -\operatorname{div}(A^T \nabla v) - \mathbf{b} \cdot \nabla v.$$

Here, A^T is the transpose of A. We note that $\mathcal{L}^*$ satisfies the assumptions in Lemma 64. Fix any function $g \in H^{-1}(\Omega) \cap L^1(\Omega)$, $g \ge 0$. Take $v \in H_0^1(\Omega)$ such that

$$\mathcal{L}^* v = g \quad \text{in } \Omega.$$

By the definition of v

$$\int_\Omega ug\, \mathrm{d}x = \langle u, \mathcal{L}^* v\rangle_\Omega = \langle \mathcal{L}u, v\rangle_\Omega = \int_\Omega fv - F \cdot \nabla v\, \mathrm{d}x.$$

Therefore, by using (50) and (51), we get

$$\begin{aligned}\left|\int_\Omega ug\ \mathrm{d}x\right| &\le \|f\|_{L^{n/2,1}(\Omega)} \|v\|_{L^{n/(n-2),\infty}(\Omega)} + \|F\|_{L^{n,1}(\Omega)} \|\nabla v\|_{L^{n/(n-1),\infty}(\Omega)} \\ &\le \left(\frac{S_\infty^2}{\nu} \|f\|_{L^{n/2,1}(\Omega)} + \frac{S_\infty}{\nu} \|F\|_{L^{n,1}(\Omega)}\right) \|g\|_{L^1(\Omega)}.\end{aligned}$$

This implies that

$$\operatorname*{ess\,sup}_\Omega u \le \frac{S_\infty^2}{\nu} \|f\|_{L^{n/2,1}(\Omega)} + \frac{S_\infty}{\nu} \|F\|_{L^{n,1}(\Omega)}.$$

According to the same argument, we can estimate $\operatorname{ess\,inf}_\Omega u$. □

Using Lemma 16 we get the following estimates:

COROLLARY 68. *Assume that* $\operatorname{div}\mathbf{b} = 0$. *Suppose that*

$$\frac{1}{q} = \frac{1}{p} - \frac{2}{n}, \quad p \in \left(1,\ \frac{n}{2}\right)$$

and $0 < r \leq \infty$. *Let* $\mu \in H^{-1}(\Omega) \cap L^{p,r}(\Omega)$ *and let* $u \in H_0^1(\Omega)$ *be a weak solution to* $\mathcal{L}u = \mu$ *in* Ω. *Then*

$$\|u\|_{L^{q,r}(\Omega)} \leq \frac{C(n,p,r)}{\nu}\|\mu\|_{L^{p,r}(\Omega)}.$$

CHAPTER 5

Potential lower bounds

In the previous chapter, we established potential upper bounds of solutions. Next, we shall derive a potential lower bound of non-negative supersolutions. If u is a non-negative superharmonic functions in $B(x_0, 2R) = \Omega$. Then, from the Riesz decomposition theorem, we have the lower bound

$$u(x) = \int_\Omega G_\Omega(x, y)\,\mathrm{d}\mu(y) + h(x) \geq \int_\Omega G_\Omega(x, y)\,\mathrm{d}\mu(y).$$

Recall the pointwise lower estimate of Green's functions

$$G_\Omega(x, y) \geq \frac{1}{C}|x - y|^{2-n} \quad \forall x \in B(y, \operatorname{dist}(y, \partial\Omega)/2).$$

Combining the two lower bounds, we arrive at

$$u(x_0) \geq \frac{1}{C}\int_{B(x_0,R)} \frac{\mathrm{d}\mu(x)}{|x_0 - x|^{n-2}} \geq \frac{1}{C}\mathbf{I}_2^\mu(x_0, R). \tag{55}$$

From results in [**53**] and [**28**], existence of the Green function and the estimate (55) also holds for equations of type (3) except for the difference of constant. Unfortunately, even under condition (38), the lower bound does not holds in general. In this chapter, considering about the relation between the lower bounds of solutions and the measure data, we establish the potential lower bound (55) under the assumptions (21), (38) and the additional assumption (56). More precisely, we derive Corollary 72. This two-sided estimate yields a necessary and sufficient condition of boundary regularity.

1. The lower potential estimate

For further properties of supersolutions, we assume that

$$\|(\operatorname{div}\mathbf{b})_-\|_{L^{n/2,1}(\Omega)} \leq \frac{\nu}{8S_\infty^2(C_H)^\alpha} \tag{56}$$

in addition to (21) and (38), where C_H is the constant as in Theorem 47 and α is a geometric constant to be chosen later (see (62)). Since C_H depends on $\mathcal{B}/\nu$, this assumption depends also on $\mathcal{B}/\nu$. Under these conditions, we have the following:

Lemma 69. *Let u be a lower semicontinuous supersolution to $\mathcal{L}u = \mu \geq 0$ in Ω. Then there exists a constant C depending only on n and $\mathcal{B}/\nu$ such that*

$$R^{2-n}\mu(\overline{B(x_0, R)}) \leq \mathcal{B}^* C\left(\inf_{B(x_0,R)} u - \inf_{B(x_0,2R)} u\right),$$

whenever $B(x_0, 2R) \subset \Omega$.

Proof. Take $\eta \in C_c^\infty(B(x_0, 5R/4))$ such that $0 \le \eta \le 1$ and $\eta = 1$ on $\overline{B(x_0, R)}$. Let $w \in H_0^1(B(x_0, 2R))$ be the solution to the Dirichlet problem

$$\mathcal{L}w = \eta\mu \quad \text{in } B(x_0, 2R).$$

By the minimum principle, $w \ge 0$ in $B(x_0, 2R)$. Let

$$M := \operatorname*{ess\,sup}_{\partial B(x_0,3R/2)} w, \quad m := \operatorname*{ess\,inf}_{\partial B(x_0,3R/2)} w.$$

Since $\mathcal{L}w = 0$ in $B(x_0, 2R) \setminus \overline{B(x_0, 5R/4)}$, these quantities are well-defined. From the maximum principle,

$$0 \le w \le M \quad \text{in } B(x_0, 2R) \setminus \overline{B(x_0, 3R/2)}.$$

Notice that according to the Harnack inequality, we have

$$M \le (C_H)^\alpha m, \tag{57}$$

where α is a constant depending only on n. Let us consider the function

$$\widetilde{w} := \min\{w, M\}.$$

From the minimum principle, $\widetilde{w} \ge m$ in $B(x_0, 3R/2)$. Choose $\phi \in C_c^\infty(B(x_0, 2R))$ such that

$$\phi \equiv 1 \quad \text{in } B(x_0, 3R/2), \quad |\nabla\phi| \le \frac{C}{R}.$$

Then, since $\widetilde{w}\phi^2 \ge m$ in $B(x_0, 3R/2) \supset \overline{B(x_0, 5R/4)}$, we have

$$\begin{aligned} &m(\eta\mu)(B(x_0, 2R)) \\ &\le \int_{B(x_0,2R)} (\widetilde{w}\phi^2)\, \mathrm{d}(\eta\mu) \\ &\le \int_{B(x_0,2R)} A\nabla w \cdot \nabla(\widetilde{w}\phi^2)\, \mathrm{d}x + \int_{B(x_0,2R)} (\mathbf{b}\cdot\nabla w)(\widetilde{w}\phi^2)\, \mathrm{d}x. \end{aligned} \tag{58}$$

Since $\operatorname{supp}\nabla\phi \subset B(x_0, 2R) \setminus \overline{B(x_0, 3R/2)}$, it follows that if $\nabla(\widetilde{w}\phi^2)(x) \ne 0$, then $\widetilde{w}(x) = w(x)$. Therefore, by using Young's inequality, we get

$$\begin{aligned} &\int_{B(x_0,2R)} A\nabla w \cdot \nabla(\widetilde{w}\phi^2)\, \mathrm{d}x \\ &\quad \le C\|A\|_{L^\infty(\Omega)} \left(\int_{B(x_0,2R)} |\nabla\widetilde{w}|^2\phi^2\, \mathrm{d}x + M^2 \int_{B(x_0,2R)} |\nabla\phi|^2\, \mathrm{d}x \right). \end{aligned} \tag{59}$$

Likewise, since

$$\begin{aligned} &\int_{B(x_0,2R)} (\mathbf{b}\cdot\nabla w)\left(\widetilde{w}\phi^2\right)\, \mathrm{d}x \\ &= -\int_{B(x_0,2R)} \left(\mathbf{b}\cdot\nabla(\widetilde{w}\phi^2)\right) w\, \mathrm{d}x - \int_{B(x_0,2R)} (\operatorname{div}\mathbf{b}) w\widetilde{w}\phi^2\, \mathrm{d}x \\ &= -\int_{B(x_0,2R)} (\mathbf{b}\cdot\nabla\widetilde{w})\widetilde{w}\phi^2\, \mathrm{d}x - 2\int_{B(x_0,2R)} (\mathbf{b}\cdot\nabla\phi)\widetilde{w}^2\phi\, \mathrm{d}x \\ &\quad - \int_{B(x_0,2R)} \operatorname{div}\mathbf{b}\, w\widetilde{w}\phi^2\, \mathrm{d}x, \end{aligned}$$

using (21), we get

$$\begin{aligned}(60)\qquad &\int_{B(x_0,2R)}(\mathbf{b}\cdot\nabla w)(\widetilde{w}\phi^2)\,\mathrm{d}x\\ &\le C|||\mathbf{b}|||_\Omega\left(\int_{B(x_0,2R)}|\nabla\widetilde{w}|^2\phi^2\,\mathrm{d}x+M^2\int_{B(x_0,2R)}|\nabla\phi|^2\,\mathrm{d}x\right)\\ &\quad+M\|(\operatorname{div}\mathbf{b})_-\|_{L^{n/2,1+}(\Omega)}\|w\|_{L^{n/(n-2),\infty}(B(x_0,2R))}.\end{aligned}$$

The second term of the right-hand side can be estimated by (39):

$$\|w\|_{L^{n/(n-2),\infty}(B(x_0,2R))}\le\frac{4S_\infty^2}{\nu}(\eta\mu)(B(x_0,2R)).$$

Since $(M-w)_+=M-\widetilde{w}$ is a subsolution to $\mathcal{L}u=0$ in $B(x_0,2R)$, Lemma 36 yields

$$(61)\qquad \int_{B(x_0,2R)}|\nabla\widetilde{w}|^2\phi^2\,\mathrm{d}x\le C_EM^2\int_{B(x_0,2R)}|\nabla\phi|^2\,\mathrm{d}x.$$

Combining these inequalities (57)-(61), we get

$$\begin{aligned}m(\eta\mu)(B(x_0,2R))&\le\mathcal{B}^*\left(CC_E(C_H)^{2\alpha}\right)m^2R^{n-2}\\ &\quad+\left((C_H)^\alpha\frac{4S_\infty^2}{\nu}\|(\operatorname{div}\mathbf{b})_-\|_{L^{n/2,1}(\Omega)}\right)m(\eta\mu)(B(x_0,2R)).\end{aligned}$$

Therefore, if $(\operatorname{div}\mathbf{b})_-$ is small enough such that

$$(62)\qquad \|(\operatorname{div}\mathbf{b})_-\|_{L^{n/2,1}(\Omega)}\le\frac{\nu}{8S_\infty^2(C_H)^\alpha},$$

then we have

$$m(\eta\mu)(B(x_0,2R))\le\mathcal{B}^*Cm^2R^{n-2},$$

where C is a constant depending only on n and $\mathcal{B}/\nu$. If $m=0$, then, from the strong minimum principle, $(\eta\mu)\equiv 0$. Thus,

$$R^{2-n}(\eta\mu)(B(x_0,2R))\le\mathcal{B}Cm.$$

On the other hand, from the comparison principle, we have

$$w\le u-\inf_{B(x_0,2R)}u\quad\text{in } B(x_0,2R).$$

Therefore,

$$m\le\inf_{B(x_0,3R/2)}(u-\inf_{B(x_0,2R)}u)\le\inf_{B(x_0,R)}(u-\inf_{B(x_0,2R)}u).$$

Consequently,

$$R^{2-n}(\eta\mu)(B(x_0,2R))\le\mathcal{B}^*C\left(\inf_{B(x_0,R)}u-\inf_{B(x_0,2R)}u\right).$$

From the definition of η, we obtain

$$R^{2-n}\mu(\overline{B(x_0,R)})\le\mathcal{B}^*C\left(\inf_{B(x_0,R)}u-\inf_{B(x_0,2R)}u\right).$$

We arrived at the desired estimate. □

REMARK 70. It is necessary to add the condition (56) to lead Lemma 69. Consider the case of $A = I$, $\mathbf{b} = -\epsilon x/|x|^2$, $\Omega = B(0,2)$. For sufficiently small positive $\epsilon > 0$, the conditions (21) and (38) hold, but (56) does not hold. For $0 < r < 1$, let us consider the non-negative functions

$$u_r(x) = \min\{|x|^{2-n-\epsilon}, r^{2-n-\epsilon}\} - 2^{2-n-\epsilon}.$$

Then, u_r are supersolutions to $\mathcal{L}u = 0$ in $B(0,2)$, moreover, their Riesz measures $\mu[u_r]$ satisfy

$$\mu[u_r](B(0,1)) = (n-2+\epsilon)n|B(0,1)|r^{-\epsilon}.$$

In particular, $\mu[u_r](B(0,1)) \to \infty$ as $r \to 0$. On the other hand, we have

$$\inf_{B(0,1)} u_r - \inf_{B(0,2)} u_r = 1 - 2^{2-n-\epsilon}.$$

Therefore, under (21) and (38), this lower bound does not hold in general.

Iterating Lemma 69, we arrive at the following pointwise lower bound estimate of non-negative supersolutions.

THEOREM 71. *Let u be a lower semicontinuous non-negative supersolution to $\mathcal{L}u = \mu \geq 0$ in Ω. Then there exists a constant C_L depending only on n and $\mathcal{B}/\nu$ such that*

$$u(x_0) \geq \frac{1}{\mathcal{B}^* C_L} \mathbf{I}_2^{\mu}(x_0, R),$$

whenever $B(x_0, 2R) \subset \Omega$.

PROOF. For $k = -1, 0, 1, \ldots$, take $R_k = 2^{-k}R$ and $B_k = B(x_0, R_k)$. Then, from Lemma 69, for all $k \geq 0$, we obtain

$$R_k^{2-n}\mu(B_k) \leq \mathcal{B}^* C \left(\inf_{B_k} u - \inf_{B_{k-1}} u \right).$$

By summing over all $k = 0, 1, \ldots$, we get

$$u(x_0) = \lim_{k\to\infty} \operatorname*{ess\,inf}_{B_k} u \geq \frac{1}{\mathcal{B}^* C} \sum_{k=0}^{\infty} R_k^{2-n}\mu(B_k).$$

We arrived at the desired estimate. □

Combining Corollary 56 and Corollary 71, we obtain the following two-sided bound:

COROLLARY 72. *Let u be a non-negative solution to $\mathcal{L}u = \mu \geq 0$ in Ω. Then there exists a constant C depending only on n and $\mathcal{B}/\nu$ such that*

$$\frac{1}{\mathcal{B}^* C} \mathbf{I}_2^{\mu}(x_0, R) \leq u(x_0) \leq C \left(\operatorname*{ess\,inf}_{B(x_0,R)} u + \frac{1}{\nu} \mathbf{I}_2^{\mu}(x_0, 2R) \right),$$

whenever x_0 is a Lebesgue point of u and $B(x_0, 2R) \subset \Omega$.

2. Applications of the potential lower bound

From Theorem 71, we obtain estimates of the Riesz measure of supersolutions to the equation $\mathcal{L}u = 0$ in Ω:

Corollary 73. *Let u be a lower semicontinuous supersolution to $\mathcal{L}u = 0$ in Ω, and let μ be its Riesz measure.*

(1) *If u is finite at $x_0 \in \Omega$, then, there exists a constant $R > 0$ such that*
$$\mathbf{I}_2^\mu(x_0, R) < \infty.$$
Moreover, if u is bounded in $D \Subset \Omega$, then, there exists a constant $R > 0$ such that
$$\sup_{x_0 \in D} \mathbf{I}_2^\mu(x_0, R) < \infty.$$

(2) *If u is continuous at $x_0 \in \Omega$, then*
$$\lim_{R \to 0} \sup_{x \in B(x_0, R)} \mathbf{I}_2^\mu(x_0, R) = 0.$$
Moreover, if u is continuous in $D \Subset \Omega$, then
$$\lim_{R \to 0} \sup_{x_0 \in D} \mathbf{I}_2^\mu(x_0, R) = 0.$$

(3) *Assume that there exist constants K and $\beta > 0$ such that*
$$|u(x_0) - u(x)| \le K|x_0 - x|^\beta$$
for any $x \in B(x_0, 2R) \subset \Omega$. Then there exists a constant C such that for any $0 < r \le R$,
$$\mu(B(x_0, r)) \le CKr^{n-2+\beta}.$$

Proof. (1) Take a small ball such that $B(x_0, 2R) \subset \Omega$. Since u is lower semicontinuous, $\inf_{B(x_0,2R)} u > -\infty$. Applying Theorem 71 for $u - \inf_{B(x_0,2R)} u$, we get the first assertion. (2) Since u is continuous at x_0, for any $\epsilon > 0$, we can choose a positive number $r > 0$ such that
$$|u(x) - u(x_0)| \le \frac{\epsilon}{2\mathcal{B}^* C_L}$$
for any $x \in B(x_0, 3r)$, where $\mathcal{B}^*$ and C_L is the same constant as in Theorem 71. Then, for any $x \in B(x_0, r)$, we have
$$\begin{aligned}\mathbf{I}_2^\mu(x, r) &\le \mathcal{B}^* C_L(u(x) - \inf_{B(x,2r)} u) \\ &\le \mathcal{B}^* C_L(|u(x) - u(x_0)| + |u(x_0) - \inf_{B(x,2r)} u|) \\ &\le \mathcal{B}^* C_L 2 \frac{\epsilon}{2\mathcal{B} C_L} = \epsilon.\end{aligned}$$
(3) Fix $0 < r \le R$. According Theorem 71, we have
$$\begin{aligned}r^{2-n}\mu(B(x_0, r)) &\le C(n) \int_0^r \rho^{2-n}\mu(B(x_0, \rho)) \frac{\mathrm{d}\rho}{\rho} \\ &\le C(n)\mathcal{B}^* C_L \left(u(x_0) - \inf_{B(x_0,2r)} u \right) \\ &\le C(n)\mathcal{B}^* C_L K r^\beta.\end{aligned}$$
This completes the proof. □

Next, we give a growth order estimate of non-negative subsolutions to $\mathcal{L}u = 0$ in Ω. This is an analog of Nevanlinna's theorem in higher dimensional Euclidean spaces (see [**32**]).

COROLLARY 74. *Let u be a non-negative solution to $\mathcal{L}u = \mu \leq 0$ in $B(x_0, R)$. Assume that $u(x_0) = \lim_{R\to 0} \fint_{B(x_0,R)} u \, \mathrm{d}x = 0$. Then there exist constants C and $0 < \lambda < 1$ depending only on n and $\mathcal{B}/\nu$ such that*

$$\frac{1}{\mathcal{B}^* C} \mathbf{I}_2^{\mu}(x_0, \lambda R/2) \leq \operatorname*{ess\,sup}_{B(x_0,\lambda R)} u \leq \frac{C}{\nu} \mathbf{I}_2^{\mu}(x_0, R).$$

PROOF. The latter inequality was proved in Corollary 59. Applying Theorem 71 for $\sup_{B(x_0,\lambda R)} u - u$ in $B(x_0, \lambda R)$, we get

$$\frac{1}{\mathcal{B}^* C} \mathbf{I}_2^{\mu}(x_0, \lambda R/2) \leq \operatorname*{ess\,sup}_{B(x_0,\lambda R)} (u - u(x_0)) = \operatorname*{ess\,sup}_{B(x_0,\lambda R)} u.$$

This completes the proof. □

EXAMPLE 75. To understand the above theorem, we recall examples of entire non-negative subharmonic functions.

(1) Let $u = \max\{1 - |x|^{2-n}, 0\}$ and $\mu = \triangle u$. Then, $\mathbf{I}_2^{\mu}(0, R) = 0$ for $0 < R \leq 1$ and

$$\mathbf{I}_2^{\mu}(0, R) = \int_1^R s^{2-n}(n-2)n|B(0,1)| \frac{\mathrm{d}s}{s} = C(n)(1 - R^{2-n})$$

for $R > 1$.

(2) If $u = \max\{x_1, 0\}$ and $\mu = \triangle u$, then,

$$\mathbf{I}_2^{\mu}(0, R) = \int_0^R s^{2-n} n|B(0,1)| s^{n-1} \frac{\mathrm{d}s}{s} = C(n)R.$$

(3) If $u = |x|^2$ and $\mu = \triangle u = 2n$, then,

$$\mathbf{I}_2^{\mu}(0, R) = \int_0^R s^{2-n} 2n s^n \frac{\mathrm{d}s}{s} = C(n)R^2.$$

3. Wiener's criterion

From Lemma 69, we can give a sufficient condition of boundary regularity of solutions. This method is due to Maz′ya [**57**]. See also [**33**].

LEMMA 76. *Assume that $D \Subset \Omega$ and $x_0 \in \partial D$. Let $0 < R < \operatorname{dist}(x_0, \partial\Omega)/2$ and*

$$u = 1 - \mathfrak{R}(\complement D \cap \overline{B(x_0, R)}, (B(x_0, 2R)).$$

Then for any $0 < \rho \leq R$,

$$u \leq \exp\left(-\frac{1}{C} \int_{\rho}^{R} s^{2-n} \operatorname{cap}(\complement D \cap \overline{B(x_0, s)}, B(x_0, 2s)) \frac{\mathrm{d}s}{s}\right)$$

in $B(x_0, \rho)$, where $C > 0$ is a constant depending only on n and $\mathcal{B}/\nu$.

PROOF. For $m = 0, 1, 2, \ldots$, we take $R_m = 2^{-m}R$ and $B_m = B(x_0, R_m)$. Let

$$v_m = \mathfrak{R}(\complement D \cap \overline{B_m}, 2B_m).$$

Then, from Lemma 69, we have

$$R_m^{2-n}\mu[v_m](\overline{B_m}) \leq \mathcal{B}^* C \inf_{B_m} v_m.$$

On the other hand, it follows from Theorem 50 that

$$\begin{aligned}\operatorname{cap}(\complement D \cap \overline{B_m}, 2B_m) &\leq \|\nabla T_1(v_m)\|^2_{L^2(2B_m)} \\ &\leq \frac{4}{\nu}\mu[v_m](2B_m) = \frac{4}{\nu}\mu[v_m](\overline{B_m}).\end{aligned}$$

Therefore,

$$R_m^{2-n}\operatorname{cap}(\complement D \cap \overline{B_m}, 2B_m) \leq C \inf_{B_m} v_m.$$

Let

$$a_m = R_m^{2-n}\operatorname{cap}(\complement D \cap \overline{B_m}, 2B_m).$$

Then, since $e^t \geq 1 + t$, we have

$$v_m \geq \frac{1}{C}a_m \geq 1 - \exp(-\frac{1}{C}a_m) \quad \text{in } B_m,$$

hence

$$1 - v_m \leq \exp(-\frac{1}{C}a_m) \quad \text{in } B_m.$$

Since $u = 1 - v_0$, we have

$$u \leq \exp(-\frac{1}{C}a_0) \quad \text{in } B_0.$$

This implies that

$$u \leq \exp(-\frac{1}{C}a_0) = \exp(-\frac{1}{C}a_0)(1 - v_1) \quad \text{on } \partial B_1.$$

On the other hand, from definition, $u = 0$ q.e. on $\complement D \cap B_0$. Therefore, combining two estimates and using the comparison principle, we get

$$u \leq \exp(-\frac{1}{C}a_0)(1 - v_1) \quad \text{in } D \cap B_1.$$

Hence,

$$u \leq \exp(-\frac{1}{C}a_0)(1 - v_1) \quad \text{in } B_1.$$

From the estimate for v_1,

$$u \leq \exp(-\frac{1}{C}a_0)\exp(-\frac{1}{C}a_1) \quad \text{in } B_1.$$

Choose an integer M such that $2^{-(M+1)}R \leq \rho \leq 2^{-M}R$. By induction, we get

$$u \leq \prod_{m=0}^{M}\exp(-\frac{1}{C}a_m) = \exp(-\frac{1}{C}\sum_{m=0}^{M}a_m) \quad \text{in } B_M \supset B(x_0, \rho).$$

By a simple calculation, we can show that

$$\sum_{m=0}^{M}a_m \geq \frac{1}{C(n)}\int_\rho^R s^{2-n}\operatorname{cap}(\complement D \cap \overline{B(x_0, s)}, B(x_0, 2s))\frac{\mathrm{d}s}{s}.$$

This completes the proof. □

THEOREM 77. *Suppose that $D \Subset \Omega$, $x_0 \in \partial D$ and $\theta \in H^1(D) \cap C(\overline{D})$. Let $u \in \theta + H^1_0(D)$ be the weak solution to $\mathcal{L}u = 0$ in D. Then, there exists a constant C depending only on n and $\mathcal{B}/\nu$ such that for any $0 < \rho \le R < \operatorname{dist}(x_0, \partial\Omega)/2$,*

$$\begin{aligned}\omega(\rho) \le \omega(2R) \exp(-\frac{1}{C}\int_\rho^R s^{2-n} \operatorname{cap}(\complement D \cap \overline{B(x_0,s)}, B(x_0,2s)) \frac{\mathrm{d}s}{s}) \\ + \omega_\theta(2R),\end{aligned}$$

where

$$\omega(R) = \operatorname*{osc}_{D\cap B(x_0,R)} u, \quad \omega_\theta(R) = \operatorname*{osc}_{\partial D \cap B(x_0,R)} \theta.$$

In particular, if

$$\int_0^R s^{2-n} \operatorname{cap}(\complement D \cap \overline{B(x_0,s)}, B(x_0,2s)) \frac{\mathrm{d}s}{s} = +\infty,$$

then u is continuous at x_0.

PROOF. Without loss of generality, we may assume that $\theta(x_0) = 0$. For fixed $R > 0$, we consider the function

$$\overline{U} = \sup_{B(x_0,2R)} u (1 - \mathfrak{R}(\complement D \cap \overline{B(x_0,R)}, B(x_0,2R))) + \max_{\partial D \cap \overline{B(x_0,2R)}} \theta.$$

Note that $\max_{\partial D\cap\overline{B(x_0,2R)}} \theta \ge 0$. Therefore, from the comparison principle

$$\overline{U} \ge \sup_{B(x_0,2R)} u \ge u \quad \text{on } D \cap \partial B(x_0,2R).$$

On the other hand,

$$\overline{U} \ge \max_{\partial D\cap\overline{B(x_0,2R)}} \theta \ge u \quad \text{on } \partial D \cap \overline{B(x_0,2R)}.$$

Hence,

$$\overline{U} \ge u \quad \text{on } \partial(D \cap B(x_0,2R)).$$

From the comparison principle,

$$\overline{U} \ge u \quad \text{in } D \cap B(x_0,2R).$$

From a similar argument, if

$$\underline{U} = \inf_{B(x_0,2R)} u (1 - \mathfrak{R}(\complement D \cap \overline{B(x_0,R)}, B(x_0,2R))) + \min_{\partial D \cap \overline{B(x_0,2R)}} \theta,$$

then

$$\underline{U} \le u \quad \text{in } D \cap B(x_0,2R).$$

Therefore,

$$\operatorname*{osc}_{B(x_0,\rho)} u \le \sup_{B(x_0,\rho)} \overline{U} - \inf_{B(x_0,\rho)} \underline{U}.$$

From Lemma 76, the assertion follows. □

COROLLARY 78. *Let $D \Subset \Omega$, and let $\theta \in H^1(D) \cap C(\overline{D})$. Let $u \in \theta + H^1_0(D)$ be the weak solution to $\mathcal{L}u = 0$ in D. Assume that D satisfies the following capacity density condition at $x_0 \in \partial D$: There exist positive constants $\alpha \in (0,1)$ and $R_0 > 0$ such that*

$$\frac{\operatorname{cap}(\complement D \cap \overline{B(x_0,R)}, B(x_0,2R))}{\operatorname{cap}(\overline{B(x_0,R)}, B(x_0,2R))} \ge \alpha \tag{63}$$

for all $0 < R \leq R_0$*. Then* u *is continuous at* x_0*. Moreover, there exist constants* C *and* $\beta \in (0,1)$ *depending only on* n*,* $\mathcal{B}/\nu$ *and* α *such that*

$$\omega(\rho) \leq C \left(\frac{\rho}{R}\right)^{\beta} \omega(2R) + \omega_\theta(2R)$$

for any $0 < \rho < R \leq \min\{R_0, \operatorname{dist}(x_0, \partial\Omega)/2\}$*.*

Remark 79. The volume density condition (33) yields the capacity density condition (63). Indeed, by taking

$$u = \mathfrak{R}(\overline{B(x_0,R)} \setminus D, B(x_0,2R); -\triangle),$$

it follows from Theorem 50 and Lemma 37 that

$$\begin{aligned} |\overline{B(x_0,R)} \setminus D|^{(n-2)/n} &\leq \|u\|_{L^{n/(n-2),\infty}(B(x_0,2R))} \\ &\leq S_\infty^2 \mu[u](B(x_0,2R)) \\ &\leq C(n) \operatorname{cap}(\complement D \cap \overline{B(x_0,R)}, B(x_0,2R)). \end{aligned}$$

The converse does not hold. Indeed,

$$D = B(0,1) \setminus \{(x,y,0);\ x \geq 0,\ y \in \mathbb{R}\} \subset \mathbb{R}^3$$

satisfies (63) at $0 \in \partial D$. However, (33) does not hold at 0.

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