Note: Overview of Stochastic Process

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A decision problem associated with a series of random variable is described. Then some major literature are reviewed. Finally, major elements composing the problem are defined, and some of the problems are modeled using the definitions.

Subject classification: Stochastic Process, Markov Process, Markov Decision Process

Introduction

Observe a random variable at periods 0, 1, 2, ... to be in one of a number of possible values. Based on the values of the variable, we will incur a state associated cost. After reviewing the value of the random variable at the period, an action must be chosen to effectively change the value of random variable in the next period so that the decision will optimize a specific criteria like minimize average periodic cost. The action chosen may incur another cost: action related cost. This is what is characterized by stochastic decision process.

For example, consider the following single item stochastic inventory problem.

- On hand stock level of a specific item is observed at the beginning of a quarter.
- Quarterly demand of the item follows a specific probability distribution function.
- Various costs are accessed:
  - Quarterly holding cost of \$100 per unit is charged based on the ending level of stock.
  - If stock-out occurs, \$10000 per unit stock-out is charged.
  - If an order is placed in a period, a fixed component of \$5,000, and variable ordering cost of \$1,000 per unit ordered are incurred.
- Order placed in a quarter is immediately available to the next quarter's demand.
- Stock level at the end of the year is salvaged \$500 per unit.
- The objective is to minimize annual total cost.

This description of a stochastic decision process is applicable in many situations in business, engineering, biology, and etc. And it includes a lot of optimization models as special cases. In this paper, we shall discuss the basic elements characterizing stochastic decision process.

In the following section, we shall review past major literature in publication as a book.

In the third section, we will present some appropriate definitions to explain the stochastic decision process at first. Then we will develop some mathematical model of the process.

Literature

In late nineteenth century, Andrey Andreyevich Markov, a Russian mathematician, develops the theory of stochastic processes. A major subject of his work later become well known as Markov property, Markov chains, Markov processes, Markov decision processes, semi-Markov decision processes naming after his significant pioneering contributions in the field.

Howard (1960) solves some problems of Markov decision process using iteration methods. Feller (1968) discusses Markov chain, the general Markov process, stochastic process, and general process. Ross (1970) explains stochastic process, Markov chain, Markov decision process, semi-Markov decision process in the optimization application in mind. Karlin, and Taylor (1975, 1981) first publishes two volume set covering stochastic Process in exhaustive manner. The first volume shows various kind of stochastic processes. The second volume emphasizes mathematical treatment of the process. Fishman (1978) presents a method called discrete event simulation. Using the technique, some solution approaches to several stochastic process problems are illustrated. Again Karlin, and Taylor (1984), write an introductory, and an abridged version of the previously available two volume set. Billingsley (1986) shows what is the stochastic process using rigorous and extensive mathematical probability theory. Wolff (1989) defines stochastic process extensively although his main emphasis of the book is on queuing theory.

Entering twenty first century, Tijms (2003) publishes similar to Karlin, and Taylor's one volume text, but it is up to date. Puterman (2005) shows the nature, and the optimization of periodic Markov Decision Processes.

Definition & Model

Discrete process is a series of values. It can be non-numeric, like weather of the date. But we will leave it as numeric for now.

Discrete deterministic process is a series of values generated by a certain known function. For example, 0, 1, 4, 9, ... is a deterministic process generated by a mathematical function $f(i) = i^2$.

A collection of random variables is a discrete stochastic process. For example, if we have less than 1 millimeter rain in twenty-four hours period, we label the date 0; otherwise 1. Then a series of the number generated in most of areas on the earth is a discrete stochastic process.

Observe a process at periods $i = 0, 1, 2, \ldots$ to be in one of a number of possible states. The set of all possible states: $\mathcal{S}$ is assumed to be countable, and each state will be exhaustively labeled by the nonnegative integers 0, 1, 2 .... Each state is mutually exclusive to each other; thus the process cannot be in state s, and t where $s \neq t$ at the same period.

This is a formalization of discrete stochastic periodic process. Note that there are three main properties of interest:

- process
- periodic index, and
- state.

Additionally, if the following holds for all period $i = 0, 1, 2, \ldots$, the process is called Markov process.

$$\begin{aligned} P\{X_{i+1} = t | X_0, X_1, \ldots, X_i = s\} &= P\{X_{i+1} = t | X_i = s\} \\ &= P_{ij}. \end{aligned}$$

After reviewing the state of the process at period i, an action must be chosen, and we let $\mathcal{A}$, assumed finite, denote the set of all permissible actions at any periods. The set $\mathcal{A}$ may include "do nothing" which does not necessarily guarantee the process being in the same state at the next period $i+1$. The choice of decision depends on a certain criteria like maximizing profit, minimizing cost, maximize the population, and etc.

This is what is characterized by stochastic decision process.

If the observed process is state $s \in \mathcal{S}$ at period i, and action $a \in \mathcal{A}$ is selected, then the following two things occur:

1. We incur a cost $C(s, a)$.

2. The next state of the system: $t \in \mathcal{S}$ given an action: $a \in \mathcal{A}$, is chosen according to the transition probability $P_{st}(a)$.

Assumption (i) implies that the incurring cost is period invariant, i.e., the cost depends only on state s, and a selected action a, but not period i. A periodic cost incurred composed of two components: a state associated cost, and an action associated cost. For example, in a typical mathematical inventory problem, holding and shortage cost is a state associated cost. On the other hand, ordering cost is a action related cost.

If we let X_i denote the state of the process at period i, and a_i the action chosen at period i, then assumption (ii) is equivalent to stating the following conditional probability:

$$\begin{aligned} P\{X_{i+1} = t | X_0, a_0, X_1, a_1, ..., X_i = s, a_i = a\} &= P\{X_{i+1} = t | X_i = s, a_i = a\} \\ &= P_{ij}(a) \end{aligned}$$

The transition probability is also period invariant. The conditional probability given an action at period i depends only on the current, and subsequent state. Note that the history up to previous periods does not matter at all which is the fundamental property of Markov chain. Thus, both the costs and the transition probabilities are functions only of the last state s, and the subsequent action a.

Furthermore, we shall suppose that the costs are bounded, and we let M be a certain sufficiently big number such that $|C(i, a)| < M$ for all i, a.

Events in a period are:

1. observation of the state,

2. choose an action,

3. incurring cost,

4. random variable may change its value.

In order to choose an action after periodic review of the state, we need a criterion to evaluate a series of actions. For example, minimizing average expected cost per period is a typical one. In order to choose actions, we must follow some policy. We shall place no restrictions on the class of allowable policies, and we therefore define a policy to be any rule for choosing actions. Thus, the action chosen by a policy may, for instance, depend on the history of the process up to that point, or it may be randomized in the sense that it chooses action a with some probability $P_a, a \in \mathcal{A}$.

An important subclass of the class of all policies is the class of stationary policies, where a policy is said to be stationary if it is nonrandomized and the action it chooses at period i only depends on the state of the process at time t, but not on the value of period t itself. Or, in other words, a stationary policy is a function f mapping the state space: $\mathcal{S}$ into the action space: $\mathcal{A}$. Also, it easily follows that if a stationary policy f is employed, then the sequence of states $X_i, i = 0, 1, 2, ...$ forms a Markov chain with transition probabilities $P_{ij} = P_{ij}[f(i)]$; and it is for this reason that the process is called a Markov Decision Process.

Conclusion

Some preliminary definitions, and models are presented. The formulated stochastic decision process is abundant in application, as it encompass many useful optimization models as special cases.

Base on this overview, we will develop further theoretical definitions and models in the next paper. In it, we shall attempt to develop a theory which will enable us to find policies that are, in some sense, optimal. However, in order to accomplish this task we need to elaborate upon some appropriate optimality criteria also.

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