H^∞ Controller Design for Preview and Delayed Systems

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***Abstract*—The H^∞ control problem of general preview/delayed systems is solved using analytic solutions of the corresponding operator Riccati equations. The solution to the problem can be applied to a broad range of input/output delayed systems and enables the handling of preview/delayed control problems. The solvability condition is characterized by the roots of the transcendental equations and the control law for the general problem is given based on a predictive compensation with an integro-differential observer. Some interpretations of typical control problems are presented based on the solvability condition and the resulting control law.**

***Index Terms*—H^∞ control, preview control, delayed system, state-space approach, operator Riccati equation.**

I. Introduction

THE design method of H^∞ control laws for a broad range of infinite-dimensional systems have been studied, and the solvability conditions and control laws for preview tracking and delayed control problems are established in an explicit form [9], [10], [12], [18], [13], [25], [19], [17], [24], [16]. Full-information (FI) and output feedback H^∞ control problems of preview tracking were initially solved under the restricted condition of the matrix Riccati equation having a stabilizing solution [9], [10], and the limitation was subsequently relaxed for general problems [12]. The fixed-lag smoothing problem, which is a dual problem of preview tracking, has been solved [18], and the results were later extended to the case of multiple delays [13]. Alternative solutions of preview tracking and fixed-lag smoothing problems have also been provided [25]. The H^∞ control problems of delayed systems have been discussed using various approaches (for e.g., see [4]), and the solutions for typical input (or output) delayed systems are characterized by the solution of the transcendental equations or differential Riccati equations [28], [19], [17], [24]. The multiple delay case has been clarified [16], and the H^∞ control problem with a generalized transmission element has also been investigated [7]. Regarding the abstract formulation of a broad range of infinite-dimensional systems, a method for designing the H^∞ control law for Pritchard-Salamon systems [21] has been studied, and a state-space solution was developed based on the abstract operator Riccati equations [26]. Approximation methods for solving the operator Riccati equations have been investigated [6], and fundamental approaches to the Hankel norm optimization of general infinite-dimensional systems have also been developed [22], [23], [1].

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A unified approach to both preview and delayed H^∞ control problems has been discussed [12], and the FI control problem has also been solved using the analytic solution of the corresponding operator Riccati equation. The operator Riccati equation approach has the advantage of dealing with the preview/delayed strategies simultaneously. However, the solvability condition is still complicated because it requires the calculation of eigenvalues to guarantee the positive semi-definiteness of the stabilizing solution, and numerical difficulties are created if the eigenvalues are in the neighborhood of the origin (see Remark 4). Although an extending H^∞ control problem [12] in an output feedback setting has been discussed [11], the solvability condition inherits the limitation of [12] and the structure of the general control law has not been clarified. Control problems that deal with preview and delayed strategies frequently arise in one-directional delayed systems. For example, the control of disturbance attenuation in a wind tunnel or rolling mill is formulated using a unilateral delayed system [3] and the coordinated control of a wind farm is also formulated using multi-path preview/delayed systems [14]. Thus, the solution of general preview/delayed control problems enables to clarify the control laws for a broad range of systems, and evaluate the performance achieved by preview/delayed compensations.

In this study, we focused on a broad range of H^∞ preview/delayed control problems and developed a solution for a general setting. The solution to the problems can be applied to multiple preview/delayed control actions based on the possibility of delayed measurement, and enables the handling of various control/filtering strategies in a unified manner. Furthermore, we establish a new solvability condition for the FI control problem, which allows input/output delays and overcomes the limitation of [12]. The condition is directly characterized by the maximal eigenvalue of the compact operator and the corresponding operator Riccati equation is analytically solved. The solvability condition for the H^∞ output feedback control problem is clarified using the feature of the analytic solution. A family of solvability conditions is derived for typical preview/delayed control problems, and some interpretations of the relevant results are also provided [18], [13], [19], [17], [24].

This paper is organized as follows. In Section II, a generalized H^∞ preview/delayed control problem is formulated and typical control problems are illustrated. In Section III, the solutions of FI and output feedback control problems are provided. Furthermore, a family of solvability conditions for preview/delayed control problems is presented. In Section IV, typical control problems are discussed and some interpretations of relevant results are given. Subsequent to a description of all proofs in Section V, the feature of preview/delayed H^∞ control problems is illustrated using numerical examples (Section VI). The conclusion of this paper is presented in Section VII.

Notation and terminology: Let X and Y be real Hilbert spaces with norms $\|\cdot\|_X$, $\|\cdot\|_Y$ and inner product $\langle\cdot,\cdot\rangle_X$, $\langle\cdot,\cdot\rangle_Y$, respectively. Let Z be dense in X and Z^* be the adjoint space. The adjoint pairing between $f \in Z$ and $g \in Z^*$ will be denoted by $\langle f, g\rangle_{Z,Z^*}$. Let $\mathcal{L}(X,Y)$ denote the set of bounded linear operators $\Gamma : X \to Y$. The adjoint of $\Gamma \in \mathcal{L}(X,Y)$ will be denoted by $\Gamma^* \in \mathcal{L}(Y^*, X^*)$. When $X = Y$, we write $\mathcal{L}(X)$ instead of $\mathcal{L}(X,Y)$. A self-adjoint operator Γ will be written $\Gamma \geq 0$ if $\langle x, \Gamma x\rangle_X \geq 0$ for all $x \in X$ and $\Gamma > 0$ if $\langle x, \Gamma x\rangle_X > 0$, $x \neq 0$. The characteristic function $\chi_{[a,b]}$ is defined by $\chi_{[a,b]}(\beta) := \begin{cases} 1 & (\beta \in [a,b]) \\ 0 & (\beta \notin [a,b]) \end{cases}$.

II. Problem Formulation

Define a generalized plant with multiple input/output delays:

$$\Sigma:\ \dot{x}(t) = Ax(t) + \sum_{i=0}^{d} B_1^i w(t-h_i) + \sum_{i=0}^{d} B_2^i u(t-h_i) \quad (1a)$$

$$z(t) = \sum_{j=0}^{\ell} C_1^j x(t-\check{h}_j) + D_{12}u(t) \quad (1b)$$

$$y(t) = \sum_{j=0}^{\ell} C_2^j x(t-\check{h}_j) + D_{21}w(t) \quad (1c)$$

where $x(t) \in \mathbb{R}^n$, $w(t) \in \mathbb{R}^{m_1}$, $u(t) \in \mathbb{R}^{m_2}$, $z(t) \in \mathbb{R}^{p_1}$, $y(t) \in \mathbb{R}^{p_2}$ are the state, disturbance, control input, regulated output, and measurement of the system, respectively. The system matrices are with appropriate dimensions and the time delays h_i $(i = 0, 1, \ldots, d)$, $\check{h}_j$ $(j = 0, 1, \ldots, \ell)$ are denoted in ascending order: $0 =: h_0 < h_1 < h_2 < \cdots < h_d := L$, $0 =: \check{h}_0 < \check{h}_1 < \check{h}_2 < \cdots < \check{h}_\ell := \check{L}$. We prepare the auxiliary matrices:

$$\begin{aligned}
&A_c := A - B_2 D_{12}^+ C_1, \quad A_f := A - B_1 D_{21}^+ C_2, \\
&B := \begin{bmatrix} B_1 & B_2 \end{bmatrix}, \ B_1 := \sum_{i=0}^{d} B_1^i, \ B_2 := \sum_{i=0}^{d} B_2^i, \\
&B^i := \begin{bmatrix} B_1^i & B_2^i \end{bmatrix} \ (i = 0, 1, \ldots, d), \\
&C := \begin{bmatrix} C_1^{\mathrm{T}} & C_2^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}, \ C_1 := \sum_{j=0}^{\ell} C_1^j, \ C_2 := \sum_{j=0}^{\ell} C_2^j, \\
&C^j := \begin{bmatrix} C_1^{j\mathrm{T}} & C_2^{j\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \ (j = 0, 1, \ldots, \ell), \\
&D_{12}^+ := (D_{12}^{\mathrm{T}} D_{12})^{-1} D_{12}^{\mathrm{T}}, \ D_{21}^+ := D_{21}^{\mathrm{T}} (D_{21} D_{21}^{\mathrm{T}})^{-1}, \\
&R_c := \begin{bmatrix} -\gamma^2 \cdot I_{m_1} & 0 \\ 0 & D_{12}^{\mathrm{T}} D_{12} \end{bmatrix}, R_f := \begin{bmatrix} -\gamma^2 \cdot I_{p_1} & 0 \\ 0 & D_{21} D_{21}^{\mathrm{T}} \end{bmatrix}, \\
&N_c := I - D_{12} D_{12}^+, \ N_f := I - D_{21}^+ D_{21}
\end{aligned} \quad (2)$$

and make the following assumptions for the system Σ:

(H1) (C_2, A, B_2) is detectable and stabilizable,

(H2) D_{12} is full column rank and D_{21} is full row rank,

(H3) $\mathrm{rank}\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix} = n + m_2$,
$\mathrm{rank}\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix} = n + p_2, \quad \forall \omega \in \mathbb{R}$,

(H4) the following conditions hold:

$$B_1^i B_1^{j\mathrm{T}} = 0, \ B_2^i (D_{12}^{\mathrm{T}} D_{12})^{-1} B_2^{j\mathrm{T}} = 0,$$
$$C_1^{i\mathrm{T}} C_1^j = 0, \ C_2^{i\mathrm{T}} (D_{21} D_{21}^{\mathrm{T}})^{-1} C_2^j = 0 \ (i \neq j), \quad (3a)$$
$$B_1^i N_f B_1^{j\mathrm{T}} = 0 \ (i \neq j),$$
$$B_1^i D_{21}^+ C_2^j = 0 \ (i \neq 0 \,\mathrm{or}\, j \neq 0), \quad (3b)$$
$$C_1^{i\mathrm{T}} N_c C_1^j = 0 \ (i \neq j),$$
$$B_2^i D_{12}^+ C_1^j = 0 \ (i \neq 0 \,\mathrm{or}\, j \neq 0). \quad (3c)$$

The H^∞ control problem is to design a feedback control law such that the resulting system satisfies the following conditions:

(C1) the closed-loop system is internally stable,

(C2) the transfer function Σ_{zw} from w to z satisfies $\|\Sigma_{zw}\|_\infty < \gamma$ for a prescribed $\gamma > 0$.

For the system Σ, the assumptions (H1)-(H3) are standard and, in the delay-free case ($L = 0$, $\check{L} = 0$), they enable to solve the problem based on matrix Riccati equations (see e.g. [27]). The assumption (H4) is additionally introduced to impose a structural condition on the delayed input/output channels. The condition (3a) requires that the differently delayed input/output channels are decoupled under the normalized setting ($D_{12}^{\mathrm{T}} D_{12} = I_{m_2}$, $D_{21} D_{21}^{\mathrm{T}} = I_{p_2}$)[1]. The conditions (3b), (3c) are extension of the orthogonal conditions and enable to formulate some tracking or estimation problems. Typical problems are illustrated by Examples 1-3.

Example 1 (Preview tracking): A preview tracking problem is formulated by Σ with (H4):

$$\begin{aligned}
\Sigma^{\mathrm{prev}} : \dot{x}(t) &= Ax(t) + B_{1,0} w_0(t) + B_{1,1} w_1(t-L) + B_2 u(t) \\
z(t) &= C_1 x(t) + D_{12} u(t) \\
y(t) &= \begin{bmatrix} \tilde{y}(t) \\ w_1(t) \end{bmatrix} = \begin{bmatrix} \tilde{C}_2 \\ 0 \end{bmatrix} x(t) + \begin{bmatrix} \tilde{D}_{21} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} w_0(t) \\ w_1(t) \end{bmatrix}
\end{aligned} \quad (4)$$

where $w(t) := [\, w_0^{\mathrm{T}}(t) \ \ w_1^{\mathrm{T}}(t) \,]^{\mathrm{T}}$, $B_1^0 := [\, B_{1,0} \ \ 0 \,]$, $B_1^1 := [\, 0 \ \ B_{1,1} \,]$, $C_2 := [\, \tilde{C}_2^{\mathrm{T}} \ \ 0 \,]^{\mathrm{T}}$, $D_{21} := \begin{bmatrix} \tilde{D}_{21} & 0 \\ 0 & I \end{bmatrix}$, and w_0, w_1 denote the system uncertainty and previewable reference respectively. Replacing by $w_1(t) = r(t+L)$, it is observed that the future information $r(t+L)$ is included in the measurement. ■

Example 2 (Fixed-lag smoothing): A fixed-lag smoothing problem is formulated with (H4):

$$\begin{aligned}
\Sigma^{\mathrm{fl}} : \dot{x}(t) &= Ax(t) + B_1 w(t) \\
z_k(t) &= C_{1,k} x(t - \check{h}_k) - u_k(t) \quad (k = 0, 1, 2, \ldots, \ell) \\
y(t) &= C_2 x(t) + D_{21} w(t)
\end{aligned} \quad (5)$$

where $z(t) = [\, z_0^{\mathrm{T}}(t) \, z_1^{\mathrm{T}}(t) \cdots z_\ell^{\mathrm{T}}(t)]^{\mathrm{T}}$, $u(t) = [\, u_0^{\mathrm{T}}(t) \, u_1^{\mathrm{T}}(t) \cdots u_\ell^{\mathrm{T}}(t)]^{\mathrm{T}}$, $C_1 := [\, C_{1,0}^{\mathrm{T}} \ C_{1,1}^{\mathrm{T}} \cdots C_{1,\ell}^{\mathrm{T}}]^{\mathrm{T}}$, $D_{12} := I$, and $u_k(t)$ stands for the estimation of $C_{1,k} x(t - \check{h})$. The solution of the H^∞ problem Σ^{fl} has been clarified by [18], [13]. ■

Example 3 (Input/output delays): An output feedback problem with input/output delays is formulated by Σ with $B_1^i = 0$

[1] The condition (3a) derives $B^i R_c^{-1} B^{j\mathrm{T}} = 0$, $C^{i\mathrm{T}} R_f^{-1} C^j = 0$ $(i \neq j)$, $\forall \gamma > 0$. These equalities are also employed.

$(i = 1, 2, \ldots, d)$, $C_1^j = 0$ $(j = 1, 2, \ldots, \ell)$. This problem has been solved by [16] and a preliminary result was also reported by [11]. The formulation Σ enables to elaborate the results along [11] and, further, deal with preview strategies simultaneously. ∎

In case that the delayed signals are imposed on the channels of (w, u) or (z, y), we are faced with broad H^∞ control problems whose solutions have not yet established. Such general structure frequently arises in the one-directional delay systems and, for example, the coordination control of wind tunnel, rolling mill, and wind farm systems are formulated along Σ [3], [14]. The problem Σ enables to clarify the H^∞ performance attained by preview/delayed strategies and provide a design method of the control law.

In the sequel, we first solve a full-information (FI) control problem $\Sigma_{\rm FI}$ which is defined by (1a), (1b) with the measurement $y(t) = [\,x^{\rm T}(t)\; w^{\rm T}(t)]^{\rm T}$. The results are utilized to solve the general problem Σ.

III. Main Results

In this section, we provide solutions for the H^∞ control problems $\Sigma_{\rm FI}$ and Σ. The key point in our approach is that the corresponding operator Riccati equation is analytically solved and, further, the positive semi-definiteness of the stabilizing solution is characterized using the expression of the analytic solution. In the following, we discuss the essential part of our approach and clarify that typical conditions for preview/delayed control problems are also derived. Details on the operator Riccati equation approach and the proofs are described in Section V (A-D).

A. Full Information Problem

Introduce a Hamiltonian matrix and a differential equation:

$$H := \begin{bmatrix} A_c & -BR_c^{-1}B^{\rm T} \\ -C_1^{\rm T}N_cC_1 & -A_c^{\rm T} \end{bmatrix}, \tag{6}$$

$$\Phi_\lambda(0) = I,$$

$$\frac{d}{dt}\Phi_\lambda(t) = H_j(\lambda)\Phi_\lambda(t),\ -L + h_j \le t \le -L + h_{j+1},$$

$$H_j(\lambda) := \begin{bmatrix} A_c & -\sum_{i=0}^{j} B^iR_c^{-1}B^{i\rm T} \\ -\frac{1}{\lambda}\cdot C_1^{\rm T}N_cC_1 & -A_c^{\rm T} \end{bmatrix} \quad (j = 0, 1, 2, \ldots, d-1), \tag{7}$$

the full-information (FI) control problem $\Sigma_{\rm FI}$ is solved by the following theorem.

Theorem 1: For a given $\gamma > 0$, the FI problem $\Sigma_{\rm FI}$ is solvable iff (a) is satisfied.

(a) The Hamiltonian matrix (6) has no eigenvalues on the imaginary axis. Furthermore,

$$V_s := [\,I\;\; 0\,]\Phi_1(-L)V \tag{8}$$

is nonsingular and the maximal root of

$$\det V_p(\lambda) = 0,$$
$$V_p(\lambda) := [\,I\;\; 0\,]\Phi_\lambda(-L)\left\{(\lambda - 1)\cdot[\,I\;\; 0\,]^{\rm T} + VV_2^{\rm T}\right\} \tag{9}$$

satisfies $\lambda_{\max} \le 1$ where $V \in \mathbb{R}^{2n\times n}$ is a full column rank matrix defined by

$$V = [\,V_1^{\rm T}\;\; V_2^{\rm T}\,]^{\rm T},\ V_1, V_2 \in \mathbb{R}^{n\times n},$$
$$HV = V\Lambda_c,\ \Lambda_c:\ \text{stable matrix.} \tag{10}$$

If (a) holds, an H^∞ control law is given by

$$u(t) = -D_{12}^{+}\sum_{j=0}^{\ell} C_1^j x(t - \check{h}_j) - (D_{12}^{\rm T}D_{12})^{-1}\sum_{i=0}^{d} B_2^{i\rm T}v_i(t)$$
$$v_i(t) = G(-L + h_i, -L)x(t) + \sum_{k=0}^{d}\int_{-h_k}^{0} G(-L + h_i, -L + h_k + \beta)B^k\begin{bmatrix} w(t+\beta) \\ u(t+\beta)\end{bmatrix}d\beta, \tag{11}$$

$$G(\xi, \beta) = [\,0\;\; I\,]\Phi_1(\xi)V_s^{\#}(\xi, \beta)\Phi_1^{-1}(\beta)[\,I\;\; 0\,]^{\rm T},$$
$$V_s^{\#}(\xi, \beta) = \begin{cases} V_s^R, & \xi > \beta \\ V_s^R - I, & \xi \le \beta \end{cases},$$
$$V_s^R := VV_s^{-1}[\,I\;\; 0\,]\Phi_1(-L). \tag{12}$$

∎

For the FI problem $\Sigma_{\rm FI}$, it is shown that the solvability is generally characterized by the root of the transcendental equation (9). A control law is given by (11) and some compensation terms are included for the delayed control and the previewable disturbance. The key point in the derivation is that the stabilizing solution of the corresponding Riccati equation is expressed as

$$\mathcal{S} = \mathcal{G}^*\mathcal{V}_2(\mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2)^{-1}\mathcal{G} \in \mathcal{L}(\mathcal{X}) \tag{13}$$

$$\mathcal{V}_1 := \begin{bmatrix} V_1 & 0 \\ 0 & \mathcal{I}\end{bmatrix},\ \mathcal{V}_2 := \begin{bmatrix} V_2 & 0 \\ 0 & \Theta\end{bmatrix} \in \mathcal{L}(\mathcal{X}^o),$$

$$(\Theta\phi^1)(\xi) := \sum_{j=0}^{\ell}\chi_{[-L-\check{h}_j, 0]}(\xi)\cdot C_1^{j\rm T}N_cC_1^j\phi^1(\xi),$$
$$\phi^1 \in L_2(-L - \check{L}, 0; \mathbb{R}^n),\ \ -L - \check{L} \le \xi \le 0 \tag{14}$$

$$\Pi := \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Pi_1 & 0 \\ 0 & 0 & 0\end{bmatrix} \in \mathcal{X},$$

$$(\Pi_1\phi^1)(\xi) := \sum_{i=0}^{d}\chi_{[-L+h_i, 0]}(\xi)\cdot B^iR_c^{-1}B^{i\rm T}\phi^1(\xi),$$
$$\phi^1 \in L_2(-L, 0; \mathbb{R}^n),\ \ -L \le \xi \le 0 \tag{15}$$

$$\mathcal{G}\phi := ((\mathcal{G}\phi)^0, (\mathcal{G}\phi)^1),\quad \phi = (\phi^0, \phi^1, \phi^2) \in \mathcal{X} \tag{16}$$

$$(\mathcal{G}\phi)^0 := e^{A_cL}\phi^0 + \int_{-L}^{0} e^{-A_c\beta}\phi^1(\beta)\,d\beta$$

$$(\mathcal{G}\phi)^1(\xi) := \begin{cases} e^{A_c(\xi+L)}\phi^0 + \int_{-L}^{\xi} e^{A_c(\xi-\beta)}\phi^1(\beta)\,d\beta & (-L \le \xi \le 0) \\ \phi^2(\xi + L) & (-L - \check{L} \le \xi \le -L)\end{cases},$$

$$\mathcal{X} := \mathbb{R}^n \times L_2(-L, 0; \mathbb{R}^n) \times L_2(-\check{L}, 0; \mathbb{R}^n),$$
$$\mathcal{X}^o := \mathbb{R}^n \times L_2(-L - \check{L}, 0; \mathbb{R}^n).$$

Thus investigating the positive semi-definiteness of (13), the solvability condition (a) is clarified.

The expression (13) also yields concise conditions for preview tracking ($B_2^i = 0, i = 1, 2, \cdots d$) or delayed control ($B_1^i = 0, i = 1, 2, \cdots d$) problems. For the case of preview tracking $B_1^i = 0$ $(i = 1, 2, \cdots d)$, the positive semi-definiteness of (13) is directly verified by the stability of the resulting closed loop system. The following condition is obtained as the closed loop system with $w = 0$ is finite-dimensional (Theorem 1).

Lemma 2 (Preview tracking case): Suppose $B_2^i = 0$ ($i = 1, 2, \cdots d$) holds. Then (a) and (a_w) are equivalent.

(a_w) The Hamiltonian matrix (6) has no eigenvalues on the imaginary axis. Furthermore, the matrix (8) is nonsingular and $A_c - B_2(D_{12}^{\rm T}D_{12})^{-1}B_2^{\rm T}G(-L,-L)$ is stable. ■

For the case of delayed control $B_1^i = 0$ ($i = 1, 2, \cdots d$), the solution (13) is expressed as

$$\mathcal{S} = \mathcal{G}^*\mathcal{M}(\mathcal{I} + \mathcal{G}\Pi\mathcal{G}^*\mathcal{M})^{-1}\mathcal{G},$$
$$\mathcal{M} := \mathcal{V}_2\mathcal{V}_1^{-1} = \begin{bmatrix} S & 0 \\ 0 & \Theta \end{bmatrix} \geq 0 \tag{17}$$

where $S \geq 0$ is the stabilizing solution of the matrix Riccati equation

$$SA_c + A_c^{\rm T}S - SBR_c^{-1}B^{\rm T}S + C_1^*N_cC_1 = 0. \tag{18}$$

Since the positive semi-definiteness $\mathcal{M} \geq 0$ is preserved in (17), the condition $\mathcal{S} \geq 0$ is clarified by investigating the eigenvalue of $\mathcal{I} + \mathcal{G}\Pi\mathcal{G}^*\mathcal{M}$.

Lemma 3 (Input delay case): Suppose $B_1^i = 0$ ($i = 1, 2, \cdots d$) holds. Then (a) and (a_u) are equivalent.

(a_u) The equation (18) has a stabilizing solution $S \geq 0$ such that $A_c - BR_c^{-1}B^{\rm T}S$ is stable. Furthermore the maximal root of

$$\det \tilde{V}_p(\lambda) = 0,$$
$$\tilde{V}_p(\lambda) := [\,I \;\; 0\,]\Phi_\lambda(-L)[\,I \;\; \lambda^{-1}\cdot S\,]^{\rm T} \tag{19}$$

satisfies $\lambda_{\max} < 1$. ■

Remark 4: For the H^∞ control problem $\Sigma_{\rm FI}$, a preliminary case $\check{L} = 0$ is discussed [12]. However, the solvability condition is still complicated because it characterizes the condition $\mathcal{S} \geq 0$ by calculating the minimal eigenvalue of (13). Since (13) involves a compact operator and has an accumulating point of eigenvalues at origin, the numerical calculation of $\lambda_{\min}(\mathcal{S})$ is prohibitive in some cases. To avoid such difficulties, the condition (9) is newly derived by transforming the condition to a maximal eigenvalue problem of an auxiliary operator (Theorem 18 in Section V-B). ■

B. Output Feedback Problem

In addition to (6), (7), introduce a Hamiltonian matrix and differential equations:

$$J := \begin{bmatrix} A_f^{\rm T} & -C^{\rm T}R_f^{-1}C \\ -B_1N_fB_1^{\rm T} & -A_f \end{bmatrix} \tag{20}$$
$$\Psi_\mu(0) = I,$$
$$\frac{d}{dt}\Psi_\mu(t) = J_j(\mu)\Psi_\mu(t),\; -\check{L} + \check{h}_j \leq t \leq -\check{L} + \check{h}_{j+1},$$
$$J_j(\mu) := \begin{bmatrix} A_f^{\rm T} & -\sum_{i=0}^{j} C^{i\rm T}R_f^{-1}C^i \\ -\frac{1}{\mu}\cdot B_1N_fB_1^{\rm T} & -A_f \end{bmatrix}$$
$$(j = 0, 1, \ldots, \ell - 1), \tag{21}$$
$$\check{\Phi}^\sigma(0) = I,$$
$$\frac{d}{dt}\check{\Phi}^\sigma(t) = \check{H}_j(\sigma)\check{\Phi}^\sigma(t),\; -L + h_j \leq t \leq -L + h_{j+1},$$
$$\check{H}_j(\sigma) := \begin{bmatrix} A_c & -\sum_{i=0}^{j} B^iR_c^{-1}B^{i\rm T} + \sum_{i=j+1}^{d}\frac{1}{\sigma^2}\cdot B_1^iN_fB_1^{i\rm T} \\ -C_1^{\rm T}N_cC_1 & -A_c^{\rm T} \end{bmatrix}$$
$$(j = 0, 1, \ldots, d - 1), \tag{22}$$
$$\check{\Psi}^\sigma(0) = I,$$
$$\frac{d}{dt}\check{\Psi}^\sigma(t) = \check{J}_j^{\rm T}(\sigma)\check{\Psi}^\sigma(t),\; -\check{h}_{j+1} \leq t \leq -\check{h}_j,$$
$$\check{J}_j(\sigma) := \begin{bmatrix} A_f^{\rm T} & -\sum_{i=0}^{j} C^{i\rm T}R_f^{-1}C^i + \sum_{i=j+1}^{\ell}\frac{1}{\sigma^2}\cdot C_1^{i\rm T}N_cC_1^i \\ -B_1N_fB_1^{\rm T} & -A_f \end{bmatrix}$$
$$(j = 0, 1, \ldots, \ell - 1). \tag{23}$$

The output feedback problem Σ is solved by the following theorem.

Theorem 5: For a given $\gamma > 0$, the H^∞ control problem Σ is solvable iff (a) in Theorem 1 and (b), (c) are satisfied.

(b) The Hamiltonian matrix (20) has no eigenvalues on the imaginary axis. Furthermore,

$$U_s := [\,I \;\; 0\,]\Psi_1(-\check{L})U \tag{24}$$

is nonsingular and the maximal root of

$$\det U_p(\mu) = 0,$$
$$U_p(\mu) := [\,I \;\; 0\,]\Psi_\mu(-\check{L})\left\{(\mu - 1)\cdot[\,I \;\; 0\,]^{\rm T} + UU_2^{\rm T}\right\} \tag{25}$$

satisfies $\mu_{\max} \leq 1$ where $U \in \mathbb{R}^{2n\times n}$ is a full column rank matrix defined by

$$U := [\,U_1^{\rm T}\; U_2^{\rm T}\,]^{\rm T},\;\; U_1, U_2 \in \mathbb{R}^{n\times n},$$
$$JU = U\Lambda_f,\; \Lambda_f \in \mathbb{R}^{n\times n} : \text{stable matrix.} \tag{26}$$

(c) Maximal root of

$$\det W(\sigma) = 0,$$
$$W(\sigma) := U^{\rm T}\check{\Psi}^\sigma(-\check{L})\begin{bmatrix} \sigma\cdot I & 0 \\ 0 & -\sigma^{-1}\cdot I \end{bmatrix}\check{\Phi}^\sigma(-L)V \tag{27}$$

satisfies $\sigma_{\max} < \gamma$.

If (a), (b), (c) hold, an H^∞ control law is given by

$$u(t) = -(D_{12}^{\rm T}D_{12})^{-1}\sum_{k=0}^{d}B_2^{k\rm T}f_k(t) - D_{12}^{+}\sum_{k=0}^{\ell}C_1^k\check{f}_k(t) \tag{28a}$$

$$f_k(t) = K_1(-L + h_k, -L)\underline{x}(t) + \sum_{i=0}^{d}\int_{-h_i}^{0}K_1(-L + h_k, \alpha - L + h_i)B^i\begin{bmatrix} D_{21}^{+}y(t+\alpha) \\ u(t+\alpha) \end{bmatrix}d\alpha + \sum_{j=0}^{\ell}\int_{-\check{h}_j}^{0}K_2(-L + h_k, \beta)C_1^{j\rm T}N_cC_1^j\underline{x}(t,\beta)\,d\beta \tag{28b}$$

$$\check{f}_k(t) = \underline{x}(t, -\check{h}_k) + \check{K}_1(-\check{h}_k, -L)\underline{x}(t) + \sum_{i=0}^{d}\int_{-h_i}^{0}\check{K}_1(-\check{h}_k, \alpha - L + h_i)B^i\begin{bmatrix} D_{21}^{+}y(t+\alpha) \\ u(t+\alpha) \end{bmatrix}d\alpha + \sum_{j=0}^{\ell}\int_{-\check{h}_j}^{0}\check{K}_2(-\check{h}_k, \beta)C_1^{j\rm T}N_cC_1^j\underline{x}(t,\beta)\,d\beta \tag{28c}$$

$$\dot{\underline{x}}(t) = A_f \underline{x}(t) + \sum_{i=0}^{d} B^i \begin{bmatrix} D_{21}^{+} y(t-h_i) \\ u(t-h_i) \end{bmatrix} - \sum_{j=0}^{\ell} F(0, -\check{h}_j) g_j(t) \tag{28d}$$

$$\underline{x}(t,\beta) = \underline{x}(t+\beta) - \sum_{j=0}^{\ell} \int_{\beta}^{0} F(\beta-\xi, -\check{h}_j) g_j(t+\xi)\, d\xi \tag{28e}$$

$$g_j(t) = C^{j\mathrm{T}} R_f^{-1} \left(C^j \underline{x}(t, -\check{h}_j) + \begin{bmatrix} D_{12} u(t) \\ -y(t) \end{bmatrix} \right) \tag{28f}$$

where K_i, $\check{K}_i$ $(i=1,2)$ and F are given as follows:

$$\begin{aligned}
K_1(\xi,\beta) &= [\,0\ \ I\,] \check{\Phi}^{\gamma}(\xi) W_s^{\#}(\xi,\beta) \check{\Phi}^{\gamma-1}(\beta) [\,I\ \ 0\,]^{\mathrm{T}}, \\
W_s^{\#}(\xi,\beta) &= \begin{cases} W_s^R, & \xi \geq \beta \\ W_s^R - I, & \xi < \beta \end{cases}, \\
K_2(\xi,\beta) &= [\,0\ \ I\,] \check{\Phi}^{\gamma}(\xi) W_s^R \check{\Phi}^{\gamma-1}(-L) \Gamma^{-1} \check{\Phi}^{\gamma-1}(\beta) [\,0\ \ I\,]^{\mathrm{T}}, \\
W_s^R &= V W_s^{-1} U^{\mathrm{T}} \check{\Psi}^{\gamma}(-\check{L}) \Gamma \check{\Phi}^{\gamma}(-L), \\
\check{K}_1(\xi,\beta) &= \tfrac{1}{\gamma^2} \cdot [\,I\ \ 0\,] \check{\Psi}^{\gamma}(\xi) (\check{W}_s^R - I) \Gamma \\
&\quad \times \check{\Phi}^{\gamma}(-L) \check{\Phi}^{\gamma-1}(\beta) [\,I\ \ 0\,]^{\mathrm{T}}, \\
\check{K}_2(\xi,\beta) &= \tfrac{1}{\gamma^2} \cdot [\,I\ \ 0\,] \check{\Psi}^{\gamma}(\xi) \check{W}_s^{\#}(\xi,\beta) \check{\Psi}^{\gamma-1}(\beta) [\,0\ \ I\,]^{\mathrm{T}}, \\
\check{W}_s^{\#}(\xi,\beta) &= \begin{cases} \check{W}_s^R, & \xi \geq \beta \\ \check{W}_s^R - I, & \xi < \beta \end{cases}, \\
\check{W}_s^R &= \Gamma \check{\Phi}^{\gamma}(-L) V W_s^{-1} U^{\mathrm{T}} \check{\Psi}^{\gamma}(-\check{L}), \\
W_s &= U^{\mathrm{T}} \check{\Psi}^{\gamma}(-\check{L}) \Gamma \check{\Phi}^{\gamma}(-L) V,\ \Gamma = \begin{bmatrix} \gamma^2 \cdot I & 0 \\ 0 & -I \end{bmatrix} \\
F(\xi,\beta) &= [\,0\ \ I\,] \Psi_1(-\xi-\check{L}) U_s^{\#}(\xi,\beta) \\
&\quad \times \Psi_1^{-1}(-\beta-\check{L}) [\,I\ \ 0\,]^{\mathrm{T}}, \\
U_s^{\#}(\xi,\beta) &= \begin{cases} U_s^R, & \xi < \beta \\ U_s^R - I, & \xi \geq \beta \end{cases}, \\
U_s^R &:= U U_s^{-1} [\,I\ \ 0\,] \Psi_1(-\check{L}).
\end{aligned} \tag{29}$$

■

By Theorem 5, the H^∞ control law for Σ is given based on the predictive compensation law (28a), (28b), (28c) and the observer (28d), (28e), (28f) whose structure arises in the estimation of delayed systems [2]. In the general problem Σ, an extended observer (28e), (28f) is embedded in the control law which updates the distributed state based on the integro-differential equations.

Along Lemma 2, the condition (b) is further simplified in the fixed-lag smoothing ($C_2^j = 0, j = 1, 2, \cdots \ell$) (Example 2) and output delay ($C_1^j = 0, j = 1, 2, \cdots \ell$) cases.

Remark 6 (Fixed-lag smoothing/output delay cases): Suppose $C_2^j = 0$ $(j = 1, 2, \cdots \ell)$ holds. Then the condition (b) is equivalent to (b$_z$):

- (b$_z$) The Hamiltonian matrix (20) has no eigenvalues on the imaginary axis. Furthermore, the matrix (24) is nonsingular and $A_f - F(0,0) C_2^{\mathrm{T}} (D_{21} D_{21}^{\mathrm{T}})^{-1} C_2$ is stable.

Suppose $C_1^j = 0$ $(j = 1, 2, \cdots \ell)$ holds. Then the condition (b) is equivalent to (b$_y$):

- (b$_y$) The matrix Riccati equation:

$$A_f P + P A_f^{\mathrm{T}} - P C^{\mathrm{T}} R_f^{-1} C P + B_1 N_f B_1^{\mathrm{T}} = 0 \tag{30}$$

has a stabilizing solution $P \geq 0$ such that $A_f - P C^{\mathrm{T}} R_f^{-1} C$ is stable. Furthermore the maximal root of

$$\det \tilde{U}_p(\mu) = 0,$$
$$\tilde{U}_p(\mu) := [\,I\ \ 0\,] \Psi_\mu(-\check{L}) [\,I\ \ \mu^{-1} \cdot P\,]^{\mathrm{T}} \tag{31}$$

satisfies $\mu_{\max} < 1$. ■

IV. Special Cases and Discussion

By Section III, general solutions for the H^∞ control problems Σ_{FI}, Σ are clarified and some concise conditions (a$_w$), (a$_u$), (b$_z$), (b$_y$) are also obtained (Lemmas 2, 3, Remark 6). In this section, we first focus on the one-side delay systems (Σ with $\check{L} = 0$ or $L = 0$) and derive an alternative condition which simplifies the design procedure. Furthermore, for the preview tracking and the input/output delay problems, the connection to the relevant results [18], [13], [19], [24], [17] is investigated and generalized. The proofs are described in Section V (E-G).

A. One-side Delay Case

For the one-side delay system defined by Σ with $\check{L} = 0$, we clarify an alternative solvability condition which merges (a) and (c) (Theorems 1, 5). Complementary condition for $L = 0$ is obtained by applying the result to the transposed system (see (72)).

Lemma 7: For a given $\gamma > 0$, the H^∞ control problem Σ with $\check{L} = 0$ is solvable iff (b$_0$), (ac) are satisfied.

- (b$_0$) The equation (30) has a stabilizing solution $P \geq 0$ such that $A_f - P C^{\mathrm{T}} R_f^{-1} C$ is stable.
- (ac) For a given $\gamma > 0$, the following full-information problem defined by

$$\Sigma_{\mathrm{FI}}^{(\mathrm{AC})}:\ \dot{x}(t) = \tilde{A}_c x(t) + \sum_{i=0}^{d} \tilde{B}_1^i w(t-h_i) + \sum_{i=0}^{d} \tilde{B}_2^i u(t-h_i)$$
$$z(t) = C_1 x(t) + D_{12} u(t) \tag{32}$$

$$\begin{aligned}
&\tilde{A}_c := \tilde{A} - \tilde{B}_2 D_{12}^{+} C_1,\ \ \tilde{A} := A + \tfrac{1}{\gamma^2} \cdot P C_1^{\mathrm{T}} C_1, \\
&\tilde{B}_1^0 := (B_1^0 D_{21}^{\mathrm{T}} + P C_2^{\mathrm{T}})(D_{21} D_{21}^{\mathrm{T}})^{-\frac{1}{2}}, \\
&\tilde{B}_2^0 := B_2^0 + \tfrac{1}{\gamma^2} \cdot P C_1^{\mathrm{T}} D_{12}, \\
&\tilde{B}_1^i := B_1^i D_{21}^{\mathrm{T}} (D_{21} D_{21}^{\mathrm{T}})^{-\frac{1}{2}}, \\
&\tilde{B}_2^i := B_2^i\ (i = 1, 2, \ldots, d),\ \ \tilde{B}_2 := \sum_{i=0}^{d} \tilde{B}_2^i
\end{aligned}$$

is solvable. ■

The solvability of $\Sigma_{\mathrm{FI}}^{(\mathrm{AC})}$ is verified by applying Theorem 1 or Lemmas 2, 3.

B. Preview Tracking Case

For the fixed-lag smoothing problem [18], [13] which is a dual problem of preview tracking, the solvability condition has been fairly characterized based on the operation of a Hamiltonian matrix. For the multiple preview tracking problem

Σ with $B_2^i = 0$ $(i = 1, 2, \ldots, d)$, a direct connection is established between Lemma 2 and [18], [13].

Lemma 8: Suppose $B_2^i = 0$ $(i = 1, 2, \ldots, d)$ holds for Σ. Then the conditions (a_w) in Lemma 2 and ($\tilde{\mathrm{a}}_w$) are equivalent[2].

($\tilde{\mathrm{a}}_w$) $\Phi_1(-L)H\Phi_1^{-1}(-L) \in \mathrm{dom}(\mathrm{Ric})$, $X(-L) := \mathrm{Ric}(\Phi_1(-L)H\Phi_1^{-1}(-L)) \geq 0$. ■

For the preview tracking problem Σ^{prev} (Example 1), the feature of the resulting control law is observed by the following example.

Example 4 (Preview tracking (contd.)): Applying Theorem 5, the H^∞ control law for Σ^{prev} is obtained as follows:

$$\begin{aligned} u(t) &= -(D_{12}^{+}C_1 + \tilde{K}(-L))\underline{x}(t) \\ &\quad - \int_{-L}^{0} \tilde{K}(\beta)B_{1,1}w_1(t+\beta)\,d\beta, \\ \dot{\underline{x}}(t) &= A\underline{x}(t) + B_{1,1}w_1(t-L) + B_2u(t) \\ &\quad + (B_{1,0}\tilde{D}_{21}^{\mathrm{T}} + P\tilde{C}_2^{\mathrm{T}})(\tilde{D}_{21}\tilde{D}_{21}^{\mathrm{T}})^{-1}(\tilde{y}(t) - \tilde{C}_2\underline{x}(t)) \\ &\quad + \tfrac{1}{\gamma^2}\cdot PC_1^{\mathrm{T}}(C_1\underline{x}(t) + D_{12}u(t)), \\ \tilde{K}(\beta) &:= (D_{12}^{\mathrm{T}}D_{12})^{-1}[\,D_{12}^{\mathrm{T}}C_1\ B_2^{\mathrm{T}}\,]\begin{bmatrix} \check{K}_1(0,\beta) \\ K_1(-L,\beta)\end{bmatrix}, \\ &\quad P : \text{defied by (30).} \end{aligned} \tag{33}$$

This case, the control law is given based on a finite-dimensional observer with a predictive compensation of w_1. The solvability condition is characterized by Lemma 7. ■

C. Input/Output Delay Case

For the input/output delay systems defined by Σ with $B_1^i = 0$ $(i = 1, 2, \ldots, d)$, $C_1^j = 0$ $(j = 1, 2, \ldots, \ell)$, we will show that the conditions (a_u), (b_y), (c) (Lemma 3, Remark 6, Theorem 5) are directly characterized by differential Riccati equations [19], [24], [17].

Lemma 9: Suppose $B_1^i = 0$ $(i = 1, 2, \ldots, d)$, $C_1^j = 0$ $(j = 1, 2, \ldots, \ell)$ hold for Σ. Then the conditions (a_u) in Lemma 3, (b_y) in Remark 6, and (c) in Theorem 5 are equivalently characterized by ($\tilde{\mathrm{a}}_u$), ($\tilde{\mathrm{b}}_y$), ($\tilde{\mathrm{c}}$).

($\tilde{\mathrm{a}}_u$) The equation (18) has a stabilizing solution $S \geq 0$. Furthermore the equation:

$$\begin{aligned} -\dot{S}(t) &= S(t)A_c + A_c^{\mathrm{T}}S(t) \\ &\qquad - S(t)\tilde{B}(t)R_c^{-1}\tilde{B}^{\mathrm{T}}(t)S(t) + C_1^{\mathrm{T}}N_cC_1, \\ S(0) &= S, \\ \tilde{B}(t) &:= [\,B_1\ \tilde{B}_2(t)\,], \\ \tilde{B}_2(t) &:= \sum_{i=0}^{d-1}\chi_{[-L+h_i,0]}(t)\cdot B_2^i \end{aligned} \tag{34}$$

has a bounded solution $S(t) \geq 0$ $(-L \leq t \leq 0)$.

($\tilde{\mathrm{b}}_y$) The equation (30) has a stabilizing solution $P \geq 0$. Furthermore the equation:

$$\begin{aligned} -\dot{P}(t) &= A_fP(t) + P(t)A_f^{\mathrm{T}} \\ &\qquad - P(t)\tilde{C}^{\mathrm{T}}(t)R_f^{-1}\tilde{C}(t)P(t) + B_1N_cB_1^{\mathrm{T}}, \\ P(0) &= P, \\ \tilde{C}(t) &:= [\,C_1^{\mathrm{T}}\ \tilde{C}_2^{\mathrm{T}}(t)\,]^{\mathrm{T}}, \\ \tilde{C}_2(t) &:= \sum_{i=0}^{\ell-1}\chi_{[-\check{L}+\check{h}_i,0]}(t)\cdot C_2^i \end{aligned} \tag{35}$$

has a bounded solution $P(t) \geq 0$ $(-\check{L} \leq t \leq 0)$.

($\tilde{\mathrm{c}}$) The inequality $\lambda_{\max}(P(-\check{L})S(-L)) < \gamma^2$ holds. ■

The general structure of the control law (28) is observed along the H^∞ controller design for the input/output delay systems.

Example 5 (Input/output delays (contd.)): Define an input/output delay system based on Example 3 with $d = \ell = 1$, $D_{12}^{\mathrm{T}}[\,C_1\ D_{12}\,] = [\,0\ I\,]$, $D_{21}[\,B_1^{\mathrm{T}}\ D_{21}^{\mathrm{T}}\,] = [\,0\ I\,]$, $B_2^0 = 0$, $C_2^0 = 0$. By Theorem 5, the H^∞ control law is given as follows:

$$\begin{aligned} u(t) &= -B_2^{1\mathrm{T}}K_1(0,-L)\underline{x}(t) \\ &\quad - \int_{-L}^{0} B_2^{1\mathrm{T}}K_1(0,\beta)B_2^1u(t+\beta)\,d\beta, \\ \dot{\underline{x}}(t) &= A\underline{x}(t) + B_2^1u(t-L) + v(t+\check{L},-\check{L}), \\ \underline{x}(t,\check{L}) &= \underline{x}(t-\check{L}) + \int_{-\check{L}}^{0} v(t,\xi)\,d\xi, \\ v(t,\xi) &= F(-\check{L}-\xi,-\check{L})C_2^{1\mathrm{T}}(y(t+\xi) - C_2^1\underline{x}(t+\xi,-\check{L})) \\ &\quad + \tfrac{1}{\gamma^2}\cdot F(-\check{L}-\xi,0)C_1^{0\mathrm{T}}C_1^0\underline{x}(t+\xi). \end{aligned} \tag{36}$$

In (36), the internal data $\underline{x}(t)$, $\underline{x}(t,-\check{L})$ is updated by integro-differential equations. Similar structure is generally observed in the control law for multiple input/output delay systems. ■

V. Proofs

A. Preliminaries

In order to solve the H^∞ control problems Σ and Σ_{FI}, we prepare a system description on an appropriate function space. Introducing a Hilbert space $\mathcal{X} := \mathbb{R}^n \times L_2(-L,0;\mathbb{R}^n) \times L_2(-\check{L},0;\mathbb{R}^n)$ endowed with the inner product

$$\begin{aligned} \langle\psi,\phi\rangle &:= \psi^{0\mathrm{T}}\phi^0 \\ &\quad + \int_{-L}^{0}\psi^{1\mathrm{T}}(\beta)\phi^1(\beta)\,d\beta + \int_{-\check{L}}^{0}\psi^{2\mathrm{T}}(\beta)\phi^2(\beta)\,d\beta, \\ &\quad \psi = (\psi^0,\psi^1,\psi^2) \in \mathcal{X},\ \phi = (\phi^0,\phi^1,\phi^2) \in \mathcal{X}, \end{aligned} \tag{37}$$

the system Σ is described by the evolution equation [20]:

$$\hat{\Sigma}:\ \dot{\hat{x}}(t) = \mathcal{A}\hat{x}(t) + \mathcal{B}_1w(t) + \mathcal{B}_2u(t) \tag{38a}$$
$$z(t) = \mathcal{C}_1\hat{x}(t) + D_{12}u(t) \tag{38b}$$
$$y(t) = \mathcal{C}_2\hat{x}(t) + D_{21}w(t). \tag{38c}$$

The operator $\mathcal{A}$ is an infinitesimal generator defined by

$$\begin{aligned} &\mathcal{A}\phi := (A\phi^0 + \phi^1(-L), \phi^{1'}, \phi^{2'}), \\ &\mathrm{D}(\mathcal{A}) = \{\phi \in \mathcal{X} : \phi^1 \in W^{1,2}(-L,0;\mathbb{R}^n), \\ &\qquad \phi^2 \in W^{1,2}(-\check{L},0;\mathbb{R}^n), \phi^1(0) = 0, \phi^2(0) = \phi^0\} \end{aligned} \tag{39}$$

where $W^{1,2}(-L,0;\mathbb{R}^n)$ denotes the Sobolev space of $\mathbb{R}^n$-valued, absolutely continuous functions with square integrable derivatives on $[-L,0]$. Let $\mathcal{V}^* := \{\psi \in \mathcal{X} : \psi^1 \in W^{1,2}(-L,0;\mathbb{R}^n), \psi^1(-L) = \psi^0\}$, $\mathcal{W} := \{\phi \in \mathcal{X} : \phi^2 \in W^{1,2}(-\check{L},0;\mathbb{R}^n), \phi^2(0) = \phi^0\}$ be subspaces of $\mathcal{X}$. Then

[2]The notation follows from [27]. In the case $C_2^j = 0$ $(j = 1, 2, \ldots, \ell)$, a corresponding condition for (b_z) is obtained by applying Lemma 8 to the transposed system of Σ (see (72)).

$\mathcal{W} = \mathrm{D}_{\mathcal{V}}(\mathcal{A})$, $\mathcal{V}^* = \mathrm{D}_{\mathcal{W}^*}(\mathcal{A}^*)$ hold and $\mathcal{W}$, $\mathcal{X}$, $\mathcal{V}$ are with continuous, dense injections satisfying $\mathcal{W} \subset \mathcal{X} \subset \mathcal{V}$ ([20], Remark 2.6). The operators $\mathcal{B}_k \in \mathcal{L}(\mathbb{R}^{m_k}, \mathcal{V})$, $\mathcal{C}_k \in \mathcal{L}(\mathcal{W}, \mathbb{R}^{p_k})$ $(k = 1, 2)$ are defined by

$$\mathcal{B}_k^*\psi := B_k^{0\mathrm{T}}\psi^0 + \sum_{i=1}^{d} B_k^{i\mathrm{T}}\psi^1(-L + h_i),\ \psi \in \mathcal{V}^*,$$
$$\mathcal{C}_k\phi := C_k^0\phi^0 + \sum_{j=1}^{\ell} C_k^j\phi^2(-\check{h}_j),\ \phi \in \mathcal{W}. \tag{40}$$

Remark 10: The state $\hat{x}(t) := (\hat{x}_t^0, \hat{x}_t^1, \hat{x}_t^2) \in \mathcal{X}$ of $\hat{\Sigma}$ corresponds to the original system Σ in the following manner:

$$\begin{aligned}
&\hat{x}_t^0 := x(t),\\
&\hat{x}_t^1(\alpha) := \sum_{i=0}^{d} \chi_{[-L,-L+h_i]}(\alpha) \cdot B^i \begin{bmatrix} w_t(\alpha + L - h_i) \\ u_t(\alpha + L - h_i) \end{bmatrix},\\
&\hat{x}_t^2(\beta) := x_t(\beta),\\
&w_t(\alpha) := w(t + \alpha),\ u_t(\alpha) := u(t + \alpha),\\
&x_t(\beta) := x(t + \beta), \qquad -L \le \alpha \le 0,\ \ -\check{L} \le \beta \le 0.
\end{aligned} \tag{41}$$

The expression (41) will be employed for describing the control law along Σ. ■

The system $\hat{\Sigma}$ is in the Pritchard-Salamon class [20], [21] and typical H^∞ control problems have been characterized by corresponding operator Riccati equations [26]. In the sequel, we introduce the following operator Riccati equations:

$$\mathcal{S}\mathcal{A}_c\phi + \mathcal{A}_c^*\mathcal{S}\phi - \mathcal{S}\mathcal{B}R_c^{-1}\mathcal{B}^*\mathcal{S}\phi + \mathcal{C}_1^*N_c\mathcal{C}_1\phi = 0, \quad \phi \in \mathcal{W} \tag{42}$$
$$\mathcal{A}_f\mathcal{P}\psi + \mathcal{P}\mathcal{A}_f^*\psi - \mathcal{P}\mathcal{C}^*R_f^{-1}\mathcal{C}\mathcal{P}\psi + \mathcal{B}_1N_f\mathcal{B}_1^*\psi = 0, \quad \psi \in \mathcal{V}^* \tag{43}$$
$$\mathcal{A}_c := \mathcal{A} - \mathcal{B}_2D_{12}^+\mathcal{C}_1,\ \mathcal{A}_f := \mathcal{A} - \mathcal{B}_1D_{21}^+\mathcal{C}_2,$$
$$\mathcal{B} := [\,\mathcal{B}_1\ \mathcal{B}_2\,],\ \mathcal{C} := [\,\mathcal{C}_1^*\ \mathcal{C}_2^*\,]^*$$

and establish a design method of H^∞ control law. The H^∞ control problems Σ, Σ_{FI} are formally characterized by Propositions 11 and 12 [26].

Proposition 11 (Output feedback case): For a given $\gamma > 0$, the H^∞ control problem $\hat{\Sigma}$ is solvable iff (A), (B), (C) are satisfied.

- (A) The equation (42) has a stabilizing solution $\mathcal{S} \ge 0$ $(\mathcal{S} \in \mathcal{L}(\mathcal{V}, \mathcal{V}^*))$ such that $\mathcal{A}_c - \mathcal{B}R_c^{-1}\mathcal{B}^*\mathcal{S}$ generates an exponentially stable semigroup on $\mathcal{W}$, $\mathcal{V}$.
- (B) The equation (43) has a stabilizing solution $\mathcal{P} \ge 0$ $(\mathcal{P} \in \mathcal{L}(\mathcal{W}^*, \mathcal{W}))$ such that $\mathcal{A}_f - \mathcal{P}\mathcal{C}^*R_f^{-1}\mathcal{C}$ generates an exponentially stable semigroup on $\mathcal{W}$, $\mathcal{V}$.
- (C) The stabilizing solutions $\mathcal{S} \ge 0$, $\mathcal{P} \ge 0$ satisfy $\lambda_{\max}(\mathcal{P}\mathcal{S}) < \gamma^2$.

If (A), (B), (C) hold, an H^∞ control law is given by

$$u(t) = -(D_{12}^{\mathrm{T}}D_{12})^{-1}(\mathcal{B}_2^*\mathcal{S} + D_{12}^{\mathrm{T}}\mathcal{C}_1) \times (\mathcal{I} - \tfrac{1}{\gamma^2} \cdot \mathcal{P}\mathcal{S})^{-1}\underline{\hat{x}}(t) \tag{44a}$$
$$\dot{\underline{\hat{x}}}(t) = \mathcal{A}\underline{\hat{x}}(t) + \mathcal{B}_2u(t) + \tfrac{1}{\gamma^2} \cdot \mathcal{P}\mathcal{C}_1^*(\mathcal{C}_1\underline{\hat{x}}(t) + D_{12}u(t)) + (\mathcal{P}\mathcal{C}_2^* + \mathcal{B}_1D_{21}^{\mathrm{T}})(D_{21}D_{21}^{\mathrm{T}})^{-1}(y(t) - \mathcal{C}_2\underline{\hat{x}}(t)). \tag{44b}$$

■

For the full-information (FI) control problem $\hat{\Sigma}_{\mathrm{FI}}$ defined by (38a), (38b) with the measurement $\hat{y}(t) = (\hat{x}(t), w(t))$, the solution is characterized by (A) [26].

Proposition 12 (Full-information case): For a given $\gamma > 0$, the H^∞ control problem $\hat{\Sigma}_{\mathrm{FI}}$ is solvable iff (A) in Proposition 11 is satisfied. If (A) holds, an H^∞ control law is given by

$$u(t) = -(D_{12}^{\mathrm{T}}D_{12})^{-1}(\mathcal{B}_2^*\mathcal{S} + D_{12}^{\mathrm{T}}\mathcal{C}_1)\hat{x}(t). \tag{45}$$

■

Remark 13: A simplified condition $D_{11} = 0$ (D_{11}: feed-through matrix from w to z) is imposed on Σ as the general relaxation technique for $D_{11} \ne 0$ is not available for multiple input/output delay systems. The relaxation technique for delay-free systems (see e.g. [28]) is applicable only if (H4) is preserved for the transformed system. It is also noted that delayed signals are not allowed in the feed-through map from w to y or from u to z as the boundedness of the corresponding operators for D_{12}, D_{21} is required in the operator Riccati equation approach. These generalizations are in the direction of future research. ■

B. Proof of Theorem 1

Begin with the following lemma which is obtained by [9] Theorem 1 with an auxiliary transformation $u(t) = -D_{12}^+C_1x(t) + \tilde{u}(t)$.

Lemma 14 ([9] Theorem 1): For a given $\gamma > 0$, the equation (42) has a stabilizing solution $\mathcal{S} \ge 0$ only if the Hamiltonian matrix (6) has no eigenvalues on the imaginary axis. ■

If the FI problem $\hat{\Sigma}_{\mathrm{FI}}$ is solvable for $\gamma > 0$, Lemma 14 guarantees that there exists a full column rank matrix $V = [\,V_1^{\mathrm{T}}\ \ V_2^{\mathrm{T}}\,]^{\mathrm{T}} \in \mathbb{R}^{2n\times n}$ $(V_1, V_2 \in \mathbb{R}^{n\times n})$ satisfying (10).

Next, we derive an auxiliary delay form of $\hat{\Sigma}_{\mathrm{FI}}$, which yields an analytic solution of (42). On a state-space $\mathcal{X}^o := \mathbb{R}^n \times L_2(-L-\check{L}, 0; \mathbb{R}^n)$, introduce an auxiliary delay system:

$$\hat{\Sigma}_{\mathrm{FI}}^o:\ \begin{aligned} \dot{\hat{x}}^o(t) &= (\mathcal{A}_c^o + \mathcal{B}_2^oD_{12}^+\mathcal{C}_1^o)\hat{x}^o(t) + \mathcal{B}_1^ow(t) + \mathcal{B}_2^ou(t)\\ z(t) &= \mathcal{C}_1^o\hat{x}^o(t) + D_{12}u(t)\\ \hat{y}^o(t) &= (\hat{x}^o(t), w(t)) \end{aligned} \tag{46}$$

and a corresponding operator Riccati equation:

$$\mathcal{S}^o\mathcal{A}_c^o\phi + \mathcal{A}_c^{o*}\mathcal{S}^o\phi - \mathcal{S}^o\mathcal{B}^oR_c^{-1}\mathcal{B}^{o*}\mathcal{S}^o\phi + \mathcal{C}_1^{o*}N_c\mathcal{C}_1^o\phi = 0,$$
$$\mathcal{B}^o := [\,\mathcal{B}_1^o\ \mathcal{B}_2^o\,],\ \ \phi \in \mathcal{W}^o. \tag{47}$$

The operator $\mathcal{A}_c^o$ is an infinitesimal generator defined by

$$\mathcal{A}_c^o\phi := (A_c\phi^0, \phi^{1'}),\ \ \mathrm{D}(A_c) = \{\phi \in \mathcal{X}^o : \phi^1 \in W^{1,2}(-L-\check{L}, 0; \mathbb{R}^n), \phi^0 = \phi^1(0)\}. \tag{48}$$

Let $\mathcal{W}^o := \mathrm{D}(A_c^o)$ be a subspace of $\mathcal{X}^o$. Then $\mathrm{D}_{\mathcal{W}^o}(A_c^{o*}) = \mathcal{X}^o$ holds and $\mathcal{W}^o$, $\mathcal{X}^o$ are with continuous, dense injections satisfying $\mathcal{W}^o \subset \mathcal{X}^o$ [20]. The operators $\mathcal{B}_1^o \in \mathcal{L}(\mathbb{R}^{m_1}, \mathcal{X}^o)$, $\mathcal{B}_2^o \in \mathcal{L}(\mathbb{R}^{m_2}, \mathcal{X}^o)$, $\mathcal{C}_1^o \in \mathcal{L}(\mathcal{W}^o, \mathbb{R}^{p_1})$ are given by

$$\mathcal{B}_1^o := \mathcal{G}\mathcal{B}_1,\ \ \mathcal{B}_2^o := \mathcal{G}\mathcal{B}_2,$$
$$\mathcal{C}_1^o\phi := \sum_{j=0}^{\ell} C_1^j\phi^1(-L - \check{h}_j),\ \ \phi \in \mathcal{W}^o \tag{49}$$

where $\mathcal{G} \in \mathcal{L}(\mathcal{X}, \mathcal{X}^o)$ is defined by (16) and satisfies $\mathcal{G} \in \mathcal{L}(\mathcal{W}, \mathcal{W}^o)$, $\mathcal{G} \in \mathcal{L}(\mathcal{V}, \mathcal{X}^o)$.

For a given $\gamma > 0$, the H^∞ control problems $\hat{\Sigma}_{\rm FI}$ and $\hat{\Sigma}^o_{\rm FI}$ share the same solvability condition.

Lemma 15:

1) Let $\hat{x}(0) \in \mathcal{W}$ and $\hat{x}^o(0) = \mathcal{G}\hat{x}(0) \in \mathcal{W}^o$ be the initial states of $\hat{\Sigma}_{\rm FI}$, $\hat{\Sigma}^o_{\rm FI}$, respectively. Then the equalities $\hat{x}^o(t) = \mathcal{G}\hat{x}(t)$ and $\mathcal{C}^o_1\hat{x}^o(t) = \mathcal{C}_1\hat{x}(t)$ $(\hat{x}(t) \in \mathcal{W})$ hold for $(w, u) \in L_2(0, t; \mathbb{R}^{m_1+m_2})$.
2) The equation (42) has a stabilizing solution $\mathcal{S} \geq 0$ iff (47) has a stabilizing solution $\mathcal{S}^o \geq 0$[3].
3) Let $\mathcal{S}^o \geq 0$ be a stabilizing solution of (47). Then the stabilizing solution $\mathcal{S} \geq 0$ of (42) is given by $\mathcal{S} = \mathcal{G}^*\mathcal{S}^o\mathcal{G}$. ■

Proof: In highlight with [12], the system $\hat{\Sigma}_{\rm FI}$ allows delayed channels in the regulated output z. We note that the following equalities are obtained via straightforward calculation.

$$\begin{aligned}&\mathcal{B}_2D^+_{12}\mathcal{C}_1\phi = (B_2D^+_{12}C_1\phi^0, 0, 0),\\ &\mathcal{G}\mathcal{A}_c\phi = \mathcal{A}^o_c\mathcal{G}\phi,\ \mathcal{C}_1\phi = \mathcal{C}^o_1\mathcal{G}\phi,\ \ \phi = (\phi^0, \phi^1, \phi^2) \in \mathcal{W}\end{aligned} \tag{50}$$

1): Since $(\mathcal{A}^o_c + \mathcal{B}^o_2D^+_{12}\mathcal{C}^o_1)\mathcal{G}\phi = \mathcal{G}\mathcal{A}\phi$ $(\phi \in \mathcal{W})$ follows from (49), (50), the equality $\hat{x}^o(t) = \mathcal{G}\hat{x}(t)$ holds for $(w, u) \in L_2(0, t; \mathbb{R}^{m_1+m_2})$. By the 3rd equality of (50), the equality $\mathcal{C}^o_1\hat{x}^o(t) = \mathcal{C}_1\hat{x}(t)$ is derived.

2): By the proof of 1) , the systems $\hat{\Sigma}_{\rm FI}$ and $\hat{\Sigma}^o_{\rm FI}$ provide equivalent map from (w, u) to z. Hence, by Lemma 4 [12], the solvability conditions of the FI problems $\hat{\Sigma}_{\rm FI}$, $\hat{\Sigma}^o_{\rm FI}$ are equivalent.

3) Let $\mathcal{S}_0 \geq 0$ be a stabilizing solution of (47). By (49) and (50), it is verified that the stabilizing solution of (42) is given by $\mathcal{S} = \mathcal{G}^*\mathcal{S}^o\mathcal{G} \geq 0$. ■

The system $\hat{\Sigma}^o_{\rm FI}$ yields a Hamiltonian operator representation and enables to solve (47).

Lemma 16: Let $V \in \mathbb{R}^{2n\times n}$ be a full column rank matrix defined by (10). Then the Hamiltonian operator $\mathcal{H}^o := \begin{bmatrix}\mathcal{A}^o_c & -\mathcal{B}^oR_c^{-1}\mathcal{B}^{o*}\\ -\mathcal{C}^{o*}_1N_c\mathcal{C}^o_1 & -\mathcal{A}^{o*}_c\end{bmatrix}$ associated with the system $\hat{\Sigma}^o_{\rm FI}$ satisfies

$$\mathcal{H}^o\begin{bmatrix}\mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2\\ \mathcal{V}_2\end{bmatrix}\phi = \begin{bmatrix}\mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2\\ \mathcal{V}_2\end{bmatrix}\mathcal{A}^o_{\Lambda_c}\phi,\ \ \phi \in {\rm D}(\mathcal{A}^o_{\Lambda_c}) \tag{51}$$

$$\begin{aligned}&\mathcal{A}^o_{\Lambda_c}\phi = (\Lambda_c\phi^0, \phi^{1'}),\ \ {\rm D}(\mathcal{A}^o_{\Lambda_c}) = \{\phi \in \mathcal{X}^o :\\ &\qquad \phi^1 \in W^{1,2}(-L-\check{L}, 0;\ \mathbb{R}^{p_1}), V_1\phi^0 = \phi^1(0)\}\\ &\Lambda_c:\ \text{stable matrix defined by (10)},\end{aligned} \tag{52}$$

where $\mathcal{V}_1$, $\mathcal{V}_2$, Π are defined by (14), (15). ■

Proof: For the auxiliary Hamiltonian operator:

$$\begin{aligned}&\mathcal{H}^o_L := \begin{bmatrix}\mathcal{A}^o_c & -\mathcal{B}^o_LR_c^{-1}\mathcal{B}^{o*}_L\\ -\mathcal{C}^{o*}_1N_c\mathcal{C}^o_1 & -\mathcal{A}^{o*}_c\end{bmatrix},\\ &\mathcal{B}^o_L := \mathcal{G}\mathcal{B}_L,\ \mathcal{B}_L := [\,\mathcal{B}_{L1}\ \mathcal{B}_{L2}\,] \in \mathcal{L}(\mathbb{R}^{m_1+m_2}, \mathcal{V})\\ &\mathcal{B}^*_{L1}\psi := B_1^{\rm T}\psi^1(0),\ \mathcal{B}^*_{L2}\psi := B_2^{\rm T}\psi^1(0),\ \psi \in \mathcal{V}^*\end{aligned} \tag{53}$$

it is verified that the equality:

$$\mathcal{H}^o_L\begin{bmatrix}\mathcal{V}_1\\ \mathcal{V}_2\end{bmatrix}\phi = \begin{bmatrix}\mathcal{V}_1\\ \mathcal{V}_2\end{bmatrix}\mathcal{A}^o_{\Lambda_c}\phi,\ \phi \in {\rm D}(\mathcal{A}^o_{\Lambda_c}) \tag{54}$$

[3]The stabilizing solution of (47) means that $\mathcal{A}^o_c - \mathcal{B}^oR_c^{-1}\mathcal{B}^{o*}\mathcal{S}^o$ generates an exponentially stable semigroup on $\mathcal{W}^o$, $\mathcal{X}^o$.

holds. Since $\mathcal{T}\mathcal{H}^o_L = \mathcal{H}^o\mathcal{T}$, $\mathcal{T} := \begin{bmatrix}\mathcal{I} & \mathcal{G}\Pi\mathcal{G}^*\\ 0 & \mathcal{I}\end{bmatrix}$ follows from $\mathcal{C}_1\Pi\phi = 0$ $(\phi \in \mathcal{X})$ and $\Pi\mathcal{A}^*_c\psi + \mathcal{A}_c\Pi\psi + \mathcal{B}_LR_c^{-1}\mathcal{B}^*_L\psi - \mathcal{B}R_c^{-1}\mathcal{B}^*\psi = 0$ $(\psi \in \mathcal{V}^*)$, the equality (51) is obtained by (54). ■

It follows from (52) that the operator $\mathcal{A}^o_{\Lambda_c}$ generates an exponentially stable semigroup. Thus (51) yields a stabilizing solution:

$$\mathcal{S}^o = \mathcal{V}_2\left(\mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2\right)^{-1} \tag{55}$$

iff $\mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2$ is invertible. Exploring the conditions such that 1) the operator $\mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2$ is invertible and 2) the operator (55) is positive semi-definite, we establish a solvability condition of (42). Based on the condition 1), the existence of the stabilizing solution is characterized by the following theorem.

Theorem 17: Let $V \in \mathbb{R}^{2n\times n}$ be a full column rank matrix defined by (10). The operator Riccati equation (47) has a stabilizing solution $\mathcal{S}^o \in \mathcal{L}(\mathcal{X}^o)$ iff the matrix (8) is nonsingular. Furthermore, the stabilizing solution is given by (55). ■

Proof: We describe a proof along the line of [12] Theorem 6, which deals with the preliminary case $(\check{L} = 0)$. On the product space: $\mathcal{X}^o = \mathcal{X}^o_1 \times \mathcal{X}^o_2$, $\mathcal{X}^o_1 := \mathbb{R}^n \times L_2(-L, 0; \mathbb{R}^n)$, $\mathcal{X}^o_2 := L_2(-L-\check{L}, -L; \mathbb{R}^n)$, the operator $\mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2$ is expressed as follows:

$$\begin{aligned}&\mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2 = \begin{bmatrix}\mathcal{N}_1 & 0\\ 0 & \mathcal{I}\end{bmatrix},\\ &\mathcal{N}_1 := \mathcal{I} + \begin{bmatrix}V_1 - I & 0\\ 0 & 0\end{bmatrix} + \mathcal{G}_1\Pi\mathcal{G}^*_1\begin{bmatrix}V_2 & 0\\ 0 & \Theta_1\end{bmatrix},\\ &\quad(\Theta_1\phi^1)(\xi) := C_1^{\rm T}N_cC_1\phi^1(\xi),\\ &\qquad\qquad -L \leq \xi \leq 0,\ \phi^1 \in L_2(-L, 0; \mathbb{R}^n)\\ &\quad\mathcal{G}_1\phi := ((\mathcal{G}_1\phi)^0, (\mathcal{G}_2\phi)^1),\ \phi = (\phi^0, \phi^1) \in \mathcal{X}^o_1\\ &\quad(\mathcal{G}_1\phi)^0 := e^{A_cL}\phi^0 + \int_{-L}^0 e^{-A_c\beta}\phi^1(\beta)\,d\beta\\ &\quad(\mathcal{G}_1\phi)^1(\xi) := e^{A_c(\xi+L)}\phi^0 + \int_{-L}^{\xi} e^{A_c(\xi-\beta)}\phi^1(\beta)\,d\beta,\\ &\qquad\qquad -L \leq \xi \leq 0.\end{aligned} \tag{56}$$

The operator $\mathcal{N}_1$ shares the same structure as the FI problem where the output delays are relaxed $(\check{L} = 0)$. Hence, along the proof of [12] Theorem 6 (a) ⇔ (b), it is verified that (47) has a stabilizing solution $\mathcal{S}_o$ iff (56) is invertible. Furthermore by [12] Theorem 6 (b) ⇔ (c), (56) is invertible iff (8) is nonsingular. The stabilizing solution (55) is obtained based on (51). ■

The positive semi-definiteness of (55) is equivalent to the condition:

$$\begin{aligned}\mathcal{Q} &:= (\mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2)^*\mathcal{S}^o(\mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2)\\ &= (\mathcal{V}_1 + \mathcal{G}\Pi\mathcal{G}^*\mathcal{V}_2)^*\mathcal{V}_2 \geq 0.\end{aligned} \tag{57}$$

Transforming the condition (57) to a maximal eigenvalue problem, the positive semi-definiteness of (55) is characterized by Theorem 18.

Theorem 18: Let $V = [\,V_1^{\rm T}\ \ V_2^{\rm T}\,]^{\rm T} \in \mathbb{R}^{2n\times n}$ be a full column rank matrix defined by (10). The stabilizing solution (55) is positive semi-definite $(\mathcal{S}^o \geq 0)$ iff the maximal root of (9) satisfies $\lambda_{\max} \leq 1$. ■

Proof: On the product space: $\mathcal{X}^o = \mathcal{X}_1^o \times \mathcal{X}_2^o$, $\mathcal{X}_1^o := \mathbb{R}^n \times L_2(-L,0;\mathbb{R}^n)$, $\mathcal{X}_2^o := L_2(-L-\check{L},-L;\mathbb{R}^n)$, the condition (57) is expressed as

$$\mathcal{Q} = \begin{bmatrix} \Xi^*(\mathcal{I}-\Xi\Delta\Xi^*)\Xi & 0 \\ 0 & \Theta_2 \end{bmatrix} \geq 0,\ \Xi := \begin{bmatrix} I & 0 \\ 0 & N_c C_1 \cdot \mathcal{I} \end{bmatrix},$$

$$(\Theta_2 \phi^2)(\xi) := \sum_{j=0}^{\ell} \chi_{[-L-\check{h}_j,-L]}(\xi) C_1^{j\mathrm{T}} N_c C_1^j \phi^2(\xi),$$
$$-L-\check{L} \leq \xi \leq -L,\ \phi^2 \in L_2(-L-\check{L},-L;\mathbb{R}^n), \quad (58)$$

$$\Delta := \begin{bmatrix} I - V_1^{\mathrm{T}} V_2 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} V_2^{\mathrm{T}} & 0 \\ 0 & \mathcal{I} \end{bmatrix} \mathcal{G}_1 \Pi \mathcal{G}_1^* \begin{bmatrix} V_2 & 0 \\ 0 & \mathcal{I} \end{bmatrix}, \quad (59)$$

where $\Theta_2 \geq 0$. We first show that the conditions (58) and

$$\Xi\Delta\Xi^* \leq \mathcal{I} \quad (60)$$

are equivalent. The condition (60) derives (58) directly. By contradiction, we verify (58) derives (60). Suppose (58) holds and there exists $y \in \mathcal{X}_1^o$ such that $\langle y, (\mathcal{I}-\Xi\Delta\Xi^*)y \rangle < 0$ holds. Then an inequality: $\langle \tilde{y}, \Xi^*(\mathcal{I}-\Xi\Delta\Xi^*)\Xi\tilde{y} \rangle = \langle y, (\Xi\Xi^+ - \Xi\Delta\Xi^*)y \rangle \leq \langle y, (\mathcal{I}-\Xi\Delta\Xi^*)y \rangle < 0$ is obtained for $\tilde{y} := \Xi^+ y$, $\Xi^+ := \begin{bmatrix} I & 0 \\ 0 & (N_c C_1)^+ \cdot \mathcal{I} \end{bmatrix}$ where $(N_c C_1)^+$ is the pseudo-inverse of $N_c C_1$ and $\Xi\Xi^+\Xi = \Xi$, $(\Xi\Xi^+)^* = \Xi\Xi^+$ hold. By contradiction, it is shown that (58) derives (60).

Next, we prove that the condition (60) holds iff the maximal root of (9) satisfies $\lambda_{\max} \leq 1$. Since $\Xi\Delta\Xi^*$ is compact, we clarify the condition $\lambda_{\max}(\Xi\Delta\Xi^*) \leq 1$ by solving the eigenvalue problem of $\Xi\Delta\Xi^*$. Based on the expression

$$\lambda v = \begin{bmatrix} I - V_1^{\mathrm{T}} V_2 & 0 \\ 0 & 0 \end{bmatrix} v - \begin{bmatrix} V_2^{\mathrm{T}} & 0 \\ 0 & N_c C_1 \cdot \mathcal{I} \end{bmatrix} \mathcal{G}_1 \Pi f,$$
$$f = \mathcal{G}_1^* \begin{bmatrix} V_2 & 0 \\ 0 & C_1^{\mathrm{T}} N_c \cdot \mathcal{I} \end{bmatrix} v \quad (61)$$

which is equivalent to $\lambda v = \Xi\Delta\Xi^* v$, we will show that there exists $v \neq 0$ in (61) iff $V_p(\lambda)$ $(\lambda \neq 0, 1)$ is singular. Introducing auxiliary variables:

$$p(\xi) := \int_{-L}^{\xi} e^{A_c(\xi-\beta)} \{-(\Pi_1 f^1)(\beta)\}\, d\beta,$$
$$q(\beta) := e^{-A_c^{\mathrm{T}}\beta} V_2 v^0 + \int_{\beta}^{0} e^{A_c^{\mathrm{T}}(\xi-\beta)} C_1^{\mathrm{T}} N_c v^1(\xi)\, d\xi \quad (62)$$

to the left and right equalities of (61), we have boundary conditions:

$$\begin{bmatrix} V_2 V_2^{\mathrm{T}} & -(\lambda-1)\cdot I - V_2 V_1^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} p(0) \\ q(0) \end{bmatrix} = 0,$$
$$\begin{bmatrix} p(0) \\ q(0) \end{bmatrix} = \Phi_\lambda^{-1}(-L) \begin{bmatrix} p(-L) \\ q(-L) \end{bmatrix},\ \begin{bmatrix} p(-L) \\ q(-L) \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix} f^0$$

and

$$\tilde{V}_p(\lambda) f^0 = 0,\ \tilde{V}_p(\lambda) :=$$
$$[\, V_2 V_2^{\mathrm{T}} \ \ -(\lambda-1)\cdot I - V_2 V_1^{\mathrm{T}} \,]\, \Phi_\lambda^{-1}(-L) [\, 0 \ \ I \,]^{\mathrm{T}}. \quad (63)$$

For $\lambda \neq 0, 1$, it is verified by (61), (62) that $f^0 \neq 0$ holds iff $v = (v^0, v^1) \neq 0$. Thus $\lambda \neq 0, 1$ is the eigenvalue of $\Gamma\Delta\Gamma^*$ iff the matrix $\tilde{V}_p(\lambda)$ is nonsingular. Substituting $\Phi_\lambda^{-1}(-L) = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \Phi_\lambda^{\mathrm{T}}(-L) \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$ to (63), the condition (9) is derived. ■

The analytic solution of (42) is clarified by the following theorem.

Theorem 19: For a given $\gamma > 0$, the conditions (A) in Proposition 11 and (a) in Theorem 1 are equivalent. If (A) or (a) holds, the stabilizing solution $\mathcal{S} \geq 0$ is given by (13) and further expressed as follows:

$$(\mathcal{S}v)^0 = G(-L,-L)v^0 + \int_{-L}^{0} G(-L,\beta) v^1(\beta)\, d\beta, \quad (64a)$$

$$(\mathcal{S}v)^1(\xi) = G(\xi,-L)v^0 + \int_{-L}^{0} G(\xi,\beta) v^1(\beta)\, d\beta, \quad (64b)$$

$$(\mathcal{S}v)^2(\beta) = \sum_{j=0}^{\ell} \chi_{[-\check{h}_j,0]}(\beta) \cdot C_1^{j\mathrm{T}} N_c C_1^j v^2(\beta), \quad (64c)$$
$$-L \leq \xi \leq 0,\ -\check{L} \leq \beta \leq 0,\ v = (v^0, v^1, v^2) \in \mathcal{X}$$

where G is defined by (12). ■

Proof: The conditions (A) and (a) are equivalent by Lemmas 14, 15 and Theorems 17, 18. Furthermore if (A) or (a) holds, Lemma 15 3) and Theorem 17 yields a positive semi-definite stabilizing solution (13). In the following, we will derive (64) from (13). By (13), the equality $f = \mathcal{S}v$ is expressed as

$$\mathcal{V}_1 w = \mathcal{G}(v - \Pi f), \quad f = \mathcal{G}^* \mathcal{V}_2 w. \quad (65)$$

Introducing auxiliary variables:

$$p(\xi) := e^{A_c(\xi+L)} v^0 + \int_{-L}^{\xi} e^{A_c(\xi-\beta)} \{v^1(\beta) - (\Pi_1 f^1)(\beta)\}\, d\beta$$
$$q(\beta) := e^{-A_c^{\mathrm{T}}\beta} V_2 w^0 + \int_{\beta}^{0} e^{A_c^{\mathrm{T}}(\xi-\beta)} (\Theta w^1)(\xi)\, d\xi$$

to the left and right equalities of (65), we have

$$f^0 = q(-L),\quad f^1(\xi) = q(\xi),$$
$$f^2(\beta) = \sum_{j=0}^{\ell} \chi_{[-\check{h}_j,0]}(\beta) \cdot C_1^{j\mathrm{T}} N_c C_1^j v^2(\beta) \quad (66)$$

and the equalities:

$$\begin{bmatrix} p(\xi) \\ q(\xi) \end{bmatrix} = \Phi_1(\xi) \begin{bmatrix} p(0) \\ q(0) \end{bmatrix} - \int_{\xi}^{0} \Phi_1(\xi)\Phi_1^{-1}(\beta) \begin{bmatrix} I \\ 0 \end{bmatrix} v^1(\beta)\, d\beta \quad (67)$$

$$\begin{bmatrix} p(-L) \\ q(-L) \end{bmatrix} = \begin{bmatrix} v^0 \\ f^0 \end{bmatrix},\quad \begin{bmatrix} p(0) \\ q(0) \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} w^0 \quad (68)$$

where $\Phi_1(\cdot)$ is defined by (7). Substituting $\xi = -L$ and (68) to (67), then pre-multiplying $[I \ \ 0]$, we obtain

$$w^0 = V_s^{-1} v^0$$
$$+ \int_{-L}^{0} V_s^{-1} [\, I \ \ 0 \,] \Phi_1(-L) \Phi_1^{-1}(\beta) [\, I \ \ 0 \,]^{\mathrm{T}} v^1(\beta)\, d\beta \quad (69)$$

where V_s is nonsingular by Theorem 17. Since (67), (69), and the right equality of (68) yield

$$q(\xi) = G(\xi,-L)v^0 + \int_{-L}^{0} G(\xi,\beta) v^1(\beta)\, d\beta, \quad (70)$$

the expression (64) is derived by (66), (70). ■

Proof of Theorem 1: By Proposition 12, Theorem 19, the H^∞ control problem Σ_{FI} is solvable iff (a) holds. The control law (11) is derived from (40), (45), (64), and Remark 10. ■

C. Proofs of Lemmas 2, 3

Proof of Lemma 2: The solution of $\dot{\hat{x}}(t) = (\mathcal{A}_c - \mathcal{B}_2(D_{12}^{\mathrm{T}}D_{12})^{-1}\mathcal{B}_2^*\mathcal{S})\hat{x}(t)$, $\hat{x}(0) \in \mathcal{W}$ is bounded in $0 \le t \le \max(L, \check{L})$ (Remark 10) and, for $t \ge \max(L, \check{L})$, it is reduced to

$$\begin{aligned}&\hat{x}(t) = (x(t), 0, 0) \in \mathcal{X},\\ &\dot{x}(t) = \{A_c - B_2(D_{12}^{\mathrm{T}}D_{12})^{-1}B_2^{\mathrm{T}}G(-L, -L)\}x(t) \qquad (71)\end{aligned}$$

where $G(-L, -L)$ is defined by (12). Since the positive semi-definiteness of (13) corresponds to the stability of (71) ([12] Lemma 21), the solvability condition is characterized by (a_w). ■

Proof of Lemma 3: For a given $\gamma > 0$, the H^∞ control problem $\hat{\Sigma}_{\mathrm{FI}}$ is solvable only if the problem with $L = 0$ is solvable. Hence, by [27], the equation (18) has a stabilizing solution $S \ge 0$ and $V = [I \;\; S]^{\mathrm{T}}$ meets (10). Since (55) is expressed as $\mathcal{S}^o = \mathcal{M}(\mathcal{I} + \mathcal{G}\Pi\mathcal{G}^*\mathcal{M})^{-1}$, $\mathcal{M} := \mathcal{V}_2\mathcal{V}_1^{-1} = \begin{bmatrix} S & 0 \\ 0 & \Theta \end{bmatrix} \ge 0$, the stabilizing solution (13) is positive semi-definite iff $\lambda_{\max}(-\mathcal{G}\Pi\mathcal{G}^*\mathcal{M}) < 1$. Along [12] Corollary 15, the condition (19) is obtained. ■

D. Proof of Theorem 5

Utilizing the fundamental results obtained by Section V-B, we will solve the H^∞ output feedback problem Σ based on Proposition 11. The condition (B) is clarified exploring the duality between (42) and (43). The condition (C) is further simplified by employing the analytic solutions of (42), (43).

In order to solve (43), introduce a transposed system of Σ:

$$\begin{aligned}\Sigma^{\mathrm{T}}:\ \dot{p}(t) &= A^{\mathrm{T}}p(t) + \sum_{i=0}^{\ell} C_1^{i\mathrm{T}}\tilde{w}(t - \check{h}_i) + \sum_{i=0}^{\ell} C_2^{i\mathrm{T}}\tilde{u}(t - \check{h}_i)\\ \tilde{z}(t) &= \sum_{i=0}^{d} B_1^{i\mathrm{T}}x(t - h_i) + D_{21}^{\mathrm{T}}\tilde{u}(t)\\ \tilde{y}(t) &= \sum_{i=0}^{d} B_2^{i\mathrm{T}}p(t - h_i) + D_{12}^{\mathrm{T}}\tilde{w}(t). \qquad (72)\end{aligned}$$

On the space $\mathcal{X}^{\mathrm{T}} := \mathbb{R}^n \times L_2(-\check{L}, 0; \mathbb{R}^n) \times L_2(-L, 0; \mathbb{R}^n)$, the system Σ^{T} is described by

$$\begin{aligned}\hat{\Sigma}^{\mathrm{T}}:\ \dot{\hat{p}}(t) &= \mathcal{A}^{\mathrm{T}}\hat{p}(t) + \mathcal{C}_1^{\mathrm{T}}\tilde{w}(t) + \mathcal{C}_2^{\mathrm{T}}\tilde{u}(t)\\ \tilde{z}(t) &= \mathcal{B}_1^{\mathrm{T}}\hat{p}(t) + D_{21}^{\mathrm{T}}\tilde{u}(t)\\ \tilde{y}(t) &= \mathcal{B}_2^{\mathrm{T}}\hat{p}(t) + D_{12}^{\mathrm{T}}\tilde{w}(t) \qquad (73)\end{aligned}$$

where $\mathcal{A}^{\mathrm{T}}$ is an infinitesimal generator defined by

$$\begin{aligned}&\mathcal{A}^{\mathrm{T}}\phi := (A^{\mathrm{T}}\phi^0 + \phi^1(-\check{L}), \phi^{1'}, \phi^{2'}),\\ &\mathrm{D}(\mathcal{A}^{\mathrm{T}}) = \{\phi \in \mathcal{X}^{\mathrm{T}} : \phi^1 \in W^{1,2}(-\check{L}, 0; \mathbb{R}^n),\\ &\quad \phi^2 \in W^{1,2}(-L, 0; \mathbb{R}^n), \phi^1(0) = 0, \phi^2(0) = \phi^0\}. \qquad (74)\end{aligned}$$

Let $\mathcal{V}^{\mathrm{T}*} := \{\psi \in \mathcal{X}^{\mathrm{T}} : \psi^1 \in W^{1,2}(-\check{L}, 0; \mathbb{R}^n), \psi^1(-\check{L}) = \psi^0\}$, $\mathcal{W}^{\mathrm{T}} := \{\phi \in \mathcal{X}^{\mathrm{T}} : \phi^2 \in W^{1,2}(-\check{L}, 0; \mathbb{R}^n), \phi^2(0) = \phi^0\}$ be subspaces of $\mathcal{X}^{\mathrm{T}}$. Then $\mathcal{W}^{\mathrm{T}} = \mathrm{D}_{\mathcal{V}^{\mathrm{T}}}(\mathcal{A}^{\mathrm{T}})$, $\mathcal{V}^{\mathrm{T}*} = \mathrm{D}_{\mathcal{W}^{\mathrm{T}*}}(\mathcal{A}^{\mathrm{T}*})$ hold and $\mathcal{W}^{\mathrm{T}}$, $\mathcal{X}^{\mathrm{T}}$, $\mathcal{V}^{\mathrm{T}}$ are with continuous, dense injections satisfying $\mathcal{W}^{\mathrm{T}} \subset \mathcal{X}^{\mathrm{T}} \subset \mathcal{V}^{\mathrm{T}}$ ([20], Remark 2.6). The operators $\mathcal{C}_k^{\mathrm{T}} \in \mathcal{L}(\mathbb{R}^{p_k}, \mathcal{V}^{\mathrm{T}})$, $\mathcal{B}_k^{\mathrm{T}} \in \mathcal{L}(\mathcal{W}^{\mathrm{T}}, \mathbb{R}^{m_k})$ $(k = 1, 2)$ are given by

$$\begin{aligned}\mathcal{C}_k^{\mathrm{T}*}\psi &:= C_k^0\psi^0 + \sum_{j=1}^{\ell} C_k^j\psi^1(-\check{L} + \check{h}_j), \psi \in \mathcal{V}^{\mathrm{T}*},\\ \mathcal{B}_k^{\mathrm{T}}\phi &:= B_k^{0\mathrm{T}}\phi^0 + \sum_{i=0}^{d} B_k^{i\mathrm{T}}\phi^2(-h_i), \phi \in \mathcal{W}^{\mathrm{T}}. \qquad (75)\end{aligned}$$

Based on the operator Riccati equation defined for $\hat{\Sigma}^{\mathrm{T}}$:

$$\begin{aligned}&\mathcal{P}^{\mathrm{T}}\mathcal{A}_f^{\mathrm{T}}\phi + \mathcal{A}_f^{\mathrm{T}*}\mathcal{P}^{\mathrm{T}}\phi - \mathcal{P}^{\mathrm{T}}\mathcal{C}^{\mathrm{T}}R_f^{-1}\mathcal{C}^{\mathrm{T}*}\mathcal{P}^{\mathrm{T}}\phi\\ &\qquad + \mathcal{B}_1^{\mathrm{T}*}N_f\mathcal{B}_1^{\mathrm{T}}\phi = 0, \;\; \phi \in \mathcal{W}^{\mathrm{T}}\\ &\mathcal{A}_f^{\mathrm{T}} := \mathcal{A}^{\mathrm{T}} - \mathcal{C}_2^{\mathrm{T}}D_{21}^{+\mathrm{T}}\mathcal{B}_1^{\mathrm{T}}, \; \mathcal{C}^{\mathrm{T}} := [\,\mathcal{C}_1^{\mathrm{T}}\; \mathcal{C}_2^{\mathrm{T}}\,], \qquad (76)\end{aligned}$$

the condition (B) is characterized by the following lemma.

Lemma 20: The condition (B) holds iff the equation (76) has a stabilizing solution $\mathcal{P}^{\mathrm{T}} \ge 0$ $(\mathcal{P}^{\mathrm{T}} \in \mathcal{L}(\mathcal{V}^{\mathrm{T}}, \mathcal{V}^{\mathrm{T}*}))$ such that $\mathcal{A}^{\mathrm{T}} - \mathcal{C}^{\mathrm{T}}R_f^{-1}\mathcal{C}^{\mathrm{T}*}\mathcal{P}^{\mathrm{T}}$ generates an exponentially stable semigroup on $\mathcal{W}^{\mathrm{T}}$ and $\mathcal{V}^{\mathrm{T}}$. Furthermore, the stabilizing solution of (43) is given by

$$\mathcal{P} = \mathcal{J}^{-1}\mathcal{P}^{\mathrm{T}}\mathcal{J}^{*-1} \ge 0 \qquad (77)$$

where $\mathcal{P}^{\mathrm{T}} \ge 0$ is the stabilizing solution of (76) and $\mathcal{J} \in \mathcal{L}(\mathcal{X}, \mathcal{X}^{\mathrm{T}})$ is an isomorphic operator:

$$\begin{aligned}&\mathcal{J}\phi := \begin{bmatrix}(\mathcal{J}\phi)^0\\(\mathcal{J}\phi)^1\\(\mathcal{J}\phi)^2\end{bmatrix}, \quad \begin{aligned}&(\mathcal{J}\phi)^0 := \phi^0,\\ &(\mathcal{J}\phi)^1(\alpha) := \phi^2(-\alpha - \check{L}),\\ &(\mathcal{J}\phi)^2(\beta) := \phi^1(-\beta - L),\end{aligned}\\ &-\check{L} \le \alpha \le 0, \; -L \le \beta \le 0, \; \phi = (\phi^0, \phi^1, \phi^2) \in \mathcal{X} \qquad (78)\end{aligned}$$

satisfying $\mathcal{J} \in \mathcal{L}(\mathcal{V}, \mathcal{W}^{\mathrm{T}*})$, $\mathcal{J} \in \mathcal{L}(\mathcal{W}, \mathcal{V}^{\mathrm{T}*})$. ■

Proof: The following relations are obtained for $\hat{\Sigma}$, $\hat{\Sigma}^{\mathrm{T}}$:

$$\mathcal{A}^{\mathrm{T}*}\mathcal{J}\phi = \mathcal{J}\mathcal{A}\phi, \; \mathcal{C}_1^{\mathrm{T}*}\mathcal{J}\phi = \mathcal{C}_1\phi, \; \mathcal{C}_2^{\mathrm{T}*}\mathcal{J}\phi = \mathcal{C}_2\phi, \; \phi \in \mathcal{W} \qquad (79a)$$

$$\mathcal{B}_1^{\mathrm{T}}\psi = \mathcal{B}_1^*\mathcal{J}^*\psi, \; \mathcal{B}_2^{\mathrm{T}}\psi = \mathcal{B}_2^*\mathcal{J}^*\psi, \; \psi \in \mathcal{W}^{\mathrm{T}}. \qquad (79b)$$

Hence, if $\mathcal{P}^{\mathrm{T}} \ge 0$ is a solution of (76), a solution $\mathcal{P} \ge 0$ of (43) is given by (77). Since $\mathcal{J}(\mathcal{A}_f - \mathcal{P}\mathcal{C}^*R_f^{-1}\mathcal{C})\phi = (\mathcal{A}_f^{\mathrm{T}} - \mathcal{C}^{\mathrm{T}}R_f^{-1}\mathcal{C}^{\mathrm{T}*}\mathcal{P}^{\mathrm{T}})^*\mathcal{J}\phi$, $\phi \in \mathcal{W}$ holds by (79), both $\mathcal{P}^{\mathrm{T}}$ and $\mathcal{P}$ are stabilizing solutions if either is a stabilizing solution. ■

Applying Theorem 19, Lemma 20 to (76), it is shown that (B) and (b) are equivalent.

Lemma 21: For a given $\gamma > 0$, the conditions (B) and (b) are equivalent. If (b) holds, the stabilizing solution $\mathcal{P} \ge 0$ is given as follows:

$$\mathcal{P} = \mathcal{J}^{-1}\mathcal{G}^{\mathrm{T}*}\mathcal{U}_2(\mathcal{U}_1 + \mathcal{G}^{\mathrm{T}}\Pi^{\mathrm{T}}\mathcal{G}^{\mathrm{T}*}\mathcal{U}_2)^{-1}\mathcal{G}^{\mathrm{T}}\mathcal{J}^{*-1}, \qquad (80)$$

where $\mathcal{G}^{\mathrm{T}} \in \mathcal{L}(\mathcal{X}^{\mathrm{T}}, \mathcal{X}^{o\mathrm{T}})$, $\mathcal{X}^{o\mathrm{T}} := \mathbb{R}^n \times L_2(-\check{L} - L, 0; \mathbb{R}^n)$ and

$$\begin{aligned}&\mathcal{U}_1 := \begin{bmatrix} U_1 & 0\\ 0 & \mathcal{I}\end{bmatrix}, \; \mathcal{U}_2 := \begin{bmatrix} U_2 & 0\\ 0 & \check{\Theta}\end{bmatrix} \in \mathcal{L}(\mathcal{X}^{o\mathrm{T}}),\\ &\Pi^{\mathrm{T}} := \begin{bmatrix} 0 & 0 & 0\\ 0 & \Pi_1^{\mathrm{T}} & 0\\ 0 & 0 & 0\end{bmatrix} \in \mathcal{L}(\mathcal{X}^{\mathrm{T}}), \qquad (81)\end{aligned}$$

are defined by the following association with the terminology (16), (14), (15):

$$\Sigma \leftarrow \Sigma^{\mathrm{T}}, \; V \leftarrow U, \; \mathcal{V} \leftarrow \mathcal{U}, \; \mathcal{G} \leftarrow \mathcal{G}^{\mathrm{T}}, \; \Pi \leftarrow \Pi^{\mathrm{T}}, \; \Theta \leftarrow \check{\Theta}. \qquad (82)$$

Furthermore, the solution (80) is expressed as

$$(\mathcal{P}v)^0 = F(0,0)v^0 + \int_{-\check{L}}^{0} F(0,\beta)v^2(\beta)\,d\beta \tag{83a}$$

$$(\mathcal{P}v)^1(\beta) = \sum_{i=0}^{d} \chi_{[-L,-L+h_i]}(\beta)\cdot B_1^i N_f B_1^{i\mathrm{T}} v^1(\beta), \tag{83b}$$

$$(\mathcal{P}v)^2(\xi) = F(\xi,0)v^0 + \int_{-\check{L}}^{0} F(\xi,\beta)v^2(\beta)\,d\beta, \tag{83c}$$

$$-L \le \beta \le 0,\ -\check{L} \le \xi \le 0,\ v = (v^0, v^1, v^2) \in \mathcal{X}$$

where F is defined by (29). ■

Based on the analytic solutions (13), (80), the spectral radius condition (C) is characterized by the maximal root of a transcendental equation.

Lemma 22: Suppose (A), (B) hold for a given $\gamma > 0$. Then (C) and (c) are equivalent. ■

Proof: We will show that the roots of (27) meet the nonzero eigenvalues of $\mathcal{PS}$. Let $\sigma^2 \neq 0$ ($\sigma > 0$) be an eigenvalue of $\mathcal{PS}$ and suppose $\sigma^2 \cdot v = \mathcal{PS}v$ or

$$\sigma \cdot v = \mathcal{P}f, \quad \sigma \cdot f = \mathcal{S}v \tag{84}$$

hold for $v \neq 0$. By (13), (80), the equalities in (84) are expressed as

$$\mathcal{V}_1 w = \mathcal{G}(v - \sigma\cdot \Pi f), \quad \sigma f = \mathcal{G}^* \mathcal{V}_2 w \tag{85a}$$

$$\sigma v = \mathcal{J}^{-1}\mathcal{G}^{\mathrm{T}*}\mathcal{U}_2 \tilde{w},\ \mathcal{U}_1 \tilde{w} = \mathcal{G}^{\mathrm{T}}\mathcal{J}^{*-1}(f - \sigma\cdot\mathcal{J}^*\Pi^{\mathrm{T}}\mathcal{J}v). \tag{85b}$$

We clarify the condition such that $v \neq 0$ exists in (85). Introducing auxiliary variables:

$$p(\xi) := e^{A_c(\xi+L)}v^0 + \int_{-L}^{\xi} e^{A_c(\xi-\beta)}(v^1(\beta) - \sigma\cdot(\Pi_1 f^1)(\beta))\,d\beta \tag{86}$$

$$q(\beta) := e^{-A_c^{\mathrm{T}}\beta}V_2 w^0 + \int_{\beta}^{0} e^{A_c^{\mathrm{T}}(\xi-\beta)}(\Theta w^1)(\xi)\,d\xi \tag{87}$$

to the left and right equalities of (85a), then similarly introducing

$$\check{p}(\beta) := e^{A_f(\beta+\check{L})}U_2\tilde{w}^0 + \int_{-\beta-\check{L}}^{0} e^{A_f(\beta+\check{L}+\xi)}(\check{\Theta}\tilde{w}^1)(\xi)\,d\xi \tag{88}$$

$$\check{q}(\xi) := -e^{-A_f^{\mathrm{T}}\xi}f^0 - \int_{\xi}^{0} e^{A_f^{\mathrm{T}}(\beta-\xi)}(f^2(\beta) - \sigma\cdot(\mathcal{J}^*\Pi^{\mathrm{T}}\mathcal{J}v)^2(\beta))\,d\beta \tag{89}$$

to the left and right equalities of (85b), we have

$$\begin{bmatrix} v^0 \\ \sigma\cdot f^0\end{bmatrix} = \begin{bmatrix} p(-L) \\ q(-L)\end{bmatrix},\ \begin{bmatrix} p(-L) \\ q(-L)\end{bmatrix} = \check{\Phi}^{\sigma}(-L)\begin{bmatrix} p(0) \\ q(0)\end{bmatrix},\ \begin{bmatrix} p(0) \\ q(0)\end{bmatrix} = Vw^0. \tag{90}$$

$$0 = U^{\mathrm{T}}\begin{bmatrix} \check{p}(-\check{L}) \\ \check{q}(-\check{L})\end{bmatrix},\ \begin{bmatrix} \check{p}(-\check{L}) \\ \check{q}(-\check{L})\end{bmatrix} = \check{\Psi}^{\sigma}(-\check{L})\begin{bmatrix} \check{p}(0) \\ \check{q}(0)\end{bmatrix},\ \begin{bmatrix} \check{p}(0) \\ \check{q}(0)\end{bmatrix} = \begin{bmatrix} \sigma\cdot v^0 \\ -f^0\end{bmatrix}. \tag{91}$$

Combining the equalities (90), (91), the condition $W(\sigma)w^0 = 0$ ($w^0 \neq 0$) is obtained.

For $\sigma \neq 0$, it is verified from (85)-(89) that $v \neq 0$ exists in (85) iff $w^0 \neq 0$ satisfies $W(\sigma)w^0 = 0$. Thus the maximal eigenvalue of $\lambda_{\max}(\mathcal{PS})$ is given by $\sigma^2_{\max}$. ■

Proof of Theorem 5: By Proposition 11, Theorem 19, Lemmas 21, 22, the solvability condition is given by (a), (b), (c). In the expression of the control law (28), we first derive (28d), (28e), (28f) by rewriting the control law (44b) in the following form:

$$\begin{aligned}
\dot{\underline{\hat{x}}}(t) &= \mathcal{A}\underline{\hat{x}}(t) + \mathcal{B}\tilde{f}(t) + \tilde{g}(t),\\
\tilde{f}(t) &= \begin{bmatrix} D_{21}^{+}(y(t) - \mathcal{C}_2\underline{\hat{x}}(t)) \\ u(t)\end{bmatrix} \in \mathbb{R}^{m_1+m_2},\\
\tilde{g}(t) &= -\mathcal{PC}^* R_f^{-1}\begin{bmatrix} \mathcal{C}_1\underline{\hat{x}}(t) + D_{12}u(t) \\ \mathcal{C}_2\underline{\hat{x}}(t) - y(t)\end{bmatrix} \in \mathcal{W}.
\end{aligned} \tag{92}$$

Since $\langle\psi, \dot{\underline{\hat{x}}}(t)\rangle_{\mathcal{V}^*,\mathcal{V}} = \langle\psi, \mathcal{A}\underline{\hat{x}}(t) + \mathcal{B}\tilde{f}(t) + \tilde{g}(t)\rangle_{\mathcal{V}^*,\mathcal{V}}$, $\forall\psi \in \mathcal{V}^*$ holds, the following representation is obtained for $\underline{\hat{x}}(t) := (\underline{x}^0(t), \underline{x}^1(t,\cdot), \underline{x}^2(t,\cdot)) \in \mathcal{W}$, $\tilde{g}(t) := (\tilde{g}^0(t), \tilde{g}^1(t,\cdot), \tilde{g}^2(t,\cdot)) \in \mathcal{W}$:

$$\dot{\underline{x}}^0(t) = A\underline{x}^0(t) + \underline{x}^1(t,-L) + \sum_{i=0}^{d} B^i\tilde{f}(t-h_i) + \tilde{g}^0(t) \tag{93a}$$

$$\frac{\partial}{\partial t}\underline{x}^1(t,\beta) = \frac{\partial}{\partial\beta}\underline{x}^1(t,\beta) + \tilde{g}^1(t,\beta) \tag{93b}$$

$$\underline{x}^1(t,-L+h_k-0) = \begin{cases} \underline{x}^1(t,-L+h_k) + B^k\tilde{f}(t), & k = 1,2,\ldots,d-1 \\ B^d\tilde{f}(t), & k = d\end{cases} \tag{93c}$$

$$\frac{\partial}{\partial t}\underline{x}^2(t,\beta) = \frac{\partial}{\partial\beta}\underline{x}^2(t,\beta) + \tilde{g}^2(t,\beta),\ \ \underline{x}^2(t,0) = \underline{x}^0(t). \tag{93d}$$

While $\tilde{f}(t)$, $\tilde{g}(t)$ in (92) are expressed as

$$\tilde{f}(t) = \begin{bmatrix} D_{21}^{+}(y(t) - \sum_{j=0}^{\ell} C_2^j \underline{x}^2(t,-h_j)) \\ u(t)\end{bmatrix}, \tag{94}$$

$$\tilde{g}^0(t) = -\sum_{j=0}^{\ell} F(0,-\check{h}_j)g_j(t),\ \ \tilde{g}^1(t,\beta) = 0,$$

$$\tilde{g}^2(t,\beta) = -\sum_{j=0}^{\ell} F(\beta,-\check{h}_j)g_j(t) \tag{95}$$

by employing Lemma 21. Hence, the solutions of (93b)-(93c) and (93d) are obtained by

$$\underline{x}^1(t,\beta) = \sum_{i=0}^{d}\chi_{[-L,-L+h_i]}(\beta)\cdot B^i\tilde{f}(t+\beta+L-h_i),$$

$$\underline{x}^2(t,\beta) = \underline{x}^0(t+\beta) + \int_{\beta}^{0}\tilde{g}^2(t+\xi,\beta-\xi)\,d\xi. \tag{96}$$

Replacing the variables by $\underline{x}(t) := \underline{x}^0(t)$, $\underline{x}(t,\beta) := \underline{x}^2(t,\beta)$, the equalities (28d), (28e), (28f) are obtained from (94)-(96).

In order to derive (28a), (28b), (28c) from (44a), we focus on the relation:

$$u = -(D_{12}^{\mathrm{T}}D_{12})^{-1}(\mathcal{B}_2^*\mathcal{S} + D_{12}^{\mathrm{T}}\mathcal{C}_1)(\mathcal{I} - \tfrac{1}{\gamma^2}\cdot\mathcal{PS})^{-1}g,\quad u \in \mathbb{R}^{m_2},\ \ g \in \mathcal{W} \tag{97}$$

and elaborate the expression of u. Employing (13), (80), the equality (97) is given by

$$u + (D_{12}^{\mathrm{T}} D_{12})^{-1} \mathcal{B}_2^* f + D_{12}^+ \mathcal{C}_1 v = 0 \tag{98a}$$
$$\mathcal{V}_1 w = \mathcal{G}(v - \Pi f), \quad f = \mathcal{G}^* \mathcal{V}_2 w \tag{98b}$$
$$\begin{aligned} &\gamma^2 \cdot (v - g) = \mathcal{J}^{-1} \mathcal{G}^{\mathrm{T}*} \mathcal{U}_2 \tilde{w}, \\ &\mathcal{U}_1 \tilde{w} = \mathcal{G}^{\mathrm{T}} \mathcal{J}^{*-1} \{ f - \gamma^2 \cdot \mathcal{J}^* \Pi^{\mathrm{T}} \mathcal{J}(v - g) \}. \end{aligned} \tag{98c}$$

Introducing (86) modified as $\sigma \leftarrow 1$ and (87) to the left and right equalities of (98b), then introducing (88) and (89) modified as $\sigma \cdot (\mathcal{J}^* \Pi^{\mathrm{T}} \mathcal{J} v)^2 \leftarrow \gamma^2 \cdot (\mathcal{J}^* \Pi^{\mathrm{T}} \mathcal{J}(v - g))^2$ to (98c), we have

$$\begin{bmatrix} p(\xi) \\ q(\xi) \end{bmatrix} = \check{\Phi}^{\gamma}(\xi) \begin{bmatrix} p(0) \\ q(0) \end{bmatrix} - \int_{\xi}^{0} \check{\Phi}^{\gamma}(\xi) \check{\Phi}^{\gamma-1}(\beta) \begin{bmatrix} I \\ 0 \end{bmatrix} g^1(\beta)\, d\beta, \tag{99}$$
$$\begin{bmatrix} p(-L) \\ q(-L) \end{bmatrix} = \begin{bmatrix} v^0 \\ f^0 \end{bmatrix}, \ \begin{bmatrix} p(0) \\ q(0) \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} w^0, \tag{100}$$
$$\begin{aligned} \begin{bmatrix} \check{p}(\xi) \\ \check{q}(\xi) \end{bmatrix} &= \check{\Psi}^{\gamma}(\xi) \begin{bmatrix} \check{p}(0) \\ \check{q}(0) \end{bmatrix} \\ &+ \sum_{i=0}^{\ell} \int_{0}^{\max(\xi, -\check{h}_i)} \check{\Psi}^{\gamma}(\xi) \check{\Psi}^{\gamma-1}(\beta) \begin{bmatrix} 0 \\ I \end{bmatrix} C_1^{i\mathrm{T}} N_c C_1^i g^2(\beta)\, d\beta, \end{aligned} \tag{101}$$
$$\begin{bmatrix} U_1^{\mathrm{T}} & U_2^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \check{p}(-\check{L}) \\ \check{q}(-\check{L}) \end{bmatrix} = 0, \ \begin{bmatrix} \check{p}(0) \\ \check{q}(0) \end{bmatrix} = \begin{bmatrix} \gamma^2 \cdot (v^0 - g^0) \\ -f^0 \end{bmatrix}. \tag{102}$$

Since u in (98a) is expressed as

$$\begin{aligned} u = &-(D_{12}^{\mathrm{T}} D_{12})^{-1} \sum_{k=0}^{d} B_2^{k\mathrm{T}} q(-L + h_k) \\ &- D_{12}^+ \sum_{k=0}^{\ell} C_1^k (\tfrac{1}{\gamma^2} \cdot \check{p}(-\check{h}_k) + g^2(-\check{h}_k)), \end{aligned} \tag{103}$$

we derive the representation of $q(\xi)$, $\check{p}(\xi)$ in terms of $g = (g^0, g^1, g^2) \in \mathcal{W}$. Combining (101), (99) with the boundary conditions (102), (100), we have

$$\begin{aligned} &U^{\mathrm{T}} \check{\Psi}^{\gamma}(-\check{L}) \Gamma \check{\Phi}^{\gamma}(-L) V w^0 = \\ &U^{\mathrm{T}} \check{\Psi}^{\gamma}(-\check{L}) \Gamma \left\{ \begin{bmatrix} I \\ 0 \end{bmatrix} g^0 + \int_{-L}^{0} \check{\Phi}^{\gamma}(-L) \check{\Phi}^{\gamma-1}(\beta) \begin{bmatrix} I \\ 0 \end{bmatrix} g^1(\beta)\, d\beta \right\} \\ &+ \sum_{i=0}^{\ell} \int_{-\check{h}_i}^{0} U^{\mathrm{T}} \check{\Psi}^{\gamma}(-\check{L}) \check{\Psi}^{\gamma-1}(\beta) \begin{bmatrix} 0 \\ I \end{bmatrix} C_1^{i\mathrm{T}} N_c C_1^i g^2(\beta)\, d\beta. \end{aligned} \tag{104}$$

Since $U^{\mathrm{T}} \check{\Psi}^{\gamma}(-\check{L}) \Gamma \check{\Phi}^{\gamma}(-L) V = \gamma \cdot W(\gamma)$ is nonsingular by Theorem 5, we obtain

$$\begin{aligned} q(\xi) = &K_1(\xi, -L) g^0 + \int_{-L}^{0} K_1(\xi, \beta) g^1(\beta)\, d\beta \\ &+ \sum_{j=0}^{\ell} \int_{-\check{h}_j}^{0} K_2(\xi, \beta) C_1^{j\mathrm{T}} N_c C_1^j g^2(\beta)\, d\beta \end{aligned} \tag{105}$$

and

$$\begin{aligned} \tfrac{1}{\gamma^2} \cdot \check{p}(-\check{h}_k) = &\tilde{K}_1(-\check{h}_k, -L) g^0 + \int_{-L}^{0} \tilde{K}_1(-\check{h}_k, \beta) g^1(\beta)\, d\beta \\ &+ \sum_{j=0}^{\ell} \int_{-\check{h}_j}^{0} \tilde{K}_2(-\check{h}_k, \beta) C_1^{j\mathrm{T}} N_c C_1^j g^2(\beta)\, d\beta \\ &(k = 0, 1, \ldots, \ell) \end{aligned} \tag{106}$$

from (99), (101), (102), (104). Substituting (105), (106) to (103), then replacing by

$$\begin{aligned} &g^0 = \underline{x}(t), \\ &g^1(\xi) = \sum_{i=0}^{d} \chi_{[-L, -L+h_i]}(\xi) \cdot B^i \tilde{f}(t + \beta + L - h_i), \\ &g^2(\beta) = \underline{x}(t, \beta), \end{aligned}$$

we finally obtain the feedback laws (28a), (28b), (28c). ■

E. Proof of Lemma 7

For the system Σ defined with $\check{L} = 0$, introduce a coupled operator Riccati equation:

$$\begin{aligned} &\tilde{\mathcal{S}} \tilde{\mathcal{A}}_c \phi + \tilde{\mathcal{A}}_c^* \tilde{\mathcal{S}} \phi - \tilde{\mathcal{S}} \tilde{\mathcal{B}} R_c^{-1} \tilde{\mathcal{B}}^* \tilde{\mathcal{S}} \phi + \mathcal{C}_1^* N_c \mathcal{C}_1 \phi = 0, \ \phi \in \mathcal{W} \\ &\tilde{\mathcal{A}}_c := \tilde{\mathcal{A}} - \tilde{\mathcal{B}}_2 D_{12}^+ \mathcal{C}_1, \ \ \tilde{\mathcal{A}} := \mathcal{A} + \tfrac{1}{\gamma^2} \cdot \mathcal{P} \mathcal{C}_1^* \mathcal{C}_1 \\ &\tilde{\mathcal{B}} := [\, \tilde{\mathcal{B}}_1 \ \ \tilde{\mathcal{B}}_2 \,] \\ &\quad = [\, (\mathcal{B}_1 D_{21}^{\mathrm{T}} + \mathcal{P} \mathcal{C}_2^*)(D_{21} D_{21}^{\mathrm{T}})^{-\frac{1}{2}} \ \ \mathcal{B}_2 + \tfrac{1}{\gamma^2} \cdot \mathcal{P} \mathcal{C}_1^* D_{12} \,] \end{aligned} \tag{107}$$

where $\mathcal{P} \geq 0$ is a stabilizing solution of (43). The following lemma provides an alternative condition which inherits (A) and (C).

Lemma 23: For a given $\gamma > 0$, suppose (B) holds and let $\mathcal{P} \geq 0$ be a stabilizing solution of (43). Then the conditions (A), (C) and (AC) are equivalent.

(AC) The equation (107) has a stabilizing solution $\tilde{\mathcal{S}} \geq 0$ ($\tilde{\mathcal{S}} \in \mathcal{L}(\mathcal{V}, \mathcal{V}^*)$) such that $\tilde{\mathcal{A}}_c - \tilde{\mathcal{B}} R_c^{-1} \tilde{\mathcal{B}}^* \tilde{\mathcal{S}}$ generates an exponentially stable semigroup on $\mathcal{W}$, $\mathcal{V}$. ■

Proof: (⇒) Suppose (A), (C) hold and $\mathcal{S} \geq 0$, $\mathcal{P} \geq 0$ be stabilizing solutions of (42), (43), respectively. Then $\mathcal{I} - \frac{1}{\gamma^2} \cdot \mathcal{P}\mathcal{S}$ has bounded inverse and

$$\tilde{\mathcal{S}} := \mathcal{S}(\mathcal{I} - \tfrac{1}{\gamma^2} \cdot \mathcal{P}\mathcal{S})^{-1} \geq 0 \tag{108}$$

holds. We will show that (108) meets a stabilizing solution of (107). Substituting (108) to the left-hand side of (107), then employing (42), (43), it is shown that the operator (108) meets a solution of (107). While

$$\begin{aligned} &(\tilde{\mathcal{A}}_c - \tilde{\mathcal{B}} R_c^{-1} \tilde{\mathcal{B}}^* \tilde{\mathcal{S}})(\mathcal{I} - \tfrac{1}{\gamma^2} \cdot \mathcal{P}\mathcal{S})\phi = \\ &\quad (\mathcal{I} - \tfrac{1}{\gamma^2} \cdot \mathcal{P}\mathcal{S})(\mathcal{A}_c - \mathcal{B} R_c^{-1} \mathcal{B}^* \mathcal{S})\phi, \ \ \phi \in \mathcal{W} \end{aligned} \tag{109}$$

is obtained from (42), (107), (108) where $\mathcal{A}_c - \mathcal{B} R_c^{-1} \mathcal{B}^* \mathcal{S}$ generates an exponentially stable semigroup on $\mathcal{W}$. Hence (108) is a stabilizing solution of (107) and (AC) holds.

(⇐) Suppose (AC) holds and $\tilde{\mathcal{S}} \geq 0$, $\mathcal{P} \geq 0$ be stabilizing solutions of (107), (43), respectively. Then $\mathcal{I} + \frac{1}{\gamma^2} \cdot \mathcal{P}\tilde{\mathcal{S}}$ is invertible and $\mathcal{S} := \tilde{\mathcal{S}}(\mathcal{I} + \frac{1}{\gamma^2} \cdot \mathcal{P}\tilde{\mathcal{S}})^{-1} \geq 0$ meets a solution of (42). Since $\mathcal{I} - \frac{1}{\gamma^2} \cdot \mathcal{P}\mathcal{S} = (\mathcal{I} + \frac{1}{\gamma^2} \cdot \mathcal{P}\tilde{\mathcal{S}})^{-1}$ holds in (109), the operator $\mathcal{S} := \tilde{\mathcal{S}}(\mathcal{I} + \frac{1}{\gamma^2} \cdot \mathcal{P}\tilde{\mathcal{S}})^{-1} \geq 0$ is a stabilizing solution of (42). While $\mathcal{P}\mathcal{S}$ is expressed as $\mathcal{P}\mathcal{S} = \mathcal{P}\tilde{\mathcal{S}}(\mathcal{I} + \frac{1}{\gamma^2} \cdot \mathcal{P}\tilde{\mathcal{S}})^{-1}$ and the inequality $\lambda_{\max}(\mathcal{P}\mathcal{S}) < \gamma^2$ holds. Thus conditions (A), (C) are derived. ■

For the system Σ with $\check{L} = 0$, the corresponding state space is defined by $\mathcal{X}' := \mathbb{R}^n \times L_2(-L, 0; \mathbb{R}^n)$ and the output operators (40) are reduced to finite-rank: $\mathcal{C}_k \phi = C_k \phi^0$,

$\phi = (\phi^0, \phi^1) \in \mathcal{X}'$ $(k = 1, 2)$. Hence, the condition (B) is characterized by a matrix Riccati equation.

Lemma 24: For a given $\gamma > 0$, the conditions (B) and (b_0) are equivalent. If (b_0) holds, the stabilizing solution $\mathcal{P} \geq 0$ of (43) is given as follows:

$$\mathcal{P} = \begin{bmatrix} P & 0 \\ 0 & \Xi \end{bmatrix} \in \mathcal{L}(\mathcal{X}'),$$
$$(\Xi\phi^1)(\xi) := \sum_{i=0}^{d} \chi_{[-L,-L+h_i]}(\xi) \cdot B_1^i N_f B_1^{i\mathrm{T}} \phi^1(\xi),$$
$$\phi^1 \in L_2(-L, 0; \mathbb{R}^n), \quad -L \leq \xi \leq 0. \tag{110}$$
■

Proof: ($\Rightarrow$) Suppose (B) holds. Then, by Lemma 21, there exists a full column rank matrix $U = [\,U_1^{\mathrm{T}}\, U_2^{\mathrm{T}}\,]^{\mathrm{T}}$ satisfying (20) and, further, $U_s = U_1$ is nonsingular in (24). Hence the stabilizing solution of (43) is given by (110) with $P = U_2 U_1^{-1}$. Since (110) is positive semi-definite iff $P \geq 0$, the condition (b_0) is derived.

($\Leftarrow$) Suppose (b_0) holds. Then a positive semi-definite solution of (43) is given by (110). The solution of the evolution equation $\dot{\hat{x}}(t) = (\mathcal{A}_f - \mathcal{P}\mathcal{C}^* R_f^{-1}\mathcal{C})\hat{x}(t)$, $\hat{x}(0) = \phi \in \mathcal{X}^r$ is bounded over $0 \leq t \leq L$ and, for $t \geq L$, it is reduced to $\hat{x}(t) = (\,x(t), 0\,) \in \mathcal{X}^r$, $\dot{x}(t) = (A_f - PC^{\mathrm{T}} R_f^{-1} C)x(t)$. Hence $\mathcal{A}_f - \mathcal{P}\mathcal{C}^* R_f^{-1}\mathcal{C}$ generates an exponentially stable semigroup. Thus (B) is derived. ■

Proof of Lemma 7: By Proposition 11 and Lemmas 23, 24, the H^∞ control problem Σ with $\check{L} = 0$ is solvable iff (b_0) and (AC) hold. Furthermore by Proposition 12 and Lemma 23, the condition (AC) is equivalent to the solvability of the FI-problem $\Sigma_{\mathrm{FI}}^{(\mathrm{AC})}$ with $\gamma > 0$. ■

F. *Proof of Lemma 8*

By Lemma 14, the condition (a_w) holds only if a full column rank matrix V exists in (10). We note that $V_s = \tilde{V}_1(-L)$ and $G(-L, -L) = \tilde{V}_2(-L)\tilde{V}_1^{-1}(-L)$ hold for the matrix function defined by $\begin{bmatrix} \tilde{V}_1(t) \\ \tilde{V}_2(t) \end{bmatrix} := \Phi_1(t)V$. Since $\Phi_1(-L)H\Phi_1^{-1}(-L)\begin{bmatrix} \tilde{V}_1(-L) \\ \tilde{V}_2(-L) \end{bmatrix} = \begin{bmatrix} \tilde{V}_1(-L) \\ \tilde{V}_2(-L) \end{bmatrix}\Lambda_c$ (Λ_c : stable matrix) follows from (10), the condition $\Phi_1(-L)H\Phi_1^{-1}(-L) \in \mathrm{dom(Ric)}$ is satisfied iff $V_s = \tilde{V}_1(-L)$ is invertible. Focus on the equality

$$\Phi_1(-L)H\Phi_1^{-1}(-L)\begin{bmatrix} I \\ X(-L) \end{bmatrix} = \begin{bmatrix} I \\ X(-L) \end{bmatrix}\tilde{V}_1(-L)\Lambda_c\tilde{V}_1^{-1}(-L) \tag{111}$$

where $X(-L) := \tilde{V}_2(-L)\tilde{V}_1^{-1}(-L) = G(-L, -L)$ is a symmetric matrix. Pre-multiplying $[\,X(-L)\ -I\,]$ to both sides of (111), we have

$$X(-L)A_c + A_c^{\mathrm{T}}X(-L) - X(-L)B_2(D_{12}^{\mathrm{T}}D_{12})^{-1}B_2^{\mathrm{T}}X(-L) + C_1^{\mathrm{T}}N_cC_1 + \tilde{\Delta} = 0,$$
$$\tilde{\Delta} = \sum_{i=0}^{d}[X(-L)\ -I]\Phi_1(-L)\Phi_1^{-1}(-L + h_i)\begin{bmatrix} \frac{1}{\gamma^2}\cdot B_1^i B_1^{i\mathrm{T}} & 0 \\ 0 & 0 \end{bmatrix} \times \Phi_1^{-\mathrm{T}}(-L + h_i)\Phi_1^{\mathrm{T}}(-L)[\,X(-L)\ -I\,]^{\mathrm{T}} \geq 0 \tag{112}$$

where $\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}\Phi_1(\tau) = \Phi_1^{-\mathrm{T}}(\tau)\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$ is employed in (112). Hence, under (H1)-(H3), $X(-L) \geq 0$ holds iff the matrix $A_c - B_2(D_{12}^{\mathrm{T}}D_{12})^{-1}B_2^{\mathrm{T}}X(-L) = A_c - B_2(D_{12}^{\mathrm{T}}D_{12})^{-1}B_2^{\mathrm{T}}G(-L, -L)$ is stable. Thus, the conditions (a_w) and ($\tilde{a}_w$) are equivalent. ■

G. *Proof of Lemma 9*

Focus on the finite-horizon full-information H^∞ control problem on $[-L, 0]$:

$$\Sigma_\lambda:\ \dot{x}(t) = Ax(t) + \tilde{B}_1(t)w(t) + \tilde{B}_2(t)u(t),\quad x(-L) = 0$$
$$z_\lambda(t) = \lambda^{-\frac{1}{2}}\cdot C_1x(t) + \lambda^{-\frac{1}{2}}\cdot D_{12}u(t),\quad \lambda > 0 \tag{113}$$
$$\tilde{B}(t) := [\,B_1\ \tilde{B}_2(t)\,],$$
$$\tilde{B}_2(t) := \sum_{i=0}^{d-1}\chi_{[-L+h_i,0]}(t)\cdot B_2^i \tag{114}$$

and introduce a differential Riccati equation:

$$-\dot{S}_\lambda(t) = S_\lambda(t)A_c + A_c^{\mathrm{T}}S_\lambda(t) - S_\lambda(t)\tilde{B}(t)R_c^{-1}\tilde{B}^{\mathrm{T}}(t)S_\lambda(t) + \lambda^{-1}\cdot C_1^{\mathrm{T}}N_cC_1,\quad S_\lambda(0) = \lambda^{-1}\cdot S \tag{115}$$

where $S \geq 0$ is the stabilizing solution of (18). By [8], the finite-horizon H^∞ performance:

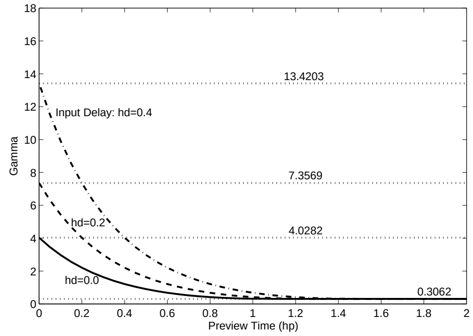
$$J_\lambda := \sup_{w\in L_2(-L,0)}\frac{\|z_\lambda\|^2_{L_2(-L,0)} + x^{\mathrm{T}}(0)S(0)x(0)}{\|w\|^2_{L_2(-L,0)}} < \gamma^2 \quad (\gamma > 0) \tag{116}$$

is attained for Σ_λ iff (115) has a bounded solution $S_\lambda(t) \geq 0$ $(-L \leq t \leq 0)$. Based on the fundamental results stated here, we first show that the condition (a_u) is characterized by a differential Riccati equation.

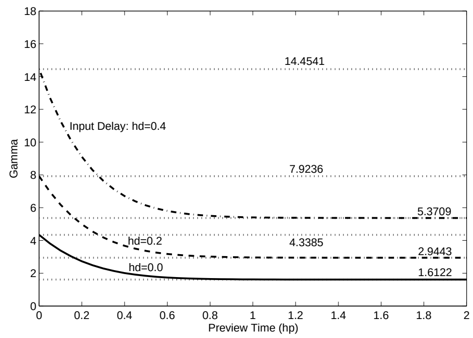
(a_u) $\Leftrightarrow$ ($\tilde{a}_u$): We note that $S_\lambda(t) := V_{2,\lambda}(t)V_{1,\lambda}^{-1}(t)$, $\begin{bmatrix} V_{1,\lambda}(t) \\ V_{2,\lambda}(t) \end{bmatrix} = \Phi_\lambda(t)\begin{bmatrix} I \\ \lambda^{-1}S \end{bmatrix}$ meets the solution of (115) (see e.g. [5]) and, further, $V_{1,\lambda}(-L) = \tilde{V}_p(\lambda)$ holds by Lemma 3.

($\Rightarrow$): Suppose $V_{1,\lambda}(-L) = \tilde{V}_p(\lambda)$ is nonsingular for $\lambda \geq 1$. We will prove by contradiction that (34) has a bounded solution $S(\cdot) \geq 0$. If (115) with $\lambda = 1$, or equivalently (34), does not have a bounded solution $S_1(\cdot) \geq 0$, the H^∞ control problem Σ_1 with $J_1 < \gamma^2$ is not solvable [8]. Let $J_1^* := \gamma_{\mathrm{opt}}^2 > \gamma^2$ be the optimal performance for the system Σ_1 and define a system Σ_{λ^*} with $\lambda^* := \gamma_{\mathrm{opt}}^2/\gamma^2 > 1$. Since $J_1 = \lambda^*\cdot J_{\lambda^*}$ holds by the definition (116), the optimal performance $J_{\lambda^*}^*$ for the system Σ_{λ^*} is given by $J_{\lambda^*}^* = \gamma^2$. For any given $\epsilon > 0$, the H^∞ control problem Σ_{λ^*} with $J_{\lambda^*} < \gamma_\epsilon^2$, $\gamma_\epsilon := \gamma + \epsilon$ is solvable and the bounded solution $S_\lambda(\cdot) \geq 0$ exists. Since $S_\lambda(\cdot) \geq 0$ is continuous and non-increasing [8], $\|S_{\lambda^*}(-L)\| \to \infty$ is derived as $\gamma_\epsilon \to \gamma + 0$. This fact implies $V_{1,\lambda^*}(-L) = \tilde{V}_p(\lambda^*)$ is singular and contradicts the assumption (a_u). Thus, (115) with $\lambda = 1$, or equivalently (34), has a bounded solution $S_1(\cdot) \geq 0$ and ($\tilde{a}_u$) is derived.

($\Leftarrow$): Suppose (115) with $\lambda = 1$ has a bounded solution $S_1(\cdot) \geq 0$. Then (115) has a bounded solution $S_\lambda(\cdot) \geq 0$ for $\lambda \geq 1$ since the H^∞ control problem for Σ_λ with $J_\lambda < \gamma^2$ is equivalent to the problem defined by Σ_1 with $J_1 < \lambda\cdot\gamma^2$. Thus (115) has bounded solutions $S_\lambda(\cdot) \geq 0$ for $\lambda \geq 1$. Since



(a) $k = 0$



(b) $k = 0.4$

Fig. 1. H^∞ performance vs. preview/delay times (full-information case).

$S_\lambda(-L) = V_{2,\lambda}(-L)V_{1,\lambda}^{-1}(-L)$ and $V_{1,\lambda}(-L) = \tilde{V}_p(\lambda)$ hold, $\tilde{V}_p(\lambda)$ $(\lambda \geq 1)$ is nonsingular and (a$_u$) is derived.

(b$_y$) $\Leftrightarrow$ ($\tilde{\text{b}}_y$): Applying the above result to the transposed system Σ^{T}, it is shown that the conditions (b$_y$), ($\tilde{\text{b}}_y$) are equivalent.

(c) $\Leftrightarrow$ ($\tilde{\text{c}}$): The matrices satisfying (10) and (26) are respectively given by $V := \begin{bmatrix} I \\ S \end{bmatrix}$, $U := \begin{bmatrix} I \\ P \end{bmatrix}$ where $S \geq 0$, $P \geq 0$ are the stabilizing solutions of (18), (30). Furthermore the solution of (35) is given by $P(t) := U_2(t)U_1^{-1}(t)$, $\begin{bmatrix} U_1(t) \\ U_2(t) \end{bmatrix} = \Psi_1(t)\begin{bmatrix} I \\ P \end{bmatrix}$. Let $\sigma_{\max} > 0$ be the maximal solution of (27) and suppose $W(\sigma_{\max})v = 0$ $(v \neq 0)$ holds. Since $\check{\Psi}^{\sigma\mathrm{T}}(-\check{L}) = \Psi_1(-\check{L})$ holds between (21) and (23), the condition $W(\sigma_{\max})v = 0$ yields $\sigma_{\max}^2 \cdot U_1^{\mathrm{T}}(-\check{L})V_{1,1}(-L)v = U_2^{\mathrm{T}}(-\check{L})V_{2,1}(-L)v$ and, further, the equality

$$\begin{aligned} \sigma_{\max}^2 \cdot \tilde{v} &= U_1^{\mathrm{T}-1}(-\check{L})U_2^{\mathrm{T}}(-\check{L})V_{2,1}(-L)V_{1,1}^{-1}(-L)\tilde{v} \\ &= P(-\check{L})S_1(-L)\tilde{v} \end{aligned} \tag{117}$$

is obtained for $\tilde{v} = V_{1,1}(-L)v \neq 0$. Thus the condition ($\tilde{\text{c}}$) is derived. If ($\tilde{\text{c}}$) holds, the equality (117) yields $W(\sigma_{\max})v = 0$ $(v \neq 0)$ for $\sigma_{\max} := \lambda_{\max}^{\frac{1}{2}}(P(-\check{L})S_1(-L))$. Thus (c) is derived. ■

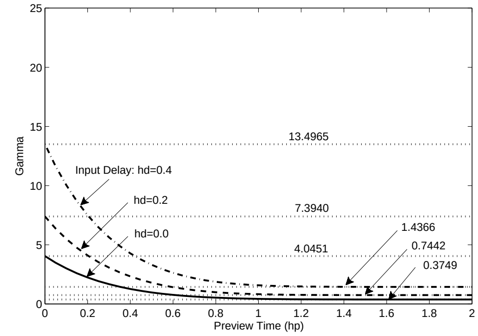
VI. Numerical Examples

Define an H^∞ preview and delayed control problem:

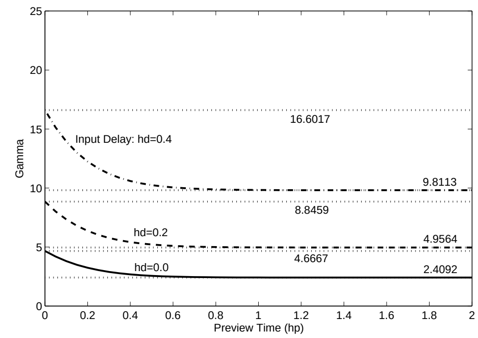
$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ k \end{bmatrix} w_0(t) \\ &\quad + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_1(t - h_p) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t - h_d) \end{aligned} \tag{118a}$$

$$z(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \quad k = 0, 0.4 \tag{118b}$$

$$y(t) = [\, x^{\mathrm{T}}(t) \;\; w_0^{\mathrm{T}}(t) \;\; w_1^{\mathrm{T}}(t) \,]^{\mathrm{T}} \tag{118c}$$



(a) $k = 0$



(b) $k = 0.4$

Fig. 2. H^∞ performance vs. preview/delay times (output feedback case).

where w_1 is the h_p unit-time previewable signal and w_0 is the uncertainty of w_1. Furthermore, h_d unit-time delay is imposed on the control u. We will investigate the H^∞ performance in terms of (h_p, h_d). Based on Theorem 1, the achievable H^∞ performance for (118a)-(118c) is obtained by Fig.1. Fig.1 (a) summarizes the performance for the case $k = 0$ and it is observed that the curves coincide by sliding aside. This feature arises from the fact that the common input delays $\min(h_p, h_d)$ can be pushed out to the regulated output. While in the case $k = 0.4$ (Fig.1 (b)), the relation between the preview and delay times is rather complicated and the H^∞ performance is not sufficiently recovered even if rich preview information is employed.

Replacing the measurement (118c) by $y(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} \epsilon \\ 0 \end{bmatrix} w_\epsilon(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_1(t)$, $\epsilon = 0.01$, we will investigate the H^∞ output feedback performance based on Theorem 5. A slight noise $(\epsilon > 0)$ is included in the measurement for satisfying (H2). Based on Theorem 5, the achievable H^∞ performance for (118a) is obtained by Fig.2 (a), (b). In the case $k = 0$, the performance in Fig.2 (a) is almost similar to Fig.1 (a) because the initial states of control systems are both relaxed in the evaluation of H^∞ performance and, further, the error system is not excessively driven by the slight measurement noise w_ϵ. While in case that the uncertainty in the preview information grows $(k = 0.4)$, the achievable performance is significantly deteriorated as the full-information is not easily recovered in the output feedback setting.

Next we focus on the preview control problem depicted by Fig.3 where $P(s)$, $K(s)$, $M(s)$ denote the plant, control law, and low-pass filter restricting the bandwidth of the control channel, respectively. The delay element e^{-hs} expresses the

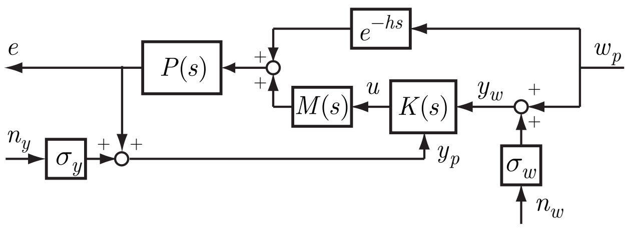
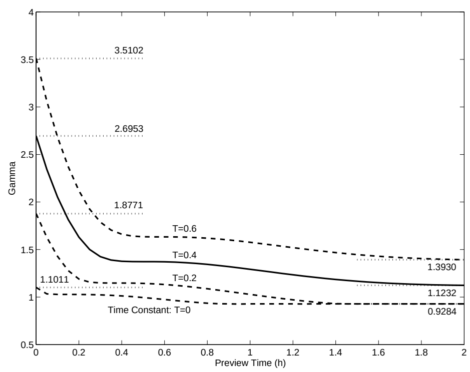
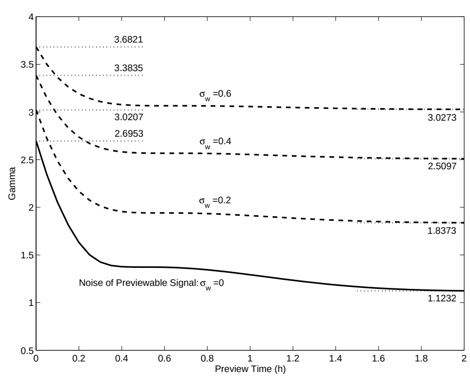


Fig. 3. H^∞ disturbance attenuation problem.



(a) $T = 0 \sim 0.4$ $(\sigma_w = 0)$



(b) $\sigma_w = 0 \sim 0.6$ $(T = 0.4)$

Fig. 4. H^∞ performance vs. control bandwidth (T) or uncertainty of previewable disturbance (σ_w).

preview time of the disturbance w_p and (n_y, n_w) denotes the uncertain noises in the measurement y_p and the previewable disturbance y_w. The control objective here is to attenuate the H^∞-norm from $w := [\, w_p,\, n_y,\, n_w]^{\mathrm{T}}$ to $z := [\, e,\, \rho \cdot u\,]^{\mathrm{T}}$ $(\rho > 0)$ by employing the information of $y := [\, y_p,\, y_w\,]^{\mathrm{T}}$. The system structure (Fig.3) frequently arises in the disturbance attenuation problem (see e.g. [15]) and the generalized plant is given by

$$\begin{bmatrix} z \\ y \end{bmatrix} = \left[\begin{array}{ccc:c} P(s) \cdot e^{-hs} & 0 & 0 & P(s)M(s) \\ 0 & 0 & 0 & \rho \\ \hdashline P(s) \cdot e^{-hs} & \sigma_y & 0 & P(s)M(s) \\ 1 & 0 & \sigma_w & 0 \end{array}\right] \begin{bmatrix} w \\ u \end{bmatrix}. \tag{119}$$

For the system (119) with $P(s) = \frac{5}{(s-1)(s-2)}$, $M(s) = \frac{1}{Ts+1}$, $\rho = 1.0$, $\sigma_y = 0.1$, we will investigate the optimal H^∞ performance in terms of (T, σ_w). Based on Theorem 5, the achievable H^∞ performance is obtained by Fig.4. Fig.4 (a) summarizes the performance for $T = 0 \sim 0.6$ $(\sigma_w = 0)$. In the cases $T = 0, 0.2$, it is observed that the H^∞ performance is recovered to the optimal level by employing preview information of w_p. While in the cases $T = 0.4, 0.6$, the H^∞ performance is not recovered to the optimal level even if any rich preview information is employed. Thus in the preview control of Fig.3, the limitation of control bandwidth is recovered to certain extent by employing the preview information of w_p. In Fig.4 (b), the achievable H^∞ performance is summarized for $\sigma_w = 0 \sim 0.6$ $(T = 0.4)$. As the uncertainty in the previewable disturbance grows, the H^∞ performance is not significantly recovered and a similar feature to the first example (118) is observed.

VII. Conclusion

A solvability condition and control law for a broad range of H^∞ preview/delayed control problems were established based on the analytic solutions of the corresponding operator Riccati equations. The solvability condition is characterized by the roots of the transcendental equations, and the control law for the general problem is given based on a predictive compensation with an integro-differential observer. The solvability conditions for typical control problems were further investigated and relevant literature were used to interpret some problems. The results are also applicable to the design of an H^2 controller because the solutions of the corresponding operator Riccati equations were clarified.

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