Formulas on preview and delayed H^∞ control

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***Abstract*—A generalized H^∞ control problem, which covers preview and delayed control strategies, is discussed based on a state-space approach. By introducing a Hamiltonian matrix, which is associated with a delay-free generalized plant, the analytic solution to the corresponding operator Riccati equation is newly established. Based on the result obtained here, the H^∞ control problem is solved and, for typical control problems (e.g. H^∞ and LQ control for multiple input delay systems, H^∞ preview control), some interpretations are provided on the resulting control system.**

***Index Terms*—H^∞ control, infinite-dimensional system, input delay system, preview control, operator Riccati equation.**

I. INTRODUCTION

THE design method of H^∞ control law has been studied for infinite-dimensional systems [6], [29], [11], [25] and, especially for a class of time delay systems, explicit solutions are obtained based on various approaches [22], [26], [17], [18], [20], [12], [13], [14]. Recently, by applying the approaches for delay systems, the effect of preview action is further investigated in terms of the H^∞ performance [19], [27], [15], [16].

In the state-space approach for infinite-dimensional systems, the abstract system theory has been discussed for a class of systems (Pritchard-Salamon systems, e.g. [24], [2]) and, if the plant is in this class, the typical control problems such as H^∞ (LQ) control or the estimation problems are characterized via corresponding operator representations [8], [10], [29]. For general infinite-dimensional systems, it should be also noted that we will face with the difficulties at the first stage to check whether the plant is in the Pritchard-Salamon systems. For time delay systems, a function-space representation is established for general retarded delay systems, which involve distributed and point delays in the control, the state, and the output, and it is clarified that the representation falls into the class of Pritchard-Salamon systems [23]. These fundamental frameworks enable us to deal with a broad class of control problems with delayed/preview strategies beyond the apparent system representation and, further, have a potential to provide an insight on the underlying property of resulting systems. However, in the paradigm of H^∞ control synthesis, the advantage of the state-space abstract theory is not brought out because, even if we can employ finite-dimensional approximation, we are faced with a huge size of finite-dimensional calculation in the repetitive procedure. Furthermore the approximation of the solution does not clarify the solvability of problem.

In this paper, we focus on a generalized class of delay systems, which covers both multiple input delay and preview control strategies, and derive explicit formulas for H^∞ (LQ) control problem by clarifying the solvability and the analytic solution to the corresponding operator Riccati equation. The generalized system enables us to discuss the H^∞ preview control, H^∞ control with multiple input delays and, further, provides analytic representation of LQ control law, which feature has not been clarified for multiple input delay systems. The key point in this approach is that the analytic solution to the operator Riccati equation is newly established based on the stable eigenspace of Hamiltonian matrix. The Hamiltonian matrix is defined with a delay-free system and enables us to provide interpretations on the feature of resulting control systems. Furthermore, in highlight with the received method for finite-dimensional systems [4], [9], the proposed approach also characterizes the limitation of the H^∞ performance, which level is not attained via causal or uncausal control strategies.

At the first stage of attacking the problem, we derive a necessary and sufficient condition on the existence of stabilizing solution to an indefinite operator Riccati equation. The condition is completely characterized by nonsingularity of a matrix, which is defined with the system parameters, and, if exists, the analytic solution is constructed with integral operators. Then we investigate the additional condition for the positive semi-definiteness of the stabilizing solution and elaborate the design method of H^∞ control law. By employing the advantage of state-space approach, the feature of H^∞ (LQ) control law is clarified from the property of the closed-loop system.

The paper is organized as follows. In Section II, the H^∞ control problem is defined for the system with preview and delayed strategies. Then we prepare preliminaries for a class of infinite-dimensional systems (Pritchard-Salamon systems) [24] and describe the generalized plant on an appropriate function-space [23]. The description is further transformed to an auxiliary form, which yields a simple structure for the analytic solution to the corresponding operator Riccati equation. In Section III, by employing the auxiliary system description, the solvability of the operator Riccati equation is completely characterized based on a Hamiltonian matrix, which is associated with the delay-free generalized plant. The analytic solution to the equation is also established based on integral operators and, from the viewpoint of the original H^∞ control problem, the design method of control law is summarized. In Section IV, the typical control problems (H^∞ preview or delayed control problems) are further discussed and it is shown that the representation of the analytic solution is

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considerably simplified. Some remarks are also provided on the behavior of the resulting system. After describing all proofs in Section V, the preview and delayed H^∞ control problems are illustrated with numerical examples (Section VI).

Notation and terminology: Let X and Y be real Hilbert spaces with norms $\|\cdot\|_X$, $\|\cdot\|_Y$ and inner product $\langle\cdot,\cdot\rangle_X$, $\langle\cdot,\cdot\rangle_Y$, respectively. Let Z be dense in X and Z^* be the adjoint space. The adjoint pairing between $f \in Z$ and $g \in Z^*$ will be denoted by $\langle f, g\rangle_{Z,Z^*}$. The space of Lebesgue measurable functions $[a,b] \to \mathbb{R}^n$, which are square integrable, will be denoted by $L_2(a,b;\mathbb{R}^n)$. Let $\mathcal{L}(X,Y)$ denote the set of bounded linear operators $\Gamma : X \to Y$. The adjoint of $\Gamma \in \mathcal{L}(X,Y)$ will be denoted by $\Gamma^* \in \mathcal{L}(Y^*,X^*)$. When $X = Y$, we write $\mathcal{L}(X)$ instead of $\mathcal{L}(X,Y)$. A self-adjoint operator Γ will be written $\Gamma \geq 0$ if $\langle x, \Gamma x\rangle_X \geq 0$ for all $x \in X$ and $\Gamma > 0$ if $\langle x, \Gamma x\rangle_X > 0$, $x \neq 0$.

II. Formulation and Preliminaries

A. Problem Formulation

Define a full information control problem (FI-problem) by the generalized plant with delays in the control and the disturbance:

$$\Sigma: \begin{aligned} \dot{x}(t) &= Ax(t) + \sum_{i=0}^{d} D_i w_i(t-h_i) + \sum_{i=0}^{d} B_i u_i(t-h_i) \\ z(t) &= Fx(t) + F_0 u(t) \\ y(t) &= \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \\ w(t) &:= \begin{bmatrix} w_0(t) \\ w_1(t) \\ \vdots \\ w_d(t) \end{bmatrix} \in \mathbb{R}^l, \quad u(t) := \begin{bmatrix} u_0(t) \\ u_1(t) \\ \vdots \\ u_d(t) \end{bmatrix} \in \mathbb{R}^m, \\ w_i(t) &\in \mathbb{R}^{l_i},\ u_i(t) \in \mathbb{R}^{m_i} \quad (i = 0, 1, \cdots, d) \\ x(t) &\in \mathbb{R}^n,\ y(t) \in \mathbb{R}^{n+l},\ z(t) \in \mathbb{R}^p \end{aligned} \tag{1}$$

where x, w, u, z, y are the state, the disturbance, the control input, the regulated output, and the measurement of the system respectively. The matrices A, F, F_0, and $B := [\,B_0, B_1, B_2, \cdots, B_d\,]$, $D := [\,D_0, D_1, D_2, \cdots, D_d\,]$ are with appropriate dimensions and h_i $(i = 0, 1, \cdots, d)$ denote time delays in the increasing order: $0 := h_0 < h_1 < h_2 < \cdots < h_d =: L$. We make following assumptions for the system Σ:

A1) $(A,\, B)$ is stabilizable.

A2) $F_0^{\mathrm{T}}[\,F\, F_0\,] = [\,0\ I\,]$.

A3) $\operatorname{rank} \begin{bmatrix} A - j\omega I & B \\ F & F_0 \end{bmatrix} = n + m, \quad \forall \omega \in \mathbb{R}$.

The H^∞ control problem is to design a feedback control law, which causally depends on y, such that the resulting closed loop system satisfies the following conditions:

C1) The closed loop system is internally stable;

C2) the resulting closed loop system Σ_{zw} from the disturbance w to the regulated output z satisfies $\|\Sigma_{zw}\|_\infty < \gamma$ for a given constant $\gamma > 0$.

The generalized plant (1) describes a broad class of H^∞ control problems and covers preview and delayed control actions. The time delays in the disturbance equivalently describe previewable reference signals and those in the control define the H^∞ control problem for input delay systems. Typical control problems are illustrated by Example 1-4.

Example 1. H^∞ preview control: Define Σ with $B = [B_0, 0]$, $D = [D_0, D_1]$, $0 = h_0 < h_1 = L$ and describe the uncertainty and the previewable reference signal by $D_0 w_0(t)$, $D_1 w_1(t - L)$, respectively. Rewriting the reference signal by $w_1(t) = r_1(t + L)$, the problem is equivalently given as follows:

$$\Sigma^{\text{prev}}: \begin{aligned} \dot{x}(t) &= Ax(t) + D_0 w_0(t) + D_1 r_1(t) + B_0 u(t) \\ z(t) &= Fx(t) + F_0 u(t) \\ y(t) &= \begin{bmatrix} x(t) \\ w_0(t) \\ r_1(t+L) \end{bmatrix}. \end{aligned} \tag{2}$$

As the future reference signal $r_1(t + L)$ is included in the measurement $y(t)$, we can deal with the preview control problem in the formulation (1). A simple case ($D_0 = 0$) is discussed by [15]. ■

Example 2. H^∞ control with input delays: The H^∞ control problem for multiple input delay systems is defined with $B = [B_0, B_1, \cdots, B_d]$, $D = [D_0, 0, \cdots, 0]$. It broadens the class of problems where analytic solutions are clarified. A related problem is independently discussed by [18]. ■

Example 3. LQ control with input delays: Define an LQ control problem by $D = 0$ with the cost-functional

$$J = \int_0^\infty \{x^{\mathrm{T}}(t)Qx(t) + u^{\mathrm{T}}(t)u(t)\}\, dt, \quad Q := F^{\mathrm{T}}F. \tag{3}$$

The formulation (1) naturally covers LQ (or H_2) control problem and enables us to solve the LQ control problem for multiple input delay systems. The LQ (or H_2) control problem with multiple input delays is independently considered in [21], [31], [32] by employing the specific structure which lies in the problem. In [21], the control law is obtained based on the fundamental property such that the impulse response is characterized with the series of delay-free control problems. In [31], [32], the LQ control problem is solved via delay-free fixed-lag smoothing problem, which is dual to the original problem. We will illustrate the LQ control along the general H^∞ control problem Σ and derive an alternative interpretation on the feature of resulting control system (Section IV) as well as the control law. ■

The control problem Σ provides a base to deal with more complicated delay systems, where sub-systems are connected with unilateral transmission delays. For example, a unilateral delay system [7] which arises in the wind-tunnel or the tandem connected processes is illustrated as follows.

Example 4. H^∞ control of unilateral delay systems: Focus on a unilateral delay system depicted by Fig.1(a), where sub-systems:

$$\Sigma_i: \begin{aligned} &\dot{\check{x}}_i(t) = \check{A}_i \check{x}_i(t) + \check{D}_i w_i(t) + \check{B}_i u_i(t) + \check{E}_i v_i(t) \\ &q_i(t) = \check{C}_i \check{x}_i(t) \\ &\check{x}_i(t) \in \mathbb{R}^{n_i}, w_i(t) \in \mathbb{R}^{l_i}, u_i(t) \in \mathbb{R}^{m_i}, \\ &q_i(t) \in \mathbb{R}^{p_i} \quad (\check{E}_2 = 0, v_2 = 0, i = 0, 1, 2), \end{aligned}$$

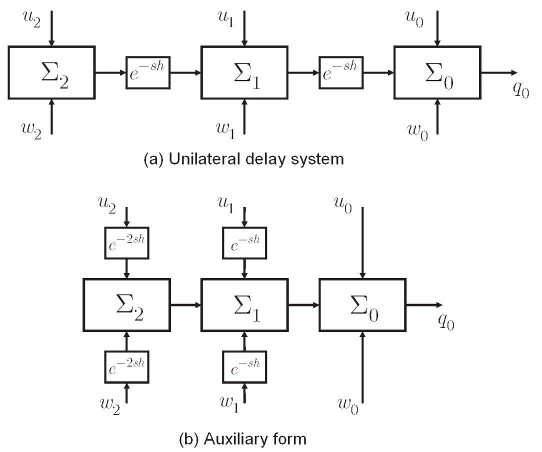

Fig. 1. Example of a unilateral delay system (3 sub-systems)

are tandem connected with the transmission delays: $v_0(t) = q_1(t-h)$, $v_1(t) = q_2(t-h)$. Rewriting the states by

$$x_i(t) = \check{x}_i(t - ih), \;\; i = 0, 1, 2,$$

the system Fig.1(a) is transformed to Fig.1(b) and defined by Σ with the following matrices:

$$\begin{aligned}
A &:= \begin{bmatrix} \check{A}_0 & \check{E}_0\check{C}_1 & 0 \\ 0 & \check{A}_1 & \check{E}_1\check{C}_2 \\ 0 & 0 & \check{A}_2 \end{bmatrix}, \\
F &:= \begin{bmatrix} \begin{bmatrix} \check{C}_0 & 0 & 0 \end{bmatrix} \\ 0_{m\times m} \end{bmatrix}, \;\; F_0 := \begin{bmatrix} 0_{p_0\times m} \\ I_m \end{bmatrix}, \\
\begin{bmatrix} D_0 & D_1 & D_2 \end{bmatrix} &:= \begin{bmatrix} \check{D}_0 & 0 & 0 \\ 0 & \check{D}_1 & 0 \\ 0 & 0 & \check{D}_2 \end{bmatrix}, \\
\begin{bmatrix} B_0 & B_1 & B_2 \end{bmatrix} &:= \begin{bmatrix} \check{B}_0 & 0 & 0 \\ 0 & \check{B}_1 & 0 \\ 0 & 0 & \check{B}_2 \end{bmatrix}, \\
m &:= m_0 + m_1 + m_2
\end{aligned}$$

where $z(t) := (q_0(t), u(t))$ and $h_0 = 0, h_1 = h, h_2 = 2h$. Thus, the H^∞ performance of the unilateral delay system Fig.1(a) is evaluated based on the system formulation (1). ■

In this paper, we will provide an explicit formula for the generalized H^∞ control problem Σ, which covers Example 1-4, and characterize the solvability and the analytic solution based on finite-dimensional operations. Before describing our approach, we will prepare an abstract system description developed by [23], [24] (Pritchard-Salamon systems) with the relation to the H^∞ control problem Σ we will solve.

B. Pritchard-Salamon Systems

Pritchard-Salamon systems describe a class of infinite-dimensional systems, which cover Σ as well as parabolic/ hyperbolic systems, and have an advantage of characterizing the LQ and H^∞ control problems based on corresponding operator representations. The detailed introduction is found in ([29], Chapter 2).

In the Pritchard-Salamon systems, the basic model is

$$\begin{aligned} \dot{x}(t) &= \mathbf{A}x(t) + \mathbf{B}u(t), \;\; x(t_0) = x_0 \\ y(t) &= \mathbf{C}x(t), \;\; t_0 \le t \le t_1, \end{aligned} \tag{4}$$

where $u(\cdot) \in L_2(t_0, t_1; \mathbf{U})$, $y(\cdot) \in L_2(t_0, t_1; \mathbf{Y})$ and $\mathbf{U}$ and $\mathbf{Y}$ are Hilbert spaces. $\mathbf{A}$ is the infinitesimal generator of a strongly continuous semigroup $\mathbf{T}(t)$ on a Hilbert space $\mathbf{X}$. In order to allow for unboundedness of the operators $\mathbf{B}$ and $\mathbf{C}$, it is assumed that $\mathbf{B} \in \mathcal{L}(\mathbf{U}, \mathbf{V})$ and $\mathbf{C} \in \mathcal{L}(\mathbf{W}, \mathbf{Y})$ where $\mathbf{W}, \mathbf{V}$ are Hilbert spaces such that

$$\mathbf{W} \subset \mathbf{X} \subset \mathbf{V} \tag{5}$$

with continuous dense injections. (4) is interpreted in the mild form

$$x(t) = \mathbf{T}(t-t_0)x_0 + \int_{t_0}^{t} \mathbf{T}(t-\sigma)\mathbf{B}u(\sigma)d\sigma, \qquad t_0 \le t \le t_1. \tag{6}$$

In order to make sure that the trajectories are well defined in all three spaces $\mathbf{W}, \mathbf{X}, \mathbf{V}$, it is assumed that $\mathbf{T}(t)$ is a strongly continuous semigroup on $\mathbf{W}$ and $\mathbf{V}$ and the following hypothesis are satisfied.

H1) There exists some constant $b > 0$ such that

$$\left\| \int_{t_0}^{t} \mathbf{T}(t-\sigma)\mathbf{B}u(\sigma)d\sigma \right\|_{\mathbf{W}} \le b\|u(\cdot)\|_{L_2(t_0,t_1;\mathbf{U})} \tag{7}$$

for all $u(\cdot) \in L_2(t_0, t_1; \mathbf{U})$.

H2) There exists some constant $c > 0$ such that

$$\|\mathbf{C}\mathbf{T}(\cdot - t_0)x\|_{L_2(t_0,t_1;\mathbf{Y})} \le c\|x\|_{\mathbf{V}} \tag{8}$$

for all $x \in \mathbf{W}$.

H3) $\mathbf{Z} = \mathrm{D}_{\mathbf{V}}(\mathbf{A}) \subset \mathbf{W}$ with continuous dense embedding where $\mathbf{Z}$ is endowed with the graph norm of $\mathbf{A}$ regarded as an unbounded closed operator on $\mathbf{V}$.

By [23], it is shown that a class of time delay systems, which involve delays in state, input, and output, are described in the framework of the Pritchard-Salamon systems. We will solve the H^∞ control problem Σ based on the abstract system description and provide explicit formulas on the solvability and the solution (control law).

C. System Description on Function-Space

For the system Σ, we first prepare a standard system representation established by [23] and precisely describe the system dynamics with the stored signals in the delay elements. Then we introduce an auxiliary system description, which preserves the solvability condition of the H^∞ control problem Σ.

Introduce a Hilbert space $\mathcal{X} := \mathbb{R}^n \times L_2(-L, 0; \mathbb{R}^l) \times L_2(-L, 0; \mathbb{R}^m)$ endowed with the inner product:

$$\begin{aligned} &\langle \psi, \phi \rangle := \\ &\psi^{0\,\mathrm{T}}\phi^0 + \int_{-L}^{0} \psi^{1\,\mathrm{T}}(\beta)\phi^1(\beta)\,d\beta + \int_{-L}^{0} \psi^{2\,\mathrm{T}}(\beta)\phi^2(\beta)\,d\beta, \\ &\qquad \psi = (\psi^0, \psi^1, \psi^2) \in \mathcal{X}, \;\; \phi = (\phi^0, \phi^1, \phi^2) \in \mathcal{X}, \end{aligned} \tag{9}$$

and describe the system Σ by the evolution equation [23].

$$\hat{\Sigma}: \begin{array}{rcl} \dot{\hat{x}}(t) &=& \mathcal{A}\hat{x}(t) + \mathcal{D}w(t) + \mathcal{B}u(t) \\ z(t) &=& \mathcal{F}\hat{x}(t) \qquad\qquad + F_0 u(t) \\ \hat{y}(t) &=& \begin{bmatrix} \hat{x}(t) \\ w(t) \end{bmatrix} \end{array} \tag{10}$$

$$\hat{x}(0) := \Xi\phi, \quad \phi \in \mathcal{X} \tag{11}$$

The operator $\mathcal{A}$ is an infinitesimal generator defined as follows:

$$\mathcal{A}\phi = \begin{bmatrix} A\phi^0 + D\phi^1(-L) + B\phi^2(-L) \\ \phi^{1\prime} \\ \phi^{2\prime} \end{bmatrix},$$

$$\mathrm{D}(\mathcal{A}) = \left\{ \phi \in \mathcal{X} : \begin{bmatrix} \phi^1 \\ \phi^2 \end{bmatrix} \in W^{1,2}(-L,0;\mathbb{R}^{l+m}), \begin{bmatrix} \phi^1(0) \\ \phi^2(0) \end{bmatrix} = 0 \right\} \tag{12}$$

where $W^{1,2}$ denotes the Sobolev space of $\mathbb{R}^{l+m}$-valued, absolutely continuous functions with square integrable derivatives on $[-L, 0]$ (see e.g. [1], Chapter 2). The adjoint operator of $\mathcal{A}$ is obtained as follows:

$$\mathcal{A}^*\psi = \begin{bmatrix} A^{\mathrm{T}}\psi^0 \\ -\psi^{1\prime} \\ -\psi^{2\prime} \end{bmatrix},$$

$$\mathrm{D}(\mathcal{A}^*) = \left\{ \psi \in \mathcal{X} : \begin{bmatrix} \psi^1 \\ \psi^2 \end{bmatrix} \in W^{1,2}(-L,0;\mathbb{R}^{l+m}), \begin{bmatrix} \psi^1(-L) \\ \psi^2(-L) \end{bmatrix} = \begin{bmatrix} D^{\mathrm{T}} \\ B^{\mathrm{T}} \end{bmatrix} \psi^0 \right\}. \tag{13}$$

Extending the state-space $\mathcal{X}$ to $\mathcal{V} := \mathrm{D}(\mathcal{A}^*)^*$, $\mathrm{D}_{\mathcal{V}}(\mathcal{A}) = \mathcal{X}$ holds and the Hilbert spaces $\mathcal{X}$, $\mathcal{V}$ are with continuous, dense injections satisfying $\mathcal{X} \subset \mathcal{V}$ [23]. Denoting the elements $\phi = (\phi^0, \phi^1, \phi^2) \in \mathcal{X}$ by

$$\begin{array}{ll} \phi^1 = (\phi^1_0, \phi^1_1, \cdots, \phi^1_d), & \phi^1_i \in L_2(-L,0;\mathbb{R}^{l_i}) \\ \phi^2 = (\phi^2_0, \phi^2_1, \cdots, \phi^2_d), & \phi^2_j \in L_2(-L,0;\mathbb{R}^{m_j}) \\ & (i,j = 0,1,\cdots d), \end{array} \tag{14}$$

input/output operators $\mathcal{D}$, $\mathcal{B}$, $\mathcal{F}$ are defined as follows:

$$\mathcal{D} : \mathbb{R}^l \to \mathcal{V}, \ \ \mathcal{D}^*\phi = \begin{bmatrix} D_0^{\mathrm{T}}\phi^0 \\ \phi^1_1(-L+h_1) \\ \vdots \\ \phi^1_i(-L+h_i) \\ \vdots \\ \phi^1_d(-L+h_d) \end{bmatrix}, \quad \phi \in \mathcal{V}^* \tag{15}$$

$$\mathcal{B} : \mathbb{R}^m \to \mathcal{V}, \ \ \mathcal{B}^*\phi = \begin{bmatrix} B_0^{\mathrm{T}}\phi^0 \\ \phi^2_1(-L+h_1) \\ \vdots \\ \phi^2_j(-L+h_j) \\ \vdots \\ \phi^2_d(-L+h_d) \end{bmatrix}, \ \phi \in \mathcal{V}^* \tag{16}$$

$$\mathcal{F} : \mathcal{X} \to \mathbb{R}^p, \ \ \mathcal{F}\phi = F\phi^0, \qquad \phi \in \mathcal{X}. \tag{17}$$

By [23], it is shown that the system $\hat{\Sigma}$ is in the Pritchard-Salamon systems with $\mathbf{W} = \mathbf{X} = \mathcal{X}$, $\mathbf{V} = \mathcal{V}$, $\mathbf{A} = \mathcal{A}$, $\mathbf{B} = [\,\tilde{\mathcal{D}}, \tilde{\mathcal{B}}\,]$, $\mathbf{C} = \mathcal{F}$.

The initial state, which corresponds to the system Σ, is given by (11) with the following operators.

$$\Xi = \begin{bmatrix} I & 0 & 0 \\ 0 & \Xi_1 & 0 \\ 0 & 0 & \Xi_2 \end{bmatrix} \in \mathcal{L}(\mathcal{X}) \tag{18}$$

$$\Xi_1 \in \mathcal{L}(L_2(-L,0;\mathbb{R}^l)), \ \Xi_2 \in \mathcal{L}(L_2(-L,0;\mathbb{R}^m)) :$$

$$(\Xi_k\phi^k)(\beta) = \begin{bmatrix} \chi_{[-L,-L+h_0]}(\beta)\cdot\phi^k_0(\beta) \\ \vdots \\ \chi_{[-L,-L+h_i]}(\beta)\cdot\phi^k_i(\beta) \\ \vdots \\ \chi_{[-L,-L+h_d]}(\beta)\cdot\phi^k_d(\beta) \end{bmatrix}, \quad -L \le \beta \le 0, \ \ k = 1,2 \tag{19}$$

where χ is a characteristic function defined by $\chi_A(\beta) = \begin{cases} 1 & (\beta \in A) \\ 0 & (\beta \notin A) \end{cases}$. It should be noted that the state $\hat{x}(t)$ $(t \ge 0)$, which is driven by u and w, satisfies $\hat{x}(t) \in \Xi\mathcal{X}$ and corresponds to the system Σ in the following manner:

$$\hat{x}(t) = \begin{bmatrix} x(t) \\ w_t \\ u_t \end{bmatrix}, \ \ w_t = \begin{bmatrix} w_{t0} \\ w_{t1} \\ \vdots \\ w_{td} \end{bmatrix}, \ \ u_t = \begin{bmatrix} u_{t0} \\ u_{t1} \\ \vdots \\ u_{td} \end{bmatrix}$$

$$w_{ti}(\beta) = \begin{cases} w_i(t+\beta+L-h_i) & -L \le \beta \le -L+h_i \\ 0 & -L+h_i \le \beta \le 0 \end{cases}$$

$$u_{tj}(\beta) = \begin{cases} u_j(t+\beta+L-h_j) & -L \le \beta \le -L+h_j \\ 0 & -L+h_j \le \beta \le 0 \end{cases}$$

$$(i,j = 0,1,\cdots,d). \tag{20}$$

Remark 1: The restriction of initial state (11) does not affect the stability condition of the system $\hat{\Sigma}$. Even if the initial state is defined by $\forall\hat{x}(0) \in \mathcal{X}$, the trajectory $\hat{x}(t)$ driven by $(w,u) \in L_2(0,t;\mathbb{R}^{l+m})$ is bounded over $[0,L]$ and $\hat{x}(L) \in \Xi\mathcal{X}$ holds. ■

Secondly, we prepare an auxiliary output delay system, which is defined with bounded input operators. Introduce a state-space $\mathcal{X}_o := \mathbb{R}^n \times L_2(-L,0;\mathbb{R}^p)$ and define an infinitesimal generator as follows:

$$\mathcal{A}_o\phi = \begin{bmatrix} A\phi^0 \\ \phi^{1\prime} \end{bmatrix}, \tag{21}$$

$$\mathrm{D}(\mathcal{A}_o) = \left\{ \phi \in \mathcal{X}_o : \ \phi^1 \in W^{1,2}(-L,0;\mathbb{R}^p), \ F\phi^0 = \phi^1(0) \right\}.$$

On the subspace $\mathcal{W}_o := \mathrm{D}(\mathcal{A}_o)$, $\mathrm{D}_{\mathcal{W}_o^*}(\mathcal{A}_o^*) = \mathcal{X}_o$ holds and the spaces $\mathcal{W}_o$, $\mathcal{X}_o$ are with continuous, dense injections satisfying $\mathcal{W}_o \subset \mathcal{X}_o$ [23]. We will define an auxiliary system with the operators $\mathcal{G} \in \mathcal{L}(\mathcal{X}, \mathcal{X}_o)$, $\mathcal{G}^* \in \mathcal{L}(\mathcal{X}_o, \mathcal{X})$:

$$\begin{bmatrix} (\mathcal{G}\phi)^0 \\ (\mathcal{G}\phi)^1(\xi) \end{bmatrix} = \begin{bmatrix} e^{AL}\phi^0 + \int_{-L}^{0} e^{-A\beta}\,[D\;B] \begin{bmatrix} \phi^1(\beta) \\ \phi^2(\beta) \end{bmatrix} d\beta \\ Fe^{A(\xi+L)}\phi^0 + F\int_{-L}^{\xi} e^{A(\xi-\beta)}\,[D\;B] \begin{bmatrix} \phi^1(\beta) \\ \phi^2(\beta) \end{bmatrix} d\beta \end{bmatrix}, \quad -L \le \xi \le 0, \quad \phi = (\phi^0,\,\phi^1,\,\phi^2) \in \mathcal{X} \tag{22}$$

$$\begin{bmatrix} (\mathcal{G}^*\psi)^0 \\ (\mathcal{G}^*\psi)^1(\beta) \\ (\mathcal{G}^*\psi)^2(\beta) \end{bmatrix} = \begin{bmatrix} e^{A^{\mathrm{T}}L}\psi^0 + \int_{-L}^{0} e^{A^{\mathrm{T}}(\xi+L)}F^{\mathrm{T}}\psi^1(\xi)\,d\xi \\ D^{\mathrm{T}}\left(e^{-A^{\mathrm{T}}\beta}\psi^0 + \int_{\beta}^{0} e^{A^{\mathrm{T}}(\xi-\beta)}F^{\mathrm{T}}\psi^1(\xi)\,d\xi\right) \\ B^{\mathrm{T}}\left(e^{-A^{\mathrm{T}}\beta}\psi^0 + \int_{\beta}^{0} e^{A^{\mathrm{T}}(\xi-\beta)}F^{\mathrm{T}}\psi^1(\xi)\,d\xi\right) \end{bmatrix}, \quad -L \le \beta \le 0, \quad \psi = (\psi^0, \psi^1) \in \mathcal{X}_o, \tag{23}$$

which satisfy the following properties.

Lemma 2: $\mathcal{G} \in \mathcal{L}(\mathcal{X}, \mathcal{W}_o)$ and $\mathcal{G} \in \mathcal{L}(\mathcal{V}, \mathcal{X}_o)$. ■

Proof: Section V-A. ■

Introduce an evolution equation.

$$\hat{\Sigma}_o: \begin{aligned} \dot{\hat{x}}_o(t) &= \mathcal{A}_o\hat{x}_o(t) + \mathcal{D}_o w(t) + \mathcal{B}_o u(t) \\ z(t) &= \mathcal{F}_o\hat{x}_o(t) + F_0 u(t) \\ \hat{y}_o(t) &= \begin{bmatrix} \hat{x}_o(t) \\ w(t) \end{bmatrix} \end{aligned} \tag{24}$$

$$\hat{x}_o(0) := \mathcal{G}\Xi\phi, \quad \phi \in \mathcal{X} \tag{25}$$

The operator $\mathcal{A}_o$ is defined by (21) and the input/output operators are given as follows:

$$\mathcal{B}_o := \mathcal{G}\mathcal{B} \in \mathcal{L}(\mathbb{R}^m, \mathcal{X}_o) \tag{26}$$

$$\mathcal{D}_o := \mathcal{G}\mathcal{D} \in \mathcal{L}(\mathbb{R}^l, \mathcal{X}_o) \tag{27}$$

$$\mathcal{F}_o \in \mathcal{L}(\mathcal{W}_o, \mathbb{R}^p) : \mathcal{F}_o\phi = \phi^1(-L), \quad \phi \in \mathcal{W}_o. \tag{28}$$

Between the systems $\hat{\Sigma}$ and $\hat{\Sigma}_o$, the following properties are preserved.

Lemma 3:

(a) Let $\mathbf{W} = \mathcal{W}_o$, $\mathbf{X} = \mathbf{V} = \mathcal{X}_o$ and $\mathbf{A} = \mathcal{A}_o$, $\mathbf{B} = [\mathcal{D}_o, \mathcal{B}_o]$, $\mathbf{C} = \mathcal{F}_o$ in (4),(5), then $\hat{\Sigma}_o$ is in the Pritchard-Salamon systems.

(b) Let $\hat{x}(0) \in \mathcal{X}$ and $\hat{x}_o(0) = \mathcal{G}\hat{x}(0) \in \mathcal{W}_o$ be the initial states of the systems $\hat{\Sigma}$, $\hat{\Sigma}_o$, respectively, then $\hat{x}_o(t) = \mathcal{G}\hat{x}(t)$ $(t \ge 0)$ holds for any $(w, u) \in L_2(0, t; \mathbb{R}^{l+m})$.

(c) For any $(w, u) \in L_2(0, t; \mathbb{R}^{l+m})$, the systems $\hat{\Sigma}$ with $\hat{x}(0) = 0$ and $\hat{\Sigma}_o$ with $\hat{x}_o(0) = 0$ generate the same output $z \in L_2(0, t; \mathbb{R}^p)$. ■

Proof: Section V-B. ■

For the Pritchard-Salamon systems, the solvability of H^∞ control problem can be described with the corresponding operator Riccati equations ([29], Theorem 4.4). For the systems $\hat{\Sigma}$, $\hat{\Sigma}_o$ with the operator Riccati equations:

$$\mathcal{S}\mathcal{A}\phi + \mathcal{A}^*\mathcal{S}\phi - \mathcal{S}\mathcal{B}\mathcal{B}^*\mathcal{S}\phi + \frac{1}{\gamma^2}\cdot\mathcal{S}\mathcal{D}\mathcal{D}^*\mathcal{S}\phi + \mathcal{F}^*\mathcal{F}\phi = 0, \quad \phi \in \mathcal{X}, \tag{29}$$

$$\mathcal{S}_o\mathcal{A}_o\phi + \mathcal{A}_o^*\mathcal{S}_o\phi - \mathcal{S}_o\mathcal{B}_o\mathcal{B}_o^*\mathcal{S}_o\phi + \frac{1}{\gamma^2}\cdot\mathcal{S}_o\mathcal{D}_o\mathcal{D}_o^*\mathcal{S}_o\phi + \mathcal{F}_o^*\mathcal{F}_o\phi = 0, \quad \phi \in \mathcal{W}_o, \tag{30}$$

the H^∞ control problems are characterized by the following propositions.

Proposition P: The H^∞ control problem $\hat{\Sigma}$ is solvable iff (29) has a stabilizing solution $\mathcal{S} \ge 0$ ($\mathcal{S} \in \mathcal{L}(\mathcal{V}, \mathcal{V}^*)$) such that $\mathcal{A} - \mathcal{B}\mathcal{B}^*\mathcal{S} + \frac{1}{\gamma^2}\cdot\mathcal{D}\mathcal{D}^*\mathcal{S}$ generates exponentially stable semigroups on $\mathcal{V}$.[1] Furthermore, if solvable, an H^∞ control law, which stabilizes $\hat{\Sigma}$ on $\mathcal{X}$, is given by

$$u(t) = -\mathcal{B}^*\mathcal{S}\hat{x}(t). \tag{31}$$

■

Proposition $\mathbf{P_o}$: The H^∞ control problem $\hat{\Sigma}_o$ is solvable iff (30) has a stabilizing solution $\mathcal{S}_o \ge 0$ ($\mathcal{S}_o \in \mathcal{L}(\mathcal{X}_o)$) such that $\mathcal{A}_o - \mathcal{B}_o\mathcal{B}_o^*\mathcal{S}_o + \frac{1}{\gamma^2}\cdot\mathcal{D}_o\mathcal{D}_o^*\mathcal{S}_o$ generates exponentially stable semigroups on $\mathcal{X}_o$. If solvable, an H^∞ control law, which stabilizes $\hat{\Sigma}_o$ on $\mathcal{X}_o$, is given by

$$u(t) = -\mathcal{B}_o^*\mathcal{S}_o\hat{x}_o(t). \tag{32}$$

■

For the operator Riccati equations (29),(30), it is noted that there exists at most one stabilizing solution in like manner as finite-dimensional case ([29], Lemma 2.33). Finally, we will verify that the H^∞ control problems for $\hat{\Sigma}$ and $\hat{\Sigma}_o$ are equivalent.

Lemma 4:

(a) If $\mathcal{S}_o \ge 0$ ($\mathcal{S}_o \in \mathcal{L}(\mathcal{X}_o)$) is a stabilizing solution to (30), the stabilizing solution $\mathcal{S} \ge 0$ ($\mathcal{S} \in \mathcal{L}(\mathcal{V}, \mathcal{V}^*)$) to (29) is given by $\mathcal{S} = \mathcal{G}^*\mathcal{S}_o\mathcal{G}$.

(b) The H^∞ control problem $\hat{\Sigma}$ is solvable iff the problem $\hat{\Sigma}_o$ is solvable. ■

Proof: Section V-C. ■

In the sequel, we pose the following problem and derive a design method of H^∞ control law for the generalized plant Σ.

Problem $\mathbf{P_o}$: For the operator Riccati equation (30), provide a necessary and sufficient condition such that there exists a stabilizing solution $\mathcal{S}_o \ge 0$ ($\mathcal{S}_o \in \mathcal{L}(\mathcal{X}_o)$). If it exists, construct the stabilizing solution $\mathcal{S}_o \ge 0$ analytically. In saying the stabilizing solution $\mathcal{S}_o$, we mean that the operator $\mathcal{A}_o - \mathcal{B}_o\mathcal{B}_o^*\mathcal{S}_o + \frac{1}{\gamma^2}\cdot\mathcal{D}_o\mathcal{D}_o^*\mathcal{S}_o$ generates exponentially stable semigroups on $\mathcal{X}_o$. ■

As follows from Lemma 4(b), the solution to $\mathbf{P_o}$ provides a necessary and sufficient condition on the solvability of the H^∞ control problem $\hat{\Sigma}$. Furthermore, the H^∞ control (31) is given by

$$u(t) = -\mathcal{B}^*\mathcal{G}^*\mathcal{S}_o\mathcal{G}\hat{x}(t) = -\mathcal{B}_o^*\mathcal{S}_o\mathcal{G}\hat{x}(t) \tag{33}$$

with the stabilizing solution $\mathcal{S}_o \ge 0$ (Lemma 4(a) and (31)).

In the next section, we will solve the problem $\mathbf{P_o}$ by exploring the analytic solution to (30) and elaborate the design method of H^∞ control law form the viewpoint of the original problem Σ.

[1] For the operator $\mathcal{S} \in \mathcal{L}(\mathcal{V}, \mathcal{V}^*)$, the positive definiteness is defined by $\forall\phi \in \mathcal{V}: \langle\phi, \mathcal{S}\phi\rangle_{\mathcal{V},\mathcal{V}^*} \ge 0$. The condition holds iff $\mathcal{S} \ge 0$ on $\mathcal{X}$. See e.g. [29], Section 2.5.

III. Main Result

Let us derive explicit formulas for the problem $\mathbf{P_o}$ and provide a design method of H^∞ control law for Σ. The key point in our approach is that the stabilizing solution to (30) is newly established based on a Hamiltonian matrix, which is defined with the delay-free system of (1) ($L = 0$). We first state a qualitative property of the Hamiltonian matrix.

Lemma 5: For a given $\gamma > 0$, the H^∞ control problem Σ is solvable only if the Hamiltonian matrix

$$H := \begin{bmatrix} A & -BB^{\mathrm{T}} + \frac{1}{\gamma^2}\cdot DD^{\mathrm{T}} \\ -F^{\mathrm{T}}F & -A^{\mathrm{T}} \end{bmatrix} \quad (34)$$

does not have eigenvalues on the imaginary axis. ■

Proof: Section V-D. ■

Lemma 5 guarantees the fact that, if (30) has a stabilizing solution $\mathcal{S}_o \in \mathcal{L}(\mathcal{X}_o)$, there exists a column full-rank matrix $V \in \mathbb{R}^{2n\times n}$ defined as follows:

$$\mathcal{X}_-(H) := \operatorname{Im} V: \quad HV = V\Lambda, \quad \Lambda: \text{ stable matrix,}$$
$$V := \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}, \quad V_1, V_2 \in \mathbb{R}^{n\times n}. \quad (35)$$

In other words, the Hamiltonian matrix H has n-dimensional stable eigenspace $\mathcal{X}_-(H)$ [4], [9] only if the H^∞ control problem Σ is solvable.

Based on Lemma 5, we first derive a necessary and sufficient condition on the existence of the stabilizing solution to (30). Then we clarify the additional condition such that the stabilizing solution turns positive semi-definite. The existence of stabilizing solution is clarified as follows.

Theorem 6: For a given $\gamma > 0$, suppose there exists a column full-rank matrix $V \in \mathbb{R}^{2n\times n}$ defined by (35). Then, the following statements (a),(b),(c) are equivalent.

(a) The operator Riccati equation (30) has a stabilizing solution $\mathcal{S}_o \in \mathcal{L}(\mathcal{X}_o)$.

(b) The operator

$$\mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2 \in \mathcal{L}(\mathcal{X}_o) \quad (36)$$

has a bounded inverse where the operators $\mathcal{V}_1, \mathcal{V}_2 \in \mathcal{L}(\mathcal{X}_o)$, $\Pi \in \mathcal{L}(\mathcal{X})$ are defined as follows:

$$\mathcal{V}_1 := \begin{bmatrix} V_1 & 0 \\ 0 & \mathcal{I} \end{bmatrix}, \quad \mathcal{V}_2 := \begin{bmatrix} V_2 & 0 \\ 0 & \mathcal{I} \end{bmatrix}, \quad (37)$$

$$\Pi = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\gamma^{-2}\cdot\Pi_1 & 0 \\ 0 & 0 & \Pi_2 \end{bmatrix} \in \mathcal{L}(\mathcal{X}):$$

$$(\Pi_k\phi^k)(\beta) = \begin{bmatrix} \chi_{[-L+h_0,0]}(\beta)\cdot\phi_0^k(\beta) \\ \vdots \\ \chi_{[-L+h_i,0]}(\beta)\cdot\phi_i^k(\beta) \\ \vdots \\ \chi_{[-L+h_d,0]}(\beta)\cdot\phi_d^k(\beta) \end{bmatrix}, \quad k = 1, 2. \quad (38)$$

(c) The matrix

$$U_0 := V^{\mathrm{T}}\Psi_d(-L)\begin{bmatrix} I \\ 0 \end{bmatrix} \quad (39)$$

is nonsingular where $\Psi_d(\cdot)$ is a fundamental solution to the differential equation

$$\begin{cases} \Psi_d(0) = I \\ \dfrac{d}{dt}\Psi_d(t) = \Psi_d(t)J_i^{\mathrm{T}}, \ -L + h_i \le t \le -L + h_{i+1}, \end{cases}$$

$$J_i := \begin{bmatrix} A & \sum_{j=0}^{i}(\frac{1}{\gamma^2}\cdot D_jD_j^{\mathrm{T}} - B_jB_j^{\mathrm{T}}) \\ -F^{\mathrm{T}}F & -A^{\mathrm{T}} \end{bmatrix}$$
$$(i = 0, 1, 2, \cdots, d-1). \quad (40)$$

If the stabilizing solution exists, it is given as follows:

$$\mathcal{S}_o := \mathcal{V}_2\left(\mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2\right)^{-1}. \quad (41)$$

■

The properties of the operators (36),(41) are noted as follows.

Remark 7: The operator (36) is further represented by

$$\mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2 = \mathcal{I} + \begin{bmatrix} V_1 - I & 0 \\ 0 & 0 \end{bmatrix} + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2$$

where $\begin{bmatrix} V_1 - I & 0 \\ 0 & 0 \end{bmatrix}$ is finite-rank and $\mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2$ is compact since Π, $\mathcal{V}_2$ are bounded and the operator $\mathcal{G} \in \mathcal{L}(\mathcal{X}, \mathcal{X}_o)$ defined by (22) is given by the sum of compact (finite-rank, Fredholm, and Volterra) operators. Thus, the condition Theorem 6(b) holds iff (36) does not have any eigenvalue at origin. ■

Remark 8: The stabilizing solution (41) is self-adjoint. The equality

$$\begin{aligned} &(\mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2)^*\,\mathcal{S}_o\,(\mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2) \\ &= \mathcal{V}_1^*\mathcal{V}_2 + \mathcal{V}_2^*\mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2 \\ &= \begin{bmatrix} V_1^{\mathrm{T}}V_2 & 0 \\ 0 & \mathcal{I} \end{bmatrix} + \mathcal{V}_2^*\mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2 \end{aligned} \quad (42)$$

follows from (41) and, further

$$V_1^{\mathrm{T}}V_2 = V_2^{\mathrm{T}}V_1 \quad (43)$$

holds for the stable eigenspace (35) ([30], Theorem 13.3). Hence (41) is self-adjoint if the stabilizing solution exists. ■

Proof of Theorem 6: Section V-E. ■

By Theorem 6(c), it is shown that the operator Riccati equation (30) has a stabilizing solution iff the matrix (39) is nonsingular.

In the derivation of Theorem 6, it is noted that the auxiliary system description $\hat{\Sigma}_o$ yields a following equality:

$$\begin{bmatrix} \mathcal{A}_o & -\mathcal{B}_o\mathcal{B}_o^* + \frac{1}{\gamma^2}\cdot\mathcal{D}_o\mathcal{D}_o^* \\ -\mathcal{F}_o^*\mathcal{F}_o & -\mathcal{A}_o^* \end{bmatrix}\begin{bmatrix} \mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2 \\ \mathcal{V}_2 \end{bmatrix}\phi$$
$$= \begin{bmatrix} \mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2 \\ \mathcal{V}_2 \end{bmatrix}\mathcal{A}_{\Lambda o}\phi, \quad \phi \in \mathrm{D}(\mathcal{A}_{\Lambda o}) \quad (44)$$

where $\mathcal{A}_{\Lambda o}$ defined by

$$\mathcal{A}_{\Lambda o}\phi = \begin{bmatrix} \Lambda\phi^0 \\ \phi^{1'} \end{bmatrix}, \quad \mathrm{D}(\mathcal{A}_{\Lambda o}) =$$
$$\left\{\phi \in \mathcal{X}_o: \ \phi^1 \in W^{1,2}(-L, 0; \mathbb{R}^p), FV_1\phi^0 = \phi^1(0)\right\}$$
$$\Lambda: \text{ stable matrix defined by (35)} \quad (45)$$

generates exponentially stable semigroups on $\mathcal{X}_o$. Therefore, in like fashion of finite-dimensional systems, the stabilizing solution (41) is constructed iff the operator $\mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2$ is invertible.

Remark 9: The auxiliary system $\hat{\Sigma}_o$ has the advantage of yielding an operator Hamiltonian representation (44), which expresses the stabilizing solution by (41). Similar approach is not available with the system $\hat{\Sigma}$ because the corresponding solution is given by $\mathcal{S} = \mathcal{G}^*\mathcal{S}_o\mathcal{G}$ and the operator $\mathcal{G}$ is compact (Lemma 4(a); see also Remark 7). ■

Employing the analytic solution (41), 1) the condition of positive semi-definiteness ($\mathcal{S}_o \geq 0$) and, 2) the representation of the H^∞ control for Σ, are clarified by Theorems 10 and 11.

Theorem 10 (Condition of $\mathcal{S}_o \geq 0$): Define a fundamental solution to the differential equation

$$\begin{cases} \tilde{\Psi}_d(0) = I \\ \dfrac{d}{dt}\tilde{\Psi}_d(t) = \tilde{\Psi}_d(t)\tilde{J}_i^{\mathrm{T}}, \ -L + h_i \leq t \leq -L + h_{i+1} \end{cases},$$

$$\tilde{J}_i := \begin{bmatrix} A & \displaystyle\sum_{j=0}^{i}\left(\frac{1}{\gamma^2}\cdot D_j D_j^{\mathrm{T}} - B_j B_j^{\mathrm{T}}\right) \\ -\frac{1}{1-\mu}\cdot F^{\mathrm{T}}F & -A^{\mathrm{T}} \end{bmatrix}, \quad (i = 0, 1, 2, \cdots, d-1) \tag{46}$$

with a scalar parameter $\mu \neq 1$. The stabilizing solution (41) is positive semi-definite ($\mathcal{S}_o \geq 0$) iff the matrix

$$\tilde{U}(\mu) := V^{\mathrm{T}}\begin{bmatrix} I & 0 \\ -\mu\cdot I & I \end{bmatrix}\tilde{\Psi}_d(-L)\begin{bmatrix} I \\ 0 \end{bmatrix} \tag{47}$$

is nonsingular for any $\mu < 0$. ■

Proof: Section V-F. ■

Theorem 11 (Control law): If the H^∞ control problem Σ is solvable, an H^∞ control for Σ is given as follows:

$$\begin{aligned} u_j(t) = &-B_j^{\mathrm{T}}W_0^j U_0^{-1} \times \\ &\Big(U_1(-L)x(t) + \sum_{i=1}^{d}\int_{-L}^{-L+h_i} U_1(\tau)\{D_i w_i(t+\tau+L-h_i) \\ &\qquad + B_i u_i(t+\tau+L-h_i)\}d\tau\Big) \\ &+ B_j^{\mathrm{T}} \times \\ &\Big(W_1^j(-L)x(t) + \sum_{i=1}^{d}\int_{-L}^{-L+\min(h_i,h_j)} W_1^j(\tau)\{D_i w_i(t+\tau+L-h_i) \\ &\qquad + B_i u_i(t+\tau+L-h_i)\}d\tau\Big) \\ &(j = 0, 1, \cdots, d) \end{aligned} \tag{48}$$

$$U_0 := V^{\mathrm{T}}\Psi_d(-L)\begin{bmatrix} I \\ 0 \end{bmatrix}, \tag{49a}$$

$$U_1(t) := V^{\mathrm{T}}\Psi_d(t)\begin{bmatrix} 0 \\ I \end{bmatrix}, \tag{49b}$$

$$W_0^j := \begin{bmatrix} I & 0 \end{bmatrix}\Psi_j(-L)\begin{bmatrix} I \\ 0 \end{bmatrix}, \tag{49c}$$

$$W_1^j(t) := \begin{bmatrix} I & 0 \end{bmatrix}\Psi_j(t)\begin{bmatrix} 0 \\ I \end{bmatrix}, \ (j = 1, 2, \cdots, d) \tag{49d}$$

$$\begin{cases} \Psi_j(-L + h_j) = I \\ \dfrac{d}{dt}\Psi_j(t) = \Psi_j(t)J_i^{\mathrm{T}}, \\ \quad -L + h_i \leq t \leq -L + h_{i+1} \ (i = 0, 1, \cdots, j-1) \end{cases} \tag{49e}$$

$$W_0^0 := I, \quad W_1^0(-L) := 0, \tag{49f}$$

where $\Psi_d(\cdot)$ is defined by (40) in Theorem 6. ■

Proof: Section V-G. ■

Based on Theorems 6,10,11, the solvability and the solution to the H^∞ control problem are summarized by the following theorem.

Theorem 12 (Main result): For a given $\gamma > 0$, the H^∞ control problem for Σ is solvable iff the following conditions (a),(b) are satisfied.

(a) The Hamiltonian matrix (34) does not have eigenvalues on the imaginary axis.

(b) The matrix (47) is nonsingular for any $\mu \leq 0$.

If the problem is solvable, an H^∞ control for Σ is given by (48),(49a)-(49f). ■

In the statement (b), the conditions obtained by Theorem 6(c) and 10 are merged as $U_0 = \tilde{U}(0)$ holds.

IV. Discussions and Interpretations

In this section, the results stated in Section III are further discussed and, for the typical control problems (Examples 1-3), it is shown that the representation of analytic solution is considerably simplified. Some interpretations are also provided on the feature of the resulting closed-loop system.

The condition of solvability for the typical control problems (Example 1-3) are obtained as follows.

Corollary 13 (H^∞ preview control): Define Σ with $B = [B_0, 0, \cdots, 0]$. For a given $\gamma > 0$, the H^∞ control problem is solvable iff the condition Theorem 6(c) holds and the following matrix is stable:

$$A - B_0 B_0^{\mathrm{T}} U_0^{-1} U_1(-L) \tag{50}$$

where $U_1(-L)$ is defined by (49b). ■

Proof: Section V-H. ■

Remark 14: Focus on an H^∞ preview control problem with $B = [B_0, 0]$, $D = [D_0, D_1]$, $h_0 = 0$, $h_1 = L$. In this case, the resulting control law is figured based on a predictive action associated with a fictitious Hamilton system.

$$u_0(t) = -B_0^{\mathrm{T}}U_0^{-1}V^{\mathrm{T}}e^{-J_0^{\mathrm{T}}L}p(t) \tag{51}$$

$$p(t) = \begin{bmatrix} 0 \\ I \end{bmatrix}x(t) + \int_{-L}^{0} e^{J_0^{\mathrm{T}}(\tau+L)}\begin{bmatrix} 0 \\ I \end{bmatrix}D_1 w_1(t+\tau)\,d\tau \tag{52}$$

The control law (51),(52) is an extension of the H^∞ preview control [15] and enables to treat both previewable references and disturbances. The output feedback case is obtained along with [16]. ■

Next, we will investigate the conditions for the problem with input delays. It is noted that the H^∞ control problem for the input delay system ($D = [D_0, 0, \cdots, 0]$) is solvable only if the problem for the delay-free system ($D = [D_0, 0, \cdots, 0]$, $h_0 = h_1 = \cdots = h_d = 0$) is solvable. In other words, the H^∞ control law

$$u = Ky \tag{53}$$

for the input delay system always provides an H^∞ control law for the delay-free system by

$$u = \tilde{K}y, \quad \tilde{K}(s) := \text{block diag}\{e^{-sh_0}I_{m_0}, e^{-sh_1}I_{m_1}, \cdots, e^{-sh_d}I_{m_d}\}K(s) \quad (54)$$

where $K(s)$ and $\tilde{K}(s)$ denote the transfer functions of K, $\tilde{K}$ respectively. Without loss of generality, we assume the existence of stabilizing solution $M = V_2V_1^{-1} \geq 0$ to the following matrix Riccati equation[2] [9]:

$$MA + A^{\mathrm{T}}M - MBB^{\mathrm{T}}M + \tfrac{1}{\gamma^2}\cdot MDD^{\mathrm{T}}M + F^{\mathrm{T}}F = 0, \quad (55)$$

which is a necessary and sufficient condition on the solvability of the delay-free H^∞ control problem Σ ($D = [D_0, 0, \cdots, 0]$, $h_0 = h_1 = \cdots = h_d = 0$). The Riccati equation (55) enables us to characterize the solvability of input-delay H^∞ control problem from the viewpoint of spectral radius condition and, further, clarifies a special structure of resulting control law.

Corollary 15 (H^∞ control with delays): Define Σ with $D = [D_0, 0, \cdots, 0]$. For a given $\gamma > 0$, the H^∞ control problem is solvable iff the maximal root of the following transcendental equation satisfies $\lambda_{\max} < 1$:

$$\begin{aligned}\lambda :\ & \det[\check{U}(\lambda)] = 0,\\ & \check{U}(\lambda) := \begin{bmatrix}\lambda\cdot I & -M\end{bmatrix}\check{\Psi}_d(-L)\begin{bmatrix}I\\0\end{bmatrix}\end{aligned} \quad (56)$$

where $M \geq 0$ is the stabilizing solution to (55) and $\check{\Psi}_d(\cdot)$ is the solution to the differential equation

$$\begin{cases}\check{\Psi}_d(0) = I\\ \dfrac{d}{dt}\check{\Psi}_d(t) = \check{\Psi}_d(t)\check{J}_i^{\mathrm{T}},\ -L + h_i \leq t \leq -L + h_{i+1}\end{cases}$$

$$\check{J}_i := \begin{bmatrix} A & \frac{1}{\gamma^2}\cdot D_0D_0^{\mathrm{T}} - \sum_{j=0}^{i} B_jB_j^{\mathrm{T}}\\ -\lambda^{-1}\cdot F^{\mathrm{T}}F & -A^{\mathrm{T}}\end{bmatrix} \quad (i = 0, 1, 2, \cdots, d-1), \quad (57)$$

which depends on a scalar parameter λ. ■

Proof: Section V-I. ■

Remark 16: Focus on an H^∞ control problem with $B = [B_0, B_1]$, $D = [D_0, 0]$, $h_0 = 0$, $h_1 = L$. By Theorem 11, the resulting control law is figured based on a predictive action associated with a Hamilton system

$$u_0(t) = K_0q(t) \quad (58)$$

$$u_1(t) = K_1q(t) \quad (59)$$

$$q(t) = \begin{bmatrix}0\\I\end{bmatrix}x(t) + \int_{-L}^{0} e^{J_0^{\mathrm{T}}(\tau+L)}\begin{bmatrix}0\\I\end{bmatrix}B_1u_1(t+\tau)\,d\tau \quad (60)$$

$$\begin{aligned}K_0 &= -B_0^{\mathrm{T}}U_0^{-1}V^{\mathrm{T}}e^{-J_0^{\mathrm{T}}L},\\ K_1 &= B_1^{\mathrm{T}}\begin{bmatrix}I & 0\end{bmatrix}\left(I - e^{-J_0^{\mathrm{T}}L}\begin{bmatrix}I\\0\end{bmatrix}U_0^{-1}V^{\mathrm{T}}\right)e^{-J_0^{\mathrm{T}}L}.\end{aligned} \quad (61)$$

[2]The stabilizing solution means that $A - BB^{\mathrm{T}}M + \frac{1}{\gamma^2}\cdot DD^{\mathrm{T}}$ is stable.

Furthermore, if a condition $FA^kB_1 = 0$ ($k = 0, 1, \cdots$) is imposed which arises in robust stabilization problems (e.g.[12]), the following equality is obtained by employing the relation $(J_0^{\mathrm{T}})^k\begin{bmatrix}0\\B_1\end{bmatrix} = \begin{bmatrix}0\\(-A)^kB_1\end{bmatrix}$ ($k = 0, 1, \cdots$).

$$\begin{aligned}q(t) &= \begin{bmatrix}0\\I\end{bmatrix}r(t),\\ r(t) &= x(t) + \int_{-L}^{0} e^{-A(\tau+L)}B_1u_1(t+\tau)\,d\tau\end{aligned} \quad (62)$$

Thus, it is observed that the control law (58)-(61) yields nominal state prediction [5], [28] for specified problems. ■

Remark 17: In the case of $D = 0$, $\gamma = \infty$, Theorem 11 provides an LQ control law for (3). In the general setting, it is observed that the nominal state prediction does not work as optimal control strategy. If we focus on a simple case $B = [0, B_1]$, $h_0 = 0$, $h_1 = L$, the representation (48) yields nominal state prediction:

$$u_1(t) = -B_1^{\mathrm{T}}Mr(t), \quad r(t):\ \text{defined by (62)} \quad (63)$$

which is obtained by employing the facts $U_0 = V_1^{\mathrm{T}}e^{-A^{\mathrm{T}}L}$, $M = V_2V_1^{-1}$, $e^{J_0^{\mathrm{T}}\tau} = \begin{bmatrix}e^{A^{\mathrm{T}}\tau} & *\\ 0 & e^{-A\tau}\end{bmatrix}$. ■

Let us provide an interpretation on the feature of resulting control system. Recall the Hamiltonian representation (44)

$$\begin{bmatrix}\mathcal{A}_o & -\mathcal{B}_o\mathcal{B}_o^* + \frac{1}{\gamma^2}\cdot\mathcal{D}_o\mathcal{D}_o^*\\ -\mathcal{F}_o^*\mathcal{F}_o & -\mathcal{A}_o^*\end{bmatrix}\begin{bmatrix}\mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2\\ \mathcal{V}_2\end{bmatrix}\phi = \begin{bmatrix}\mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2\\ \mathcal{V}_2\end{bmatrix}\mathcal{A}_{\Lambda o}\phi,\ \phi \in \mathrm{D}(\mathcal{A}_{\Lambda o}) \quad (64)$$

$$\mathcal{A}_{\Lambda o}\phi = \begin{bmatrix}\Lambda\phi^0\\ \phi^{1\prime}\end{bmatrix}, \quad \mathrm{D}(\mathcal{A}_{\Lambda o}) = \left\{\phi \in \mathcal{X}_o:\ \phi^1 \in W^{1,2}(-L, 0; \mathbb{R}^p), FV_1\phi^0 = \phi^1(0)\right\} \quad (65)$$

which is employed in Theorem 6, and investigate the property of the resulting control system. If (30) has a stabilizing solution $\mathcal{S}_o \in \mathcal{L}(\mathcal{X}_o)$ or, equivalently, the operator $\mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2$ is invertible (Theorem 6(b)), we have the following equality

$$(\mathcal{A}_o - \mathcal{B}_o\mathcal{B}_o^*\mathcal{S}_o + \tfrac{1}{\gamma^2}\cdot\mathcal{D}_o\mathcal{D}_o^*\mathcal{S}_o)(\mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2)\phi = (\mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2)\mathcal{A}_{\Lambda o}\phi, \quad \phi \in \mathrm{D}(\mathcal{A}_{\Lambda o}) \quad (66)$$

for the system

$$\dot{\hat{x}}_o(t) = (\mathcal{A}_o - \mathcal{B}_o\mathcal{B}_o^*\mathcal{S}_o + \tfrac{1}{\gamma^2}\cdot\mathcal{D}_o\mathcal{D}_o^*\mathcal{S}_o)\hat{x}_o(t) \quad (67)$$

which is obtained by applying (32) and $w(t) = \frac{1}{\gamma^2}\cdot\mathcal{D}_o^*\mathcal{S}_o\hat{x}_o(t)$ to $\hat{\Sigma}_o$.

If the stabilizing solution holds $\mathcal{S}_o \geq 0$, $w(t) = \frac{1}{\gamma^2}\cdot\mathcal{D}_o^*\mathcal{S}_o\hat{x}_o(t)$ plays as the worst-case disturbance for the H^∞ control system and (67) is equivalently transformed to a fictitious form

$$\dot{\hat{x}}_o(t) = \mathcal{A}_{\Lambda o}\hat{x}_o(t), \quad \hat{x}_o \in \mathrm{D}(\mathcal{A}_{\Lambda_o}) \quad (68)$$

where $\mathcal{A}_{\Lambda_o}$ is defined by (45). Denoting the state by $\hat{x}_o(t) := (x_o(t), v_t) \in \mathcal{X}_o$, $v_t(\beta) := v(t+\beta)(-L \leq \beta \leq 0)$, the system

(68) is further described as follows:

$$\begin{aligned} \dot{x}_o(t) &= \Lambda x_o(t) \\ v(t) &= FV_1 x_o(t) \\ v_t(\beta) &:= v(t+\beta), \quad -L \le \beta \le 0. \end{aligned} \tag{69}$$

Thus, for the generalized plants Σ with any multiple time delays, the worst-case system (67) yields identical pole configuration as far as the H^∞ control exists. For the LQ control problem ($D=0$), this property provides the pole configuration of the resulting closed loop system.

Corollary 18 (LQ control): Define an LQ control problem by $D = [\,D_0, D_1, D_2, \cdots, D_d\,] = 0$ with the cost-functional (3), the pole configuration of the resulting closed loop system coincides with the eigenvalues of Λ, which is defined by (35). In other words, the value of time delays $h_0 < h_1 < \cdots < h_d$ does not affect the pole configuration of the resulting closed loop system. ■

V. Proofs

A. Proof of Lemma 2

We first derive $\mathcal{G} \in \mathcal{L}(\mathcal{X}, \mathcal{W}_o)$. For $\phi \in \mathcal{X}$, it is verified that $\mathcal{G}\phi \in \mathcal{W}_o = \mathrm{D}(\mathcal{A}_o)$ holds since $(\mathcal{G}\phi)^1 \in W^{1,2}(-L, 0; \mathbb{R}^p)$ and $F(\mathcal{G}\phi)^0 = (\mathcal{G}\phi)^1(0)$ are satisfied by (22). Furthermore $\mathcal{G}\phi$ in $\mathcal{W}_o$ depends continuously on $\phi \in \mathcal{X}$. Hence $\mathcal{G} \in \mathcal{L}(\mathcal{X}, \mathcal{W}_o)$.

Based on (23), we will derive $\mathcal{G}^* \in \mathcal{L}(\mathcal{X}_o, \mathcal{V}^*)$, where $\mathcal{V}^* = \mathrm{D}(\mathcal{A}^*)$ is defined by (13). For $\psi \in \mathcal{X}_o$, $\mathcal{G}^*\psi \in \mathcal{V}^* = \mathrm{D}(\mathcal{A}^*)$ holds since $((\mathcal{G}^*\psi)^1, (\mathcal{G}^*\psi)^2) \in W^{1,2}(-L, 0; \mathbb{R}^{l+m})$ and $(\mathcal{G}^*\psi)^1(-L) = D^{\mathrm{T}}\psi^0$, $(\mathcal{G}^*\psi)^2(-L) = B^{\mathrm{T}}\psi^0$ are satisfied by (23). Furthermore $\mathcal{G}^*\psi$ in $\mathcal{V}^*$ depends continuously on $\psi \in \mathcal{X}_o$. Hence $\mathcal{G}^* \in \mathcal{L}(\mathcal{X}_o, \mathcal{V}^*)$ is derived. ■

B. Proof of Lemma 3

(a) We first note that a part of operators, which describe the system $\hat{\Sigma}_o$, shares a similar structure to output delay systems. Introduce alternative input operators

$$\tilde{\mathcal{D}}_o \in \mathcal{L}(\mathbb{R}^l, \mathcal{X}_o): \ \tilde{\mathcal{D}}_o v = \begin{bmatrix} Dw \\ 0 \end{bmatrix}, \ \ w \in \mathbb{R}^l \tag{70}$$

$$\tilde{\mathcal{B}}_o \in \mathcal{L}(\mathbb{R}^m, \mathcal{X}_o): \ \tilde{\mathcal{B}}_o u = \begin{bmatrix} Bu \\ 0 \end{bmatrix}, \ \ u \in \mathbb{R}^m \tag{71}$$

and define an evolution equation as follows:

$$\hat{\Sigma}'_o: \begin{aligned} \dot{\hat{x}}_o(t) &= \mathcal{A}_o\hat{x}_o(t) + \tilde{\mathcal{D}}_o w(t) + \tilde{\mathcal{B}}_o u(t) \\ z(t) &= \mathcal{F}_o\hat{x}_o(t) \qquad\qquad\qquad + F_0 u(t) \\ \hat{x}_o(0) &:= \mathcal{G}\Xi\phi, \quad \phi \in \mathcal{X}. \end{aligned} \tag{72}$$

The system $\hat{\Sigma}'_o$ describes the output delay system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Dw(t) + Bu(t) \\ z(t) &= Fx(t-L) \qquad + F_0 u(t) \end{aligned}$$

and, by [23], it is shown that $\hat{\Sigma}'_o$ belongs to the Pritchard-Salamon systems with $\mathbf{W} = \mathcal{W}_o$, $\mathbf{X} = \mathbf{V} = \mathcal{X}_o$, $\mathbf{A} = \mathcal{A}_o$, $\mathbf{B} = [\,\tilde{\mathcal{D}}_o, \tilde{\mathcal{B}}_o\,]$, $\mathbf{C} = \mathcal{F}_o$. In other words, $\hat{\Sigma}'_o$ satisfies the conditions H1)-H3). Since both the operators $\tilde{\mathcal{D}}_o$, $\tilde{\mathcal{B}}_o$ in $\hat{\Sigma}'_o$ and $\mathcal{D}_o$, $\mathcal{B}_o$ in $\hat{\Sigma}_o$ are bounded, H1) still holds for $\hat{\Sigma}_o$. While $\hat{\Sigma}_o$ satisfies H2),H3) as the corresponding operators are not perturbed. Thus $\hat{\Sigma}_o$ are in the Pritchard-Salamon systems.

(b) We first prepare the following equalities for $\hat{\Sigma}$ and $\hat{\Sigma}_o$.

$$\mathcal{A}_o\mathcal{G}\phi = \mathcal{G}\mathcal{A}\phi, \quad \phi \in \mathcal{X} \tag{73}$$

$$\mathcal{F}_o\mathcal{G}\phi = \mathcal{F}\phi, \quad \phi \in \mathcal{X} \tag{74}$$

The equality (73) follows from

$$\langle \psi, \mathcal{A}_o\mathcal{G}\phi\rangle_{\mathcal{X}_o} = \langle \mathcal{A}^*\mathcal{G}^*\psi, \phi\rangle_{\mathcal{X}}, \ \ \forall\psi \in \mathcal{X}_o, \phi \in \mathcal{X}, \tag{75}$$

which is verified by straightforward calculation with (13),(21),(22),(23). While (74) is obtained with (17),(22),(28).

By (26),(27),(73),(74), the states of $\hat{\Sigma}$, $\hat{\Sigma}_o$ satisfy

$$\hat{x}_o(t) = \mathcal{G}\hat{x}(t), \quad \hat{x}(t) \in \mathcal{X}. \tag{76}$$

Thus, for $(w,u) \in L_2(0,t;\, \mathbb{R}^{l+m})$, the mild solutions of $\hat{\Sigma}$ with $\hat{x}(0) \in \mathcal{X}$ and $\hat{\Sigma}_o$ with $\hat{x}_o(0) = \mathcal{G}\hat{x}(0)$ satisfies (76).

(c) Since $\mathcal{F}\hat{x}(t) = \mathcal{F}_o\hat{x}_o(t)$ by (74) and (b) holds, the systems $\hat{\Sigma}$ with $\hat{x}(0) = 0$ and $\hat{\Sigma}_o$ with $\hat{x}_o(0) = 0$ generate the same output $z \in L_2(0,t;\, \mathbb{R}^p)$ for any $(w,u) \in L_2(0,t;\, \mathbb{R}^{l+m})$. ■

C. Proof of Lemma 4

(a) Let $\mathcal{S}_o \ge 0$ be the stabilizing solution to (30), then the system

$$\begin{aligned} \dot{\hat{x}}_o(t) &= \left(\mathcal{A}_o - \mathcal{B}_o\mathcal{B}_o^*\mathcal{S}_o + \tfrac{1}{\gamma^2}\cdot\mathcal{D}_o\mathcal{D}_o^*\mathcal{S}_o\right)\hat{x}_o(t), \\ \hat{x}_o(0) &= \mathcal{G}\Xi\phi, \ \phi \in \mathcal{X} \end{aligned} \tag{77}$$

or

$$\begin{aligned} \dot{\hat{x}}_o(t) &= \mathcal{A}_o\hat{x}_o(t) + \mathcal{D}_o w(t) + \mathcal{B}_o u(t), \\ \hat{x}_o(0) &= \mathcal{G}\Xi\phi, \quad \phi \in \mathcal{X} \\ w(t) &= \tfrac{1}{\gamma^2}\cdot\mathcal{D}_o^*\mathcal{S}_o\hat{x}_o(t), \quad u(t) = -\mathcal{B}_o^*\mathcal{S}_o\hat{x}_o(t) \end{aligned} \tag{78}$$

is exponentially stable on $\mathcal{X}_o$ and the following inequalities hold for $k > 0$:

$$\begin{aligned} &\|\hat{x}_o(t)\|_{\mathcal{X}_o} \le c_1\cdot e^{-kt}, \ \|w(t)\|_{\mathbb{R}^l} \le c_2\cdot e^{-kt}, \\ &\|u(t)\|_{\mathbb{R}^m} \le c_3\cdot e^{-kt} \ \ (c_1, c_2, c_3 > 0). \end{aligned} \tag{79}$$

In order to verify (a), we will show that the system

$$\dot{\hat{x}}(t) = \mathcal{A}\hat{x}(t) + \mathcal{D}w(t) + \mathcal{B}u(t), \ \hat{x}(0) = \Xi\phi, \ \phi \in \mathcal{X} \tag{80}$$

$$w(t) = \tfrac{1}{\gamma^2}\cdot\mathcal{D}^*\mathcal{S}\hat{x}(t), \quad u(t) = -\mathcal{B}^*\mathcal{S}\hat{x}(t), \quad \mathcal{S} = \mathcal{G}^*\mathcal{S}_o\mathcal{G}$$

is exponentially stable. Since (26),(27) and Lemma 3(b) yield

$$w(t) = \tfrac{1}{\gamma^2}\cdot\mathcal{D}^*\mathcal{S}\hat{x}(t) = \tfrac{1}{\gamma^2}\cdot\mathcal{D}_o^*\mathcal{S}_o\hat{x}_o(t) \tag{81}$$

$$u(t) = -\mathcal{B}^*\mathcal{S}\hat{x}(t) = -\mathcal{B}_o^*\mathcal{S}_o\hat{x}_o(t) \tag{82}$$

for the systems (78) and (80), the following inequality is obtained by Lemma 3(b).

$$\|\mathcal{G}\hat{x}(t)\|_{\mathcal{X}_o} = \|\hat{x}_o(t)\|_{\mathcal{X}_o} \le c_1\cdot e^{-kt} \tag{83}$$

By (20),(22),(79), the inequalities

$$\begin{aligned} &\|\mathcal{G}\hat{x}(t)\|_{\mathcal{X}_o} \ge \\ &\left\| e^{AL}x(t) + \int_{-L}^{0} e^{-A\beta}\,[D\,B]\begin{bmatrix} w_t(\beta) \\ u_t(\beta) \end{bmatrix} d\beta \right\|_{\mathbb{R}^n} \end{aligned} \tag{84}$$

$$\left\| \int_{-L}^{0} e^{-A\beta}\,[D\,B]\begin{bmatrix} w_t(\beta) \\ u_t(\beta) \end{bmatrix} d\beta \right\|_{\mathbb{R}^n} \le c_4\cdot e^{-kt} \ (c_4 > 0) \tag{85}$$

are obtained. Hence, by triangle inequality with (83),(84),(85), the following inequality is derived.

$$
\begin{aligned}
&\|e^{AL}x(t)\|_{\mathbb{R}^n} \\
&\le \left\| e^{AL}x(t) + \int_{-L}^{0} e^{-A\beta}\,[D\;B]\begin{bmatrix} w_t(\beta) \\ u_t(\beta)\end{bmatrix} d\beta \right\|_{\mathbb{R}^n} \\
&\quad + \left\| -\int_{-L}^{0} e^{-A\beta}\,[D\;B]\begin{bmatrix} w_t(\beta) \\ u_t(\beta)\end{bmatrix} d\beta \right\|_{\mathbb{R}^n} \\
&\le (c_1 + c_4)\cdot e^{-kt}
\end{aligned}
\tag{86}
$$

As follows from (79),(86), $\|\hat{x}(t)\|_{\mathcal{X}} \le c_5 \cdot e^{-kt}$ ($c_5 > 0$) holds for the system (80).

(b) ($\Rightarrow$) Suppose the H^∞ control problem $\hat{\Sigma}$ is solvable and an H^∞ control law is given by (31). Then the system

$$
\begin{aligned}
\dot{\hat{x}}(t) &= (\mathcal{A} + \mathcal{B}\mathcal{K})\hat{x}(t) + \mathcal{D}w(t), \\
&\qquad \mathcal{K} := -\mathcal{B}^*\mathcal{S}, \quad \hat{x}(0) = \Xi\phi, \quad \phi \in \mathcal{X} \qquad (87) \\
z(t) &= (\mathcal{F} + F_0\mathcal{K})\hat{x}(t)
\end{aligned}
$$

is exponentially stable and defines the mild solution as follows:

$$
\hat{x}(t) = T_{\mathcal{K}}(t)\Xi\phi + \int_0^T T_{\mathcal{K}}(t-\sigma)\mathcal{D}w(\sigma)\,d\sigma \tag{88}
$$

where $T_{\mathcal{K}}(t)$ is the strongly continuous semigroup generated by $\mathcal{A} + \mathcal{B}\mathcal{K}$. It follows from Lemma 3(b),(c) that a control

$$
u(t) = \mathcal{K}\left(T_{\mathcal{K}}(t)\Xi\phi + \int_0^T T_{\mathcal{K}}(t-\sigma)\mathcal{D}w(\sigma)\,d\sigma \right), \tag{89}
$$

which is causal of $\hat{y}_o(t)$, exponentially stabilizes $\hat{\Sigma}_o$ and the resulting system provides equivalent mapping to (87). Hence an H^∞ control exists for $\hat{\Sigma}_o$.

($\Leftarrow$) proved by (a). ■

D. Proof of Lemma 5

Define a generalized plant

$$
\tilde{\Sigma}: \begin{aligned}
\dot{x}(t) &= Ax(t) + Dw(t-L) + Bu(t) \\
z(t) &= Fx(t) + F_0u(t) \\
y(t) &= \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}
\end{aligned} \tag{90}
$$

associated with the system (1). Then the H^∞ control problem $\tilde{\Sigma}$ is solvable only if the problem Σ is solvable since all control delays are removed from control channels and maximal delays are imposed on the disturbances of $\tilde{\Sigma}$. In other words, any H^∞ control for Σ can be applied to $\tilde{\Sigma}$ by including fictitious input delays in the control. By Theorem 1 [15], it is shown that the H^∞ control problem $\tilde{\Sigma}$ is solvable only if the matrix (34) does not have eigenvalues on the imaginary axis. Thus Lemma 5 is proved. ■

E. Proof of Theorem 6

We first prepare some operator equalities, which enable to solve the operator Riccati equation (30).

Lemma 19: The following equality holds for $\mathcal{V}_1$, $\mathcal{V}_2$:

$$
\mathcal{H}_{Lo}\begin{bmatrix} \mathcal{V}_1 \\ \mathcal{V}_2 \end{bmatrix}\phi = \begin{bmatrix} \mathcal{V}_1 \\ \mathcal{V}_2 \end{bmatrix}\mathcal{A}_{\Lambda o}\phi, \quad \phi \in \mathrm{D}(\mathcal{A}_{\Lambda o}) \tag{91}
$$

$$
\mathcal{H}_{Lo} \in \mathcal{L}(\mathcal{W}_o \times \mathcal{X}_o, \mathcal{X}_o \times \mathcal{W}_o^*):
$$

$$
\mathcal{H}_{Lo} := \begin{bmatrix} \mathcal{A}_o & -\mathcal{B}_{Lo}\mathcal{B}_{Lo}^* + \frac{1}{\gamma^2}\cdot\mathcal{D}_{Lo}\mathcal{D}_{Lo}^* \\ -\mathcal{F}_o^*\mathcal{F}_o & -\mathcal{A}_o^* \end{bmatrix} \tag{92a}
$$

$$
\mathcal{A}_{\Lambda o}\phi = \begin{bmatrix} \Lambda\phi^0 \\ \phi^{1'} \end{bmatrix}, \quad \mathrm{D}(\mathcal{A}_{\Lambda o}) = \{\phi \in \mathcal{X}_o : \phi^1 \in W^{1,2}(-L,0;\, \mathbb{R}^p), FV_1\phi^0 = \phi^1(0)\} \tag{92b}
$$

$$
\mathcal{D}_{Lo} := \mathcal{G}\mathcal{D}_L \in \mathcal{L}(\mathbb{R}^{m_1}, \mathcal{X}_o), \tag{92c}
$$

$$
\mathcal{B}_{Lo} := \mathcal{G}\mathcal{B}_L \in \mathcal{L}(\mathbb{R}^{m_2}, \mathcal{X}_o), \tag{92d}
$$

$$
\mathcal{D}_L \in \mathcal{L}(\mathbb{R}^{m_1}, \mathcal{V}):\ \mathcal{D}_L^*\psi = \psi^1(0),\ \psi \in \mathcal{V}^* \tag{92e}
$$

$$
\mathcal{B}_L \in \mathcal{L}(\mathbb{R}^{m_2}, \mathcal{V}):\ \mathcal{B}_L^*\psi = \psi^2(0),\ \psi \in \mathcal{V}^* \tag{92f}
$$

where Λ is a stable matrix defined by (35). ■

Proof: It is noted that the operators $\mathcal{D}_{Lo}$, $\mathcal{B}_{Lo}$ are explicitly given based on (92c)-(92f),(23).

$$
\mathcal{D}_{Lo}^*\psi = \mathcal{D}_L^*\mathcal{G}^*\psi = (\mathcal{G}^*\psi)^1(0) = D^{\mathrm{T}}\psi^0, \quad \psi \in \mathcal{X}_o \tag{93a}
$$

$$
\mathcal{D}_{Lo}w = \begin{bmatrix} Dw \\ 0 \end{bmatrix} \in \mathcal{X}_o, \quad w \in \mathbb{R}^{m_1} \tag{93b}
$$

$$
\mathcal{B}_{Lo}^*\psi = \mathcal{B}_L^*\mathcal{G}^*\psi = (\mathcal{G}^*\psi)^2(0) = B^{\mathrm{T}}\psi^0, \quad \psi \in \mathcal{X}_o \tag{93c}
$$

$$
\mathcal{B}_{Lo}u = \begin{bmatrix} Bu \\ 0 \end{bmatrix} \in \mathcal{X}_o, \quad u \in \mathbb{R}^{m_2} \tag{93d}
$$

In order to derive (91), we verify the following equalities.

$$
\mathcal{A}_o\mathcal{V}_1\phi + (-\mathcal{B}_{Lo}\mathcal{B}_{Lo}^* + \tfrac{1}{\gamma^2}\cdot\mathcal{D}_{Lo}\mathcal{D}_{Lo}^*)\mathcal{V}_2\phi = \mathcal{V}_1\mathcal{A}_{\Lambda o}\phi \tag{94}
$$

$$
-\mathcal{F}_o^*\mathcal{F}_o\mathcal{V}_1\phi - \mathcal{A}_o^*\mathcal{V}_2\phi = \mathcal{V}_2\mathcal{A}_{\Lambda o}\phi, \quad \phi \in \mathrm{D}(\mathcal{A}_{\Lambda o}) \tag{95}
$$

By (34),(35),(93a)-(93d) with the fact $\mathcal{V}_1\phi \in \mathrm{D}(\mathcal{A}_o) = \mathcal{W}_o$ ($\phi \in \mathrm{D}(\mathcal{A}_{\Lambda o})$), (94) is obtained as follows:

$$
\begin{aligned}
&\mathcal{A}_o\mathcal{V}_1\phi + (-\mathcal{B}_{Lo}\mathcal{B}_{Lo}^* + \tfrac{1}{\gamma^2}\cdot\mathcal{D}_{Lo}\mathcal{D}_{Lo}^*)\mathcal{V}_2\phi \\
&= \begin{bmatrix} (AV_1 - BB^{\mathrm{T}}V_2 + \frac{1}{\gamma^2}\cdot DD^{\mathrm{T}}V_2)\phi^0 \\ \phi^{1'} \end{bmatrix} \\
&= \begin{bmatrix} V_1\Lambda\phi^0 \\ \phi^{1'} \end{bmatrix} = \mathcal{V}_1\mathcal{A}_{\Lambda o}\phi, \quad \phi \in \mathrm{D}(\mathcal{A}_{\Lambda o}).
\end{aligned} \tag{96}
$$

While, we have following equalities on the left hand side of (95).

$$
\begin{aligned}
&\langle \psi, -\mathcal{F}_o^*\mathcal{F}_o\mathcal{V}_1\phi - \mathcal{A}_o^*\mathcal{V}_2\phi \rangle_{\mathcal{W}_o,\mathcal{W}_o^*} \\
&= -\langle \mathcal{F}_o\psi, \mathcal{F}_o\mathcal{V}_1\phi \rangle_{\mathbb{R}^p} - \langle \mathcal{A}_o\psi, \mathcal{V}_2\phi \rangle_{\mathcal{X}_o} \\
&= -\psi^{1\mathrm{T}}(-L)\phi^1(-L) - \left\langle \begin{bmatrix} A\psi^0 \\ \psi^{1'} \end{bmatrix}, \begin{bmatrix} V_2\phi^0 \\ \phi^1 \end{bmatrix} \right\rangle_{\mathcal{X}_o}, \\
&\qquad \psi \in \mathrm{D}(\mathcal{A}_o) = \mathcal{W}_o, \quad \phi \in \mathrm{D}(\mathcal{A}_{\Lambda_o})
\end{aligned} \tag{97}
$$

$$
\int_{-L}^{0} \psi^{1\mathrm{T}'}(\beta)\phi^1(\beta)\,d\beta = -\int_{-L}^{0} \psi^{1\mathrm{T}}(\beta)\phi^{1'}(\beta)\,d\beta + \psi^{1\mathrm{T}}(0)\phi^1(0) - \psi^{1\mathrm{T}}(-L)\phi^1(-L) \tag{98}
$$

$$
\psi^{1\mathrm{T}}(0)\phi^1(0) = \psi^{0\mathrm{T}}F^{\mathrm{T}}FV_1\phi^0 \tag{99}
$$

Hence, by (97),(98),(99) and (35), the following equality is obtained for $\psi \in \mathrm{D}(\mathcal{A}_o) = \mathcal{W}_o$, $\phi \in \mathrm{D}(\mathcal{A}_{\Lambda_o})$:

$$
\begin{aligned}
&\langle \psi, -\mathcal{F}_o^*\mathcal{F}_o\mathcal{V}_1\phi - \mathcal{A}_o^*\mathcal{V}_2\phi \rangle_{\mathcal{W}_o,\mathcal{W}_o^*} \\
&= \left\langle \psi, \begin{bmatrix} (-F^{\mathrm{T}}FV_1 - A^{\mathrm{T}}V_2)\phi^0 \\ \phi^{1'} \end{bmatrix} \right\rangle_{\mathcal{W}_o,\mathcal{W}_o^*} \\
&= \left\langle \psi, \begin{bmatrix} V_2\Lambda\phi^0 \\ \phi^{1'} \end{bmatrix} \right\rangle_{\mathcal{W}_o,\mathcal{W}_o^*} = \langle \psi, \mathcal{V}_2\mathcal{A}_{\Lambda o}\phi \rangle_{\mathcal{W}_o,\mathcal{W}_o^*}.
\end{aligned} \tag{100}
$$

Thus (95) holds. ∎

In Lemma 19, it should be noted that the operator $\mathcal{A}_{\Lambda o}$ generates exponentially stable semigroups on $\mathcal{X}_o$ since the structure of $\mathcal{A}_{\Lambda o}$ is same as (21) and the matrix Λ defined by (35) is stable. Based on Lemma 19, an Hamiltonian representation for (30) is obtained as follows.

Lemma 20: For the operator Riccati equation (30), the Hamiltonian representation

$$\mathcal{H}_o \begin{bmatrix} \mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2 \\ \mathcal{V}_2 \end{bmatrix} \phi = \begin{bmatrix} \mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2 \\ \mathcal{V}_2 \end{bmatrix} \mathcal{A}_{\Lambda o}\phi, \quad \phi \in \mathrm{D}(\mathcal{A}_{\Lambda o}) \tag{101}$$

$$\mathcal{H}_o \in \mathcal{L}(\mathcal{W}_o \times \mathcal{X}_o, \mathcal{X}_o \times \mathcal{W}_o^*): \quad \mathcal{H}_o := \begin{bmatrix} \mathcal{A}_o & -\mathcal{B}_o\mathcal{B}_o^* + \frac{1}{\gamma^2}\cdot\mathcal{D}_o\mathcal{D}_o^* \\ -\mathcal{F}_o^*\mathcal{F}_o & -\mathcal{A}_o^* \end{bmatrix} \tag{102}$$

holds. ∎

Proof: We first verify the following equalities.

$$\mathcal{F}\Pi\phi = 0, \quad \phi \in \mathcal{X} \tag{103}$$

$$\Pi\mathcal{A}^*\psi + \mathcal{A}\Pi\psi + \frac{1}{\gamma^2}\cdot(\mathcal{D}\mathcal{D}^* - \mathcal{D}_L\mathcal{D}_L^*)\psi - (\mathcal{B}\mathcal{B}^* - \mathcal{B}_L\mathcal{B}_L^*)\psi = 0, \quad \psi \in \mathcal{V}^* \tag{104}$$

Equation (103) is obtained by (17),(38). On the left hand side of (104), we have a following equality.

$$\begin{aligned} &\langle\tilde{\psi}, \{\Pi\mathcal{A}^* + \mathcal{A}\Pi + \tfrac{1}{\gamma^2}\cdot(\mathcal{D}\mathcal{D}^* - \mathcal{D}_L\mathcal{D}_L^*) - (\mathcal{B}\mathcal{B}^* - \mathcal{B}_L\mathcal{B}_L^*)\}\psi\rangle_{\mathcal{V}^*,\mathcal{V}} \\ &= \langle\Pi\tilde{\psi}, \mathcal{A}^*\psi\rangle_{\mathcal{X}} + \langle\mathcal{A}^*\tilde{\psi}, \Pi\psi\rangle_{\mathcal{X}} \\ &\quad -\tfrac{1}{\gamma^2}\cdot\langle\mathcal{D}_L^*\tilde{\psi}, \mathcal{D}_L^*\psi\rangle_{\mathbb{R}^{m_1}} + \tfrac{1}{\gamma^2}\cdot\langle\mathcal{D}^*\tilde{\psi}, \mathcal{D}^*\psi\rangle_{\mathbb{R}^{m_1}} \\ &\quad +\langle\mathcal{B}_L^*\tilde{\psi}, \mathcal{B}_L^*\psi\rangle_{\mathbb{R}^{m_2}} - \langle\mathcal{B}^*\tilde{\psi}, \mathcal{B}^*\psi\rangle_{\mathbb{R}^{m_2}}, \quad \tilde{\psi}, \psi \in \mathcal{V}^* \end{aligned} \tag{105}$$

By (12),(13),(38), some terms in (105) are calculated as follows:

$$\begin{aligned} &\langle\Pi\tilde{\psi}, \mathcal{A}^*\psi\rangle_{\mathcal{X}} + \langle\mathcal{A}^*\tilde{\psi}, \Pi\psi\rangle_{\mathcal{X}} \\ &= \tfrac{1}{\gamma^2}\cdot\sum_{i=0}^{d}\int_{-L+h_i}^{0}(\tilde{\psi}_i^{1\mathrm{T}}(\beta)\psi_i^{1'}(\beta) + \tilde{\psi}_i^{1\mathrm{T}'}(\beta)\psi_i^1(\beta))\,d\beta \\ &\quad -\sum_{i=0}^{d}\int_{-L+h_i}^{0}(\tilde{\psi}_i^{2\mathrm{T}}(\beta)\psi_i^{2'}(\beta) + \tilde{\psi}_i^{2\mathrm{T}'}(\beta)\psi_i^2(\beta))\,d\beta \\ &= \tfrac{1}{\gamma^2}\cdot\sum_{i=0}^{d}\left[\tilde{\psi}_i^{1\mathrm{T}}(\beta)\psi_i^1(\beta)\right]_{\beta=-L+h_i}^{\beta=0} \\ &\quad -\sum_{i=0}^{d}\left[\tilde{\psi}_i^{2\mathrm{T}}(\beta)\psi_i^2(\beta)\right]_{\beta=-L+h_i}^{\beta=0}, \quad \tilde{\psi}, \psi \in \mathcal{V}^* \end{aligned} \tag{106}$$

and, further, the equalities

$$-\tfrac{1}{\gamma^2}\cdot\langle\mathcal{D}_L^*\tilde{\psi}, \mathcal{D}_L^*\psi\rangle_{\mathbb{R}^{m_1}} + \tfrac{1}{\gamma^2}\cdot\langle\mathcal{D}^*\tilde{\psi}, \mathcal{D}^*\psi\rangle_{\mathbb{R}^{m_1}} = -\tfrac{1}{\gamma^2}\cdot\sum_{i=0}^{d}\left[\tilde{\psi}_i^{1\mathrm{T}}(\beta)\psi_i^1(\beta)\right]_{\beta=-L+h_i}^{\beta=0}, \tag{107}$$

$$\langle\mathcal{B}_L^*\tilde{\psi}, \mathcal{B}_L^*\psi\rangle_{\mathbb{R}^{m_2}} - \langle\mathcal{B}^*\tilde{\psi}, \mathcal{B}^*\psi\rangle_{\mathbb{R}^{m_2}} = \sum_{i=0}^{d}\left[\tilde{\psi}_i^{2\mathrm{T}}(\beta)\psi_i^2(\beta)\right]_{\beta=-L+h_i}^{\beta=0}, \quad \tilde{\psi}, \psi \in \mathcal{V}^* \tag{108}$$

are obtained from (15),(16),(92e),(92f) where the components of $\tilde{\psi}$, ψ are denoted along (14). Hence, by (105)-(108), (104) holds.

Next, applying an invertible operator

$$\mathcal{J} := \begin{bmatrix} \mathcal{I} & \mathcal{G}\Pi\mathcal{G}^* \\ 0 & \mathcal{I} \end{bmatrix} \in \mathcal{L}(\mathcal{X}_o, \mathcal{W}_o^*) \tag{109}$$

to (91), we will show that

$$\mathcal{J}\mathcal{H}_{Lo}\mathcal{J}^{-1}\mathcal{J}\begin{bmatrix} \mathcal{V}_1 \\ \mathcal{V}_2 \end{bmatrix}\phi = \mathcal{J}\begin{bmatrix} \mathcal{V}_1 \\ \mathcal{V}_2 \end{bmatrix}\mathcal{A}_{\Lambda o}\phi, \phi \in \mathrm{D}(\mathcal{A}_{\Lambda o}) \tag{110}$$

yields (101). The equality $\mathcal{J}\begin{bmatrix} \mathcal{V}_1 \\ \mathcal{V}_2 \end{bmatrix} = \begin{bmatrix} \mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^* \\ \mathcal{V}_2 \end{bmatrix}$ holds and, by (73),(74),(92c),(92d),(103),(104),

$$\mathcal{J}\mathcal{H}_{Lo}\mathcal{J}^{-1} = \mathcal{H}_o \in \mathcal{L}(\mathcal{W}_o \times \mathcal{X}_o, \mathcal{X}_o \times \mathcal{W}_o^*) \tag{111}$$

is verified directly. Thus (101),(102) are obtained. ∎

Proof of Theorem 6:

(a) ⇔ **(b):** (⇒) Applying $[-\mathcal{S}_o,\ \mathcal{I}]$ to (101), then using (30), we have

$$\mathcal{A}_{So}^*\mathcal{T}\phi + \mathcal{T}\mathcal{A}_{\Lambda o}\phi = 0, \quad \phi \in \mathrm{D}(\mathcal{A}_{\Lambda o}), \tag{112a}$$

$$\mathcal{T} := \mathcal{S}_o\mathcal{V}_1 - \mathcal{V}_2 + \mathcal{S}_o\mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2, \tag{112b}$$

$$\mathcal{A}_{So} := \mathcal{A}_o - \mathcal{B}_o\mathcal{B}_o^*\mathcal{S}_o + \tfrac{1}{\gamma^2}\cdot\mathcal{D}_o\mathcal{D}_o^*\mathcal{S}_o. \tag{112c}$$

Since $\mathcal{A}_{So}$ and $\mathcal{A}_{\Lambda o}$ generate exponentially stable semigroups on $\mathcal{X}_o$, the Lyapunov equation (112a) requires $\mathcal{T} = 0$ ([3],[29], Lemma 2.32). By Remark 7, there exists a bounded inverse of (36) iff it does not have any eigenvalue at origin. We verify by contradiction that the operator $\mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2$ is invertible. Suppose that $\mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2$ has an eigenvalue 0 and

$$(\mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2)v = 0 \tag{113}$$

holds for $v = (v^0, v^1) \neq 0$. Then the equalities

$$\mathcal{T}v = -\mathcal{V}_2 v = 0, \quad \mathcal{V}_1 v = 0 \tag{114}$$

hold by (112b) and (113). Furthermore (114), (37) imply that there exists $v^0 \in \mathbb{R}^n$ such that

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} v^0 = 0, \quad v^0 \neq 0, \quad v^1 = 0 \tag{115}$$

holds for $v = (v^0, v^1)$. This contradicts the fact that the matrix $V = [\,V_1^{\mathrm{T}}\ V_2^{\mathrm{T}}\,]^{\mathrm{T}} \in \mathbb{R}^{2n\times n}$ is column full-rank. Thus (b) is derived.

(⇐) Since (101),(102) hold and (b) assumes the existence of bounded inverse for $\mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2$, (41) is one of the bounded solution to (30). Applying $[\mathcal{I},\ 0]$ to (101), we have

$$\mathcal{A}_{So}(\mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2)\phi = (\mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2)\mathcal{A}_{\Lambda o}\phi, \quad \phi \in \mathrm{D}(\mathcal{A}_{\Lambda o}) \tag{116}$$

where $\mathcal{A}_{\Lambda o}$ generates exponentially stable semigroups on $\mathcal{X}_o$. Thus $\mathcal{A}_{So}$ generates exponentially stable semigroups on $\mathcal{X}_o$ and (41) provides the stabilizing solution.

(b)⇔(c): By contraposition, we will show that the operator $\mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2$ has an eigenvalue 0 iff the matrix U_0 is singular. By the equality $(\mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2)v = 0$, $\mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2$ has an eigenvalue 0 iff there exists $v \neq 0$ such that

$$\mathcal{V}_1 v = \mathcal{G}u, \quad -u = \Pi\mathcal{G}^*\mathcal{V}_2 v \tag{117}$$

hold. In the following, we first show that the matrix U_0 is singular if $u \neq 0$, $v \neq 0$ exist. Introducing auxiliary variables

$$p(\beta) := -e^{-A^{\mathrm{T}}\beta} V_2 v^0 - \int_{\beta}^{0} e^{A^{\mathrm{T}}(\xi-\beta)} F^{\mathrm{T}} v^1(\xi)\, d\xi \quad (118a)$$

$$q(\xi) := e^{A(\xi+L)} u^0 + \int_{-L}^{\xi} e^{A(\xi-\beta)}\, [D\, B] \begin{bmatrix} u^1(\beta) \\ u^2(\beta) \end{bmatrix} d\beta \quad (118b)$$

to the right and left equalities in (117) respectively, we have the following relations:

$$u^0 = 0 \quad (119a)$$

$$u_i^1(\beta) = -\tfrac{1}{\gamma^2} \cdot \chi_{[-L+h_i,0]}(\beta) \cdot D_i^{\mathrm{T}} p(\beta) \quad (119b)$$

$$u_i^2(\beta) = \chi_{[-L+h_i,0]}(\beta) \cdot B_i^{\mathrm{T}} p(\beta) \quad (i = 0, 1, \cdots, d) \quad (119c)$$

$$p'(\beta) = -A^{\mathrm{T}} p(\beta) + F^{\mathrm{T}} v^1(\beta), \quad -L \leq \beta \leq 0 \quad (119d)$$

$$p(0) = -V_2 v^0 \quad (119e)$$

$$V_1 v^0 = q(0) \quad (119f)$$

$$v^1(\xi) = F q(\xi) \quad (119g)$$

$$q'(\xi) = A q(\xi) + D u^1(\xi) + B u^2(\xi), \quad -L \leq \xi \leq 0 \quad (119h)$$

$$q(-L) = u^0. \quad (119i)$$

Combining the conditions (119b),(119c),(119d),(119g),(119h), we have

$$\begin{bmatrix} p'(\xi) \\ q'(\xi) \end{bmatrix} = -J_i^{\mathrm{T}} \begin{bmatrix} p(\xi) \\ q(\xi) \end{bmatrix}, \quad \begin{matrix} -L + h_i \leq \xi \leq -L + h_{i+1} \\ (i = 0, 1, \cdots, d-1) \end{matrix} \quad (120)$$

and, further, (120) and (119a),(119e),(119f),(119i) yield the following conditions

$$\begin{bmatrix} p(0) \\ q(0) \end{bmatrix} = \Psi_d(-L) \begin{bmatrix} p(-L) \\ q(-L) \end{bmatrix} \quad (121)$$

$$V_1^{\mathrm{T}} p(0) + V_2^{\mathrm{T}} q(0) = 0, \quad q(-L) = 0 \quad (122)$$

where (43) is imposed in (122). By (121),(122), we finally obtain the equality:

$$\begin{bmatrix} V_1^{\mathrm{T}}\, V_2^{\mathrm{T}} \end{bmatrix} \Psi_d(-L) \begin{bmatrix} I \\ 0 \end{bmatrix} p(-L) = U_0 p(-L) = 0. \quad (123)$$

If $p(-L) = 0$, (122),(120) yield $p(\cdot) \equiv 0$, $q(\cdot) \equiv 0$ and, further with (119f),(118a),(119g), we have

$$[\, V_1^{\mathrm{T}}\ V_2^{\mathrm{T}}\,]^{\mathrm{T}} v^0 = 0, \quad v^1 = 0 \quad (124)$$

for $v = (v^0, v^1)$, where $V = [\, V_1^{\mathrm{T}}\ V_2^{\mathrm{T}}\,]^{\mathrm{T}}$ is column full-rank by (35). Hence it is shown that $v = 0$ if $p(-L) = 0$. In other words, the matrix U_0 must be singular if $\mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2$ has an eigenvalue 0. Next we will show $v \neq 0$ if $p(-L) \neq 0$ exists in (123). Suppose $v = 0$ holds even if $p(-L) \neq 0$. Then (119d),(119e) derive $p(\cdot) \equiv 0$. Hence, by contradiction, $v \neq 0$ if $p(-L) \neq 0$.

Thus, $\mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2$ does not have any eigenvalue at origin iff the matrix U_0 is nonsingular. The representation (41) is obtained in the proof (a)⇐(b). ■

F. Proof of Theorem 10

Based on Remark 8, we will show that $\mathcal{S}_o \geq 0$ holds for the operator (41) iff the matrix (47) is nonsingular for $\mu < 0$.

We first verify that $\lambda =: \mu^{-1} \neq 0$ is an eigenvalue of $\mathcal{S}_o$ iff there exists $f \neq 0$ such that

$$(\mathcal{V}_1 - \mu \cdot \mathcal{V}_2) f = \mathcal{G} g, \quad -g = \Pi \mathcal{G}^* \mathcal{V}_2 f \quad (125)$$

hold. Let $v \neq 0$ be the corresponding eigenvector of $\mathcal{S}_o$ and

$$\lambda v = \mathcal{V}_2 (\mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2)^{-1} v \quad (126)$$

holds. Then $f = (\mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2)^{-1} v \neq 0$, $g = -\Pi\mathcal{G}^*\mathcal{V}_2 f$ satisfy (125). Hence $f \neq 0$ exists. Conversely, let $f \neq 0$ exists for (125). Then we have an equality

$$\lambda \mathcal{V}_2 f = \mathcal{V}_2 (\mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2)^{-1} \mathcal{V}_2 f \quad (127)$$

from (125). If $\mathcal{V}_2 f = 0$, (125) requires $\mathcal{V}_1 f = 0$ and, further, $f = 0$ since $\mathcal{V}_1 f = \mathcal{V}_2 f = 0$ for $f = (f^1, f^2) \in \mathcal{X}_o$ yields $[V_1^{\mathrm{T}} V_2^{\mathrm{T}}]^{\mathrm{T}} f^0 = 0$, $f^1 = 0$, where $V = [V_1^{\mathrm{T}} V_2^{\mathrm{T}}]^{\mathrm{T}}$ defined by (35) is column full-rank. Hence the eigenvector $v = \mathcal{V}_2 f \neq 0$ exists for (126).

For the eigenvalue $\lambda = \mu^{-1} \neq 0, 1$ of $\mathcal{S}_o$, we will show that there exists $f \neq 0$ to (125) iff $\tilde{U}(\mu)$ is singular. Introducing auxiliary variables

$$\tilde{p}(\beta) := -e^{-A^{\mathrm{T}}\beta} V_2 f^0 - \int_{\beta}^{0} e^{A^{\mathrm{T}}(\xi-\beta)} F^{\mathrm{T}} f^1(\xi)\, d\xi \quad (128a)$$

$$\tilde{q}(\xi) := e^{A(\xi+L)} g^0 + \int_{-L}^{\xi} e^{A(\xi-\beta)}\, [D\, B] \begin{bmatrix} g^1(\beta) \\ g^2(\beta) \end{bmatrix} d\beta \quad (128b)$$

to the right and left equalities in (125) respectively, then combining the relations along the manipulation in the proof of Theorem 6 (b)⇔(c), we have

$$\begin{bmatrix} \tilde{p}'(\beta) \\ \tilde{q}'(\beta) \end{bmatrix} = -\tilde{J}_i^{\mathrm{T}} \begin{bmatrix} \tilde{p}(\beta) \\ \tilde{q}(\beta) \end{bmatrix}, \quad \begin{matrix} -L + h_i \leq \beta \leq -L + h_{i+1} \\ (i = 0, 1, \cdots, d-1) \end{matrix} \quad (129)$$

$$\text{i.e.} \quad \begin{bmatrix} \tilde{p}(0) \\ \tilde{q}(0) \end{bmatrix} = \tilde{\Psi}_d(-L) \begin{bmatrix} \tilde{p}(-L) \\ \tilde{q}(-L) \end{bmatrix} \quad (130)$$

and the following equalities:

$$g^0 = 0 \quad (131a)$$

$$\tilde{p}(0) = -V_2 f^0 \quad (131b)$$

$$(V_1 - \mu V_2) f^0 = \tilde{q}(0) \quad (131c)$$

$$\tilde{q}(-L) = g^0. \quad (131d)$$

Since (131b),(131c),(43) and (131a)(131d) yield the boundary conditions

$$(V_1 - \mu V_2)^{\mathrm{T}} \tilde{p}(0) + V_2^{\mathrm{T}} \tilde{q}(0) = 0, \quad \tilde{q}(-L) = 0, \quad (132)$$

we finally obtain an equality:

$$V^{\mathrm{T}} \begin{bmatrix} I & 0 \\ -\mu \cdot I & I \end{bmatrix} \tilde{\Psi}_d(-L) \begin{bmatrix} I \\ 0 \end{bmatrix} \tilde{p}(-L) = \tilde{U}(\mu) \tilde{p}(-L) = 0 \quad (133)$$

from (130),(132) for $\mu = \lambda^{-1}$.

If $\tilde{p}(-L) = 0$, (132),(129) yield $\tilde{p}(\cdot) \equiv 0$, $\tilde{q}(\cdot) \equiv 0$, and further with (131b),(131c), we have

$$[\, V_1^{\mathrm{T}}\ V_2^{\mathrm{T}}\,]^{\mathrm{T}} f^0 = 0, \quad f^1 = 0 \quad (134)$$

for $f = (f^1, f^2)$, where $V = [\, V_1^{\mathrm{T}}\ V_2^{\mathrm{T}}\,]^{\mathrm{T}}$ is column full-rank by (35). Hence $v = 0$ if $p(-L) = 0$. In other words, $\tilde{U}(\mu)$ must be singular if there exists an eigenvalue $\lambda = \mu^{-1}$.

Next we will show $f \neq 0$ if $\tilde{p}(-L) \neq 0$ exists in (133). Suppose $f = (f^0, f^1) = 0$ holds even if $\tilde{p}(-L) \neq 0$. Then (128a) derives $\tilde{p}(\cdot) \equiv 0$. Hence, by contradiction, $f \neq 0$ if $\tilde{p}(-L) \neq 0$.

Thus, for a given $\mu \neq 0, 1$, $\tilde{U}(\mu)$ is singular iff $f \neq 0$ exists for (125). Since $\lambda = \mu^{-1}$ is an eigenvalue of (41) iff $\tilde{U}(\mu)$ is singular, it is shown that the stabilizing solution (41) is positive semi-definite iff $\tilde{U}(\mu)$ is nonsingular for $\mu < 0$. ■

G. Proof of Theorem 11

In order to provide a representation of the control law $u(t) = -\mathcal{B}_o^* \mathcal{S}_o \mathcal{G} \hat{x}(t) = -\mathcal{B}^* \mathcal{G}^* \mathcal{S}_o \mathcal{G} \hat{x}(t)$, we first define the relations $u = -\mathcal{B}^* f$, $f = \mathcal{G}^* \mathcal{S}_o \mathcal{G} \Xi g$, $g := \hat{x}(t) = (x(t), w_t, u_t)$ ($u \in \mathbb{R}^m, f, g \in \mathcal{X}$) and elaborate the representation of u. Based on (20),(41), the equality $f = \mathcal{G}^* \mathcal{S}_o \mathcal{G} \Xi g$ is equivalently given by following relations:

$$f = \mathcal{G}^* \mathcal{V}_2 v, \quad \mathcal{V}_1 v = \mathcal{G}(\Xi g - \Pi f). \tag{135}$$

Eliminating the variable v, we will derive the representation (48), (49a)-(49f). Define $r_i^1 \in L_2(-L, 0; \mathbb{R}^{l_i})$ and $r_i^2 \in L_2(-L, 0; \mathbb{R}^{m_i})$ ($i = 0, 1, \cdots, d$):

$$r_i^1(\beta) = \begin{cases} g_i^1(\beta) & -L \le \beta \le -L + h_i \\ \gamma^{-2} \cdot f_i^1(\beta) & -L + h_i \le \beta \le 0 \end{cases} \tag{136a}$$

$$r_i^2(\beta) = \begin{cases} g_i^2(\beta) & -L \le \beta \le -L + h_i \\ -f_i^2(\beta) & -L + h_i \le \beta \le 0 \end{cases} \tag{136b}$$

then introducing auxiliary variables

$$\bar{p}(\beta) = -e^{-A^{\mathrm{T}}\beta} V_2 v^0 - \int_\beta^0 e^{A^{\mathrm{T}}(\xi - \beta)} F^{\mathrm{T}} v^1(\xi)\, d\xi \tag{137a}$$

$$\bar{q}(\xi) = e^{A(\xi+L)} g^0 + \int_{-L}^{\xi} e^{A(\xi-\beta)} \sum_{i=0}^{d} (D_i r_i^1(\beta) + B_i r_i^2(\beta))\, d\beta \tag{137b}$$

to the left and right equalities in (135), we have

$$\begin{aligned} \begin{bmatrix} \bar{p}(\xi) \\ \bar{q}(\xi) \end{bmatrix} &= \Phi(\xi)\Phi^{-1}(-L) \begin{bmatrix} 0 \\ I \end{bmatrix} g^0 \\ &+ \Phi(\xi)\Phi^{-1}(-L) \begin{bmatrix} I \\ 0 \end{bmatrix} \bar{p}(-L) \\ &+ \sum_{i=1}^{d} \int_{-L}^{\min(\xi, -L+h_i)} \Phi(\xi)\Phi^{-1}(\beta) \begin{bmatrix} 0 \\ I \end{bmatrix} \\ &\quad \times (D_i g_i^1(\beta) + B_i g_i^2(\beta))\, d\beta \end{aligned} \tag{138}$$

$$\begin{cases} \Phi(0) = I \\ \dfrac{d}{dt}\Phi(t) = -J_i^{\mathrm{T}} \Phi(t), \ -L + h_i \le t \le -L + h_{i+1} \\ \qquad\qquad (i = 0, 1, \cdots, d-1) \end{cases} \tag{139}$$

with the following equalities:

$$V_1^{\mathrm{T}} \bar{p}(0) + V_2^{\mathrm{T}} \bar{q}(0) = 0 \tag{140}$$

$$u_j = B_j^{\mathrm{T}} \bar{p}(-L + h_j) \quad (j = 0, 1, \cdots, d). \tag{141}$$

Pre-multiplying either $[I\ 0]$ or V^{T} to both sides of (138), we obtain the equalities:

$$\begin{aligned} \bar{p}(-L + h_j) &= W_0^j \bar{p}(-L) + W_1^j(-L) g^0 \\ &+ \sum_{i=1}^{d} \int_{-L}^{-L+\min(h_i, h_j)} W_1^j(\beta)(D_i g_i^1(\beta) + B_i g_i^2(\beta))\, d\beta, \\ &\qquad (j = 0, 1, \cdots, d) \end{aligned} \tag{142}$$

$$0 = U_0 \bar{p}(-L) + U_1(-L) g^0 + \sum_{i=1}^{d} \int_{-L}^{-L+h_i} U_1(\beta)(D_i g_i^1(\beta) + B_i g_i^2(\beta))\, d\beta. \tag{143}$$

Hence, eliminating $\bar{p}(-L + h_j)$ and $\bar{p}(-L)$ in (141)-(143), the control law is obtained by (48),(49a)-(49f). ■

H. Proof of Corollary 13

Since the solvability of the operator Riccati equations (29) and (30) are equivalent (Lemma 4(b)), we first characterize the solvability of (30) in terms of the stability of the closed loop system.

Lemma 21: On the stabilizing solution $\mathcal{S} \in \mathcal{L}(\mathcal{V}, \mathcal{V}^*)$, the following statements are equivalent.

(a) The stabilizing solution to the equation (30) is positive semi-definite ($\mathcal{S} \ge 0$).
(b) The operator $\mathcal{A} - \mathcal{B}\mathcal{B}^*\mathcal{S}$ generates exponentially stable semigroups on $\mathcal{X}$.
(c) The matrix $A - BB^{\mathrm{T}} U_0^{-1} U_1(-L)$ is stable. ■

Proof: (a)⇔(b): By Lemma 4 and Propositions $\mathbf{P}$, $\mathbf{P_o}$, it is shown that the stabilizing solution to (30) is positive semi-definite ($\mathcal{S} \ge 0$) iff the stabilizing solution to (29) is positive semi-definite. Suppose the stabilizing solution $\mathcal{S} \ge 0$ exists to (29), then the H^∞ control law (31) exponentially stabilizes $\hat{\Sigma}$ on $\mathcal{X}$. Hence (b) holds. Suppose (b) holds. By rewriting (29), we have an operator Lyapunov equation

$$\begin{aligned} &\mathcal{S}(\mathcal{A} - \mathcal{B}\mathcal{B}^*\mathcal{S})\phi + (\mathcal{A} - \mathcal{B}\mathcal{B}^*\mathcal{S})^*\mathcal{S}\phi + \mathcal{Q}\phi = 0, \ \ \phi \in \mathcal{X}, \\ &\qquad \mathcal{Q} := \mathcal{S}\mathcal{B}\mathcal{B}^*\mathcal{S} + \tfrac{1}{\gamma^2}\mathcal{S}\mathcal{D}\mathcal{D}^*\mathcal{S} + \mathcal{F}^*\mathcal{F} \ge 0, \end{aligned} \tag{144}$$

where $\mathcal{A} - \mathcal{B}\mathcal{B}^*\mathcal{S}$ generates exponentially stable semigroups on $\mathcal{X}$. Hence, by [3], there exists a stabilizing solution $\mathcal{S} \ge 0$. **(b)⇔(c):** Focus on the closed-loop system

$$\dot{\hat{x}}(t) = (\mathcal{A} - \mathcal{B}\mathcal{B}^*\mathcal{S})\hat{x}(t), \quad \hat{x}(0) \in \mathrm{D}(\mathcal{A}). \tag{145}$$

The solution $\hat{x}(t)$ is bounded over the interval $0 \le t \le L$ and, after the time $t = L$, the behavior $\hat{x}(t)$ is reduced to

$$\begin{aligned} &\hat{x}(t) = (x(t), 0, 0) \in \mathcal{X}, \\ &\dot{x}(t) = (A - BB^{\mathrm{T}} U_0^{-1} U_1(-L)) x(t). \end{aligned} \tag{146}$$

Thus the closed loop system (146) is exponentially stable iff (c) holds. ■

By Lemma 21, it is shown that the stabilizing solution to (29) is positive semi-definite iff the condition (c) holds. Hence the H^∞ control problem ($B_0 = [B_0, 0, \cdots, 0]$) is solvable iff the conditions stated in Corollary 13 holds. ■

I. Proof of Corollary 15

By using the solution $M = V_2 V_1^{-1} \geq 0$, the stabilizing solution (41) is represented as follows:

$$\mathcal{S}_o := \begin{bmatrix} M & 0 \\ 0 & \mathcal{I} \end{bmatrix} \left(\mathcal{I} + \mathcal{G}\Pi\mathcal{G}^* \begin{bmatrix} M & 0 \\ 0 & \mathcal{I} \end{bmatrix} \right)^{-1}. \tag{147}$$

Hence the stabilizing solution is positive semi-definite iff the maximal eigenvalue of $\mathcal{Q} := \mathcal{G}\Pi\mathcal{G}^* \begin{bmatrix} M & 0 \\ 0 & \mathcal{I} \end{bmatrix}$ is smaller than 1. In the following, we will show that the nonzero eigenvalues of $\mathcal{Q}$ are given by the roots of the transcendental equation (56). By the relations

$$\lambda v = -\mathcal{G}u, \quad u = -\Pi\mathcal{G}^* \begin{bmatrix} M & 0 \\ 0 & \mathcal{I} \end{bmatrix} v, \tag{148}$$

λ is an eigenvalue of $\mathcal{Q}$ iff there exists $v \neq 0$ such that the equalities (148) hold. Introducing auxiliary variables

$$\check{p}(\beta) := -e^{-A^{\mathrm{T}}\beta} M v^0 - \int_\beta^0 e^{A^{\mathrm{T}}(\xi-\beta)} F^{\mathrm{T}} v^1(\xi)\, d\xi \tag{149a}$$

$$\check{q}(\xi) := e^{A(\xi+L)} u^0 + \int_{-L}^{\xi} e^{A(\xi-\beta)} \left[D\ B\right] \begin{bmatrix} u^1(\beta) \\ u^2(\beta) \end{bmatrix} d\beta \tag{149b}$$

to the right and left equalities in (148) respectively, we have

$$\begin{bmatrix} \check{p}'(\beta) \\ \check{q}'(\beta) \end{bmatrix} = -\check{J}_i^{\mathrm{T}}(\mu) \begin{bmatrix} \check{p}(\beta) \\ \check{q}(\beta) \end{bmatrix},$$
$$-L + h_i \leq \beta \leq -L + h_{i+1} \quad (i = 0, 1, \cdots, d-1)$$
$$\text{i.e.} \quad \begin{bmatrix} \check{p}(0) \\ \check{q}(0) \end{bmatrix} = \check{\Psi}_d(-L) \begin{bmatrix} \check{p}(-L) \\ \check{q}(-L) \end{bmatrix} \tag{150}$$

with the boundary conditions

$$\lambda\check{p}(0) - M\check{q}(0) = 0, \quad \check{q}(-L) = 0. \tag{151}$$

By (150) and (151), the equality

$$\begin{bmatrix} \lambda \cdot I & -M \end{bmatrix} \check{\Psi}_d(-L) \begin{bmatrix} I \\ 0 \end{bmatrix} \check{p}(-L) = 0 \tag{152}$$

is obtained. Therefore, by employing the same approach as the proof of Theorem 6 (Section V-E), the stabilizing solution (41) is positive semi-definite iff the maximal root to (56) satisfies $\lambda_{\max} < 1$. ■

VI. Illustrative Examples

Based on the results stated in Section III,IV, we will illustrate the achievable H^∞ performance for preview control and unilateral delay systems. The relation between the effect of preview/delayed action and the resulting H^∞ performance is discussed.

A. Preview Control System

Define an H^∞ preview control problem:

$$\dot{x}(t) = \begin{bmatrix} 1 & 0 \\ 1 & 4 \end{bmatrix} x(t) + \begin{bmatrix} d \\ 0 \end{bmatrix} w_0(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w_1(t-h) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$
$$z(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \quad d = 0, 0.4, 0.8, \tag{153}$$

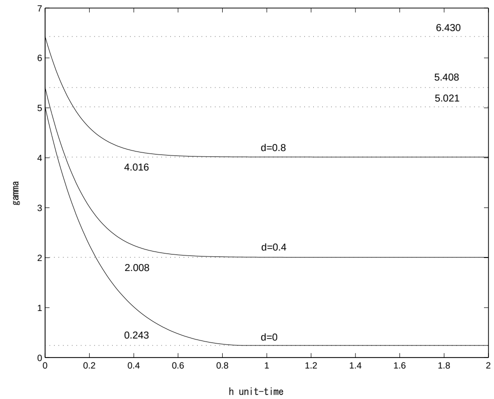

Fig. 2. h-γ in preview control

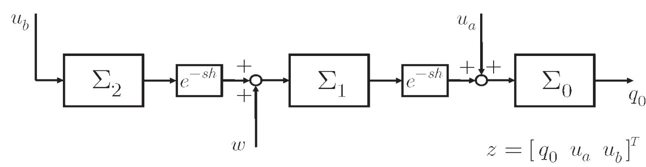

Fig. 3. A unilateral delay system (3 sub-systems)

where w_1 is h unit-time previewable reference and w_0 is the uncertainty which affect the information of previewable signal (see also (1),(2)). We will investigate the relation between the preview time h and the optimal performance $\gamma_{\mathrm{opt}}(h)$.

Based on Theorem 12 (or Corollary 13), the relation between h and $\gamma_{\mathrm{opt}}(h)$ for the cases $d = 0, 0.4, 0.8$ are obtained by Fig.2. For example in the case $d = 0$, the optimal value $\gamma_{\mathrm{opt}}(h)$ decreases monotonically as the preview time h increases and reaches the performance limit $\gamma_{\mathrm{low}}(h) = 0.243$. Therefore, by employing previewable information, the H^∞ performance is improved from $\gamma_{\mathrm{opt}}(0) = 5.021$ to $\gamma_{\mathrm{low}} = 0.243$ (95.1% reduced). In the cases $d = 0.4, 0.8$, where the previewable information involves uncertainty, the relation between h and $\gamma_{\mathrm{opt}}(h)$ are similarly depicted in Fig.2. For $d = 0.4, 0.8$, the H^∞ norm between w and z are reduced by 62.8% and 37.5% respectively. Thus it is observed that the effect of preview strategy decreases as the uncertainty in the information increases.

By [19], it is shown via dual filtering problem that the H^∞ preview performance typically reaches the optimal value γ_{low} in finite preview time. In Fig.2, the case $d = 0$ corresponds to this result. Similar phenomena are observed in the cases $d = 0.4, 0.8$.

B. Unilateral Delay System

Focus on the unilateral delay system depicted by Fig.3. In this system, the control input (u_a, u_b) is applied to the sub-systems Σ_0, Σ_2 and the disturbance w is applied to Σ_1. We define the sub-systems by

$$\Sigma_i(s) := \frac{1}{Ts+1} \quad (i = 0, 1, 2)$$

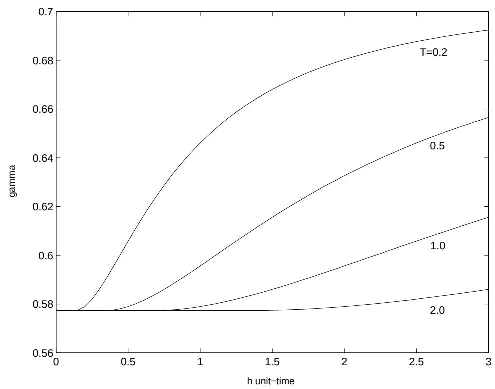

Fig. 4. h-γ in unilateral control system

and investigate the relation between the transmission delay h and the optimal performance $\gamma_{\rm opt}(h)$ in the full-information problem. Transforming the unilateral system Fig.3 to the auxiliary form Fig.1(b), then applying Theorem 12, the relation between h and $\gamma_{\rm opt}(h)$ for $T = 0.2, 0.5, 1.0, 2.0$ are obtained by Fig.4.

In this example, the relation is rather complicated than the preview control problem (Section VI-A) and the resulting H^∞ performance depends on the trade-off between the strength of preview action in u_a and the limitation of delayed action in u_b. In the case $T = 2.0$, the optimal value $\gamma_{\rm opt}(h)$ decreases slightly over $0 \le h \le 1.5$ and turns to increase around $h = T \simeq 2.0$. For the cases $T = 0.2, 0.5, 1.0$, it is observed that the optimal values turn to increase around the delay time, which meets the time constant of sub-systems.

VII. Conclusion

A generalized H^∞ control problem, which covers preview and delayed control strategies, is discussed based on a state-space approach. By introducing a Hamiltonian matrix, which is defined with a delay-free system, the analytic solution to corresponding operator Riccati equation is newly established. Based on the results obtained here, explicit formulas are derived for the H^∞ control problem and, further, some interpretations are provided on the property of the resulting control systems. The solution to the output feedback problem is obtained by introducing a solution to filtering Riccati equation (e.g. [16]).

In the formulation of the generalized plant, the orthonormal condition A2) plays a key role to establish an analytic solution to the operator Riccati equation and, in case when the condition is relaxed or the term of disturbance is included in the regulated output, the analytic representation is not trivial. The generalization of formulation is a direction of future research.

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