Stabilization of Angular Velocity of Asymmetrical Rigid Body Using Two Constant Torques

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Introduction

The problem of stabilization of angular velocity using less than three control torques has been investigated by several authors[1-11]. The present system can be stabilized by nonlinear control schemes, which may be categorized into three types: time-varying control schemes, discontinuous control schemes, and time-invariant control schemes. In those studies, it is assumed that an arbitrary magnitude of the torques can be fed to a satellite to attenuate its rotational motion using gas jet thrusters employing the Pulse Width Pulse Frequency (PWPF) Modulator. If thrusters provide two control torques without the use of PWPF, then the magnitude of the torque does not linearly respond to the input magnitude, and, indeed, is constant. No previous studies have considered this special case. If a control scheme could be derived for this case, then the resulting control method can facilitate a reduction in the use of PWPF modulation from satellites, or can form a backup system in the event of malfunction of the PWPF modulators, subject to the two control torque problem.

In this Note, provided that the model uncertainties and external disturbances are neglected, a constant control torque method to attenuate the rotational motion of an asymmetric rigid body is proposed. The proposed control method is classed as a discontinuous and open loop control method. Because the magnitude of the control torque is constant, that is, it can only be applied On-Off, and the control timing is pre-determined. In this Note, the set of angular velocities of an asymmetrical rigid body achievable by employing a single constant control torque is defined as “constant-torque-manifold” or simply referred to as the manifold. In the case when only a single constant control torque is used, this manifold can be analytically obtained by integrating the equations of motion backwards in time from the angular velocities that are accessible to the origin by employing a single constant control torque, where the set of angular velocities that can access the origin by a single constant control torque is hereafter referred to as a transient goal. A trajectory resulting from the proposed control method consists of three steps: If the polhode starting from initial angular velocities has intersection points with the manifolds, then the rotational motion does not need to be either boosted or damped. On the other hand, if the polhode starting from the initial angular velocities has no intersection points with the manifold, the rotational motion must be boosted around the maximum or minimum principal moment of inertia and damped around the middle principal moment of inertia to ensure that the trajectory has at least one point intersecting the manifolds. The first step is thus given by the trajectory of the angular velocities to an intersection point with the manifold, without the control torques, or boosted/damped by the control torques if necessary. The second step is a trajectory sliding on the manifold, by means of a single constant torque, until the transient goal is reached. The final step is a trajectory from the transient goal to the origin. The control timings, durations and the sign of control torques can then be determined by calculating the intersection points between the manifold and polhode or the boosted/damped trajectory, between the trajectory sliding on the manifold and the transient goal, and between the single spin motion and the origin. A schematic representation of these trajectories is shown in Fig. 1. The idea of the constant-torque-manifold for stabilization of the rotational motion of an asymmetric rigid body is inspired by Ref [12], which describes a method to obtain a trajectory of the angular velocities of an asymmetric rigid body when a single constant torque is employed along either the maximum, minimum, or middle principal moment of inertia axis. Contrary to the robust feedback schemes[9-11], a demerit of the proposed method is that it is not robust to the modeling errors, and external disturbances. On the other hand, a merit of the proposed method is that it can easily estimate the convergence time. This is because the manifolds are obtained analytically and the dynamic control problem is converted into a kinematics problem of the calculation of intersection points between the manifolds and polhode or boosted/damped trajectory, and this converted problem can be solved by the bi-section method.

Results of a numerical simulation of the present method applied to a test problem are given later in this Note to show that the complete attenuation can be achieved by using the proposed manifold, provided that external disturbances and modeling uncertainties are absent, and the intersection point between the manifolds and polhode or boosted/damped trajectories is completely obtained.

Problem Statement and Equations of Motion

Throughout this study, it is assumed that the principal axes of a satellite are coincident with its body frame coordinates. Because this Note treats only the case of an asymmetrical rigid body, principal moments of inertia satisfy $j_1 \neq j_2 \neq j_3$. Without loss of generality, it is assumed that $j_1 > j_2 > j_3$. Referring to Ref.[12], the equation of motion can be rewritten in the form

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} x_2 x_3 \\ -x_3 x_1 \\ x_1 x_2 \end{bmatrix} + \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} \tag{1}$$

where ()’ denotes the derivative operator $d()/d\tau$ where τ is the scaled time, (x_1, x_2, x_3) is the scaled angular velocity vector and (μ_1, μ_2, μ_3) is the scaled constant torque vector. The detail of the variable changes is described in Ref.[12].

This Note considers the derivation of a control method using less than three constant control torques that can attenuate the rotational motion of an asymmetric rigid body.

Constant Control Torque Method

A. Transient Goals and Constant Torque Patterns

If the angular velocities along the uncontrolled axis and one of the remaining controlled axes are both zero, then the system is both controllable and accessible to the origin. This situation, which is a single spin motion around the one of the controllable axes, can be therefore treated as a transient goal. In this Note, the control torques are assumed to be generated by gas jet thrusters without using PWPF modulators. In this case, the torque takes a plus or minus sign, or can be zero in magnitude (i.e. switched off), and the total number of combinations of the signs of the thrusters is nine. Although all the above sets of signs of the control torques can be used to generate manifolds, the analytical manifolds can be obtained only for the case when a single constant control torque is not along one of the principal axes corresponding to the single spin motion. Therefore in this Note, only this case will be considered.

Because there exist three possibilities for the uncontrolled axis; the maximum, middle, or minimum principal moment of inertia, and there are two cases for the single spin motion around the controllable axis, there thus exist six cases for generating the manifolds, as listed in Table 1.

B. Manifolds

The manifolds can be obtained by integrating the equations of motion backward in time from transient goals with the employment of a single constant control torque. With reference to Ref.[12], and taking the angular velocities of the transient goals into account, the manifold for each case in Table 1 can then be analytically obtained as follows:

(I) $$x_1^2 = 2\mu_1\theta - A^2 \sin^2\theta \,, \qquad (2a)$$

$$x_2 = -A\sin\theta, \quad (2b)$$

$$x_3 = A\cos\theta. \quad (3c)$$

(II) $$x_1^2 = 2\mu_1\theta + A^2\sin^2\theta, \quad (3a)$$

$$x_2 = A\cos\theta, \quad (3b)$$

$$x_3 = A\sin\theta. \quad (3c)$$

(III) $$x_1 = A\sinh\theta, \quad (4a)$$

$$x_2^2 = 2\mu_2\theta - A^2\sinh^2\theta, \quad (4b)$$

$$x_3 = A\cosh\theta. \quad (4c)$$

where θ is the parameter determined by integrating the angular velocity along the constant control torque backward in time associated with the initial condition $\theta(0) = 0$, and A is the parameter to describe the scaled angular velocity of the transient goal. Note that the manifolds for cases (IV), (V), and (VI) are omitted here, because when x_1 is swapped with x_3, the manifolds for cases (IV), (V), and (VI) are the same as those for cases (III), (I), and (II), respectively. In cases (II) and (VI), θ is not limited and the trajectories on the manifolds are open and non-periodic. In cases (III) and (IV), θ is limited and the trajectories on the manifolds are always closed and periodic. In cases (I) and (V), the trajectories on the manifolds are open or closed, depending on the parameter. Let β be a parameter defined as $\beta := |\mu_i| / A^2$ to simply obtain the separatrix between the closed and open trajectories on the manifolds for cases (I) and (V). Solving the equations $f(\beta,\theta) := 2\beta\theta - \sin^2\theta = 0$ and $f_\theta(\beta,\theta) := 2\beta - 2\sin\theta\cos\theta = 0$ with respect to β and θ yields $\beta^* \cong 0.362306$ and $\theta^* \cong 1.16556$. The scaled angular velocity of the transient goal corresponding to the separatrix, A^*, is given by $A^* = \sqrt{|\mu_i| / \beta^*}$. Note that if $\beta > \beta^*$, then θ for cases (I) and (V) is not limited, that is, the trajectories on the manifold are open and non-periodic. On the other hand, if $\beta < \beta^*$, then θ for cases (I) and (V) is limited, that is, the trajectories on the manifold are closed and periodic. The upper bounded value of parameter θ can be determined by imposing the condition that the angular velocity around the constant control torque equals zero with respect to θ for cases (III) and (IV), and under the condition $\beta < \beta^*$ for cases (I) and (V). Figures 2(a), 2(b) and 2(c) show the manifolds for cases (I), (III) and (VI), respectively, where the scaled control torques are assumed to be unit for the sake of simplicity. Note that the periodic parts of the manifolds are shown in Figs. 2(a) and 2(b) as open ones due to the limitation of the programming code based on Mathematica ®.

C. Calculation of the Points of Intersection with the Manifolds

Let α be the parameter given by $\alpha = H^2/2E$, where H is the magnitude of the angular momentum given by $H = \sqrt{(j_1\omega_1)^2 + (j_2\omega_2)^2 + (j_3\omega_3)^2}$, and E is the rotational energy of the rigid body given by $E = \left(j_1\omega_1^2 + j_2\omega_2^2 + j_3\omega_3^2\right)/2$. The intersection points between the polhode or the boosted or damped trajectory and the manifolds can be obtained numerically by using the bi-section method. There may exist several intersection points; one of these points should be selected according to control criteria, such as settling time or energy optimality. Because the control input is assumed to be constant in this Note, the point with the minimum control duration can be chosen as the energy optimal solution.

C-1. Calculation of the Intersection Point between the Polhode and the Manifolds

By replacing the constant control torque with no control torque, and introducing a new variable ψ, the polhode scaled by the variable changes given in Ref.[12] can be expressed in the form of a function of parameter ψ as follows:

For the case $\alpha \geq j_2$,

$$x_{p1} = \mathrm{sgn}(x_{p1}(0))\sqrt{x_{p1}^2(0) + \frac{D^2}{2}\left\{\cos 2\eta - \cos\left(2(\psi + \eta)\right)\right\}}, \tag{5a}$$

$$x_{p2} = D\cos\left(\psi + \eta\right), \tag{5b}$$

$$x_{p3} = D\sin\left(\psi + \eta\right), \tag{5c}$$

where

$$D = \sqrt{x_{p2}^2(0) + x_{p3}^2(0)}, \tag{6a}$$

$$\eta = \mathrm{a}\tan 2\left(x_{p3}(0), x_{p2}(0)\right), \tag{6b}$$

$$\psi(\tau) = \int_0^\tau x_{p1}(\xi) d\xi. \tag{6c}$$

For the case $\alpha < j_2$, the expression for the polhode can be obtained by swapping x_{p1} with x_{p3} in Eqs.(5) and (6). Note that for the case $\alpha = j_2$, the angular velocities converge to the one around the middle principal moment of inertia, and in this case, by solving $x_{p1} = x_{p3} = 0$ with respect to ψ, the range of parameter ψ is limited between 0 and $\left(\mathrm{sgn}(\eta) + \mathrm{sgn}(x_{p1}(0))\right)\pi / 2 - \eta$.

An intersection point between the manifolds and the polhode can be obtained numerically by solving three equations $x_{p1} = x_1, x_{p2} = x_2$ and $x_{p3} = x_3$ with respect to A, θ, and ψ. Note that, as mentioned earlier, the range of phase parameter θ is limited for cases (III) and (IV), and under the condition $A > A^*$ for cases (I) and (V), and that the range of ψ is limited for the case of $\alpha = j_2$, This calculation process is repeated until all combinations of the manifolds corresponding to the constant control torque along the controllable axis are completed.

C-2. Calculation of Intersection Point between Boosted or Damped Trajectory and the Manifolds

If the scaled polhode trajectory has no intersection points with the manifolds, then, under the proposed method, the rotational motion has to be boosted or damped until it has at least one intersection point with the manifold. A typical example is the case given by a single spin motion around the uncontrollable axis. If the intersection points exist, then it is obvious that the angular velocity of an asymmetric rigid body can be stabilized to the origin by the presented piecewise steps. A problem left open is whether or not an intersection point exists between the boosted/damped trajectory and the manifolds. This problem is briefly discussed here. Note that hereafter for the purpose of simplicity, the scaled constant torque is assumed to be unit.

The rotational motion of a rigid body with damping around either the middle or minimum principal moment of inertia and boosting around the maximum principal moment of inertia is likely to converge to a flat spin motion around the maximum principal moment of inertia. The polhode near the flat spin motion is a closed loop trajectory around the axis of x_1, the manifold for case (I) is connected with the axis of x_1, and its radius around the axis of x_1 is limited within $\left|A^*\right| \cong 1.66135$, as shown in Fig. 2(a). The manifold for case (II) is also connected with the axis of x_1, but its radius around the axis of x_1 is not limited. This implies that the polhode satisfying $\alpha > j_2$ and $\sqrt{x_2^2 + x_3^2} < \left|A^*\right|$ always has an intersection point with the manifold for cases (I) and (II), and that the polhode satisfying $\alpha > j_2$ but not $\sqrt{x_2^2 + x_3^2} < \left|A^*\right|$ always intersects the manifold for case (II). Therefore, if the axis of the maximum principal moment of inertia is controllable, then to easily have an intersection point with the manifold for case (II), the following control method, which boosts around the maximum principal moment of inertia and damps around the other controllable axis, should be conducted.

$\mu_1 = \text{sgn}(x_1)$, $\mu_3 = -\text{sgn}(x_3)$ (angular velocity around the axis of x_3 is controllable) (7a)

$\mu_1 = \text{sgn}(x_1)$, $\mu_2 = -\text{sgn}(x_2)$ (angular velocity around the axis of x_2 is controllable) (7b)

On the other hand, if the axis around the maximum principal moment of inertia is uncontrollable, then the radius of the manifold around the axis of x_3 is not limited, and the manifold is connected with the axis of x_3, as shown in Fig. 2(c). Beside, the polhode for the case of $\alpha < j_2$ is a closed loop trajectory around the axis of x_3. This implies that polhode for the case of $\alpha < j_2$ always has an intersection point with the manifold for case (VI). Therefore, in this case, to have at least one intersection point with the manifold, the following signs should be selected for the constant control torques.

$$\mu_2 = -\text{sgn}(x_2)\,,\quad \mu_3 = \text{sgn}(x_3) \tag{8}$$

Test Problem

An example numerical simulation is conducted to demonstrate the validity of the proposed control method. It is assumed that the uncontrolled axis is around the minimum principal moment of inertia. The parameters for numerical simulation are as follows: the moments of inertia (j_1, j_2, j_3) = (15, 10, 7) [kgm^2], the constant control torque is $T_{1,2} = \pm 20$ [Nm], and the initial angular velocity vector is $(\omega_1(0),\ \omega_2(0),\ \omega_3(0)) = (1.54,\ \text{-1.95 -0.58})\ [\text{rad/s}]$.

Firstly, to determine if boosting process is needed, it is checked if the polhode intersects the manifolds. Two intersection points between the polhode and the manifold are found, and they are the angular velocity vectors (1.6296, 1.6329, -1.1622) [rad/s], and (1.6296, -1.6329, 1.1622) [rad/s], respectively. This means that no control torque is needed until reaching the manifold. The time response of the angular velocities, the time history of the control torques, the trajectory of the time and energy optimal solution along with the manifold, and the trajectory of the energy optimal but not time optimal solution are shown in Figs. 3(a), 3(b), 3(c), and 3(d), respectively. It can be seen that the angular velocities are successfully controlled to the origin by the proposed method. The time required to reach each intersection point from the initial angular velocities is determined as 1.6848[sec], and 4.1845[sec], respectively. The energy consumption of the second solution is the same as that of the first solution, but the first solution was selected from the viewpoint of the settling time in this Note. When the angular velocity vector reaches the first intersect point, the sign of the control torques is determined to be (-, 0). That is, a trajectory sliding on the manifold is generated by a negative constant control torque along the maximum principal moment of inertia. The constant control torque is employed until the uncontrolled angular velocity ω_3 and the one of the controlled angular velocity ω_1 converge to zero. The time required to reach the transient goal from the intersection point is found to be 1.1003[sec]. After the two angular velocities become zero, the angular velocity of the remaining controllable axis ω_2 is controlled until reaching the origin by a negative constant torque along the middle principal moment of inertia. The time required to converge to the origin from the transient goal is found to be 1.0222[sec]. The total time required to converge from the initial angular velocity to the origin is, therefore, given by approximately 3.8073 (1.6848 +1.1003+1.0222)[sec].

Conclusions

This Note has proposed a constant control torque method for attenuating the rotational motion of an asymmetric rigid body. The manifold is defined as a set of the angular velocities of an asymmetrical rigid body that can approach the transient goal by employing a constant control torque, and can be obtained analytically by integrating the equations of motion backward in time from the transient goal, which is accessible from the origin by means of a single constant control torque. The trajectory resulting from the proposed method consists of three steps: First, a trajectory boosted around the maximum or minimum principal moments of inertia, and damped around the middle principal moment of inertia by control torques if necessary (if unnecessary, a trajectory of torque-free motion (polhode)) until an intersection point with the manifold is reached). Second, a trajectory sliding on the manifold, and finally a trajectory along the one controllable axis until the origin is reached. Thanks to the analytically obtained manifolds and polhode, the time required for convergence to the origin can be obtained numerically by calculating the intersection point between the manifolds and trajectories.

To delete ambiguities of multiple solutions, the energy optimality and settling time is considered. The results of an example numerical simulation showed that the complete attenuation of the angular velocities of an asymmetrical rigid body can be achieved by the proposed method, provided that internal and external disturbances and modeling uncertainties are absent, and the intersection point between the trajectory and the manifold is completely obtained.

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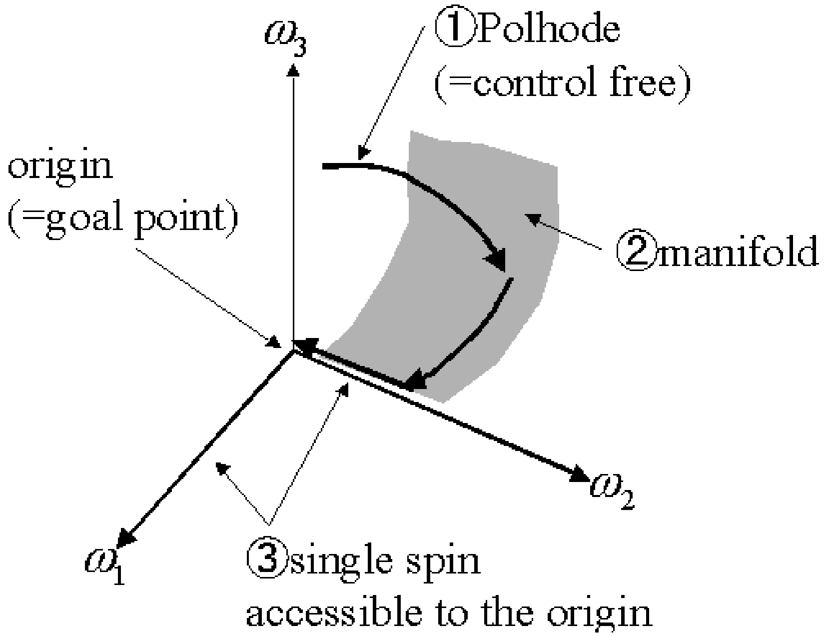
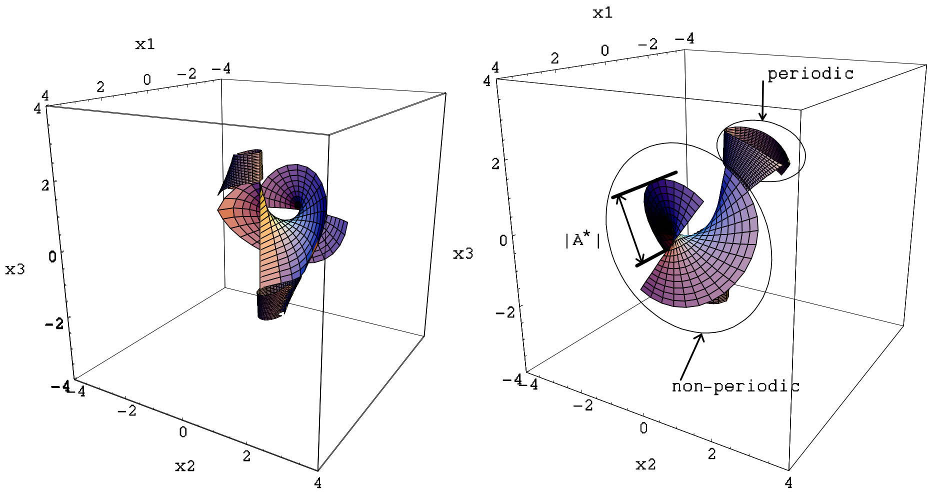
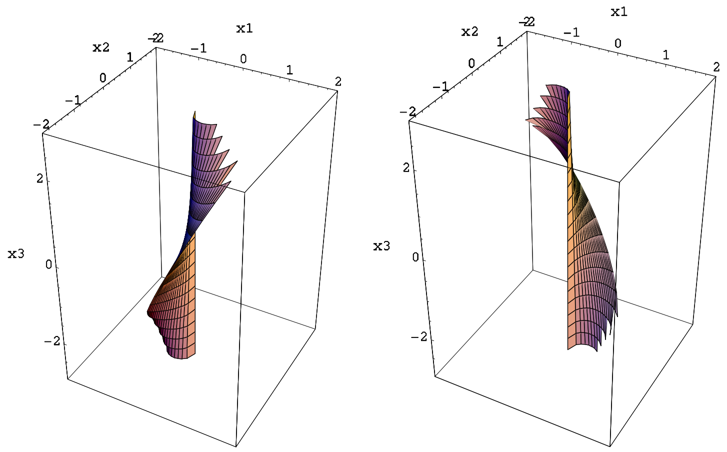


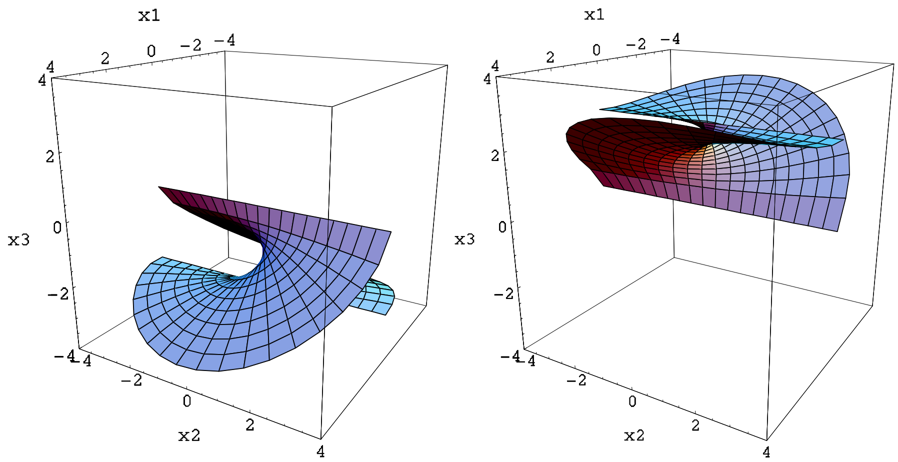
Fig. 1 Schematic view of the trajectory resulting from the constant control torque method.



(a) $\mu_1 = +1$ $\mu_1 = -1$



(b) $\mu_2 = +1$ $\mu_2 = -1$



(c) $\mu_3 = +1$ $\mu_3 = -1$

Fig. 2 Scaled constant-control manifold for case (I) (a), for case (III) (b), and for case (VI)(c).

Table 1. Transient Goals.

maximum	middle	minimum	transient goal*
(I) constant torque	uncontrollable	controllable	$x_1(0)=0$, $x_2(0)=0$, $x_3(0)\neq 0$
(II) constant torque	controllable	uncontrollable	$x_1(0)=0$, $x_2(0)\neq 0$, $x_3(0)=0$
(III) uncontrollable	constant torque	controllable	$x_1(0)=0$, $x_2(0)=0$, $x_3(0)\neq 0$
(IV) controllable	constant torque	uncontrollable	$x_1(0)\neq 0$, $x_2(0)=0$, $x_3(0)=0$
(V) controllable	uncontrollable	constant torque	$x_1(0)\neq 0$, $x_2(0)=0$, $x_3(0)=0$
(VI) uncontrollable	controllable	constant torque	$x_1(0)=0$, $x_2(0)\neq 0$, $x_3(0)=0$

*axis 1: maximum, 2: middle, 3: minimum

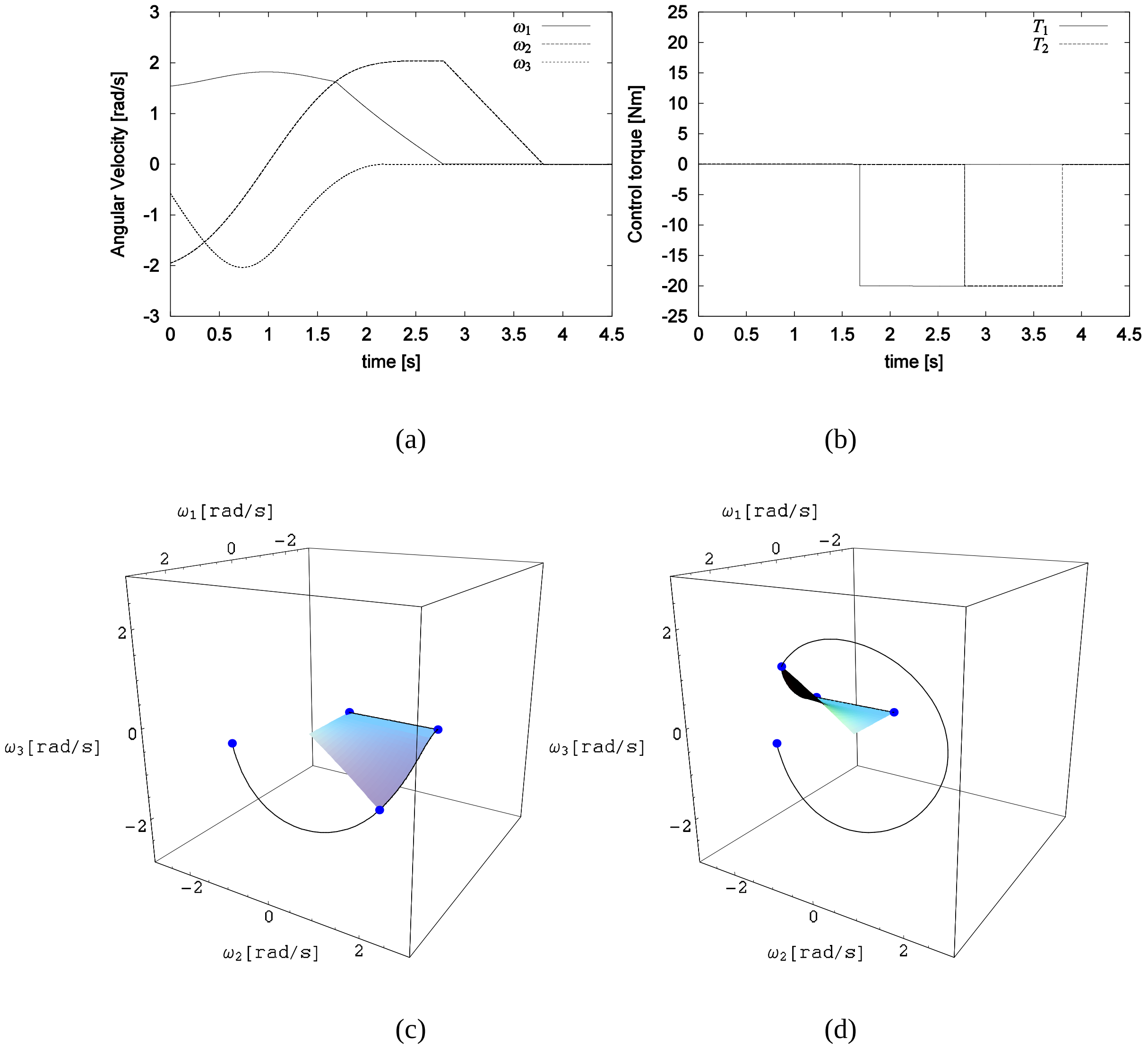


Fig.3 Time response of the angular velocities (a), time histories of the constant control torques (b), the time and energy optimal trajectory along with the intersected manifold (c), and the energy optimal but not time optimal trajectory along with the intersected manifold (d) (The start and endpoints of each segment of the trajectory are indicated by the markers).