H^2/H^∞ controller design for input-delay and preview systems based on state decomposition approach

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Abstract

This thesis concentrates on the efficient solution methods of H^2/H^∞ optimal control problems for input-delay and preview systems. Although the problems can be reformulated to the ones for delay-free systems by augmenting the state space of the controlled systems, the numerical solution of the Riccati/KYP (Kalman-Yakubovich-Popov) equations for the augmented systems requires special efforts, and complicates controller tuning. On the other hand, it is known that the optimal control laws for certain classes of time-delay systems can be constructed without solving the augmented Riccati/KYP equations. Such design problems are called reduced-order construction problems in this thesis. The solutions of the reduced-order construction problems are still limited in theoretical and practical perspectives. The main purpose of the thesis is to propose a new approach for the reduced-order construction problems, which enables to derive the optimal output feedback controllers for input-delayed and preview systems in a unified manner. We focus on the internal dynamics of the overall systems, and decompose it toward the H^2 and H^∞ performance objectives.

The fundamental idea of our approach is first introduced for the discrete-time input-delayed H^2/H^∞ control problems. The state decomposition enables to solve the output feedback problem through the simpler ones, namely, the full information and output estimation problems. The discrete-time optimal controllers are obtained in the Smith predictor form. They are constructed from the Riccati/KYP equations for the delay-free systems.

The solution procedure is further extended to the continuous-time preview H^2/H^∞ control problems in an output feedback setting. The optimal utilization of the preview information is exploited at the full information and output estimation problems. The clear structures of the optimal controllers are revealed as the combination of the finite-dimensional observers and preview-feedforward compensation.

In the H^∞ control problems for the input-delayed and preview systems, the J-spectral factorization techniques in the literature are employed. Their interconnection to the augmented Riccati/KYP equations is clarified by reviewing the techniques from a view point of the internal state dynamics.

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Contents

Notation

- M_{ij}: The $(i,\, j)$ block of a linear operator or matrix M.
- $\sigma(M)$: The set of the spectra of a linear operator M or set of the eigenvalues of a square matrix M.
- $J_{\rm s}$: A block matrix which defines a symplectic inner product; $J_{\rm s} := \begin{bmatrix} O & -I \\ I & O \end{bmatrix}$.
- $\mathcal{H}(z^{-l})$: The orthogonal complement of $z^{-l}H^2$ in H^2.
- $W^{2,1}([0,l])$: The Sobolev space of the functions weakly differentiable in $L^2([0,l])$.
- $T_{a\,b}$: The transfer function from the input b to the output a.
- $G^{\sim}$: The para-conjugate of the transfer function G.
- $P_1 \star P_2$: The Redheffer's star product of two-input-output systems P_1 and P_2; For $P_1 : (w_1,\, u_1) \to (z_1,\, y_1)$ and $P_2 : (w_2,\, u_2) \to (z_2,\, y_2)$, $P_1 \star P_2$ denotes the system obtained by connecting y_1 and z_2 to w_2 and u_1, respectively.
- $\mathcal{C}(P)$: The chain scattering representation of a two-input-output system P; $\mathcal{C}(\cdot)$ defines the two-input-output system $\mathcal{C}(P) : (y,\, u) \to (w,\, z)$ from $P : (w,\, u) \to (z,\, y)$.
- $\mathcal{C}^{-1}(G)$: The inverse chain scattering representation of a two-input-output system G; $\mathcal{C}^{-1}(\cdot)$ is the inverse mapping of $\mathcal{C}(\cdot)$.
- $\mathcal{S}(\Phi)$: The Schur complementation transform of a two-input-output system Φ; $\mathcal{S}(\cdot)$ defines the two-input-output system $\mathcal{S}(\Phi) : (h,\, u) \to (-y,\, k)$ from $\Phi : (y,\, u) \to (h,\, k)$.
- $\mathcal{S}^{-1}(\Omega)$: The inverse Schur complementation transform of a two-input-output system Ω; $\mathcal{S}^{-1}(\cdot)$ is the inverse mapping of $\mathcal{S}(\cdot)$.
- $\delta_{\phi,\,\theta}$: The Kronecker's delta; $\delta_{\phi,\,\theta} = 1$ if $\phi = \theta$, and $\delta_{\phi,\,\theta} = 0$ if $\phi \neq \theta$.
- Γ_θ: The trace operator; It evaluates the values of $f(\cdot) \in W^{2,1}([0,\, l], \mathbb{R}^n)$ at θ: $\Gamma_\theta f = f(\theta)$.
- δ_θ: The delta function which has its support at θ.
- $H(\theta)$: The step function; $H(\theta) = 1$ if $\theta > 0$, and $H(\theta) = 0$ if $\theta < 0$.

Chapter 1 Introduction

1.1 Background of research

Dynamical systems such as chemical, transport, and flexible structure systems are often modeled as input-delayed systems. Predictive control strategies [11], [43] are systematic ways to compensate the adverse effects of the input-delays. Time-delays are also employed to formulate preview control problems, where efficient use of future information is investigated to improve control performances [60]. They have applications in active suspension of vehicles, positioning of cranes, and shape control of rolling mills.

The state-space augmentation is known as the standard and straightforward technique which allows us to apply general theories of delay-free systems to state or input-output delayed systems. The technique regards the past history of state or input-output signals as state variables, and rewrites the delayed systems into the delay-free forms. Since the dimensions of the overall systems including the past history increases according to the delay lengths, the overall systems in the delay-free forms are called the augmented systems.

In the discrete-time settings, the H^2/H^∞-optimal control laws can be determined, in principle, by solving the matrix Riccati/KYP (Kalman-Yakubovich-Popov) equations for the augmented systems. However, the numerical computation of their solutions suffers from complexity and inaccuracy due to the increased orders and special structures of the delay elements [4]. In the continuous-time setting, the overall system including preview lags is an infinite-dimensional system, and the associated Riccati equation turns out to be a couple of nonlinear PDEs (partial differential equations). Although they can be approximately solved by gridding spatial domains and by selecting interpolating functions, such approximations raise stability concerns on the resulting controlled systems [25].

1.2 Motivation for reduced-order construction methods

To overcome the difficulties associated with the state augmentation, a state-prediction approach for the discrete-time input-delayed LQ control is proposed in [43]. The optimal state feedback gain and control cost are constructed from the standard Riccati equation for the corresponding delay-free system. It is also shown that the Hermitian matrix defining the optimal control cost is in fact the stabilizing solution of the standard Riccati equation for the augmented system. In the continuous-time settings, it has been known that the solutions of the operator Riccati equations can be found in some control problems [61]. The class of input-output delayed systems for which the operator Riccati equations are explicitly solvable has been expanding [28], [30].

In this thesis, we define the reduced-order construction problems as those of constructing the optimal control laws for time-delay systems without solving the augmented Riccati/KYP equations. The reduced-order construction methods enable to avoid the numerical difficulties and lead to simplification of controller tuning process. The studies in [23], [47] indicate the reduced-order construction methods developed for the control of input-delayed systems can be generalized for that of the infinite-dimensional systems with inner functions at the control input. The optimal control problems for them have been studied from the operator theoretic perspectives [9], [22], and the explicit solutions are obtained in terms of finite-dimensional matrix operations. We cite reduced-order construction methods relevant for input-delayed and preview control problems in the two subsections below.

1.2.1 Previous results for input-delayed control systems

The results in [34] and [47] can be regarded as the extensions of that in [43] to the discrete-time LQG and H^2 control problems, respectively. The former deals with multiple input-output delays entering different channels. The result requires solving the control Riccati equation with its order increasing according to the time delay lengths when the number of the input delays is more than one. The latter studies the H^2 control problem where the input delay is generalized to a general inner function, and revealed the relationship between the inner function and achievable control performance. However, no causally implementable form of the optimal controller is provided. The relevant results for delayed-measurement systems with its performance index as the closed-loop H^∞ norm are reported in [17], [56], [8]. In [17], [56], the solution of the H^∞ filtering problem under one-step delayed measurement is constructed from the standard Riccati equation for the delay-free case. However, generalization of their construction methods to the multi-step delayed case is not straightforward. In [8], an alternative approach based on the J-spectral factorization theory is developed to the H^∞ filtering problem under multi-step delayed measurement. It requires constructing interactor matrices as many times as the delay length to cancel infinite zeros due to the delay element, and hence the resulting solution is not as simple as that in [43].

The continuous-time theory is more mature than the discrete-time one in that it can handle the H^∞ performance objective in output feedback settings. In [38], [40], an H^2 control problem for a generalized rational plant with a single input-delay is solved finding constraints on the associated finite-dimensional problem. In [36], the fundamental technique of reducing an delayed J-spectral density to delay-free one is proposed . In the subsequent papers [37], [38], this technique is enhanced to treat the continuous-time H^∞ problem for a generalized rational plant with a single input-delay. The dynamic programming solutions for the H^2 and H^∞ performance objectives are given in [44] and in [52], respectively. In [28], explicit solutions of operator H^∞-type Riccati equations are constructed via state transformation and integral equations. The corresponding operator H^2-type Riccati equations can be solved using similar techniques.

Among the above continuous-time results, the resulting H^∞ controller is parameterized using the Riccati equations in the standard H^∞ problem in [38]. And the structure of them is identified as the Smith-type one, which consists of the finite-dimensional H^∞ controller for the unstable dynamics and the measurement compensation part based on the past history of the control input. However, it is not a trivial question whether the discrete-time H^∞ controller can be also parameterized using the standard Riccati or KYP equations and implemented in the Smith form for the following reasons: 1) the discrete-time KYP equations have more complicated structures than the continuous-time counterparts; 2) The approach in [37], [38] involves the preliminary step where the inner function at the control input channel is regarded as a part of the controller. The step requires auxiliary arguments to guarantee the causality of the controller and elaborate manipulation of transfer functions to obtain the final implementation form.

1.2.2 Previous results for preview control systems

The reduce-order construction methods for the continuous-time H^2/H^∞ preview control problems have been obtained in parallel approaches for the input-delayed problems. However, most of them are limited to the full information settings. For brevity, we cite mainly the results in the continuous-time settings. In [31], [38], the continuous-time fixed-lag smoothing problems are solved by reducing the infinite-dimensional J-spectral density to a finite-dimensional one. That technique can be regarded an extension of the J-spectral factorization technique for the input delay systems in [36]. The problem setting can be regarded as dual to the full information preview control problems. In [53], [54], both continuous- and discrete-time preview control problems are tackled via a unified game theoretical method by splitting the optimization intervals. In [45], the idea similar to [53], [54] is applied to the H^2 controller design. In [28], the established state transformation technique is capable of dealing with the input delay and preview information at the same time.

In preview control systems, the control input is allowed to utilize the future information of external signals. Such situations can be artifically realized with sensors reading the incoming signals. In active suspension control [31], the road profiles in front of vehicles can be available with displacement sensors. They are regarded as preview information and utilized for improving riding comfort or road handling. In rolling mill control [7], rough steel is fed through a series of rollers to shape it into desired forms. Upstream variation in steel thickness and roll gaps can be given to downstream controllers as preview information. As for large and complex control systems, it is difficult to prepare sensors enough to measure all the state variables and disturbances. In active suspension control, either the displacement or velocity of dumpers is often measured [18], [35]. In rolling mill control, the tension variation around rollers is estimated using disturbance observes when accurate measurement is difficult under the rolling environment [19], [62]. From the viewpoint of the practical applications, it is important to identify the measurement information patterns based on which optimal preview control problems are explicitly solvable.

The output feedback H^2 controller design methods are reported in [32], [57]. However, the exact optimality of the overall closed-loop system is not guaranteed because the available preview information is not considered in their first design stages. In the former method, the state-space Youla parameterization is employed for the stabilization of the system at the first stage. At this stage, the stabilizing feedback gains can be chosen arbitrarily, only if they are stabilizing. The Youla parameter is determined by the orthogonality principle in H^2 space at the second stage. In the latter method, the standard finite-dimensional H^2 controller for the non-preview case is constructed at the first stage. At this state, the fact that the control input can act in advance of the disturbance is ignored. The preview information is incorporated into the additional compensation [27] at the second stage.

H^∞ preview controller synthesis based on partial information on state variables and disturbances is addressed in [26], where the partial information on the state variables is free from measurement noise. The state transformation in [28] enabled to solve the operator Riccati equations explicitly. A competing approach with the operator Riccati equation approach is the J-spectral factorization approach originated in [36]. The preview control problem in the full information setting can be easily solved by dualizing the results in [31], [38]. However, the application of the technique to the output feedback setting needs further investigation. This is because, in contrast to the input-delay problem [38], the preview action is difficult to be regarded as the constraint on the controller.

1.3 Contributions of this work

The thesis proposes alternative reduce-order construction methods of the H^2/H^∞ optimal controllers for the input and preview systems. We focus on the internal dynamics of the overall systems, and decomposes it to derive the delay-free systems which play a key role for the H^2/H^∞ performance optimization. Our approach enables to derive the optimal output feedback controller in a unified manner. The output feedback problem is solved through the simpler ones, namely the full information and output estimation ones. The internal state decomposition approach is applicable in both continuous-time and discrete-time settings. Below we explain how the proposed approach achieves the contributions, and reveals the new insights in the reduced-oder construction problems.

1.3.1 Discrete-time H^2 input-delayed control

The output feedback stabilization of a discrete-time single-input input-delayed system is described as a preliminary result. We focus on the state-space representation of the input delay element, and introduce a state transformation to derive a Smith predictor.

Motivated by the state transformation, the internal state decomposition approach is introduced to derive the reduced-order construction method for the discrete-time input-delayed H^2 control problem. The causality and stability constraints imposed by the delay element are resolved more simply in comparison with the previous approach [23], [40], [47].

We obtain the optimal controller in the Smith predictor form from the Riccati equations for the corresponding delay-free system. The resulting controller has the different structure from those in the state predictor form [34]. It is implemented with the observer for the possibly unstable dynamics and a measurement compensation part based on the past history of the control input.

1.3.2 Discrete-time H^∞ input-delayed control

We pursues a parameterization of the discrete-time H^∞ suboptimal controllers for the input-delayed systems. A discrete-time counterpart of the J-spectral factorization technique in [36] and [37] is developed as the main tool. In contrast to the previous research [37, 38, 21], the first step is to simplify the original problem to an one-sided model matching problem, and to apply the J-spectral factorization technique to the latter. The second step is to formulate an output estimation problem in order to realize the control law based on partial information. Focusing on the relationship between the state variables of the irrational and reduced rational J-spectral densities, the measured output is modified in order to reduce the output estimation problem to finite-dimensional one. It is revealed that the parameterization of the H^∞ controllers is obtained only by solving the standard KYP equations and checking the matrix eigenvalues. They are implemented in the Smith form using the past history of the control input.

The above-mentioned J-spectral factorizability condition composes a part of the solvability conditions for the output feedback problem. The min-max optimization approach is also considered in the full information setting to obtain an alternative solvability condition. We extend the optimization technique in [52] into the discrete-time setting, and construct the stabilizing solution of the standard KYP equation for the augmented system from that for the delay-free systems. Furthermore, the J-spectral factorizability condition is proved to be equivalent to the H^∞ disturbance attenuation condition by analyzing the initial finite-time response of the input-delay system, and consequently that approach is confirmed to yield the same control law as the min-max optimization approach.

1.3.3 Continuous-time H^2 preview control

We investigate the effectiveness of the state decomposition in designing H^2 preview controllers. The output feedback controller is constructed through the full information and output estimation problems. Contrary to the previous design methods [32], [57], we exploit the available preview information at all the design stages, and derive the output feedback controller guaranteeing the exact optimality. The preview controller can be given the interpretation as the finite-dimensional observer combined with preview-feedforward compensation.

In the full information problem, we utilize the fact that the optimal state feedback law is obtained as the solution of the corresponding model matching problem. We introduce alternative state transformations to solve the model matching problem via the spectral factorization theory [17]. One of the state transformations defines the state decomposition parallel to that in the discrete-time input-delayed H^2 control problem. In the output estimation problem, we employ the newly introduced state variable to describe the infinite-dimensional generalized plant in the form amenable to the explicit solution.

1.3.4 Continuous-time H^∞ preview control

We extend the design procedure of solving the full information and output estimation problems to the H^∞ control case. The clear structure of the preview controller is again identified as the combination of the finite-dimensional observer and preview-feedforward compensation. It is shown to be possible to construct the H^∞ preview control law based on the information pattern different from those in [26].

In the full information problem, the technique of reducing an irrational J-spectral density to rational one [38] is employed. We clarify the relationship between the state variables of the infinite-dimensional and reduced finite-dimensional J-spectral densities by introducing the state transformations. The explicit solution to the control-type operator Riccati equation is found by combining the proposed state transformations.

1.4 Organization of this thesis

The thesis consists of six chapters including this chapter. The rest of the thesis is organized as follows:

Chapter 2 addresses the H^2 control problem for the discrete-time input-delayed systems. The fundamental idea of the state decomposition approach is introduced based on the derivation of a Smith predictor for a single input-delay system. The efficient implementation of the optimal controller in the Smith predictor form is provided.

Chapter 3 pursues a parameterization of the discrete-time H^∞ suboptimal controllers. A discrete-time counterpart of the J-spectral factorization technique in [36] is developed. A discrete-time counterpart of the min-max optimization technique in [52] is also considered. The stabilizing solution of the augmented KYP equation and another characterization of the solvability are provided using that for the delay-free case.

Chapter 4 shows that the internal state decomposition approach is also effective to derive the H^2 preview output feedback controller. The clear structure of the optimal controller is identified as the combination of the finite-dimensional observer and preview-feedforward compensation. Contrary to the previous design methods [32], [57], the proposed one enables to derive the output feedback controller archiving the exact optimal performance.

Chapter 5 extends the preview control design method to the H^∞ control case. Our design method leads to the H^∞ preview controller based on the information pattern different from those in the previous research [28]. As an additional note, the relationship between the J-spectral factorization technique in [31], [38] and the operator Riccati equation is presented.

Chapter 6 reviews the H^2/H^∞ controller design methods obtained in the previous chapters, and summerizes the features of them. The future subjects of research are stated with reference to the strength and limitations of the proposed approach.

Chapter 2 Discrete-time H^2 input-delayed control

2.1 Introduction

This chapter is concerned with the discrete-time H^2 control problem under the setup in Fig. 2.1, where P_+ is an input-delayed generalized plant and K_+ is a controller to design. The generalized plant P_+ consists of the l-step delay function z^{-l} and delay-free generalized plant P. The continuous-time counterpart of the problem is solved in [40], where the delay function e^{-sl} exists at the control input. The more general continuous- and discrete-time problems are studied in [23], [47], where the delay functions e^{-sl} and z^{-l} are generalized to the continuous- and discrete-time inner functions $m(s)$ and $m(z)$, respectively. The previous approach in [23], [40], [47] regards the existence of the delay or inner function as the constraint on the stabilizing controller for the delay-free generalized plant P, and restrict the Youla parameter for the delay-free problem so that the causality and stability are preserved. The approach successfully characterizes the increase of the optimal cost due to the delay or inner functions in a closed form, and allows us to interpret the deteriorating effect of the time-delay or unstable zeros. However, it is difficult to see how the unstable dynamics of P_+ is stabilized, and in the discrete-time setting [47] the explicit optimal controller is not given.

We derive a new solution method by decomposing the state variable of P into the sum of the possibly unstable and stable ones. The former has the dimension independent of the delay length l. The latter has the the dimension equal to l multiplied by the number of the control inputs. Our solution method does not require considering the causality and stability constraints imposed by the input delay. The state decomposition approach reveals a new interpretation of stabilization process, and leads to the explicit form of the discrete-time H^2 optimal controller. For the derivation of the solution method, two preparations are made; 1) To describe the fundamental idea of the state decomposition approach, a discrete-time Smith-predictor for a single input-delay system is derived based on state-space representation; 2) To utilize the H^2 orthogonality principle effectively, an alternative discrete-time truncation operator to the literature [47] is introduced.

This chapter is organized as follows. In Section 2.2, the problem formulation and assumptions are stated. In Section 2.3, the discrete-time Smith-predictor is derived based on state-space representation. In Section 2.4, the alternative discrete-time truncation operator is introduced. In Section 2.5, the state decomposition approach is developed to cope with the input delay, and the explicit optimal controller is derived. In Section 2.7, the input-delay effect on the sampled-data H^2 control performance is analyzed using the obtained results. In Section 2.8, the proofs left in the previous sections are given.

2.2 Problem formulation

We address the H^2 control problem for the input-delayed plant P_+. The focus is on the parameterization of the stabilizing controller K_+ through the delay-free control problem. The

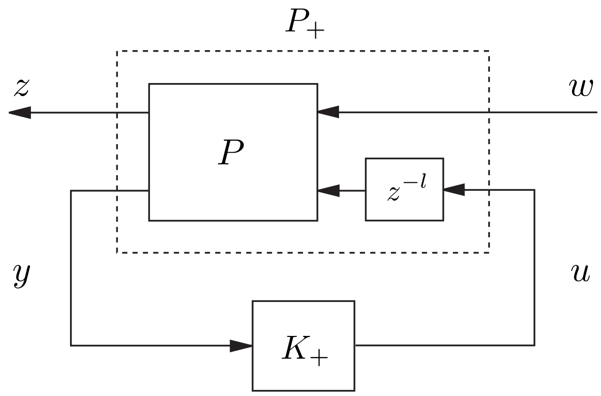

Figure 2.1: Control system with input delay.

delay-free part P is assumed to be

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & O \end{array}\right], \tag{2.1}$$

and satisfies the same assumptions (A1)-(A2) as in the standard H^2 control problem [20].

(A1) $(A,\, B_2)$ and $(A,\, C_2)$ are stabilizable and detectable, respectively.

(A2) For $\forall\,\theta \in [-\pi,\, \pi]$, $\begin{bmatrix} A - e^{j\theta}I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ and $\begin{bmatrix} A - e^{j\theta}I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ are of full column rank and of full row rank, respectively.

The above assumptions ensure the existence of the positive semidefinite stabilizing solutions X and Y of the following control- and filtering-type Riccati equations:

$$Q + A^*XA - X - (S^* + B_2^*XA)^*\, R_{\mathrm{c}}^{-1}\, (S^* + B_2^*XA) = O,\ R_{\mathrm{c}} := R + B_2^*XB_2 > O,$$
$$\acute{Q} + AYA^* - Y - \left(\acute{S}^* + AYC_2^*\right) \acute{R}_{\mathrm{c}}^{-1} \left(\acute{S}^* + AYC_2^*\right)^* = O,\ \acute{R}_{\mathrm{c}} := \acute{R} + C_2YC_2^* > O,$$

where the following definitions are used for simplicity:

$$\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} := \begin{bmatrix} C_1 & D_{12} \end{bmatrix}^* \begin{bmatrix} C_1 & D_{12} \end{bmatrix},\ \begin{bmatrix} \acute{Q} & \acute{S}^* \\ \acute{S} & \acute{R} \end{bmatrix} := \begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} \begin{bmatrix} B_1 \\ D_{21} \end{bmatrix}^*.$$

Furthermore, the nonsingularity of the state matrix A is required in our approach.

(H) The matrix A is nonsingular.

Based on the preliminaries in Sections 2.3 and 2.4, the discrete-time H^2 optimal controller is derived via the full information and output estimation problems.

2.3 Consideration on the Smith predictor

Two well-known stabilization methods for time-delay systems are the Smith's method and the finite spectrum assignment method. In the former method, the adverse effects of time-delays are eliminated from the closed-loop systems via loop-shifting arguments with internal models [46], [48], and the resulting controller is called the Smith predictor. In the latter method, the feedback control laws which relocate the finite number of unstable spectra of the time-delay systems are constructed via state-predictive transformations [11] and the resulting controller is called the finite spectrum assignment controller. In [40], it is revealed that the H^2 optimal controller for the continuous-time single input-delay system has a Smith predictor structure, and can be recast as the observer-predictor-based finite spectrum assignment controller.

In this section, we derive a discrete-time Smith predictor and observer-predictor-based controller for a single input-delay system via Krein's formula [13], which expresses the solutions of Sylvester equations by complex integration. The presentation in this section has an important implication for deriving our solution method.

Consider the discrete-time single input-delay system $P(z)\,z^{-l}$ (l is a positive integer):

$$\begin{aligned} x(n+1) &= Ax(n) + Bu(n-l) \\ y(n) &= Cx(n). \end{aligned}$$

For technical reasons, we assume that A is nonsingular and the pairs (A, B) and (A, C) are stabilizable and detectable, respectively. Let us define the collection of the past control inputs:

$$\upsilon(n) := \begin{bmatrix} \upsilon(1,n) \\ \upsilon(2,n) \\ \vdots \\ \upsilon(l,n) \end{bmatrix} := \begin{bmatrix} u(n-1) \\ u(n-2) \\ \vdots \\ u(n-l) \end{bmatrix} \in \mathbb{R}^{l\cdot\dim u}.$$

By augmenting the state space from $\mathbb{R}^{\dim x}$ to $\mathbb{R}^{\dim x} \times \mathbb{R}^{l\cdot\dim u}$, the input-delay system can be rewritten as the delay-free system:

$$\begin{aligned} \begin{bmatrix} x(n+1) \\ \upsilon(n+1) \end{bmatrix} &= \begin{bmatrix} A & B\Gamma_1 \\ O & \mathcal{S} \end{bmatrix} \begin{bmatrix} x(n) \\ \upsilon(n) \end{bmatrix} + \begin{bmatrix} O \\ \Delta_l \end{bmatrix} u(n), \\ y(n) &= Cx(n), \end{aligned} \tag{2.2}$$

where

$$\mathcal{S} := \begin{bmatrix} O & I & & & \\ O & & I & & \\ \vdots & & & \ddots & \\ \vdots & & & & I \\ O & O & \cdots & \cdots & O \end{bmatrix} \in \mathbb{R}^{l\cdot\dim u \times l\cdot\dim u},$$

$$\Delta_l := \begin{bmatrix} O^{\mathrm{T}} & O^{\mathrm{T}} & \cdots & O^{\mathrm{T}} & I^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{l\cdot\dim u \times \dim u},$$

$$\Gamma_1 := \begin{bmatrix} I & O & O & \cdots & O \end{bmatrix} \in \mathbb{R}^{\dim u \times l\cdot\dim u}.$$

In Eq. (2.2), the possibly unstable mode of $x(n)$ is influenced by the stable mode of $\upsilon(n)$ through $B\Gamma_1$. To eliminate $B\Gamma_1$, we consider the following transformation:

$$\begin{bmatrix} I & U_x \\ O & I \end{bmatrix} \begin{bmatrix} A - zI & B\Gamma_1 \\ O & \mathcal{S} - zI \end{bmatrix} \begin{bmatrix} I & -U_x \\ O & I \end{bmatrix} = \begin{bmatrix} A - zI & B\Gamma_1 - (AU_x - U_x\mathcal{S}) \\ O & \mathcal{S} - zI \end{bmatrix}, \tag{2.3}$$

where $U_x \in \mathbb{R}^{\dim x} \times \mathbb{R}^{l \cdot \dim u}$ is a matrix, which should satisfy the Sylvester equation

$$AU_x - U_x\mathcal{S} = B\Gamma_1$$

to make the $(1,2)$ block in Eq. (2.3) zero. Using Krein's formula, we can find the solution:

$$\begin{aligned} U_x &= \frac{1}{2\pi j} \oint_{\partial\sigma(A)} (zI - A)^{-1} B\Gamma_1 (zI - \mathcal{S})^{-1} \, dz \\ &= A^{-1}B\Gamma_1 + A^{-2}B\Gamma_1\mathcal{S} + \cdots + A^{-l}B\Gamma_1\mathcal{S}^{l-1}, \end{aligned}$$

where the following identity is used:

$$(zI - \mathcal{S})^{-1} = \frac{I}{z} + \frac{\mathcal{S}}{z^2} + \cdots + \frac{\mathcal{S}^{l-1}}{z^l}.$$

Introducing a new state variable $x^{\mathrm{R}}(n)$ by the equation

$$\begin{bmatrix} x(n) \\ \upsilon(n) \end{bmatrix} = \begin{bmatrix} I & -U_x \\ O & I \end{bmatrix} \begin{bmatrix} x^{\mathrm{R}}(n) \\ \upsilon(n) \end{bmatrix}, \tag{2.4}$$

the input-delay system is represented as

$$\begin{aligned} \begin{bmatrix} x^{\mathrm{R}}(n+1) \\ \upsilon(n+1) \end{bmatrix} &= \begin{bmatrix} A & O \\ O & \mathcal{S} \end{bmatrix} \begin{bmatrix} x^{\mathrm{R}}(n) \\ \upsilon(n) \end{bmatrix} + \begin{bmatrix} A^{-l}B \\ \Delta_l \end{bmatrix} u(n), \\ y(n) &= C \begin{bmatrix} I & -U_x \end{bmatrix} \begin{bmatrix} x^{\mathrm{R}}(n) \\ \upsilon(n) \end{bmatrix}, \end{aligned}$$

where its state-transition matrix is block-diagonalized.

Temporarily, let us assume all the state variables are available. We choose the following state feedback gain for $x^{\mathrm{R}}(n)$:

$$F^{\mathrm{R}} := FA^l,$$

where F is such that $A + A^{-l}BF^{\mathrm{R}} = A^{-l}(A + BF)A^l$ is stable. Since $\mathcal{S}$ is stable, the following state feedback law stabilizes the dynamics of $x^{\mathrm{R}}(n)$ and $\upsilon(n)$:

$$u(n) = \begin{bmatrix} F^{\mathrm{R}} & O \end{bmatrix} \begin{bmatrix} x^{\mathrm{R}}(n) \\ \upsilon(n) \end{bmatrix}.$$

Next, we introduce the following standard observer to estimate $x^{\mathrm{R}}(n)$ and $\upsilon(n)$:

$$\begin{bmatrix} x_{\mathrm{c}}^{\mathrm{R}}(n+1) \\ \upsilon_{\mathrm{c}}(n+1) \end{bmatrix} = \begin{bmatrix} A & O \\ O & \mathcal{S} \end{bmatrix} \begin{bmatrix} x_{\mathrm{c}}^{\mathrm{R}}(n) \\ \upsilon_{\mathrm{c}}(n) \end{bmatrix} + \begin{bmatrix} A^{-l}B \\ \Delta_l \end{bmatrix} u(n) + \begin{bmatrix} L_x^{\mathrm{R}} \\ L_\upsilon^{\mathrm{R}} \end{bmatrix} \left(C \begin{bmatrix} I & -U_x \end{bmatrix} \begin{bmatrix} x_{\mathrm{c}}^{\mathrm{R}}(n) \\ \upsilon_{\mathrm{c}}(n) \end{bmatrix} - y(n) \right).$$

Let $L_x^{\mathrm{R}} := L$ be such that $A+LC$ is stable and $L_v^{\mathrm{R}} := O$, then the above observer is rearranged into the following form:

$$\begin{bmatrix} x_{\mathrm{c}}^{\mathrm{R}}(n+1) \\ \upsilon_{\mathrm{c}}(n+1) \end{bmatrix} = \begin{bmatrix} A+LC & -LCU_x \\ O & \mathcal{S} \end{bmatrix} \begin{bmatrix} x_{\mathrm{c}}^{\mathrm{R}}(n) \\ \upsilon_{\mathrm{c}}(n) \end{bmatrix} + \begin{bmatrix} A^{-l}B \\ \Delta_l \end{bmatrix} u(n) - \begin{bmatrix} L \\ O \end{bmatrix} y(n),$$
$$u(n) = \begin{bmatrix} F^{\mathrm{R}} & O \end{bmatrix} \begin{bmatrix} x_{\mathrm{c}}^{\mathrm{R}}(n) \\ \upsilon_{\mathrm{c}}(n) \end{bmatrix}.$$

Since the choice of the observer gains preserves the stable dynamics of $\upsilon_{\mathrm{c}}(n)$, the state estimation error converges to zero. The obtained observer is depicted in Fig. 2.2, where the FIR (finite impulse response) operator $\Pi(z)$ is defined by

$$\Pi(z) := -\left(A^{-1}z^{-l} + A^{-2}z^{-(l-1)} + \cdots + A^{-l}z^{-1}\right) = (zI-A)^{-1}\left(z^{-l}I - A^{-l}\right).$$

The observer has the structure of the Smith predictor in the sense that the measured output $y(n)$ is modified to another measured output $y^{\mathrm{R}}(n)$ using the past history of the control input:

$$y^{\mathrm{R}}(n) = y(n) - C\Pi(z)Bu(n).$$

The Smith predictor can be recast as the observer-predictor-based controller if the state variables of the Smith predictor is changed by the equation

$$\begin{bmatrix} x_{\mathrm{c}}(n) \\ \upsilon_{\mathrm{c}}(n) \end{bmatrix} = \begin{bmatrix} I & -U_x \\ O & I \end{bmatrix} \begin{bmatrix} x_{\mathrm{c}}^{\mathrm{R}}(n) \\ \upsilon_{\mathrm{c}}(n) \end{bmatrix}.$$

The structure of the observer-predictor-based controller is depicted in Fig. 2.3. The variables $x_{\mathrm{c}}(n)$ and $p(n)$ in Fig. 2.3 can be regarded as the estimation and l-step ahead prediction of the state variable $x(n)$, respectively.

$$\begin{bmatrix} x_{\mathrm{c}}(n+1) \\ \upsilon_{\mathrm{c}}(n+1) \end{bmatrix} = \begin{bmatrix} A+LC & B\Gamma_1 \\ O & \mathcal{S} \end{bmatrix} \begin{bmatrix} x_{\mathrm{c}}(n) \\ \upsilon_{\mathrm{c}}(n) \end{bmatrix} + \begin{bmatrix} O \\ \Delta_l \end{bmatrix} u(n) - \begin{bmatrix} L \\ O \end{bmatrix} y(n),$$
$$u(n) = F\begin{bmatrix} A^l & A^lU_x \end{bmatrix} \begin{bmatrix} x_{\mathrm{c}}(n) \\ \upsilon_{\mathrm{c}}(n) \end{bmatrix}.$$

The key points observed here are that the Smith predictor stabilizes the possibly unstable dynamics of $x^{\mathrm{R}}(n)$ ignoring the stable dynamics of $\upsilon(n)$, and that it is equivalently transformed into the state predictor form.

Remark 1. *If Fig. 2.2 and Fig. 2.3 are compared with Fig. 5 and Fig. 4 in [41], respectively, it can be verified that the structures of the resulting controllers are similar to those of the continuous-time Smith predictor and observer-predictor-based controller. In the following, it is noted briefly that the same idea as in the discrete-time setting is also applicable to deriving the continuous-time Smith predictor and observer-predictor-based controller for the single input-delay system* $P(s)\,e^{-ls}$ *(*l *is a positive real number):*

$$\dot{x}(t) = Ax(t) + Bu(t-l),$$
$$y(t) = Cx(t).$$

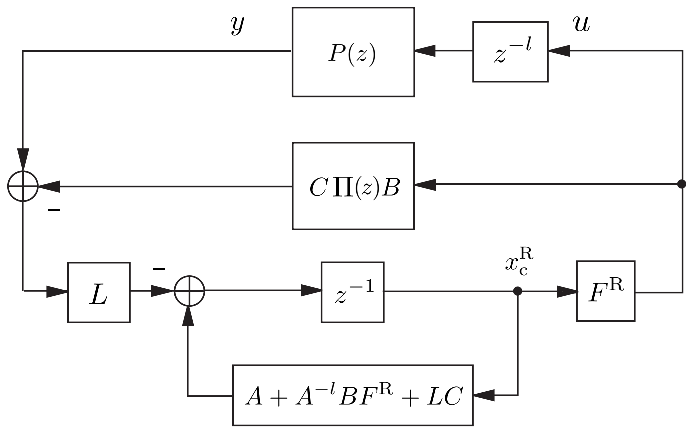

Figure 2.2: Smith predictor.

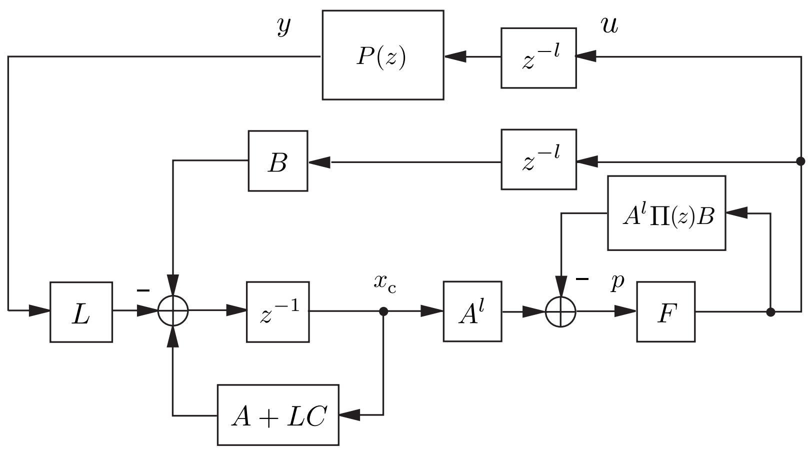

Figure 2.3: Observer-predictor-based controller.

Based on the state-space framework in [50], we rewrite the above input-delay system as the infinite-dimensional descriptor system:

$$\frac{d}{dt}\begin{bmatrix} I & O \\ O & E_\upsilon \end{bmatrix}\begin{bmatrix} x(t) \\ \upsilon(\theta,t) \end{bmatrix} = \begin{bmatrix} A & B\Gamma_0 \\ O & A_\upsilon \end{bmatrix}\begin{bmatrix} x(t) \\ \upsilon(\theta,t) \end{bmatrix} + \begin{bmatrix} O \\ \Delta_l \end{bmatrix} u(t),$$

where

$$E_\upsilon := \begin{bmatrix} I \\ O \end{bmatrix},\ A_\upsilon := \begin{bmatrix} \frac{\partial}{\partial\theta} \\ -\Gamma_l \end{bmatrix},\ \Delta_l := \begin{bmatrix} O \\ I \end{bmatrix},$$

and Γ_0, Γ_l are operators which evaluate the values of $\upsilon(\theta,t)$ at $\theta = 0,\, l$, respectively. The domains of the above operators are defined as follows:

$$\mathcal{D}\left(E_\upsilon\right),\ \mathcal{D}\left(A_\upsilon\right),\ \mathcal{D}(\Gamma_0) := W^{2,1}([0,\, l],\, \mathbb{R}^{\dim u}).$$

The following transformation corresponds to that in Eq. (2.3):

$$\begin{bmatrix} I & U_x \\ O & I \end{bmatrix} \begin{bmatrix} A - sI & B\Gamma_0 \\ O & A_\upsilon - sE_\upsilon \end{bmatrix} \begin{bmatrix} I & -U_x E_\upsilon \\ O & I \end{bmatrix} = \begin{bmatrix} A - sI & B\Gamma_0 - (AU_x E_\upsilon - U_x A_\upsilon) \\ O & A_\upsilon - sE_\upsilon \end{bmatrix}, \quad (2.5)$$

where $U_x : L^2\left([0,l],\, \mathbb{R}^{\dim u}\right) \times \mathbb{R}^{\dim u} \to \mathbb{R}^{\dim x}$ *is an operator, which should satisfy the Sylvester equation*

$$AU_x E_\upsilon - U_x A_\upsilon = B\Gamma_0$$

to make the $(1,2)$ *block of Eq. (2.5) zero. The operator* U_x *can be found explicitly via Krein's formula and employed to introduce the new state variable* $x^{\mathrm{R}}(t)$*:*

$$\begin{bmatrix} x(t) \\ \upsilon(t) \end{bmatrix} = \begin{bmatrix} I & -U_x E_\upsilon \\ O & I \end{bmatrix} \begin{bmatrix} x^{\mathrm{R}}(t) \\ \upsilon(t) \end{bmatrix}.$$

Stabilizing the dynamics of $x^{\mathrm{R}}(t)$ *by the same reasoning as in the discrete-time setting, the continuous-time Smith predictor can be constructed. Furthermore, it is internally equivalent to the observer-predictor-based controller.* □

2.4 Truncation operator

In this section, we introduce an alternative discrete-time truncation operator to simplify the derivation of the discrete-time H^2 optimal solution. In contrast to the previous research [47], the proposed truncation operator satisfies a stricter orthogonality condition.

We denote the orthogonal complement of $z^{-l}H^2$ in H^2 by $\mathcal{H}(z^{-l})$. This space is characterized by

$$\mathcal{H}(z^{-l}) = \left\{ f \in H^2 \;\middle|\; z^l f \in H^{2\perp} \right\}.$$

The elements of $\mathcal{H}(z^{-l})$ are matrix polynomials in z^{-1} of $(l-1)$-th order. The alternative discrete-time truncation operator is defined in the following definition.

Definition 1. *For a given* $G(z) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ *with the nonsingular state matrix* A*, an alternative truncation operator* $\tau_{z^{-l}}\left[G\right](z)$ *applied to* $G(z)$ *is defined as follows:*

$$\tau_{z^{-l}}\left[G\right](z) := G(z) - z^{-l} G_{z^{-l}}(z) \in \mathcal{H}(z^{-l}),\;\; G_{z^{-l}}(z) := \left[\begin{array}{c|c} A & A^l B \\ \hline C & CA^{l-1}B \end{array}\right].$$

□

The term "truncation" comes from the fact that $\tau_{z^{-l}}\left[G\right](z)$ represents the first l impulse responses of $G(z)$ (Fig. 2.4):

$$\tau_{z^{-l}}\left[G\right](z) = D + \frac{CB}{z} + \cdots + \frac{CA^{l-2}B}{z^{l-1}}. \quad (2.6)$$

In the previous research [47], the discrete-time truncation operator is defined for a general inner function $m(z)$. When the inner function is restricted to z^{-l}, it is defined as in the following definition.

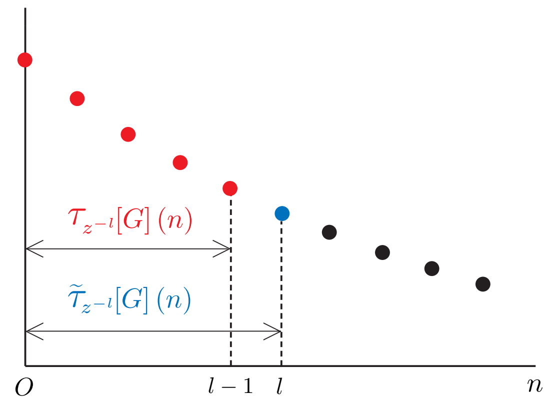

Figure 2.4: Impulse responses of $\tau_{z^{-l}}[G]$ and $\widetilde{\tau}_{z^{-l}}[G]$.

Definition 2. *For a given* $G(z) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ *with the nonsingular state matrix* A*, the previous truncation operator* $\widetilde{\tau}_{z^{-l}}[G](z)$ *applied to* $G(z)$ *is defined as follows:*

$$\widetilde{\tau}_{z^{-l}}[G](z) := G(z) - z^{-l}\widetilde{G}_{z^{-l}}(z) \in \mathcal{H}(z^{-(l+1)}),\ \widetilde{G}_{z^{-l}}(z) := \left[\begin{array}{c|c} A & A^l B \\ \hline C & O \end{array}\right].$$

□

It can be verified that

$$\widetilde{\tau}_{z^{-l}}[G](z) = D + \frac{CB}{z} + \cdots + \frac{CA^{l-2}B}{z^{l-1}} + \frac{CA^{l-1}B}{z^l}, \tag{2.7}$$

and therefore $\tau_{z^{-l}}[G](z)$ represents the first $l+1$ impulse responses of $G(z)$. Moreover, the relationship between $\widetilde{\tau}_{z^{-l}}[G](z)$ and $\tau_{z^{-l}}[G](z)$ is given by

$$\widetilde{\tau}_{z^{-l}}[G](z) = \tau_{z^{-(l+1)}}[G](z) \in \mathcal{H}(z^{-(l+1)}).$$

2.5 Solution via closed-loop reduction

2.5.1 Full information problem

The state variable $x(n)$ of P follows the difference equation

$$x(n+1) = Ax(n) + B_1 w(n) + B_2 u(n-l).$$

Referring to the state transformation used in Section 2.3, we introduce new state variables $x^{\mathrm{R}}(n)$ and $\epsilon(n)$, which follow Eq. (2.8) and Eq. (2.9), respectively:

$$x^{\mathrm{R}}(n+1) = Ax^{\mathrm{R}}(n) + B_1 w(n) + A^{-l}B_2 u(n), \tag{2.8}$$

$$\epsilon(n+1) = A\epsilon(n) + \left(z^{-l}I - A^{-l}\right) B_2 u(n). \tag{2.9}$$

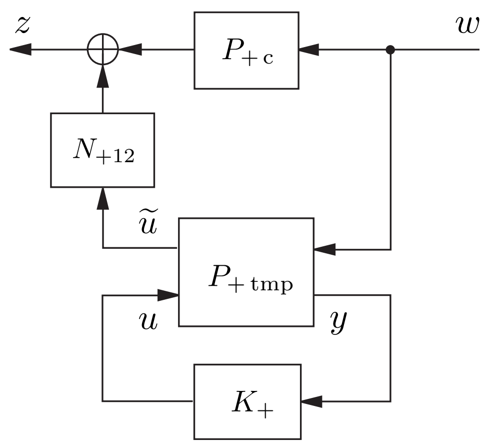

Figure 2.5: Decomposition of P_+ by state feedback.

Note that $x(n)$ is decomposed into the sum of $x^{\mathrm{R}}(n)$ and $\epsilon(n)$:

$$x(n) = x^{\mathrm{R}}(n) + \epsilon(n).$$

The above $x^{\mathrm{R}}(n)$ and $\epsilon(n)$ correspond to $x^{\mathrm{R}}(n)$ and $-Vv(n)$ in Section 2.3, respectively.

Since the transfer function from $B_2u(z)$ to $\epsilon(z)$:

$$\Pi(z) := (zI - A)^{-1}\left(z^{-l}I - A^{-l}\right)$$

is strictly causal and stable, we try to stabilize the dynamics of $x^{\mathrm{R}}(n)$ by state feedback. Let us make the change of the control input

$$\widetilde{u}(n) := -F^{\mathrm{R}}x^{\mathrm{R}}(n) - F_{21}^{\mathrm{R}}w(n) + u(n),\ \ F^{\mathrm{R}} := FA^l, \tag{2.10}$$

where F and F_{21}^{R} are to be determined in Lemma 1. We represent the regulated output $z(z)$ by $w(z)$ and $\widetilde{u}(z)$ (Fig. 2.5):

$$z(z) = P_{+\mathrm{c}}(z)w(z) + N_{+12}(z)\widetilde{u}(z). \tag{2.11}$$

Lemma 1. *The transfer functions $P_{+\mathrm{c}}$ and N_{+12} defined by*

$$P_{+\mathrm{c}}(z) = \tau_{z^{-l}}\left[P_{11}\right](z) + z^{-l}\left[\begin{array}{c|c} A + B_2F & A^lB_1 + B_2F_{21}^{\mathrm{R}} \\ \hline C_1 + D_{12}F & D_{11}^{\mathrm{R}} + D_{12}F_{21}^{\mathrm{R}} \end{array}\right],$$

$$N_{+12}(z) = z^{-l}\left[\begin{array}{c|c} A + B_2F & B_2 \\ \hline C_1 + D_{12}F & D_{12} \end{array}\right],$$

$$D_{11}^{\mathrm{R}} := C_1A^{l-1}B_1, \tag{2.12}$$

$$F_{21}^{\mathrm{R}} := -R_{\mathrm{c}}^{-1}\left(D_{12}^*D_{11}^{\mathrm{R}} + B_2^*XA^lB_1\right), \tag{2.13}$$

$$F := -R_{\mathrm{c}}^{-1}\left(S^* + B_2^*XA\right)$$

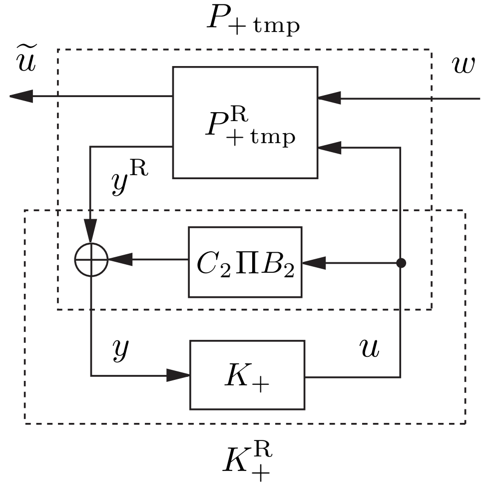

Figure 2.6: Output transformation.

satisfy the orthogonality condition

$$N^{\sim}_{+12}(z)P_{+\mathrm{c}}(z) \in H^{2\perp},$$

and the isometry condition

$$N^{\sim}_{+12}(z)N_{+12}(z) = R_{\mathrm{c}}.$$

□

Proof. Subsection 2.8.1. □

From Lemma 1, under the condition $T_{\widetilde{u}\,w} \in H^2$, the H^2 norm of $T_{z\,w}$ can be decomposed as follows:

$$\|T_{z\,w}\|_2^2 = \|P_{+\mathrm{c}}\|_2^2 + \left\|R_{\mathrm{c}}^{1/2}T_{\widetilde{u}\,w}\right\|_2^2. \tag{2.14}$$

Moreover, the following state feedback law is H^2-optimal:

$$u(n) = F^{\mathrm{R}}x^{\mathrm{R}}(n) + F^{\mathrm{R}}_{21}w(n) = F^{\mathrm{R}}x(n) - F^{\mathrm{R}}\Pi(z)B_2u(n) + F^{\mathrm{R}}_{21}w(n).$$

2.5.2 Output estimation problem

To minimize the scaled H^2 norm of $T_{\widetilde{u}\,w}$ in Eq. (2.14), we consider the output estimation problem for the auxiliary generalized plant $P_{+\,\mathrm{tmp}}$ (Fig. 2.5). The realization of $P_{+\,\mathrm{tmp}}$ is given as follows:

$$\begin{aligned}
\epsilon(n+1) &= A\epsilon(n) + \left(z^{-l}I - A^{-l}\right)B_2u(n),\\
x^{\mathrm{R}}(n+1) &= Ax^{\mathrm{R}}(n) + B_1w(n) + A^{-l}B_2u(n),\\
\widetilde{u}(n) &= -F^{\mathrm{R}}x^{\mathrm{R}}(n) - F^{\mathrm{R}}_{21}w(n) + u(n),\\
y(n) &= C_2x^{\mathrm{R}}(n) + C_2\epsilon(n) + D_{21}w(n).
\end{aligned}$$

Since $\epsilon(n)$ is determined by $u(n)$ strictly causally, we consider the generalized plant $P^{\mathrm{R}}_{+\mathrm{tmp}}$ of which the measured output is given by

$$y^{\mathrm{R}}(n) := y(n) - C_2\epsilon(n) = C_2 x^{\mathrm{R}}(n) + D_{12}w(n)$$

instead of $y(n)$ (Fig. 2.6). The realization of $P^{\mathrm{R}}_{+\mathrm{tmp}}$ is given as follows:

$$\begin{aligned} x^{\mathrm{R}}(n+1) &= Ax^{\mathrm{R}}(n) + B_1 w(n) + A^{-l}B_2 u(n), \\ \widetilde{u}(n) &= -F^{\mathrm{R}}x^{\mathrm{R}}(n) - F^{\mathrm{R}}_{21}w(n) + u(n), \\ y^{\mathrm{R}}(n) &= C_2 x^{\mathrm{R}}(n) + D_{21}w(n). \end{aligned}$$

By applying the standard solution method to the H^2 output estimation problem for $P^{\mathrm{R}}_{+\mathrm{tmp}}$, we obtain the following theorem.

Theorem 1. *Under the assumptions (A1)-(A2) and (H), the stabilizing controller K_+ is parameterized as shown in Fig. 2.7. It consists of the measurement compensation part:*

$$y^{\mathrm{R}}(z) = y(z) - C_2\Pi(z)B_2u(z),$$

and the observer-based controller K^{R}_+ estimating the state variable $x^{\mathrm{R}}(n)$:

$$K^{\mathrm{R}}_+ = \mathcal{F}_l(J^{\mathrm{R}}_+,\, L^{\mathrm{R}}_{22} + Q_+(z)),\;\; L^{\mathrm{R}}_{22} := \left(F^{\mathrm{R}}_{21}D^*_{21} + F^{\mathrm{R}}YC^*_2\right)\acute{R}_{\mathrm{c}}^{-1},$$

where $\forall\, Q_+ \in H^2$ is the Youla parameter. The realization of J^{R}_+ is given as follows:

$$\begin{aligned} x^{\mathrm{R}}_{\mathrm{c}}(n+1) &= \left(A + A^{-l}B_2F^{\mathrm{R}} + LC_2\right)x^{\mathrm{R}}_{\mathrm{c}}(n) - Ly^{\mathrm{R}}(n) + A^{-l}B_2\mu(n), \\ u(n) &= F^{\mathrm{R}}x^{\mathrm{R}}_{\mathrm{c}}(n) + \mu(n), \\ \nu(n) &= -C_2x^{\mathrm{R}}_{\mathrm{c}}(n) + y^{\mathrm{R}}(n), \end{aligned}$$

where $L := -\left(\acute{S}^* + AYC^*_2\right)\acute{R}_{\mathrm{c}}^{-1}$.

Furthermore, the H^2-optimal controller is obtained when $\widetilde{Q}(z) = O$ and the corresponding optimal cost E_+ is given by

$$E^2_+ = \|P_{+\mathrm{c}}\|^2_2 + \left\| R^{1/2}_{\mathrm{c}} \left[\begin{array}{c|c} A + LC_2 & B_1 + LD_{21} \\ \hline -F^{\mathrm{R}} + L^{\mathrm{R}}_{22}C_2 & -F^{\mathrm{R}}_{21} + L^{\mathrm{R}}_{22}D_{21} \end{array}\right] \right\|^2_2, \tag{2.15}$$

where

$$\|P_{+\mathrm{c}}\|^2_2 = \|\tau_{z^{-l}}\left[P_{11}\right]\|^2_2 + \left\| \left[\begin{array}{c|c} A + B_2F & A^{-l}B_1 + B_2F^{\mathrm{R}}_{21} \\ \hline C_1 + D_{12}F & D^{\mathrm{R}}_{11} + D_{12}F^{\mathrm{R}}_{21} \end{array}\right] \right\|^2_2. \tag{2.16}$$

□

The controller given in Theorem 1 is in the Smith form shown in Fig. 2.7. The first and second terms on the right-hand side of Eq. (2.15) represent control and estimation costs, respectively. Furthermore, $\|\tau_{z^{-l}}\left[P_{11}\right]\|^2_2$ in Eq. (2.16) represents the control cost which cannot be affected by any control.

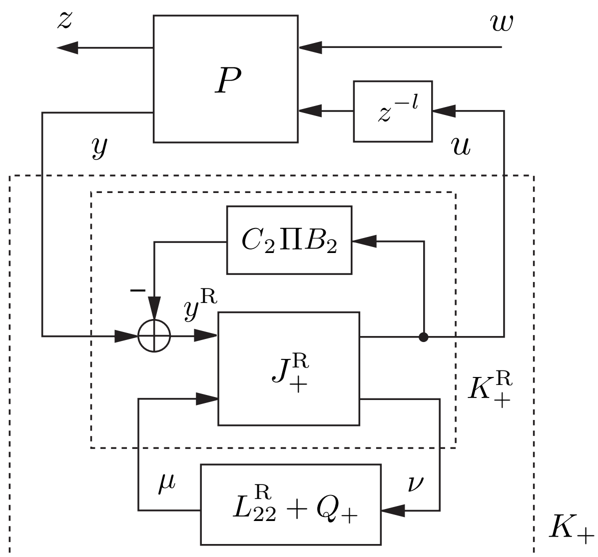

Figure 2.7: Stabilizing controller in Smith form.

Remark 2. *In the derivation of the optimal controller, the delay element* $C_2 \Pi B_2$ *of* $P_{+\,\mathrm{tmp}}$ *is canceled out of the closed-loop interconnection by the internal model technique [46]. If there exists a delay mismatch between the delay element and its internal model, the obtained controller fails in stabilizing the closed-loop system. It inherits the intrinsic weakness to the delay mismatch like other classes of Smith predictors [16].*

One possible remedy for the uncertain or time-varying delay is to apply the robust H^∞ *design method [1], [46] where the delay variation is regarded as the multiplicative perturbation and its gain is covered by an appropriate weighting function. However, the design method neglects the phase margin information, and cannot reflect the possible trajectories of the time-varying delay on the resulting controller. Utilization of both the gain and phase margin information is expected to yield better robust stability conditions [12], [16], [33]. Practical design methods are also proposed to incorporate delay-scheduling rules into the convex LPV controllers for bilateral teleoperation [10] and canal flow control [5].* □

2.6 Increase of the optimal cost

Theorem 1 in the previous section derived the causally implementable form of the optimal controller, which was not obtained in the previous research [47]. The previous approach, however, revealed how the Youla parameter for the delay-free problem should be constrained by the existence of the input delay. This section shows that Theorem 1 leads to the same result as in [47]

The following theorem is cited from [47]. It is stated using the alternative truncation operator introduced in Section 2.4. We provide another proof based on Theorem 1.

Theorem 2 (Nishio-Kashima [47])**.** $z^{-l}K_+(z)$ *has the structure of the stabilizing controller* *of the standard H^2 problem (Fig. 2.8):*

$$z^{-l}K_+(z) = \mathcal{F}_l(J,\, L_{22} + Q(z)), \tag{2.17}$$

where

$$J := \left[\begin{array}{c|cc} A + B_2F + LC_2 & -L & B_2 \\ \hline F & O & I \\ -C_2 & I & O \end{array}\right],$$

$$L_{22} := \left(F_{21}D_{21}^* + FYC_2^*\right)\acute{R}_{\mathrm{c}}^{-1},\ F_{21} := -R_{\mathrm{c}}^{-1}\left(D_{12}^*D_{11} + B_2^*XB_1\right),$$

and the Youla parameter $Q(z)$ is constrained by

$$Q(z) = \tau_{z^{-l}}\left[\Theta\right](z) + z^{-l}\widetilde{Q}(z), \tag{2.18}$$

$$\forall \widetilde{Q}(z) \in H^\infty\left(\subset H^2\right),\ \Theta(s) := \left[\begin{array}{c|c} A & L \\ \hline F & -L_{22} \end{array}\right].$$

Furthermore, the optimal cost E_+ is given by

$$E_+^2 = E^2 + \left\|R_{\mathrm{c}}^{1/2}\tau_{z^{-l}}\left[\Theta\right]\acute{R}_{\mathrm{c}}^{1/2}\right\|_2^2, \tag{2.19}$$

where E is the optimal cost of the standard delay-free H^2 control problem. □

Proof. Subsection 2.8.2. □

Eq. (2.19) shows that the increase of the optimal cost incurred by z^{-l} is given by the scaled H^2 norm of the alternative truncation operator. In [47], the Youla parameter $Q(z)$ in Eq. (2.17) is constrained by

$$Q(z) = \widetilde{\tau}_{z^{-l}}\left[\Theta\right](z) + z^{-l}\widetilde{Q}_+(z),\ \forall \widetilde{Q}_+(z) \in H^\infty\left(\subset H^2\right). \tag{2.20}$$

Since the two terms on the right-hand side of Eq. (2.20) are not orthogonal to each other, the optimal cost cannot be attained with $\widetilde{Q}_+(z) = O$.

Remark 3. *The state decomposition approach proposed in this chapter can be extended to the class of multiple input delay systems where the differently delayed channels are decoupled each other:*

$$\begin{aligned} x(n+1) &= Ax(n) + B_1w(n) + \sum_{i=1}^{N} B_{2/i}u_i(n-l_i), \\ z(n) &= C_1x(n) + \sum_{i=1}^{N} D_{12/i}u_i(n-l_i), \\ y(n) &= C_2x(n) + D_{21}w(n). \end{aligned}$$

In the multiple input-delay case, the state decomposition should be introduced more carefully to determine the state feedback gains for each of the control input channels. One of promising guidelines is to consider the spectral factorization of $\Phi_{+\,22} := P_{+\,12}^{\sim}P_{+\,12}$ where $P_{+\,12}$ is

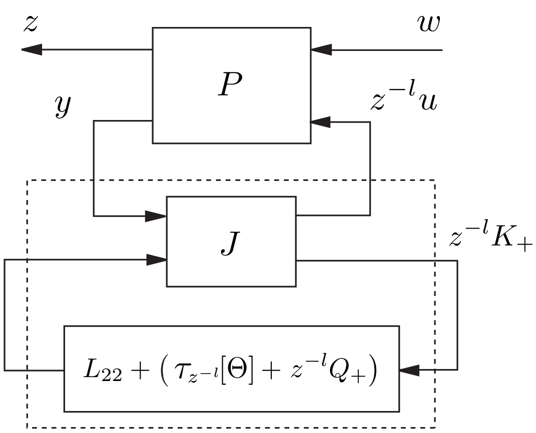

Figure 2.8: Constraint on the Youla parameter.

the transfer function from the control input $u(z) := \begin{bmatrix} u_1(z)^{\mathrm{T}} & u_2(z)^{\mathrm{T}} & \dots & u_N(z)^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$ *to the regulated output* $z(z)$*:*

$$P_{+12} := \left[\begin{array}{c|c} A & \begin{bmatrix} B_{2/1} & B_{2/2} & \dots & B_{2/N} \end{bmatrix} \\ \hline C_1 & \begin{bmatrix} D_{12/1} & D_{12/2} & \dots & D_{12/N} \end{bmatrix} \end{array}\right] \begin{bmatrix} z^{-l_1}I & & & O \\ & z^{-l_2}I & & \\ & & \ddots & \\ O & & & z^{-l_N}I \end{bmatrix}.$$

The appropriate state decomposition can be derived by transforming the state-variables of the spectral density Φ_{+22} *as in the Chapters 4 and 5.* □

Remark 4. *The discrete-time state matrix* A *is assumed to be invertible in Chapters 2 and 3 to implement the* H^2 *and* H^∞ *controllers in the Smith predictor form. However, the assumption is restrictive in purely discrete-time control problems, and can be avoided by realizing the optimal controller in the predictor form [43], [34]. In [43], the optimal LQ state feedback law is constructed from the Riccati equation for the delay-free system by regarding the following* l*-step prediction as the new state variable:*

$$x(n+l) = A^l x(n) + A^{l-1}B_2 u(n) + A^{l-2}B_2 u(n-1) + \ldots + B_2 u(n-l).$$

If the exogenous disturbance is ignored, the state variable $x^{\mathrm{R}}(n)$ *introduced in this chapter and* l*-step prediction is related by* $A^l x^{\mathrm{R}}(n) = x(n+l)$*. In [34], the results in [43] are extended to the LQG control problem. The multiple input and output delays are dealt with provided that each of input and output channels is delayed by the respective one time length.*

The optimal output feedback controllers in [34] and this chapter are obtained by the different routes and implemented in the different forms. In [34], the first step is to estimate the future state variable under the noise. The second step is to replace the noise-free prediction for the optimal state feedback (initially found in [43]) with the optimal estimation based on the separation theorem. The procedure provides the observer-predictor-based controller as shown in Fig. 2.3. The LQ control laws in [43], [34] indicate room for improvement in the input-delayed H^∞ *controller design method.* □

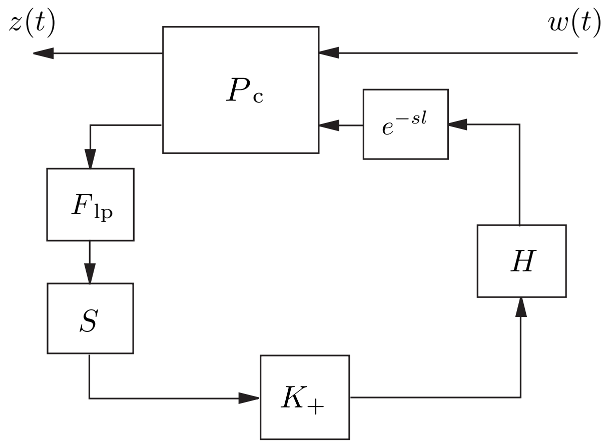

Figure 2.9: Setup of sampled-data control.

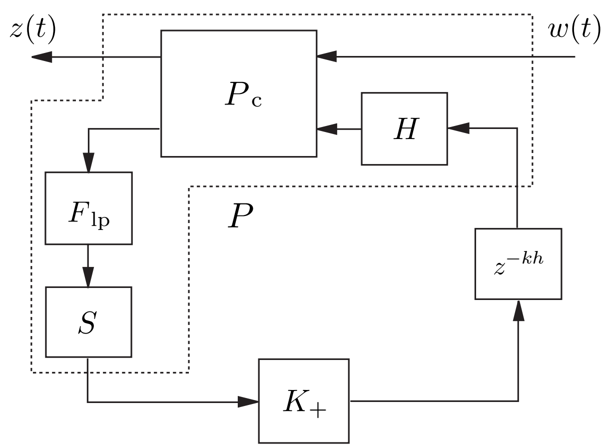

Figure 2.10: Replacement of e^{-sl} with z^{-kh}.

2.7 Example

This section illustrates the input-delay effect on the sampled-data H^2 control performance. In the sampled-data setting, we design the discrete-time controller K_+ for the following continuous-time input-delayed generalized plant $P_{+\mathrm{c}}$ (Fig. 2.9):

$$P_{+\mathrm{c}} := P_{\mathrm{c}} \begin{bmatrix} I & O \\ O & e^{-sl} I \end{bmatrix}, \; P_{\mathrm{c}} = \left[\begin{array}{c|cc} 0.09 & \begin{bmatrix} 0.05 & 0.03 \end{bmatrix} & 4.8 \\ \hline \begin{bmatrix} 0.6 \\ 0.16 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0.95 \\ 0.6 \end{bmatrix} \\ 0.8 & \begin{bmatrix} 0.03 & 0.7 \end{bmatrix} & 0 \end{array} \right].$$

The control input to $P_{+\mathrm{c}}$ is interpolated with the zero-order hold H. The measured output from $P_{+\mathrm{c}}$ is filtered with the low-pass filter $F_{\mathrm{lp}}(s) = 1/(1+s)$ and then discretized with the

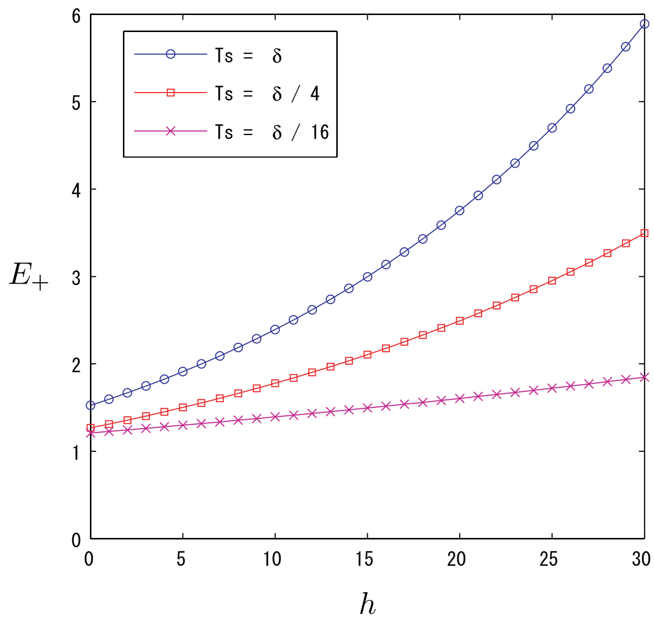

Figure 2.11: Input-delayed H^2 optimal cost.

impulse sampler S.

We assume that the time-delay length l and sampling period T_{s} are discretized by the following equations:

$$l = \delta h \ \ (h = 0,\, 1,\, 2,\, \ldots)\,,\ T_{\mathrm{s}} = \frac{\delta}{k} \ \ (k = 1,\, 2,\, 3,\, \ldots)\,,$$

where $\delta = 0.5$ is a scaling parameter. This assumption yields the following commuting relationship:

$$e^{-sl} H = H z^{-kh}. \tag{2.21}$$

Note that the increase of the time-delay length or decrease of the sampling period results in the larger discrete-time input-delay length kh.

By Eq. (2.21), we obtain the following discrete-time input-delay system (Fig. 2.10):

$$P_+ := \begin{bmatrix} I & O \\ O & SF_{\mathrm{lp}} \end{bmatrix} P_{\mathrm{c}} \begin{bmatrix} I & O \\ O & e^{-sl} H \end{bmatrix} = P \begin{bmatrix} I & O \\ O & z^{-kh} I \end{bmatrix}, \tag{2.22}$$

where

$$P := \begin{bmatrix} I & O \\ O & SF_{\mathrm{lp}} \end{bmatrix} P_{\mathrm{c}} \begin{bmatrix} I & O \\ O & H \end{bmatrix}.$$

The optimal controller K_+ can be determined as that for P_+ by Theorem 1.

The optimal H^2 control cost determined by Theorem 2 is depicted in Fig. 2.11. From Fig. 2.11, we observe that the optimal cost is monotonically increasing with respect to the time-delay length, while it is monotonically decreasing with respect to the sampling period.

Remark 5. *The time-delay length l is assumed to be the multiple of the sampling T_{s} in the example. Let us mention the general case where*

$$l = mT_{\mathrm{s}} + p \ \ (m : \text{nonnegative integer},\ 0 \le p < T_{\mathrm{s}}),$$

and the continuous-time input $u_{\mathrm{c}}(t)$ is the zero-order interpolation of the discrete-time input $u(n)$: $u_{\mathrm{c}}(t) = Hu(t)$. In this case, the continuous-time evolution equation

$$\dot{x}_{\mathrm{c}}(t) = A_{\mathrm{c}} x_{\mathrm{c}}(t) + B_{\mathrm{c}\,1} w_{\mathrm{c}}(t) + B_{\mathrm{c}\,2} u_{\mathrm{c}}(t-l),$$

$$x_{\mathrm{c}}(0) = 0,\ \ w_{\mathrm{c}}(t) = \delta(t) w \ \ (w : \text{constant vector})$$

is equivalently discretized into the following discrete-time one:

$$x(n+1) = Ax(n) + B_1 w(n) + B_{2\,\mathrm{m}} u(n-m) + B_{2\,\mathrm{r}} u(n-m-1),$$

$$x(n) = x_{\mathrm{c}}(nT_{\mathrm{s}}),\ \ w(n) = \delta(n+1) w,$$

$$A := e^{A_{\mathrm{c}} T_{\mathrm{s}}},\ \ B_{2\,\mathrm{m}} := \int_0^{T_{\mathrm{s}}-p} e^{A\theta} d\theta B_2,\ \ B_{2\,\mathrm{r}} := \int_{T_{\mathrm{s}}-p}^{T_{\mathrm{s}}} e^{A\theta} d\theta B_2.$$

Note that the two lengths of delay enter the same control input channel simultaneously. Nevertheless, in [42], the LQ state feedback law for the above input-delay system is also constructed from the delay-free Riccati equation. Unfortunately, the derivation is based on direct algebraic manipulation and the underlying theme is unclear. Therefore, the extension of the result in [42] to the output feedback case is not straightforward. □

2.8 Proofs

2.8.1 Proof of Lemma 1

By direct calculation, $P_{+\mathrm{c}}$ and $N_{+12}(z)$ in Eq. (2.11) are expressed as follows:

$$P_{+\mathrm{c}}(z) = z^{-l} \left[\begin{array}{c|c} A + A^{-l} B_2 F^{\mathrm{R}} & B_1 + A^{-l} B_2 F_{21}^{\mathrm{R}} \\ \hline C_1 A^l + D_{12} F^{\mathrm{R}} & D_{12} F_{21}^{\mathrm{R}} \end{array} \right] + D_{11} - C_1 \Pi(z) A^l B_1, \tag{2.23}$$

$$N_{+12}(z) = z^{-l} \left[\begin{array}{c|c} A + A^{-l} B_2 F^{\mathrm{R}} & A^{-l} B_2 \\ \hline C_1 A^l + D_{12} F^{\mathrm{R}} & D_{12} \end{array} \right].$$

Noting that $A + A^{-l} B_2 F^{\mathrm{R}} = A^{-l} (A + B_2 F) A^l$, $N_{+12}(z)$ is further expressed as

$$N_{+12}(z) = z^{-l} N_{12}(z),\ \ N_{12}(z) = \left[\begin{array}{c|c} A + B_2 F & B_2 \\ \hline C_1 + D_{12} F & D_{12} \end{array} \right].$$

Since $N_{12}(z)$ is the inner function which appears in the standard delay-free H^2 control problem, $N_{+12}(z)$ satisfies the isometry condition.

Next, let us substitute the right-hand side of the following equation into Eq. (2.23) to determine the matrix D_{11}^{R}:

$$D_{11} = z^{-l} D_{11}^{\mathrm{R}} + \left(D_{11} - z^{-l} D_{11}^{\mathrm{R}} \right).$$

The substitution yields

$$P_{+c}(z) = \left\{ \left[\begin{array}{c|c} A & B_1 \\ \hline C_1 & D_{11} \end{array} \right] - z^{-l} \left[\begin{array}{c|c} A & A^l B_1 \\ \hline C_1 & D_{11}^{\mathrm{R}} \end{array} \right] \right\} + z^{-l} \left[\begin{array}{c|c} A + B_2 F & A^l B_1 + B_2 F_{21}^{\mathrm{R}} \\ \hline C_1 + D_{12} F & D_{11}^{\mathrm{R}} + D_{12} F_{21}^{\mathrm{R}} \end{array} \right]. \tag{2.24}$$

By choosing D_{11}^{R} as defined in Eq. (2.12), the first term on the right-hand side of Eq. (2.24) becomes $\tau_{z^{-l}}[P_{11}](z)$:

$$P_{+c}(z) = \tau_{z^{-l}}[P_{11}](z) + z^{-l} \left[\begin{array}{c|c} A + B_2 F & A^l B_1 + B_2 F_{21}^{\mathrm{R}} \\ \hline C_1 + D_{12} F & D_{11}^{\mathrm{R}} + D_{12} F_{21}^{\mathrm{R}} \end{array} \right].$$

We further determine F_{21}^{R} to satisfy the orthogonality condition. We calculate $N_{+12}^{\sim}(z)P_{+c}(z)$ as follows:

$$\begin{aligned} N_{+12}^{\sim}(z)P_{+c}(z) &= \left[\begin{array}{c|c} A + B_2 F & B_2 \\ \hline C_1 + D_{12} F & D_{12} \end{array} \right]^{\sim} \left(z^l \tau_{z^{-l}}[P_{11}](z) + \left[\begin{array}{c|c} A + B_2 F & A^l B_1 + B_2 F_{21}^{\mathrm{R}} \\ \hline C_1 + D_{12} F & D_{11}^{\mathrm{R}} + D_{12} F_{21}^{\mathrm{R}} \end{array} \right] \right) \\ &= \left[\begin{array}{c|c} A + B_2 F & B_2 \\ \hline C_1 + D_{12} F & D_{12} \end{array} \right]^{\sim} z^l \tau_{z^{-l}}[P_{11}](z) \\ &\quad + \left[\begin{array}{c|c} A + B_2 F & B_2 \\ \hline D_{11}^{\mathrm{R}*}(C_1 + D_{12} F) + B_1^* A^{l*} X (A + B_2 F) & O \end{array} \right]^{\sim} \\ &\quad + \left\{ \left(D_{12}^* D_{11}^{\mathrm{R}} + B_2^* X A^l B_1 \right) + R_{\mathrm{c}} F_{21}^{\mathrm{R}} \right\}. \end{aligned} \tag{2.25}$$

Since the first and second terms on the right-hand side of Eq. (2.25) belong to $H^{2\perp}$, we choose F_{21}^{R} as defined in Eq. (2.13) so that the constant term in Eq. (2.25) becomes zero.

2.8.2 Proof of Theorem 2

Using the manipulation employed in [38], [40], we can show that

$$\begin{aligned} z^{-l} K_+(z) &= \mathcal{F}_l\left(\begin{bmatrix} O & z^{-l} I \\ I & -C_2 \Pi(z) B_2 \end{bmatrix}, \mathcal{F}_l(J_+^{\mathrm{R}}, L_{22}^{\mathrm{R}} + \widetilde{Q}(z)) \right) \\ &= \mathcal{F}_l(J, -F\Pi(z) A^l L + z^{-l} \left(L_{22}^{\mathrm{R}} + Q_+(z) \right)). \end{aligned} \tag{2.26}$$

Referring to Eq. (2.26), we define the Youla parameter $Q(z)$ by

$$\begin{aligned} Q(z) :=& -L_{22} - F\Pi(z) A^l L + m(z) \left(L_{22}^{\mathrm{R}} + \widetilde{Q}(z) \right) \\ =& \left\{ \left[\begin{array}{c|c} A & L \\ \hline F & -L_{22} \end{array} \right] - z^{-l} \left[\begin{array}{c|c} A & L \\ \hline F A^l & -L_{22}^{\mathrm{R}} \end{array} \right] \right\} + z^{-l} Q_+(z). \end{aligned} \tag{2.27}$$

At this point, the equality $-L_{22}^{\mathrm{R}} = F A^{l-1} L$ can be verified by direct calculation. Therefore, we have Eq. (2.18). From Eqs. (2.18), (2.26) and (2.27), we obtain Eq. (2.17).

Since the first and second terms of the right-hand side of Eq. (2.18) belong to $\mathcal{H}(z^{-l})$ and $z^{-l}H^2$, respectively, the following identity holds:

$$\left\| R_{\mathrm{c}}^{1/2} Q \acute{R}_{\mathrm{c}}^{1/2} \right\|_2^2 = \left\| R_{\mathrm{c}}^{1/2} \tau_{z^{-l}}[\Theta] \acute{R}_{\mathrm{c}}^{1/2} \right\|_2^2 + \left\| R_{\mathrm{c}}^{1/2} \widetilde{Q} \acute{R}_{\mathrm{c}}^{1/2} \right\|_2^2. \tag{2.28}$$

Furthermore, the result of the standard H^2 control problem yields the following identity:

$$\|T_{zw}\|_2^2 = E^2 + \left\| R_{\mathrm{c}}^{1/2} Q \acute{R}_{\mathrm{c}}^{1/2} \right\|_2^2 . \tag{2.29}$$

From Eqs. (2.28) and (2.29), the optimal cost $E_+^2 = \min_{\widetilde{Q} \in H^\infty} \|T_{zw}\|_2^2$ is attained with $\widetilde{Q}(z) = O$ and given by Eq. (2.19).

2.9 Conclusion

This chapter considered the H^2 control problem for the discrete-time input-delay system. We developed the closed-loop reduction solution method, and revealed the Smith structure of the optimal controller. The main point of our approach is to decompose the dynamics of the state variable of the controlled plant, and to stabilize that of the delay-independent order. The decomposition is implied by the state transformation of the single input-delay system.

The overall stabilizing controller was parameterized with the internal model $C_2 \Pi B_2$ and observer K_+^{R} estimating the newly introduced state variable. Although the orders of the Riccati equations required for the implementation are reduced, the resulting controller remains of the same order as the input-delayed controlled plant. Since the observer K_+^{R} is constructed for the delay-free generalized plant $P_{\mathrm{tmp}}^{\mathrm{R}}$, the order of it can be reduced further, for example, by LMI techniques [51] at the expense of the exact optimality. In [41], the technique for approximating the distributed control law for the continuous-time Smith predictor is developed based on the small gain theorem. The approximation technique can be extended to reduce the order of the internal model $C_2 \Pi B_2$.

We can also derive the optimal controller and cost for the corresponding continuous-time H^2 control problem in a similar way. Contrary to the previous approach [23] and [40], our approach can avoid the preliminary procedure of finding the causality and stability constraints on the Youla parameter for the delay-free problem.

Chapter 3 Discrete-time H^∞ input-delayed control

3.1 Introduction

This chapter pursues a parameterization of the H^∞ suboptimal controllers for the discrete-time input-delay system by decomposing the internal state variables in a similar way to Chapter 2. The output feedback problem is solved through the full information and output estimation problems. A discrete-time counterpart of the J-spectral factorization technique in [36] is developed to deal with the full information problem. Focusing on the relationship between the state variables of the input-delayed and delay-free J-spectral densities, the measured output is modified in order to reduce the output estimation problem to the delay-free one. The J-spectral factorization approach yields the parameterization of the discrete-time H^∞-suboptimal controllers in the Smith form. The J-spectral factorizability condition composes a part of the solvability conditions for the output feedback problem. It requires verifying the regularity of the subblock of the symplectic matrix on the unbounded interval.

In the full information problem, we also consider the min-max optimization approach to construct the H^∞ state feedback law. The min-max optimization approach yields the H^∞ disturbance attenuation condition which requires tracing the input-delay parameter on the finitely many points, and hence is more tractable than the J-spectral factorizability condition. Furthermore, the interpretation between the J-spectral and min-max optimization approaches are provided via finite-horizon ℓ^2-gain analysis.

The min-max optimization is adopted for the H^∞ disturbance attenuation along the lines of [52]. The essential idea is to partition the cost functional into two terms according to whether they are affected by the control input. We extend it into the discrete-time setting, and construct the stabilizing solution of the standard KYP equation for the augmented system from that for the delay-free case. The LQ reduced-order construction [43] is always possible if there exists the positive semidefinite stabilizing solution of the delay-free Riccati equation. Contrary to it, we newly reveal that the H^∞ disturbance attenuation condition requires the the additional positive definiteness on the solution of the delay-free KYP equation.

This chapter is organized as follows. In Section 3.2, the problem formulation and assumptions are stated. In Section 3.3, as a preliminary, a discrete-time counterpart of the completion operator [37] employed for the continuous-time J-spectral factorization is introduced. In Section 3.4, the original output feedback problem is first simplified to the full information problem. The discrete-time J-spectral factorization technique is developed after recasting the full information problem as the one-sided model matching problem. Furthermore, it is described how the output feedback problem is solved based on the state decomposition approach. In Section 3.5, the H^∞ state feedback law is constructed in another way via the min-max optimization. Moreover, the equivalence to the J-spectral factorization approach is provided explicitly. In Section 3.6, the numerical example of calculating the achievable H^∞ performance is considered. In Section 3.7, the proofs left in the previous sections are given.

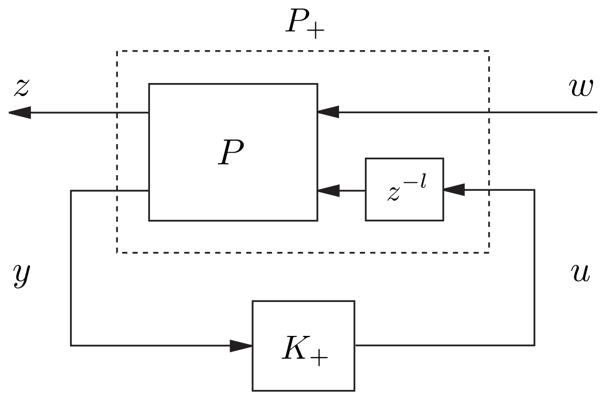

Figure 3.1: Control system with input delay.

3.2 Problem formulation

Consider the discrete-time input-delay system depicted in Fig. 3.1. The input-delayed generalized plant P_+ consists of the l-step input delay z^{-l} and delay-free generalized plant P. Our objectives are to derive a tractable solvability condition of the H^∞ control problem for P_+, and to reveal the transparent structure of the controller K_+ rendering the closed-loop L^2 gain less than a given γ:

$$\|T_{zw}\|_\infty < \gamma,$$

where T_{zw} is the transfer function from the disturbance w to the regulated output z.

We assume that the delay-free generalized plant P in Fig. 3.1 is given by

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & O & D_{12} \\ C_2 & D_{21} & O \end{array}\right],$$

and make the following assumptions (X), (Y), and (H).

(X) The standard H^∞ full information problem in the delay-free case is solvable. In other words, for $\forall\lambda \geq \gamma$, the KYP equation

$$\begin{aligned} F_\lambda^* R_{\mathrm{mm\,c}\,\lambda} F_\lambda &= Q + A^* X_\lambda A - X_\lambda, \\ -R_{\mathrm{mm\,c}\,\lambda} F_\lambda &= S^* + B^* X_\lambda A, \\ R_{\mathrm{mm\,c}\,\lambda} &= R_{\mathrm{mm}\,\lambda} + B^* X_\lambda B, \end{aligned}$$

$$B := \begin{bmatrix} B_1 & B_2 \end{bmatrix},\ S := \begin{bmatrix} O & S_2 \end{bmatrix},\ R_{\mathrm{mm}\,\lambda} := \begin{bmatrix} -\lambda^2 I & O \\ O & R_2 \end{bmatrix},$$

$$\begin{bmatrix} Q & S_2 \\ S_2^* & R_2 \end{bmatrix} := \begin{bmatrix} C_1 & D_{12} \end{bmatrix}^* \begin{bmatrix} C_1 & D_{12} \end{bmatrix}$$

has the positive semidefinite stabilizing solution X_λ such that

$$A_{\mathrm{c}\,\lambda} := A + B_1 F_{1\,\lambda} + B_2 F_{2\,\lambda}$$

is stable, and $R_{\mathrm{mm\,c}\,\lambda}$ satisfies the following definiteness conditions:

$$\begin{aligned} R_{\mathrm{mm\,c}\,22\,\lambda} &> O, \\ -\lambda^2\Lambda_{\mathrm{c}\,\lambda} := R_{\mathrm{mm\,c}\,11\,\lambda} - R_{\mathrm{mm\,c}\,12\,\lambda}R^{-1}_{\mathrm{mm\,c}\,22\,\lambda}R_{\mathrm{mm\,c}\,21\,\lambda} &< O. \end{aligned}$$

(Y) The standard H^∞ full control problem in the delay-free case is solvable. Therefore, the KYP equation

$$\begin{aligned} L_\gamma \acute{R}_{\mathrm{mm\,c}\,\gamma}L^*_\gamma &= \acute{Q} + AY_\gamma A^* - Y_\gamma, \\ -\acute{R}_{\mathrm{mm\,c}\,\gamma}L^*_\gamma &= \acute{S} + CY_\gamma A^*, \\ \acute{R}_{\mathrm{mm\,c}\,\gamma} &= \acute{R}_{\mathrm{mm}\,\gamma} + CY_\gamma C^*, \end{aligned}$$

$$C := \begin{bmatrix} C_1 \\ C_2 \end{bmatrix},\ \acute{S} := \begin{bmatrix} O \\ \acute{S}_2 \end{bmatrix},\ \acute{R}_{\mathrm{mm}\,\gamma} := \begin{bmatrix} -\gamma^2 I & O \\ O & \acute{R}_2 \end{bmatrix},$$

$$\begin{bmatrix} \acute{Q} & \acute{S}^*_2 \\ \acute{S}_2 & \acute{R}_2 \end{bmatrix} := \begin{bmatrix} B_1 \\ D_{21} \end{bmatrix}\begin{bmatrix} B_1 \\ D_{21} \end{bmatrix}^*$$

has the positive semidefinite stabilizing solution Y_γ such that the following matrix is stable:

$$\acute{A}_{\mathrm{c}\,\gamma} := A + L_{1\,\gamma}C_1 + L_{2\,\gamma}C_2.$$

(H) The matrix A is invertible, and hence the following symplectic matrix $H_{\mathrm{FH}\,\lambda}$ is well-defined:

$$H_{\mathrm{FH}\,\lambda} := H^{-1}_{\mathrm{FH}\,\delta\,\lambda}H_{\mathrm{FH}\,\sigma} = \begin{bmatrix} A - \frac{1}{\lambda^2}B_1B_1^*A^{-*}Q & -\frac{1}{\lambda^2}B_1B_1^*A^{-*} \\ A^{-*}Q & A^{-*} \end{bmatrix},$$

where $(H_{\mathrm{FH}\,\delta\,\lambda},\ H_{\mathrm{FH}\,\sigma})$ is the symplectic pair given by

$$H_{\mathrm{FH}\,\delta\,\lambda} := \begin{bmatrix} I & \frac{1}{\lambda^2}B_1B_1^* \\ O & A^* \end{bmatrix},\ H_{\mathrm{FH}\,\sigma} := \begin{bmatrix} A & O \\ Q & I \end{bmatrix}.$$

From the nth power of $H_{\mathrm{FH}\,\lambda}$, we define the following symplectic matrices for simplicity:

$$\mathcal{E}_\lambda(n) := (H_{\mathrm{FH}\,\lambda})^n,\ \mathcal{E}_{X\,\lambda}(n) := \begin{bmatrix} I & O \\ X_\lambda & I \end{bmatrix}\mathcal{E}_\lambda(n)\begin{bmatrix} I & O \\ -X_\lambda & I \end{bmatrix}. \tag{3.1}$$

Furthermore, we introduce the following definitions using $\mathcal{E}_\lambda(n)$:

$$\begin{bmatrix} B^{\mathcal{D}}_{2\,\lambda}(n) \\ S^{\mathcal{D}}_{2\,\lambda}(n) \end{bmatrix} := \mathcal{D}^{-1}_\lambda(n)\begin{bmatrix} B_2 \\ S_2 \end{bmatrix},\ \mathcal{D}_\lambda(n) := H_{\mathrm{FH}\,\delta\,\lambda}\mathcal{E}_\lambda(n)H^{-1}_{\mathrm{FH}\,\delta\,\lambda}.$$

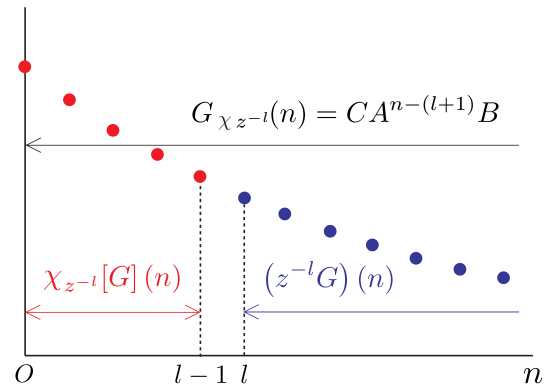

Figure 3.2: Impulse response of $\chi_{z^{-l}}[G](z)$.

3.3 Completion operator

The following continuous-time completion operator is defined in [37] for the delay function e^{-sl} (l is a positive number) in relation with the J-spectral factorization technique.

Definition 3. *For a given continuous-time delay function e^{-sl} and state-space system $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & O \end{array}\right]$, the continuous-time completion operator $\chi_{e^{-sl}}[G](s)$ is defined as follows:*

$$\chi_{e^{-sl}}[G](s) := G_{\chi e^{-sl}}(s) - e^{-sl}G(s), \ G_{\chi e^{-sl}}(s) := \left[\begin{array}{c|c} A & e^{-Al}B \\ \hline C & O \end{array}\right].$$

□

Referring to Definition 3, the discrete-time completion operator is introduced below. It is employed for the J-spectral factorization in Section 3.4.1.

Definition 4. *For a given discrete-time delay function z^{-l} and state-space system $G(z) = \left[\begin{array}{c|c} A & B \\ \hline C & CA^{-1}B \end{array}\right]$, the discrete-time completion operator $\chi_{z^{-l}}[G](z)$ is defined as follows:*

$$\chi_{z^{-l}}[G](z) := G_{\chi z^{-l}}(z) - z^{-l}G(z) \in \mathcal{H}(z^{-l}), \ G_{\chi z^{-l}}(z) := \left[\begin{array}{c|c} A & A^{-l}B \\ \hline C & CA^{-(l+1)}B \end{array}\right].$$

□

The impulse response of $\chi_{z^{-l}}[G](z)$ is given by

$$\chi_{z^{-l}}[G](n) = \begin{cases} CA^{n-(l+1)}B & (n = 0, 1, \cdots, l-1) \\ O & (n = l, l+1, \cdots) \end{cases}.$$

Like the continuous-time counterpart, this impulse response completes that of $z^{-l}G(z)$ in the interval $0 \leq n < l$ as shown in Fig. 3.2.

3.4 Solution via closed-loop reduction

3.4.1 Full information problem

Our approach focuses on the model matching problem associated with the full information problem, where the internal state and exogenous disturbance are available for the control purpose. To find an H^∞ control law, the discrete-time J-spectral factorization technique is developed. In the previous research [37], [38], the continuous-time J-spectral factorization technique is applied to the one-block problem formulated after parameterizing the standard H^∞ controller for the delay-free generalized plant. In contrast to [37], [38], the model matching problem treated in this section is a two-block problem. Consequently, an alternative kind of J-lossless factors of the input-delayed generalized plant P_+ is identified.

Denote the available disturbance by $y_{\mathrm{mm}} := w$, then the generalized plant $P_{+\,\mathrm{mm}}$ from $(w,\, u)$ to $(z,\, y_{\mathrm{mm}})$ is given by

$$P_{+\,\mathrm{mm}} = P_{\mathrm{mm}} \begin{bmatrix} I & O \\ O & z^{-l}I \end{bmatrix},\ P_{\mathrm{mm}} := \begin{bmatrix} I & O \\ P_{11} & P_{12} \end{bmatrix}.$$

If the original control problem in Section 3.2 is solvable, then for $\forall\lambda \geq \gamma$, the following model matching problem for $P_{+\,\mathrm{mm}}$ should also be solvable.

[MM] Find a causal transfer function $T_{u\,w}$ from the disturbance w to the control input u such that

$$P_{11} + P_{12} m\, T_{u\,w} \in H^\infty \text{ and } \|P_{11} + P_{12} m\, T_{u\,w}\|_\infty < \lambda.$$

The model matching problem [MM] can be recast as the following J-spectral factorization problem [SF] [17], and the solution of [MM] is parameterized as follows:

$$T_{u\,w} = \mathcal{F}_l(\mathcal{C}^{-1}\,(M_{+\,\lambda}),\, T_{\widetilde{u}\,\widetilde{w}}),\quad \forall T_{\widetilde{u}\,\widetilde{w}} \in H^\infty \text{ such that } \left\| R^{1/2}_{\mathrm{mm\,c\,22}\,\lambda} T_{\widetilde{u}\,\widetilde{w}} \Lambda^{\mathrm{R}\,(-1/2)}_{\mathrm{c}\,\lambda} \right\|_\infty < \lambda,$$

where $M_{+\,\lambda}$ is the J-spectral factor of the J-spectral density and $T_{\widetilde{u}\,\widetilde{w}}$ denotes the family of the transfer functions from $\widetilde{w}$ to $\widetilde{u}$.

[SF] Define the J-spectral density $\Phi_{+\,\lambda}$: $(y_{\mathrm{mm}},\, u) \to (w,\, z)$ by the equation

$$\Phi_{+\,\lambda} := \mathcal{C}\,(P_{+\,\mathrm{mm}})^\sim \begin{bmatrix} -\lambda^2 I & O \\ O & I \end{bmatrix} \mathcal{C}\,(P_{+\,\mathrm{mm}})\,.$$

Then, find the J-spectral factorization of $\Phi_{+\,\lambda}$ which satisfies the following conditions (SF1) and (SF2):

(SF1) There exist positive definite matrices $\Lambda^{\mathrm{R}}_{\mathrm{c}\,\lambda}$, $R^{\mathrm{R}}_{\mathrm{mm\,c\,22}\,\lambda}$ and a stable J-spectral factor $M_{+\,\lambda} : (\widetilde{w},\, \widetilde{u}) \to (y_{\mathrm{mm}},\, u)$ with strictly causal $\left(M^{-1}_{+\,\lambda}\right)_{12}$ such that

$$\Phi_{+\,\lambda} = M^{-\sim}_{+\,\lambda} \begin{bmatrix} -\lambda^2 \Lambda^{\mathrm{R}}_{\mathrm{c}\,\lambda} & O \\ O & R^{\mathrm{R}}_{\mathrm{mm\,c\,22}\,\lambda} \end{bmatrix} M^{-1}_{+\,\lambda}. \tag{3.2}$$

(SF2) The transfer function $N_{+\lambda} : (\widetilde{w}, \widetilde{u}) \to (w, z)$ defined below is a J-inner function.

$$N_{+\lambda} := \mathcal{C}\,(P_{+\,\mathrm{mm}})\, M_{+\lambda}. \tag{3.3}$$

The main purpose of this section is to solve the problem [SF] for $\lambda = \gamma$, or equivalently for $\forall\lambda \geq \gamma$. By analyzing the structure of $\Phi_{+\lambda}$, the J-spectral factorization problem is reduced to that for some delay-free J-spectral density. The reduction technique is initially introduced in the continuous-time setting [36]. In the subsequent analysis, the relationship between the state variables of $\Phi_{+\lambda}$ and $\Phi^{\mathrm{R}}_{+\lambda}$ is clarified from a viewpoint of decomposition of the internal state dynamics as in Chapter 2.

Before proceeding, the following causal and stable function is defined via the discrete-time completion operator:

$$\chi_{z^{-l}\lambda}(z) := \chi_{z^{-l}} \left[\left(P_{11}^{\sim} P_{11} - \lambda^2 I \right)^{-1} P_{11}^{\sim} P_{12} \right](z),$$

and is decomposed into the constant matrix $\chi_{z^{-l}0\,\lambda}$ and the strictly causal function $\chi_{z^{-l}\ominus\,\lambda}(z)$ as follows:

$$\chi_{z^{-l}\lambda}(z) = \chi_{z^{-l}0\,\lambda} + \chi_{z^{-l}\ominus\,\lambda}(z).$$

It is verified that $\chi_{z^{-l}0\,\lambda} = \frac{1}{\lambda^2} B_1^* A^{-*} S^{\mathrm{R}}_{2\lambda}$ and

$$\begin{aligned}\chi_{z^{-l}\ominus\,\lambda}(z) = &- \begin{bmatrix} -\frac{1}{\lambda^2} B_1 \\ O \end{bmatrix}^* H_{\mathrm{FH}\,\sigma}^{-*} J_{\mathrm{s}} \left(zI - H_{\mathrm{FH}\,\lambda} \right)^{-1} \left(z^{-l} I - \mathcal{E}_{\lambda}^{-1}(l) \right) H_{\mathrm{FH}\,\delta\,\lambda}^{-1} \begin{bmatrix} B_2 \\ S_2 \end{bmatrix} \\ &- \frac{1}{\lambda^2} B_1^* A^{-*} S_2 z^{-l},\end{aligned}$$

where $\begin{bmatrix} B^{\mathrm{R}}_{2\lambda} \\ S^{\mathrm{R}}_{2\lambda} \end{bmatrix} := \mathcal{D}_{\lambda}^{-1}(l) \begin{bmatrix} B_2 \\ S_2 \end{bmatrix}$ and $\mathcal{D}_{\lambda}(l) := H_{\mathrm{FH}\,\delta\,\lambda} \mathcal{E}_{\lambda}(l) H_{\mathrm{FH}\,\delta\,\lambda}^{-1} = H_{\mathrm{FH}\,\sigma} \mathcal{E}_{\lambda}(l) H_{\mathrm{FH}\,\sigma}^{-1}$.

In the following lemma, we transform the input-delayed $\Phi_{+\lambda}$ into the delay-free $\Phi^{\mathrm{R}}_{+\lambda}$. For the transformation, we use $\chi_{z^{-l}\ominus\,\lambda}(z)$ rather than $\chi_{z^{-l}\lambda}(z)$ to meet the strict causality of $\left(M_{+\lambda}^{-1} \right)_{12}$ required in (SF1).

Lemma 2. *The J-spectral density $\Phi^{\mathrm{R}}_{+\lambda}$ defined by*

$$\Phi^{\mathrm{R}}_{+\lambda} := \begin{bmatrix} I & \chi_{z^{-l}\ominus\,\lambda} \\ O & I \end{bmatrix}^{\sim} \Phi_{+\lambda} \begin{bmatrix} I & \chi_{z^{-l}\ominus\,\lambda} \\ O & I \end{bmatrix} \tag{3.4}$$

is a delay-free J-spectral density. Furthermore, the state variables (x, p) of Φ_+ and $(x^{\mathrm{R}}, p^{\mathrm{R}})$ of Φ^{R}_+ are related as follows:

$$\begin{bmatrix} x(z) \\ p(z) \end{bmatrix} = \begin{bmatrix} x^{\mathrm{R}}(z) \\ p^{\mathrm{R}}(z) \end{bmatrix} + \begin{bmatrix} \Pi_{x\,\lambda}(z) \\ \Pi_{p\,\lambda}(z) \end{bmatrix} u(z), \tag{3.5}$$

where $\Pi_{\lambda}(z) := \begin{bmatrix} \Pi_{x\,\lambda}(z)^{\mathrm{T}} & \Pi_{p\,\lambda}(z)^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$ is the strictly causal and stable transfer function defined by

$$\begin{bmatrix} \Pi_{x\,\lambda}(z) \\ \Pi_{p\,\lambda}(z) \end{bmatrix} := \left(zI - H_{\mathrm{FH}\,\lambda} \right)^{-1} \left(z^{-l} I - \mathcal{E}_{\lambda}^{-1}(l) \right) H_{\mathrm{FH}\,\delta\,\lambda}^{-1} \begin{bmatrix} B_2 \\ S_2 \end{bmatrix}.$$

□

Proof. Define Ω_+: $(h_{\rm mm},\, u) \to (-y_{\rm mm},\, k)$ as the Schur complementation transform of Φ_+: $\Omega_+ := \mathcal{S}(\Phi_+)$. Its realization is given by

$$\begin{bmatrix} x(n+1) \\ p(n+1) \end{bmatrix} = H_{\mathrm{FH}\,\lambda} \begin{bmatrix} x(n) \\ p(n) \end{bmatrix} + H_{\mathrm{FH}\,\delta\,\lambda}^{-1} \begin{bmatrix} -\frac{1}{\lambda^2}B_1 & B_2 z^{-l} \\ O & S_2 z^{-l} \end{bmatrix} \begin{bmatrix} h_{\rm mm}(n) \\ u(n) \end{bmatrix},$$
$$\begin{bmatrix} -y_{\rm mm}(n) \\ k(n) \end{bmatrix} = \begin{bmatrix} -\frac{1}{\lambda^2}B_1 & B_2 z^{-l} \\ O & S_2 z^{-l} \end{bmatrix}^{\sim} H_{\mathrm{FH}\,\sigma}^{-*} J_{\rm s} \begin{bmatrix} x(n) \\ p(n) \end{bmatrix} \begin{bmatrix} -\frac{1}{\lambda^2}I & \frac{1}{\lambda^2}B_1^* A^{-*} S_2 z^{-l} \\ O & R_2 - B_2^* A^{-*} S_2 \end{bmatrix} \begin{bmatrix} h_{\rm mm}(n) \\ u(n) \end{bmatrix}.$$

Referring to the above realization, we define the delay-free $\Omega^{\rm R}_{+\,\lambda} : (h_{\rm mm},\, u) \to (-y^{\rm R}_{\rm mm},\, k^{\rm R})$ by the following state-space realization:

$$\begin{bmatrix} x^{\rm R}(n+1) \\ p^{\rm R}(n+1) \end{bmatrix} = H_{\mathrm{FH}\,\lambda} \begin{bmatrix} x^{\rm R}(n) \\ p^{\rm R}(n) \end{bmatrix} + H_{\mathrm{FH}\,\delta\,\lambda}^{-1} \begin{bmatrix} -\frac{1}{\lambda^2}B_1 & B^{\mathcal{D}}_{2\,\lambda}(l) \\ O & S^{\mathcal{D}}_{2\,\lambda}(l) \end{bmatrix} \begin{bmatrix} h_{\rm mm}(n) \\ u(n) \end{bmatrix},$$
$$\begin{bmatrix} -y^{\rm R}_{\rm mm}(n) \\ k^{\rm R}(n) \end{bmatrix} = \begin{bmatrix} -\frac{1}{\lambda^2}B_1 & B^{\mathcal{D}}_{2\,\lambda}(l) \\ O & S^{\mathcal{D}}_{2\,\lambda}(l) \end{bmatrix}^{*} H_{\mathrm{FH}\,\sigma}^{-*} J_{\rm s} \begin{bmatrix} x^{\rm R}(n) \\ p^{\rm R}(n) \end{bmatrix}$$
$$+ \begin{bmatrix} -\frac{1}{\lambda^2}I & \frac{1}{\lambda^2}B_1^* A^{-*} S^{\mathcal{D}}_{2\,\lambda}(l) - \chi_{z^{-l}0\,\lambda} \\ -\chi^*_{z^{-l}0\,\lambda} & R_2 - B_2^* A^{-*} S_2 \end{bmatrix} \begin{bmatrix} h_{\rm mm}(n) \\ u(n) \end{bmatrix}.$$

By comparing the realizations of Ω_+ and $\Omega^{\rm R}_+$, the following properties are observed:

1. The difference $\begin{bmatrix} x(z) \\ p(z) \end{bmatrix} - \begin{bmatrix} x^{\rm R}(z) \\ p^{\rm R}(z) \end{bmatrix}$ is independent of $h_{\rm mm}(z)$ and determined by $u(z)$, namely, the identity in Eq. (3.5) holds.

2. The difference $z^l \begin{bmatrix} x(z) \\ p(z) \end{bmatrix} - \mathcal{E}_\lambda(l) \begin{bmatrix} x^{\rm R}(z) \\ p^{\rm R}(z) \end{bmatrix}$ is independent of $u(z)$ and determined by $h_{\rm mm}(z)$ as follows:

$$z^l \begin{bmatrix} x(z) \\ p(z) \end{bmatrix} - \mathcal{E}_\lambda(l) \begin{bmatrix} x^{\rm R}(z) \\ p^{\rm R}(z) \end{bmatrix} = \left(z^l I - \mathcal{E}_\lambda(l)\right) (zI - H_{\rm FH})^{-1} H_{\rm FH}^{-1} \begin{bmatrix} -\dfrac{1}{\lambda^2}B_1 \\ O \end{bmatrix} h_{\rm mm}(z).$$

These properties yield the following identity:

$$\Omega^{\rm R}_{+\,\lambda} = \Omega_{+\,\lambda} + \begin{bmatrix} O & \chi_{z^{-l}\ominus\,\lambda} \\ \chi^{\sim}_{z^{-l}\ominus\,\lambda} & O \end{bmatrix}.$$

Taking the inverse Schur complementation transform $\mathcal{S}^{-1}(\cdot)$ of this identity, we obtain Eq. (3.4). □

Remark 6. *In Lemma 2, it is explicitly stated that the state variables $(x,\, p)$ should be decomposed as in Eq. (3.5) when the J-spectral density $\Phi_{+\,\lambda}$ is transformed according to Eq. (3.4). Note that as $\lambda \to \infty$, $\mathcal{E}_\lambda(l)$ becomes the lower triangular matrix, and the first row in Eq. (3.5) coincides with the internal state decomposition proposed in the corresponding H^2 control problem in Chapter 2.* □

By Lemma 2, the delay-free $\Phi^{\mathrm{R}}_{+\lambda}$ is obtained by multiplying $\Phi_{+\lambda}$ by the bi-stable transfer function and its para-conjugate from the right and left, respectively. Moreover, $\chi_{m\ominus\lambda}$ in Eq. (3.4) is strictly causal. Therefore, it is necessary for (SF1) that there exists a stable J-spectral factor $M^{\mathrm{R}}_{+\lambda}$ with strictly causal $\left(M^{\mathrm{R}\,(-1)}_{+\lambda}\right)_{12}$ such that

$$\Phi^{\mathrm{R}}_{+\lambda} = M^{\mathrm{R}\,(-\sim)}_{+\lambda} \begin{bmatrix} -\lambda^2 \Lambda^{\mathrm{R}}_{\mathrm{c}\,\lambda} & O \\ O & R^{\mathrm{R}}_{\mathrm{mm\,c\,22}\,\lambda} \end{bmatrix} M^{\mathrm{R}\,(-1)}_{+\lambda}. \tag{3.6}$$

This necessity is equivalent to the following conditions (C1) and (C2).

(C1) The KYP equation

$$\begin{aligned} F^{\mathrm{R}\,*}_{\lambda} R^{\mathrm{R}}_{\mathrm{mm\,c}\,\lambda} F^{\mathrm{R}}_{\lambda} &= Q + A^* X^{\mathrm{R}}_{\lambda} A - X^{\mathrm{R}}_{\lambda}, \\ -R^{\mathrm{R}}_{\mathrm{mm\,c}\,\lambda} F^{\mathrm{R}}_{\lambda} &= \left[O \;\; S^{\mathcal{D}}_{2\,\lambda}(l)\right]^* + \left[B_1 \;\; B^{\mathcal{D}}_{2\,\lambda}(l) - B_1 \chi_{m0\,\lambda}\right]^* X^{\mathrm{R}}_{\lambda} A, \\ R^{\mathrm{R}}_{\mathrm{mm\,c}\,\lambda} &= R^{\mathrm{R}}_{\mathrm{mm}\,\lambda} + \left[B_1 \;\; B^{\mathcal{D}}_{2\,\lambda}(l) - B_1 \chi_{m0\,\lambda}\right]^* X^{\mathrm{R}}_{\lambda} \left[B_1 \;\; B^{\mathcal{D}}_{2\,\lambda}(l) - B_1 \chi_{m0\,\lambda}\right], \end{aligned} \tag{3.7}$$

where $R^{\mathrm{R}}_{\mathrm{mm}\,\lambda} := \begin{bmatrix} -\lambda^2 I & \lambda^2 \chi_{z^{-l}0\,\lambda} \\ \lambda^2 \chi^*_{z^{-l}0\,\lambda} & R^{\mathrm{R}}_{\mathrm{mm\,22}\,\lambda} \end{bmatrix}$ and

$$R^{\mathrm{R}}_{\mathrm{mm\,22}\,\lambda} := R_2 - B_2^* A^{-*} S_2 + B^{\mathcal{D}\,*}_{2\,\lambda}(l) A^{-*} S^{\mathcal{D}}_{2\,\lambda}(l) - \lambda^2 \chi^*_{z^{-l}0\,\lambda} \chi_{z^{-l}0\,\lambda},$$

has the stabilizing solution X^{R}_{λ} such that the following matrix is stable:

$$A^{\mathrm{R}}_{\mathrm{c}\,\lambda} := A + B_1 F^{\mathrm{R}}_{1\,\lambda} + \left(B^{\mathrm{R}}_{2\,\lambda} - B_1 \chi_{m0\,\lambda}\right) F^{\mathrm{R}}_{2\,\lambda}.$$

(C2) The Hermitian matrix $R^{\mathrm{R}}_{\mathrm{mm\,c}\,\lambda}$ in Eq. (3.7) satisfies the following definiteness conditions:

$$R^{\mathrm{R}}_{\mathrm{mm\,c\,22}\,\lambda} > O, \;\; -\lambda^2 \Lambda^{\mathrm{R}}_{\mathrm{c}\,\lambda} := R^{\mathrm{R}}_{\mathrm{mm\,c\,11}\,\lambda} - R^{\mathrm{R}}_{\mathrm{mm\,c\,12}\,\lambda} R^{\mathrm{R}\,(-1)}_{\mathrm{mm\,c\,22}\,\lambda} R^{\mathrm{R}}_{\mathrm{mm\,c\,21}\,\lambda} < O.$$

Remark 7. *Since $\mathcal{E}_{\lambda}(\theta)$ is a symplectic matrix, the following identity holds:*

$$\mathcal{D}^{-*}_{\lambda}(\theta) \begin{bmatrix} O & -A^{-*} \\ A^{-1} & O \end{bmatrix} \mathcal{D}^{-1}_{\lambda}(\theta) = \begin{bmatrix} O & -A^{-*} \\ A^{-1} & O \end{bmatrix}.$$

Therefore, $-B_2^* A^{-*} S_2 + B^{\mathcal{D}\,*}_{2\,\lambda}(\theta) A^{-*} S^{\mathcal{D}}_{2\,\lambda}(\theta)$ is an Hermitian matrix. □

The following lemma represents the stabilizing solution of the KYP equation in (C1) using that of the standard KYP equation in (X).

Lemma 3. *Suppose that for any fixed $\lambda \geq \gamma$, the KYP equation in (X) has the stabilizing solution X_{λ}. Then, the condition (C1) is equivalent to the following condition (*E*).*

(E) *The* (2, 2) *block of $\mathcal{E}_{X\,\lambda}(n)$ defined by Eq. (3.1) is regular for $n = l$:*

$$\det \mathcal{E}_{X\,22\,\lambda}(l) = \det \left(X_{\lambda} \mathcal{E}_{12\,\lambda}(l) + \mathcal{E}_{22\,\lambda}(l)\right) \neq 0.$$

If the above condition (E) is satisfied, X_λ^{R}, $A_{\mathrm{c}\,\lambda}^{\mathrm{R}}$ *and* $F_\lambda^{\mathrm{R}} =: \begin{bmatrix} F_{1\,\lambda}^{\mathrm{R}\,\mathrm{T}} & F_{2\,\lambda}^{\mathrm{R}\,\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{(\dim w + \dim u)\times \dim x}$ *are constructed as follows:*

$$X_\lambda^{\mathrm{R}} = \left(X_\lambda \mathcal{E}_{12\,\lambda}(l) + \mathcal{E}_{22\,\lambda}(l)\right)^{-1}\left(X_\lambda \mathcal{E}_{11\,\lambda}(l) + \mathcal{E}_{21\,\lambda}(l)\right), \tag{3.8}$$

$$A_{\mathrm{c}\,\lambda}^{\mathrm{R}} = \mathcal{E}_{X\,22\,\lambda}^{*}(l) A_{\mathrm{c}\,\lambda} \mathcal{E}_{X\,22\,\lambda}^{-*}(l), \tag{3.9}$$

$$F_{1\,\lambda}^{\mathrm{R}} = \frac{1}{\lambda^2} B_1^* X_\lambda^{\mathrm{R}} A_{\mathrm{c}\,\lambda}^{\mathrm{R}} + \chi_{z^{-l}0\,\lambda} F_{2\,\lambda}^{\mathrm{R}}, \tag{3.10}$$

$$F_{2\,\lambda}^{\mathrm{R}} = F_{2\,\lambda} \mathcal{E}_{X\,22\,\lambda}^{-*}(l). \tag{3.11}$$

□

Proof. Subsection 3.7.1. □

The following theorem gives a constructive solution of [SF], and the parameterization of the solutions of [MM] are obtained from it.

Theorem 3. *Suppose that the assumptions (X) and (H) are satisfied. The* J*-spectral factorization of* $\Phi_{+\,\gamma}$ *satisfying (SF1)-(SF2) exists if and only if the following conditions (J1)-(J2) are satisfied.*

(J1) *For* $\forall\lambda \geq \gamma$*, the condition (E) in Lemma 3 is satisfied.*

(J2) *For* $\forall\lambda \geq \gamma$*, the condition (C2) before Lemma 3 is satisfied.*

Under the existence conditions, the J*-spectral factor* $M_{+\,\lambda}$ *in (SF1) is given as follows:*

$$M_{+\,\lambda} = \begin{bmatrix} I & \chi_{z^{-l}\ominus\,\lambda} \\ O & I \end{bmatrix} M_{+\,\lambda}^{\mathrm{R}}, \tag{3.12}$$

where

$$M_{+\,\lambda}^{\mathrm{R}} := \left[\begin{array}{c|cc} A_{\mathrm{c}\,\lambda}^{\mathrm{R}} & B_1 & B_{2\,\lambda}^{\mathrm{R}} - B_1\chi_{z^{-l}0\,\lambda} \\ \hline F_{1\,\lambda}^{\mathrm{R}} & I & O \\ F_{2\,\lambda}^{\mathrm{R}} & O & I \end{array}\right] \begin{bmatrix} I & O \\ F_{21\,\lambda}^{\mathrm{R}} & I \end{bmatrix}, \tag{3.13}$$

$$F_{21\,\lambda}^{\mathrm{R}} := -R_{\mathrm{mm\,c}\,22\,\lambda}^{\mathrm{R}\,(-1)} R_{\mathrm{mm\,c}\,21\,\lambda}.$$

Moreover, the J*-inner function* $N_{+\,\lambda}$ *in (SF2) is given as follows:*

$$N_{+\,\lambda} = N_\lambda N_{+\,\lambda}^{z^{-l}},$$

where

$$N_\lambda := \left[\begin{array}{c|cc} A_{\mathrm{c}\,\lambda} & B_1 & B_2 \\ \hline F_{1\,\lambda} & I & O \\ C_1 + D_{12}F_{2\,\lambda} & O & D_{12} \end{array}\right] \begin{bmatrix} I & O \\ F_{21\,\lambda} & I \end{bmatrix}, \tag{3.14}$$

$$F_{21\,\lambda} := -R_{\mathrm{mm\,c}\,22\,\lambda}^{-1} R_{\mathrm{mm\,c}\,21\,\lambda}, \tag{3.15}$$

$$N_{+\,\lambda}^{z^{-l}} := \begin{bmatrix} I & O \\ -F_{21\,\lambda} & I \end{bmatrix} \left\{ - \begin{bmatrix} \frac{1}{\lambda^2} B_1^* A^{-*} Q + F_{1\,\lambda} & \frac{1}{\lambda^2} B_1^* A^{-*} \\ F_{2\,\lambda} & O \end{bmatrix} \left(z^{-l} I - \mathcal{E}_\lambda^{-1}(l)\right) \right.$$
$$\left. \cdot \left(zI - H_{\mathrm{FH}\,\lambda}\right)^{-1} \begin{bmatrix} (*1) & (*2) \end{bmatrix} + \begin{bmatrix} I & -\frac{1}{\lambda^2} B_1^* A^{-*} S_2 z^{-l} \\ O & z^{-l} I \end{bmatrix} \right\} \begin{bmatrix} I & O \\ F_{21\,\lambda}^{\mathrm{R}} & I \end{bmatrix}, \tag{3.16}$$

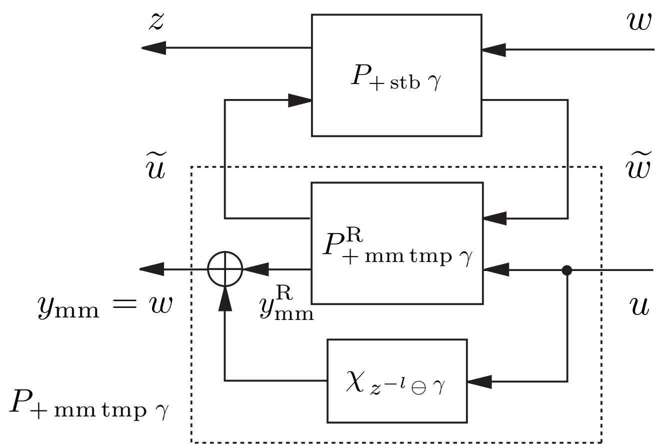

Figure 3.3: Factorization of $P_{+\,\mathrm{mm}}$.

$$(*1) := -\begin{bmatrix} I \\ -X_\lambda \end{bmatrix} \mathcal{E}^{-*}_{X\,22\,\lambda}(l) B_1, \ (*2) := H^{-1}_{\mathrm{FH}\,\delta\,\lambda} \begin{bmatrix} B_2 \\ S_2 \end{bmatrix} - \begin{bmatrix} I \\ -X_\lambda \end{bmatrix} \mathcal{E}^{-*}_{X\,22\,\lambda}(l) \left(B^{\mathcal{D}}_{2\,\lambda}(l) - B_1 \chi_{z^{-l}0\,\lambda} \right).$$

□

Proof. Subsection 3.7.2. □

By taking the inverse chain scattering representation of the equality $\mathcal{C}\left(P_{+\,\mathrm{mm}}\right) = N_{+\,\gamma} M^{-1}_{+\,\gamma}$, $P_{+\,\mathrm{mm}}$ is factorized as

$$P_{+\,\mathrm{mm}} = P_{+\,\mathrm{stb}\,\gamma} \star P_{+\,\mathrm{mm\,tmp}\,\gamma},$$

where $P_{+\,\mathrm{stb}\,\gamma} := \mathcal{C}^{-1}\left(N_{+\,\gamma}\right)$ and $P_{+\,\mathrm{mm\,tmp}\,\gamma} := \mathcal{C}^{-1}\left(M^{-1}_{+\,\gamma}\right)$. By Eq. (3.12), the first output $y^{\mathrm{R}}_{\mathrm{mm}}$ of $M^{\mathrm{R}}_{+\,\lambda}$ is given by

$$y^{\mathrm{R}}_{\mathrm{mm}}(n) = y_{\mathrm{mm}}(n) - \chi_{z^{-l}\ominus\gamma}(z) u(n), \tag{3.17}$$

and $P_{+\,\mathrm{mm\,tmp}\,\gamma}$ has the structure shown in Fig. 3.3. In the figure, $P^{\mathrm{R}}_{+\,\mathrm{mm\,tmp}\,\gamma}$ is defined by $P^{\mathrm{R}}_{+\,\mathrm{mm\,tmp}\,\gamma} := \mathcal{C}^{-1}\left(M^{\mathrm{R}\,(-1)}_{+\,\gamma}\right)$. Moreover, $P_{+\,\mathrm{stb}\,\gamma}$ is J-lossless because $N_{+\,\gamma}$ is a J-inner function. Corollary 1 below represents a solution of [MM] in a form of state feedback law. It is derived by focusing on the relationship between the state variables $x(n)$ and $x^{\mathrm{R}}(n)$.

Corollary 1. *Suppose that the J-spectral factorizability conditions (J1)-(J2) are satisfied. Then, the following state feedback law is stabilizing, and suppresses the H^∞ norm of $T_{z\,w}$ below the given γ:*

$$u(n) = F^{\mathrm{p\,R}}_{2\,\gamma} x(n) + F^{\mathrm{R}}_{21\,\gamma} w(n) - \left\{ F^{\mathrm{p\,R}}_{2\,\gamma} \Pi_{x\,\gamma}(z) + F^{\mathrm{R}}_{21\,\gamma} \chi_{z^{-l}\ominus\gamma}(z) \right\} u(n), \tag{3.18}$$

where $F^{\mathrm{p\,R}}_{2\,\lambda}$ is defined by

$$F^{\mathrm{p\,R}}_{2\,\lambda} := F^{\mathrm{R}}_{2\,\lambda} - F^{\mathrm{R}}_{21\,\lambda} F^{\mathrm{R}}_{1\,\lambda}.$$

□

Proof. By Fig. 3.3, the control law

$$u(n) = \mathcal{F}_l(\mathcal{C}^{-1}\left(M^{\mathrm{R}}_{+\gamma}\right),\, O)y^{\mathrm{R}}_{\mathrm{mm}}(n)$$

makes the transfer function from $\widetilde{w}$ to $\widetilde{u}$ zero, and is realized in the following form:

$$u(n) = F^{\mathrm{p\,R}}_{2\,\gamma} x^{\mathrm{R}}(n) + F^{\mathrm{R}}_{21\,\gamma} y^{\mathrm{R}}_{\mathrm{mm}}(n). \tag{3.19}$$

It is an H^∞ control law since $P_{+\,\mathrm{stb}\,\gamma}$ is J-lossless. Furthermore, Eq. (3.19) is rewritten as in Eq. (3.18) using Eqs. (3.5) and (3.17), and recalling that $y_{\mathrm{mm}}(n) = w(n)$. □

3.4.2 Output estimation problem

In the previous section, we solved the J-spectral factorization problem [SF] and factorized the generalized plant $P_{+\,\mathrm{mm}}$, which is associated with the model matching problem [MM], as shown in Fig. 3.3. Based on the factorization, we solve the original output feedback problem by reducing it to a delay-free output estimation problem.

Recall that the state variable $x(z)$ of the delay-free generalized plant P is decomposed as the sum of $x^{\mathrm{R}}(z)$ and $\Pi_{x\,\lambda}(z)u(z)$ in Eq. (3.5). Our principle is to estimate x^{R} instead of x along the corresponding H^2 solution method in Chapter 2.

The original measured output y is constructed with the state variable x^{R} and measured output $y^{\mathrm{R}}_{\mathrm{mm}}$ of $P^{\mathrm{R}}_{+\,\mathrm{mm\,tmp}\,\gamma}$ as follows (Fig. 3.4):

$$\begin{aligned}
y(n) &= C_2 x(n) + D_{21} w(n) \\
&= C_2 x^{\mathrm{R}}(n) + D_{21} y^{\mathrm{R}}_{\mathrm{mm}}(n) + \left\{C_2 \Pi_{x\,\gamma}(z) + D_{21}\chi_{z^{-l}\ominus\,\gamma}(z)\right\} u(n) \ (\because \text{Eqs. (3.5) and (3.17)}) \\
&= \left(C_2 + D_{21} F^{\mathrm{R}}_{1\,\gamma}\right) x^{\mathrm{R}}(n) + D_{21}\widetilde{w}(n) + \left\{C_2 \Pi_{x\,\gamma}(z) + D_{21}\chi_{z^{-l}\ominus\,\gamma}(z)\right\} u(n).
\end{aligned} \tag{3.20}$$

In Eq. (3.20), the following equation from Eq. (3.13) is used:

$$y^{\mathrm{R}}_{\mathrm{mm}}(n) = F^{\mathrm{R}}_{1\,\gamma} x^{\mathrm{R}}(n) + \widetilde{w}(n).$$

Let $P_{+\,\mathrm{tmp}\,\gamma} : (\widetilde{w},\, u) \to (\widetilde{u},\, y)$ be the generalized plant, which is derived by replacing the measured output y_{mm} of $P_{+\,\mathrm{mm\,tmp}\,\gamma} : (\widetilde{w},\, u) \to (\widetilde{u},\, y_{\mathrm{mm}})$ with the measured output y of P_+. From Eq. (3.20), it is seen that $P_{+\,\mathrm{tmp}\,\gamma}$ consists of the delay-free generalized plant $P^{\mathrm{R}}_{+\,\mathrm{tmp}\,\gamma} : (\widetilde{w},\, u) \to (\widetilde{u},\, y^{\mathrm{R}})$ and the following part:

$$y(n) = y^{\mathrm{R}}(n) + \left\{C_2 \Pi_{x\,\gamma}(z) + D_{21}\chi_{z^{-l}\ominus\,\gamma}(z)\right\} u(n). \tag{3.21}$$

The structure of $P_{+\,\mathrm{tmp}}$ is also depicted in Fig. 3.5. The generalized plant $P_{+\,\mathrm{tmp}\,\gamma}$ is in output estimation form, and its state-space realization is given as follows:

$$\begin{aligned}
x^{\mathrm{R}}(n+1) &= \left(A + B_1 F^{\mathrm{R}}_{1\,\gamma}\right) x^{\mathrm{R}}(n) + B_1 \widetilde{w}(n) + \left(B^{\mathcal{D}}_{2\,\gamma}(l) - B_1 \chi_{z^{-l}0\,\gamma}\right) u(n), \\
\widetilde{u}(n) &= -F^{\mathrm{R}}_{2\,\gamma} x^{\mathrm{R}}(n) - F^{\mathrm{R}}_{21\,\gamma}\widetilde{w}(n) + u(n), \\
y^{\mathrm{R}}(n) &= \left(C_2 + D_{21} F^{\mathrm{R}}_{1\,\gamma}\right) x^{\mathrm{R}}(n) + D_{21}\widetilde{w}(n).
\end{aligned}$$

The discussion so far shows that P_+ is factorized as $P_+ = P_{+\,\mathrm{stb}\,\gamma} \star P_{+\,\mathrm{tmp}\,\gamma}$. Since $P_{+\,\mathrm{stb}\,\gamma}$ is J-lossless, the following equivalences hold by Redheffer's lemma [59]:

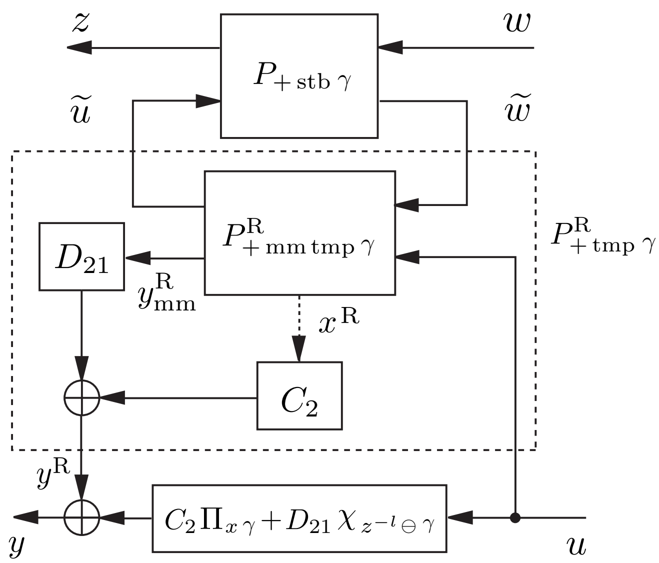

Figure 3.4: Construction of measured output y.

1. Stabilization of $P_+ \iff$ Stabilization of $P_{+\,\mathrm{tmp}\,\gamma}$.
2. $\|T_{z\,w}\|_\infty < \gamma \iff \left\| R^{\mathrm{R}\,(1/2)}_{\mathrm{mm\,c}\,\gamma} T_{\widetilde{u}\,\widetilde{w}} \Lambda^{\mathrm{R}\,(-1/2)}_{\mathrm{c}\,\gamma} \right\|_\infty < \gamma$.

Therefore, the H^∞ control problem for P_+ is reduced to that for $P_{+\,\mathrm{tmp}\,\gamma}$.

The second term on the right-hand side of Eq. (3.21) is the strictly causal and stable transfer function multiplied to u. We include its copy into the controller K_+ as shown in Fig. 3.6, where K^{R}_+ is the undetermined part of K_+. Then the problem of parameterizing the controller K_+ for $P_{+\,\mathrm{tmp}\,\gamma}$ is reduced to that of parameterizing the controller K^{R}_+ for $P^{\mathrm{R}}_{+\,\mathrm{tmp}\,\gamma}$.

In the sequel, we solve the delay-free H^∞ control problem for $P^{\mathrm{R}}_{+\,\mathrm{tmp}\,\gamma}$, and obtain the parameterization of K^{R}_+. Noting that $P^{\mathrm{R}}_{+\,\mathrm{tmp}\,\gamma}$ is in the output estimation form, it is done via the factorization of the J-spectral density $\acute{\Phi}^{\mathrm{R}}_{+\,\mathrm{tmp}}$:

$$\acute{\Phi}^{\mathrm{R}}_{+\,\mathrm{tmp}\,\gamma} := \acute{P}^{\mathrm{R}\,\sim}_{+\,\tau\,\gamma} \begin{bmatrix} -\gamma^2 \acute{\Lambda}^{\mathrm{R}}_{\mathrm{tmp}\,\gamma} & O \\ O & \acute{W}^{\mathrm{R}}_{\mathrm{tmp}\,\gamma} \end{bmatrix} \acute{P}^{\mathrm{R}}_{+\,\tau\,\gamma},\ \acute{\Lambda}^{\mathrm{R}}_{\mathrm{tmp}\,\gamma} := R^{\mathrm{R}\,(-1)}_{\mathrm{mm\,c\,22}\,\gamma},\ \acute{W}^{\mathrm{R}}_{\mathrm{tmp}\,\gamma} := \Lambda^{\mathrm{R}\,(-1)}_{\mathrm{c}\,\gamma},$$

where

$$\acute{P}^{\mathrm{R}}_{+\,\tau\,\gamma} := \left[\begin{array}{c|c} \acute{A}^{\mathrm{R}\,*}_{\tau\,\gamma} & \acute{C}^{\mathrm{R}\,*}_{\tau\,\gamma} \\ \hline \acute{B}^{\mathrm{R}\,*}_{\tau\,\gamma} & \acute{D}^{\mathrm{R}\,*}_{\tau\,\gamma} \end{array}\right],$$

$$\acute{A}^{\mathrm{R}}_{\tau\,\gamma} := A^{\mathrm{R}}_{\mathrm{c}\,\gamma},\ \acute{C}^{\mathrm{R}}_{\tau\,\gamma} := \begin{bmatrix} -F^{\mathrm{R}}_{2\,\gamma} \\ C_2 + D_{21} F^{\mathrm{R}}_{1\,\gamma} \end{bmatrix},\ \acute{D}^{\mathrm{R}}_{\tau\,\gamma} := \begin{bmatrix} I & -F^{\mathrm{R}}_{21\,\gamma} \\ O & D_{21} \end{bmatrix},$$

$$\acute{B}^{\mathrm{R}}_{\tau\,\gamma} := \left[-\left(B^{\mathcal{D}}_{2\,\gamma}(l) - B_1 \chi_{z^{-l}0\,\gamma} \right) \quad B_1 + \left(B^{\mathcal{D}}_{2\,\gamma}(l) - B_1 \chi_{z^{-l}0\,\gamma} \right) F^{\mathrm{R}}_{21\,\gamma} \right].$$

The solvability of the H^∞ output estimation problem for $P^{\mathrm{R}}_{+\,\mathrm{tmp}\,\gamma}$, or equivalently the J-spectral factorizability of $\acute{\Phi}^{\mathrm{R}}_{+\,\mathrm{tmp}\,\gamma}$ coincides with the following conditions (T1) and (T2).

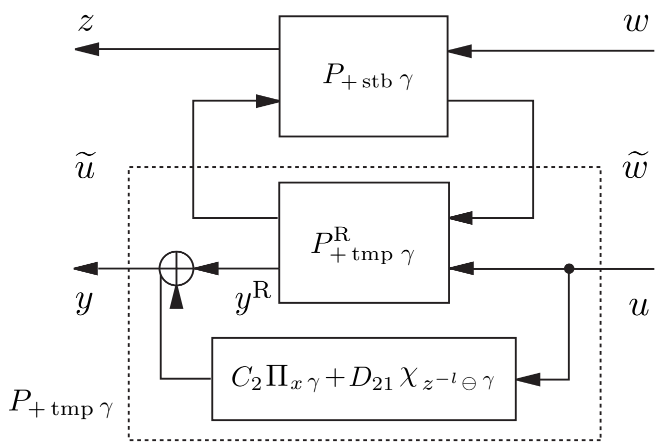

Figure 3.5: Structure of $P_{+\,\mathrm{tmp}}$.

(T1) The KYP equation

$$\begin{aligned} L^{\mathrm{R}}_{\mathrm{tmp}\,\gamma}\acute{R}^{\mathrm{R}}_{\mathrm{tmp\,c}\,\gamma}L^{\mathrm{R}\,*}_{\mathrm{tmp}\,\gamma} &= \acute{Q}^{\mathrm{R}}_{\mathrm{tmp}\,\gamma} + \acute{A}^{\mathrm{R}}_{\tau\,\gamma}Y^{\mathrm{R}}_{\mathrm{tmp}\,\gamma}\acute{A}^{\mathrm{R}\,*}_{\tau\,\gamma} - Y^{\mathrm{R}}_{\mathrm{tmp}\,\gamma}, \\ -\acute{R}^{\mathrm{R}}_{\mathrm{tmp\,c}\,\gamma}L^{\mathrm{R}\,*}_{\mathrm{tmp}\,\gamma} &= \acute{S}^{\mathrm{R}}_{\mathrm{tmp}\,\gamma} + \acute{C}^{\mathrm{R}}_{\tau\,\gamma}Y^{\mathrm{R}}_{\mathrm{tmp}\,\gamma}\acute{A}^{\mathrm{R}\,*}_{\tau\,\gamma}, \\ \acute{R}^{\mathrm{R}}_{\mathrm{tmp\,c}\,\gamma} &= \acute{R}^{\mathrm{R}}_{\mathrm{tmp}\,\gamma} + \acute{C}^{\mathrm{R}}_{\tau\,\gamma}Y^{\mathrm{R}}_{\mathrm{tmp}\,\gamma}\acute{C}^{\mathrm{R}\,*}_{\tau\,\gamma}, \end{aligned} \tag{3.22}$$

$$\begin{bmatrix} \acute{Q}^{\mathrm{R}}_{\mathrm{tmp}\,\gamma} & \acute{S}^{\mathrm{R}\,*}_{\mathrm{tmp}\,\gamma} \\ \acute{S}^{\mathrm{R}}_{\mathrm{tmp}\,\gamma} & \acute{R}^{\mathrm{R}}_{\mathrm{tmp}\,\gamma} \end{bmatrix} := \begin{bmatrix} \acute{B}^{\mathrm{R}}_{\tau\,\gamma} \\ \acute{D}^{\mathrm{R}}_{\tau\,\gamma} \end{bmatrix} \begin{bmatrix} -\gamma^2\acute{\Lambda}^{\mathrm{R}}_{\mathrm{tmp}\,\gamma} & O \\ O & \acute{W}^{\mathrm{R}}_{\mathrm{tmp}\,\gamma} \end{bmatrix} \begin{bmatrix} \acute{B}^{\mathrm{R}}_{\tau\,\gamma} \\ \acute{D}^{\mathrm{R}}_{\tau\,\gamma} \end{bmatrix}^*$$

has the positive semidefinite stabilizing solution $Y^{\mathrm{R}}_{\mathrm{tmp}\,\gamma}$ such that the following matrix is stable:

$$\acute{A}^{\mathrm{R}}_{\tau\,\mathrm{c}\,\gamma} := \acute{A}^{\mathrm{R}}_{\tau\,\gamma} + L^{\mathrm{R}}_{\mathrm{tmp1}\,\gamma}\acute{C}^{\mathrm{R}}_{\tau 1\,\gamma} + L^{\mathrm{R}}_{\mathrm{tmp2}\,\gamma}\acute{C}^{\mathrm{R}}_{\tau 2\,\gamma}.$$

(T2) The Hermitian matrix $\acute{R}^{\mathrm{R}}_{\mathrm{tmp\,c}\,\gamma}$ in Eq. (3.22) satisfies the following definiteness conditions:

$$\acute{R}^{\mathrm{R}}_{\mathrm{tmp\,c\,22}\,\gamma} > O,\ -\gamma^2\acute{\Lambda}^{\mathrm{R}}_{\mathrm{tmp\,c}\,\gamma} := \acute{R}^{\mathrm{R}}_{\mathrm{tmp\,c\,11}\,\gamma} - \acute{R}^{\mathrm{R}}_{\mathrm{tmp\,c\,12}\,\gamma}\acute{R}^{\mathrm{R}\,(-1)}_{\mathrm{tmp\,c\,22}\,\gamma}\acute{R}^{\mathrm{R}}_{\mathrm{tmp\,c\,21}\,\gamma} < O.$$

The relationship between the stabilizing solutions of the KYP equations in (Y) and (T1) are given in the following lemma.

Lemma 4. *Under the condition (Y), the existence of the positive semidefinite stabilizing solution $Y^{\mathrm{R}}_{\mathrm{tmp}\,\gamma}$ in (T1) is equivalent to the following condition (Z):*

(Z) *The maximal eigenvalue of $Y_\gamma X^{\mathrm{R}}_\gamma$ is less than γ^2: $\lambda_{\max}\left(Y_\gamma X^{\mathrm{R}}_\gamma\right) < \gamma^2$.*

If the above condition is satisfied, $Y^{\mathrm{R}}_{\mathrm{tmp}\,\gamma}$, $\acute{A}^{\mathrm{R}}_{\tau\,\mathrm{c}\,\gamma}$ and $L^{\mathrm{R}}_{\mathrm{tmp}\,\gamma} =: \begin{bmatrix} L^{\mathrm{R}}_{\mathrm{tmp1}\,\gamma} & L^{\mathrm{R}}_{\mathrm{tmp2}\,\gamma} \end{bmatrix}$ are given by

$$Y^{\mathrm{R}}_{\mathrm{tmp}\,\gamma} = Z^{\mathrm{R}\,(-1)}_\gamma Y_\gamma,\ \acute{A}^{\mathrm{R}}_{\tau\,\mathrm{c}\,\gamma} = Z^{\mathrm{R}\,(-1)}_\gamma \acute{A}_{\mathrm{c}\,\gamma} Z^{\mathrm{R}}_\gamma,$$

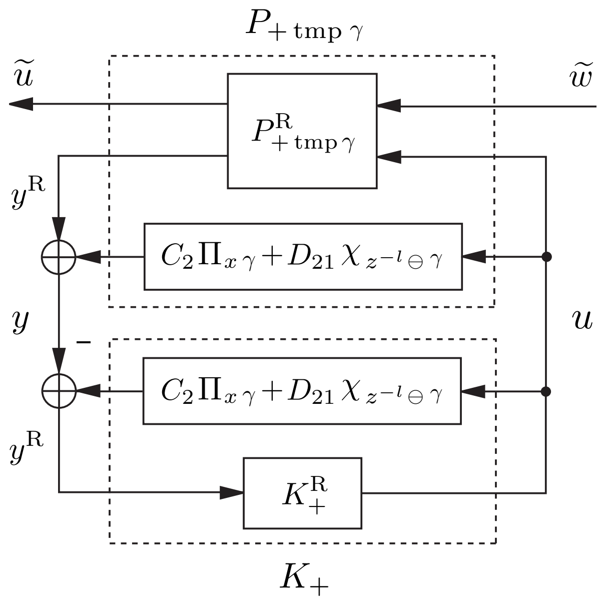

Figure 3.6: Internal model structure of K_+.

$$L^{\mathrm{R}}_{\mathrm{tmp1}\,\gamma} = Z^{\mathrm{R}\,(-1)}_{\gamma}\left(B^{\mathcal{D}}_{2\,\gamma}(l) + \frac{1}{\gamma^2}\acute{A}_{\mathrm{c}\,\gamma}Y_{\gamma}S^{\mathcal{D}}_{2\,\gamma}(l) + L_{2\,\gamma}D_{21}\chi_{z^{-l}0\,\gamma}\right),\ L^{\mathrm{R}}_{\mathrm{tmp2}\,\gamma} = Z^{\mathrm{R}\,(-1)}_{\gamma}L_{2\,\gamma},$$

where $Z^{\mathrm{R}}_{\gamma} := I - \frac{1}{\gamma^2}Y_{\gamma}X^{\mathrm{R}}_{\gamma}$. ☐

Proof. Subsection 3.7.3. ☐

After parameterizing K^{R}_{+} from the J-spectral factor of $\acute{\Phi}^{\mathrm{R}}_{+\,\mathrm{tmp}\,\gamma}$, the complete structure of K_+ is identified as shown in Theorem 4 below. In the theorem, the condition (T1) is replaced with the condition (Z) by Lemma 4.

Theorem 4. *Under the assumptions (X), (H) and (Y), the discrete-time* H^{∞} *controller* K_+ *for* P_+ *exists if and only if the* J*-spectral factorizability conditions (J1)-(J2), and the output estimation conditions (Z) and (T2) are satisfied.*

If the existence conditions are satisfied, the H^{∞} *controller* $K_+ : (y,\, \mu) \to (u,\, \nu)$ *is parameterized in the Smith predictor form (Fig. 3.7). It consists of the measurement compensation part:*

$$y^{\mathrm{R}}(n) = y(n) - \left\{C_2\Pi_{x\,\gamma}(z) + D_{21}\chi_{z^{-l}\ominus\,\gamma}(z)\right\}u(n)$$

and the observer-based controller K^{R}_{+} *estimating the state variable estimating* x^{R}*:*

$$K^{\mathrm{R}}_{+} = \mathcal{F}_l(J^{\mathrm{R}}_{+\,\gamma},\, L^{\mathrm{R}}_{\mathrm{tmp12}} + Q_+(z)),\ L^{\mathrm{R}}_{\mathrm{tmp12}\,\gamma} := -\acute{R}^{\mathrm{R}}_{\mathrm{tmp\,c\,12}\,\gamma}\acute{R}^{\mathrm{R}\,(-1)}_{\mathrm{tmp\,c\,22}\,\gamma},$$

where $\forall\, Q_+(z) \in H^{\infty}$ *is the Youla parameter such that*

$$\left\|\acute{\Lambda}^{\mathrm{R}\,(-1/2)}_{\mathrm{tmp\,c}\,\gamma}Q_+\acute{R}^{\mathrm{R}\,(1/2)}_{\mathrm{tmp\,c\,22}\,\gamma}\right\|_{\infty} < \gamma.$$

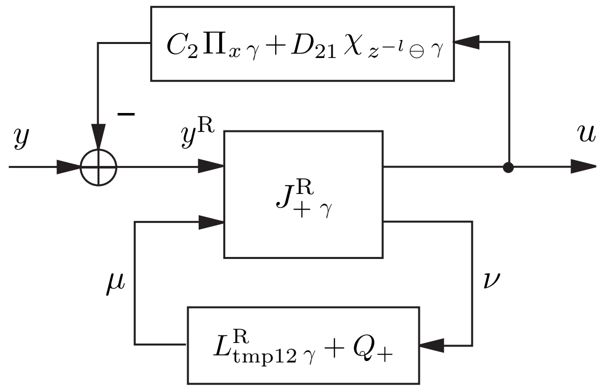

Figure 3.7: Discrete-time H^∞ suboptimal controllers in Smith form.

The realization of $J^{\mathrm{R}}_{+\,\gamma}$ is given by

$$J^{\mathrm{R}}_{+\,\gamma} := \left[\begin{array}{c|cc} A^{\mathrm{R}}_{\mathrm{c}\gamma} + L^{\mathrm{R}}_{\mathrm{tmp2}\,\gamma}\left(C_2 + D_{21}F^{\mathrm{R}}_{1\,\gamma}\right) & -L^{\mathrm{R}}_{\mathrm{tmp2}\,\gamma} & L^{\mathrm{R}}_{\mathrm{tmp1}\,\gamma} \\ \hline F^{\mathrm{R}}_{2\,\gamma} & O & I \\ -\left(C_2 + D_{21}F^{\mathrm{R}}_{1\,\gamma}\right) & I & O \end{array}\right].$$

□

Remark 8. *Theorem 4 claims that the output feedback controller exists if and only if both of the full information and output estimation problems are solvable. The solvability of the full information problem requires the J-spectral factorizability conditions (J1)-(J2). The solvability of the output estimation problem requires the conditions (Z) and (T2). Note that the conditions (E) and (C2) should be checked for infinitely many points of λ. Therefore, the implementation of the H^∞ requires more computational burden than the H^2 control case where the H^2 controller in the Smith form was implementable under the solvability of the delay-free H^2 control problem.* □

3.5 Alternative solvability condition via min-max optimization

The J-spectral factorizability conditions (J1)-(J2) compose a part of the solvability conditions for the output feedback problem. It requires verifying the regularity of the subblock of the symplectic matrix on the unbounded interval. In this section, we consider the min-max optimization approach in the full information problem. While the J-spectral factorization theory enables to generalize the delay function to general inner functions in the frequency-domain representation [21], the min-max optimization theory is suitable for considering time-domain specifications such as the initial-condition uncertainty and finite-horizon control [17].

The min-max optimization approach yields the H^∞ disturbance attenuation condition which requires checking the matrix positive definiteness only on finitely many points. Furthermore, the J-spectral factorizability condition is proved to be equivalent to the H^∞ disturbance attenuation condition, and consequently that approach is confirmed to yield the same control law as the min-max optimization approach.

Consider the multi-step input-delayed plant $P_{+\,\mathrm{FI}}$:

$$\begin{aligned} x(n+1) &= Ax(n) + B_1 w(n) + B_2 u(n-l), \\ z(n) &= C_1 x(n) + D_{12} u(n-l), \end{aligned} \tag{3.23}$$

where the state variable $x(n)$ and disturbance $w(n)$ are assumed to be available for the control purpose, and the delay length l is a non-negative integer. First, let us review the technique of the state-space augmentation. According to it, the state variable $\upsilon(n)$ maintaining the history of the past control input are introduced:

$$\upsilon(n) := \begin{bmatrix} \upsilon(1,n) \\ \upsilon(2,n) \\ \vdots \\ \upsilon(l,n) \end{bmatrix} := \begin{bmatrix} u(n-1) \\ u(n-2) \\ \vdots \\ u(n-l) \end{bmatrix} \in \mathbb{R}^{l\cdot \dim u}.$$

Then, $P_{+\,\mathrm{FI}}$ is rewritten into the delay-free form:

$$\begin{aligned} \begin{bmatrix} x(n+1) \\ \upsilon(n+1) \end{bmatrix} &= \widetilde{A} \begin{bmatrix} x(n) \\ \upsilon(n) \end{bmatrix} + \widetilde{B}_1 w(n) + \widetilde{B}_2 u(n), \\ z(n) &= \widetilde{C}_1 \begin{bmatrix} x(n) \\ \upsilon(n) \end{bmatrix} + \widetilde{D}_{12} u(n), \end{aligned}$$

$$\widetilde{A} := \begin{bmatrix} A & B_2\Gamma_1 \\ O & \mathcal{S} \end{bmatrix},\ \widetilde{B}_1 := \begin{bmatrix} B_1 \\ O \end{bmatrix},\ \widetilde{B}_2 := \begin{bmatrix} O \\ \Delta_l \end{bmatrix},\ \widetilde{C}_1 := \begin{bmatrix} C_1 & D_{12}\Gamma_1 \end{bmatrix},\ \widetilde{D}_{12} := O,$$

$$\mathcal{S} := \begin{bmatrix} O & I & & & \\ O & O & I & & \\ \vdots & & \ddots & \ddots & \\ \vdots & & & O & I \\ O & O & \cdots & \cdots & O \end{bmatrix} \in \mathbb{R}^{l\cdot\dim u \times l\cdot \dim u},$$

$$\begin{aligned} \Delta_l &:= \begin{bmatrix} O^{\mathrm{T}} & O^{\mathrm{T}} & \cdots & O^{\mathrm{T}} & I^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{l\cdot\dim u\times \dim u}, \\ \Gamma_1 &:= \begin{bmatrix} I & O & O & \cdots & O \end{bmatrix} \in \mathbb{R}^{\dim u \times l\cdot\dim u}. \end{aligned}$$

Referring to the above augmented state-space representation, let us introduce the following assumptions (A_+1)-(A_+2).

(A_+1) $\left(\widetilde{A},\, \widetilde{B}_2\right)$ is stabilizable.

(A_+2) For $\forall\,\theta \in [-\pi,\,\pi]$, $\begin{bmatrix} \widetilde{A} - e^{j\theta}I & \widetilde{B}_2 \\ \widetilde{C}_1 & \widetilde{D}_{12} \end{bmatrix}$ is of full column rank.

Under the assumptions (A_+1)-(A_+2), the H^∞ control problem with the performance bound γ is solvable if and only if the solution of the standard KYP equation

$$\begin{aligned} \widetilde{F}_\lambda^* \widetilde{R}_{\mathrm{mm\,c}\,\lambda} \widetilde{F}_\lambda &= \widetilde{Q} + \widetilde{A}^* \widetilde{X}_\lambda \widetilde{A} - \widetilde{X}_\lambda, \\ -\widetilde{R}_{\mathrm{mm\,c}\,\lambda} \widetilde{F}_\lambda &= \widetilde{S}^* + \widetilde{B}^* \widetilde{X}_\lambda \widetilde{A}, \\ \widetilde{R}_{\mathrm{mm\,c}\,\lambda} &= R^+_{\mathrm{mm}\,\lambda} + \widetilde{B}^* \widetilde{X}_\lambda \widetilde{B}, \end{aligned} \tag{3.24}$$

$$\widetilde{B} := \begin{bmatrix} \widetilde{B}_1 & B_{+2} \end{bmatrix}, \ \widetilde{S} := \begin{bmatrix} O & O \end{bmatrix},$$
$$\widetilde{Q} := \begin{bmatrix} Q & S_2\Gamma_1 \\ \Gamma_1^* S_2^* & \Gamma_1^* R_2 \Gamma_1 \end{bmatrix}, \ \widetilde{R}_{\mathrm{mm}\,\lambda} := \begin{bmatrix} -\lambda^2 I & O \\ O & O \end{bmatrix},$$
$$\begin{bmatrix} Q & S_2 \\ S_2^* & R_2 \end{bmatrix} := \begin{bmatrix} C_1 & D_{12} \end{bmatrix}^* \begin{bmatrix} C_1 & D_{12} \end{bmatrix}$$

for the augmented system satisfies the following conditions (FI$_+$1)-(FI$_+$2) for $\lambda = \gamma$.

(FI$_+$1) The solution $\widetilde{X}_\lambda$ is a positive semidefinite matrix such that $\widetilde{A} + \widetilde{B}\widetilde{F}_\lambda$ is stable.

(FI$_+$2) $\widetilde{R}_{\mathrm{mm\,c}\,\lambda}$ satisfies the definiteness conditions:

$$\widetilde{R}_{\mathrm{mm\,c}\,22\,\lambda} > O, \ -\lambda^2 \Lambda_{\mathrm{c}\,\lambda}^+ := \widetilde{R}_{\mathrm{mm\,c}\,11\,\lambda} - \widetilde{R}_{\mathrm{mm\,c}\,12\,\lambda} \widetilde{R}_{\mathrm{mm\,c}\,22\,\lambda}^{-1} \widetilde{R}_{\mathrm{mm\,c}\,21\,\lambda} < O.$$

If the conditions (FI$_+$1)-(FI$_+$2) are satisfied for $\lambda = \gamma$, then the H^∞ state feedback law is given by

$$u(n) = \left(\widetilde{F}_{2x\,\gamma} - \widetilde{F}_{21\,\gamma}\widetilde{F}_{1x\,\gamma}\right) x(n) + \left(\widetilde{F}_{2\upsilon\,\gamma} - \widetilde{F}_{21\,\gamma} F_{1\upsilon\,\gamma}\right) \upsilon(n) + \widetilde{F}_{21\,\gamma} w(n), \tag{3.25}$$

where $\widetilde{F}_{ix\,\lambda}$ and $\widetilde{F}_{i\upsilon\,\lambda}$ ($\lambda \geq \gamma$, $i = 1, 2$) are the subblocks of $\widetilde{F}_\lambda$ partitioned as

$$\widetilde{F}_\lambda = \begin{bmatrix} \widetilde{F}_{1\,\lambda} \\ \widetilde{F}_{2\,\lambda} \end{bmatrix} \in \mathbb{R}^{(\dim w + \dim u) \times (\dim x + l \cdot \dim u)},$$

$$\widetilde{F}_{1\,\lambda} = \begin{bmatrix} \widetilde{F}_{1x\,\lambda} & \widetilde{F}_{1\upsilon\,\lambda} \end{bmatrix} \in \mathbb{R}^{\dim w \times (\dim x + l \cdot \dim u)}, \ \widetilde{F}_{2\,\lambda} = \begin{bmatrix} \widetilde{F}_{2x\,\lambda} & \widetilde{F}_{2\upsilon\,\lambda} \end{bmatrix} \in \mathbb{R}^{\dim u \times (\dim x + l \cdot \dim u)}$$

conformably with $w(n)$, $u(n)$, $x(n)$ and $\upsilon(n)$, and $\widetilde{F}_{21\,\lambda}$ ($\lambda \geq \gamma$) is defined by

$$\widetilde{F}_{21\,\lambda} := -\widetilde{R}_{\mathrm{mm\,c}\,22\,\lambda}^{-1} \widetilde{R}_{\mathrm{mm\,c}\,21\,\lambda}. \tag{3.26}$$

Note that the augmented matrix $\widetilde{A}$ is sparse and singular even if A is not. Moreover, its order is given by $\dim x + l$, and increases linearly with respect to the delay length. Those properties of $\widetilde{A}$ make it difficult to compute $\widetilde{X}_\lambda$ for large delay lengths [4], [49].

In the sequel, we consider the min-max optimization approach for the efficient construction of the discrete-time H^∞ state feedback law. The computation of the stabilizing solution of the augmented KYP equation is reduced to that of the standard KYP equation

$$\begin{aligned} F_\lambda^* R_{\mathrm{mm\,c}\,\lambda} F_\lambda &= Q + A^* X_\lambda A - X_\lambda, \\ -R_{\mathrm{mm\,c}\,\lambda} F_\lambda &= S^* + B^* X_\lambda A, \\ R_{\mathrm{mm\,c}\,\lambda} &= R_{\mathrm{mm}\,\lambda} + B^* X_\lambda B, \end{aligned}$$

$$B := \begin{bmatrix} B_1 & B_2 \end{bmatrix}, \ S := \begin{bmatrix} O & S_2 \end{bmatrix}, \ R_{\mathrm{mm}\,\lambda} := \begin{bmatrix} -\lambda^2 I & O \\ O & R_2 \end{bmatrix}$$

associated with the H^∞ full information problem in the delay-free case.

The assumption (X) ensures the assumptions (A_+1)-(A_+2), and that, in the delay-free case ($l = 0$), the H^∞ state feedback law is obtained as follows:

$$u(n) = (F_{2\,\gamma} - F_{21\,\gamma} F_{1\,\gamma})\, x(n) + F_{21\,\gamma} w(n),$$

where $F_{1\,\lambda}$ and $F_{2\,\lambda}$ ($\lambda \geq \gamma$) are the subblocks of F_λ partitioned as

$$F_\lambda = \begin{bmatrix} F_{1\,\lambda} \\ F_{2\,\lambda} \end{bmatrix} \in \mathbb{R}^{(\dim w + \dim u)\times \dim x}$$

conformably with $w(n)$ and $u(n)$, and $F_{21\,\lambda}$ ($\lambda \geq \gamma$) is defined by $F_{21\,\lambda} := -R_{\mathrm{mm\,c}\,22\,\lambda}^{-1} R_{\mathrm{mm\,c}\,21\,\lambda}$.

3.5.1 Min-max optimization approach

Let us focus on the following min-max optimization problem:

$$\max_{w \in \ell^2[0,\,\infty)} \min_{u \in \ell^2[0,\,\infty)} J_\lambda(x(0), \upsilon(0);\, w, u),$$

where the infinite-horizon functional $J_\lambda(x(0), \upsilon(0);\, w, u)$ is defined by

$$J_\lambda(x(0), \upsilon(0);\, w, u) := \sum_{n=0}^{\infty} \|z(n)\|_2^2 - \lambda^2 \|w(n)\|_2^2$$

for $w \in \ell^2([0,\,\infty),\, \mathbb{R}^{\dim w})$ and $u \in \ell^2([0,\,\infty),\, \mathbb{R}^{\dim u})$. We reduce the infinite-horizon optimization to the maximization of the finite-horizon cost functional

$$J_{\mathrm{FH}\,\lambda}(x(0), \upsilon(0);\, w) := x^*(l) X_\lambda x(l) + \sum_{n=0}^{l-1} \|z(n)\|_2^2 - \lambda^2 \|w(n)\|_2^2 .$$

This enables to characterize the discrete-time H^∞ disturbance attenuation condition in terms of the following backward Riccati difference equation:

$$X_\lambda(n-1) = Q + A^* \left(I - \frac{1}{\lambda^2} X_\lambda(n) B_1 B_1^*\right)^{-1} X_\lambda(n) A,\ \ (n = l,\, l-1,\, \ldots,\, 1)\,,\ \ X_\lambda(l) = X_\lambda. \tag{3.27}$$

The following preliminary lemma ensures the solution of the finite-horizon optimization.

Lemma 5. *The following conditions* (N) *and* (R) *are equivalent:*

(N) *The cost functional $J_{\mathrm{FH}\,\lambda}(x(0), \upsilon(0);\, w)$ is negatively coercive under the zero initial condition:*

$$J_{\mathrm{FH}\,\lambda}(0, 0;\, w) \leq -\epsilon^2 \|w\|_{\ell^2[0,\,l-1]}^2 \tag{3.28}$$

for $\exists\, \epsilon > 0$ *and* $\forall\, w(\cdot) \in \ell^2([0,\, l-1],\, \mathbb{R}^{\dim w})$.

(R) *The solution $X_\lambda(n)$ of the Riccati difference equation satisfies the definiteness conditions:*

$$\lambda^2 I - B_1^* X_\lambda(n+1) B_1 > O \quad (n = l-1,\, l-2,\, \ldots, 0)\,.$$

If either of them is satisfied, the finite-horizon optimization admits the unique maximizer, and its value is given by the quadratic form

$$\max_{w\in\ell^2[0,\,l-1]} J_{\mathrm{FH}\,\lambda}(x(0),\upsilon(0);\ w) = \begin{bmatrix} x(0) \\ \upsilon(0) \end{bmatrix}^* \widetilde{X}_\lambda \begin{bmatrix} x(0) \\ \upsilon(0) \end{bmatrix},$$

where $\widetilde{X}_\lambda$ *is some positive semidefinite matrix.* □

Proof. The equivalence of the conditions (N) and (R) is shown by the dynamic programming argument [15], [17].

The condition (R) enables to complete the square in $J_{\mathrm{FH}\,\lambda}(x(0),\upsilon(0);\ w)$ as follows:

$$J_{\mathrm{FH}\,\lambda}(x(0),\upsilon(0);\ w) = \begin{bmatrix} x(0) \\ \upsilon(0) \end{bmatrix}^* \widetilde{X}_\lambda \begin{bmatrix} x(0) \\ \upsilon(0) \end{bmatrix} - \sum_{n=0}^{l-1} \left\| \left(\lambda^2 I - B_1^* X_\lambda(n) B_1\right)^{1/2} (w(n) - w_*(n)) \right\|_2^2, \tag{3.29}$$

where $w_*(n)$ and $\widetilde{X}_\lambda$ are constructed using the Riccati difference equation. By Eq. (3.29), the unique maximizer is w_*.

Choosing the disturbance $w(n) = 0$ in Eq. (3.29), we have the identity

$$\begin{bmatrix} x(0) \\ \upsilon(0) \end{bmatrix}^* \widetilde{X}_\lambda \begin{bmatrix} x(0) \\ \upsilon(0) \end{bmatrix} = J_{\mathrm{FH}\,\lambda}(x(0),\upsilon(0);\ 0) + \sum_{n=0}^{l-1} \left\| \left(\lambda^2 I - B_1^* X_\lambda(n) B_1\right)^{1/2} w_*(n) \right\|_2^2.$$

Therefore, $\widetilde{X}_\lambda$ is positive semidefinite. □

The following lemma shows that the value of the mim-max optimization is equal to that of the finite-horizon optimization.

Lemma 6. *The H^∞ full information problem with the performance bound λ is solvable, only if the condition* (R) *in Lemma 5 is satisfied. Moreover, the state feedback strategies:*

$$w_*(n) = \begin{cases} \arg\max\limits_{w\in\ell^2[0,\,l-1]} J_{\mathrm{FH}\,\lambda}(x(0),\upsilon(0);\ w) & (0 \le n \le l-1) \\ F_{1\,\lambda} x(n) & (n \ge l) \end{cases}, \tag{3.30}$$

$$u_*(n) = F_{2\,\lambda} x(n+l) \qquad (n \ge 0) \tag{3.31}$$

attain the optimal value of the min-max optimization, and it is expressed as

$$J_\lambda(x(0),\upsilon(0);\ w_*,u_*) = \max_{w\in\ell^2[0,\,l-1]} J_{\mathrm{FH}\,\lambda}(x(0),\upsilon(0);\ w).$$

Proof. Let us split $J_\lambda(x(0),\upsilon(0);\ w,u)$ as follows:

$$J_\lambda(x(0),\upsilon(0);\ w,u) = J_{\mathrm{FH}\,\lambda}(x(0),\upsilon(0);\ w) + \sum_{n=0}^{\infty} \|z(n+l)\|_2^2 - \lambda^2 \|w(n+l)\|_2^2,$$

where the first term on the right-hand side is independent of the control input as in [52]. By the assumption (X), the above equation is rewritten as

$$J_\lambda(x(0),\upsilon(0);\, w,u) = J_{\mathrm{FH}\,\lambda}(x(0),\upsilon(0);\, w) + \sum_{n=0}^{\infty} \left\| R^{1/2}_{\mathrm{mm\,c\,22}\,\lambda} \{u(n) - u_{\min}(n)\} \right\|_2^2 - \lambda^2 \sum_{n=0}^{\infty} \left\| \Lambda^{1/2}_{\mathrm{c}\,\lambda} \{w(n+l) - F_{1\,\lambda} x(n+l)\} \right\|_2^2,$$

where $u_{\min}(n)$ is temporarily defined by

$$u_{\min}(n) := (F_{2\,\lambda} - F_{21\,\lambda}F_{1\,\lambda})\, x(n+l) + F_{21\,\lambda} w(n+l). \tag{3.32}$$

Hence, the control input $u(n) := u_{\min}(n)$ minimizes $J_\lambda(x(0),\upsilon(0); w,u)$, and the minimum is given by

$$J_\lambda(x(0),\upsilon(0);\, w,u_{\min}) = J_{\mathrm{FH}\,\lambda}(x(0),\upsilon(0);\, w) - \lambda^2 \sum_{n=0}^{\infty} \left\| \Lambda^{1/2}_{\mathrm{c}\,\lambda} \{w(n+l) - F_{1\,\lambda} x(n+l)\} \right\|_2^2. \tag{3.33}$$

The following disturbance on the horizon $[l,\, \infty)$ maximizes the second term of the right-hand side of Eq. (3.33):

$$w_*(n+l) := F_{1\,\lambda} x(n+l) \ \ (n = 0,\, 1,\, \ldots)\,. \tag{3.34}$$

Substituting Eq. (3.34) into Eq. (3.32) with $w(n+l) := w_*(n+l)$, we find the control input in Eq. (3.31).

If an H^∞ control law exists, for $\exists\, \epsilon > 0$, the following inequalities should hold under the zero initial condition:

$$\min_{u\in \ell^2[0,\infty)} J_\lambda(0,0;\, w,u) \le -\epsilon^2 \|w\|^2_{\ell^2[0,\,\infty)} \le -\epsilon^2 \|w\|^2_{\ell^2[0,\,l-1]}\,.$$

Substituting $w_*(n+l)$ in Eq. (3.34) into the above inequalities leads to Eq. (3.28). This implies, by Lemma 5, the necessity of the condition (R) and the existence of the maximizer of $J_{\mathrm{FH}\,\lambda}(x(0),\upsilon(0);\, w)$. □

The optimal strategies in Eqs. (3.30) and (3.31) are in time-varying form. We prepare the following lemma to rewrite them into time-invariant form in Lemma 9.

Lemma 7. *Introduce the auxiliary variables $p(n) \in \mathbb{R}^{\dim x}$ and*

$$\kappa(n) := \begin{bmatrix} \kappa(1,n)^{\mathrm{T}} & \kappa(2,n)^{\mathrm{T}} & \cdots & \kappa(l,n)^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{l\cdot \dim u}.$$

Then $w_*(n)$ and $u_*(n)$ in Eqs. (3.30) and (3.31) are generated from the following time-invariant equations:

$$x(n+1) = Ax(n) + B_1 w_*(n) + B_2\Gamma_1 \upsilon(n), \tag{3.35}$$

$$A^* p(n+1) = Qx(n) + p(n) + S_2\Gamma_1\upsilon(n), \tag{3.36}$$

$$\upsilon(n+1) = \mathcal{S}\upsilon(n) + \Delta_l u_*(n), \tag{3.37}$$

$$\mathcal{S}^*\kappa(n+1) = \Gamma_1^* S_2^* x(n) - \Gamma_1^* B_2^* p(n+1) + \kappa(n), \tag{3.38}$$

$$0 = -B_1^* p(n+1) - \lambda^2 w_*(n), \tag{3.39}$$

$$0 = \Delta_l^* \kappa(n+1) + R_2 u_*(n), \tag{3.40}$$

$$p(n+l) = -X_\lambda x(n+l). \tag{3.41}$$

□

Proof. First we remark on the generation of $w_*(n)$ $(0 \le n \le l-1)$. Introduce $p(n)$ as the Lagrange multiplier for the state variable $x(n)$ on the horizon $[0,\ l-1]$, then the first-order conditions for the extremum of $J_{\mathrm{FH}\,\lambda}(x(0), \upsilon(0);\ w)$ are given by Eqs. (3.35)-(3.37), (3.39) and the following equation:

$$p(l) = -X_\lambda x(l). \tag{3.42}$$

Hence, $w_*(n)$ $(0 \le n \le l-1)$ is generated from Eqs. (3.35)-(3.37), (3.39) and (3.42).

Second we remark on the generation of $w_*(n)$ $(n \ge l)$ and $u_*(n)$ $(n \ge 0)$. Noting that $\kappa(n)$ in Eqs. (3.38) and (3.40) acts as the lead element which transfers the future signal $S_2^* x(n+l) - B_2^* p(n+l+1)$ to the current control input $u_*(n)$, it is verified that $w_*(n)$ $(n \ge l)$ and $u_*(n)$ $(n \ge 0)$ are generated from Eqs. (3.35)-(3.40) and (3.42).

In summary, $w_*(n)$ and $u_*(n)$ are generated from Eqs. (3.35)-(3.40) and (3.42). We can replace this Eq. (3.42) with Eq. (3.41), since the constraint in Eq. (3.42) is positively invariant under the time evolution defined by Eqs. (3.35)-(3.40). □

The following lemma is introduced to solve the two-point boundary-value problem defined by Eqs. (3.35), (3.36) and (3.41). It enables to derive the optimal strategies in time-invariant form and the corresponding value of $J_\lambda(x(0), \upsilon(0);\ w, u)$ in Lemma 9. Furthermore, the positive semidefinite matrix defining the optimal value is assured to be the stabilizing solution of the augmented KYP equation in Lemma 10.

Lemma 8. *Under the assumption (H), the following conditions (D1) and (D2) are equivalent:*

(D1) *The Riccati difference equation (3.27) is well-defined for* $n = l-1,\ l-2,\ \ldots,\ 0$*:*

$$\det\left(I - \frac{1}{\lambda^2} X_\lambda(n+1) B_1 B_1^*\right) \neq 0. \tag{3.43}$$

(D2) *The* $(2,\ 2)$ *block of* $\mathcal{E}_{X\,\lambda}(l-n)$ *defined by Eq. (3.1) is regular for* $n = l-1,\ l-2,\ \ldots,\ 0$*:*

$$\det\left(X_\lambda \mathcal{E}_{12\,\lambda}(l-n) + \mathcal{E}_{22\,\lambda}(l-n)\right) \neq 0. \tag{3.44}$$

If either of the above conditions is satisfied, the solution $X_\lambda(n)$ *of the backward Riccati difference equation is represented as*

$$X_\lambda(n) = \left(X_\lambda \mathcal{E}_{12\,\lambda}(l-n) + \mathcal{E}_{22\,\lambda}(l-n)\right)^{-1} \left(X_\lambda \mathcal{E}_{11\,\lambda}(l-n) + \mathcal{E}_{21\,\lambda}(l-n)\right). \tag{3.45}$$

□

Proof. Subsection 3.7.4. □

Lemma 9. *The optimal strategies in Eqs. (3.30) and (3.31) are represented in the following time-invariant forms for any time instant* $n \ge 0$*:*

$$w_*(n) = \widetilde{F}_{1x\,\lambda} x(n) + \widetilde{F}_{1\upsilon\,\lambda} \upsilon(n), \tag{3.46}$$

$$u_*(n) = \widetilde{F}_{2x\,\lambda} x(n) + \widetilde{F}_{2\upsilon\,\lambda} \upsilon(n), \tag{3.47}$$

where the feedback gains are constructed as follows:

$$\widetilde{F}_{1x\,\lambda} := -\frac{1}{\lambda^2}B_1^* A^{-*}\left\{Q - X_\lambda(0)\right\},$$

$$\widetilde{F}_{1\upsilon\,\lambda} := \begin{bmatrix}\widetilde{F}_{1\upsilon\,\lambda}(1) & \widetilde{F}_{1\upsilon\,\lambda}(2) & \cdots & \widetilde{F}_{1\upsilon\,\lambda}(l)\end{bmatrix},$$

$$\widetilde{F}_{1\upsilon\,\lambda}(\theta) := -\frac{1}{\lambda^2}B_1^* A^{-*}\left\{S_2\ \delta_{1,\theta} - \begin{bmatrix}X_\lambda(0) & I\end{bmatrix} H_{\mathrm{FH}\,\delta\,\lambda}^{-1}\begin{bmatrix}B_{2\,\lambda}^{\mathcal{D}}(\theta)\\ S_{2\,\lambda}^{\mathcal{D}}(\theta)\end{bmatrix}\right\}\ (\theta = 1,\,2,\,\ldots,\,l),$$

$$\widetilde{F}_{2x\,\lambda} := \widetilde{F}_{2\lambda}\left\{\mathcal{E}_{11\,\lambda}(l) - \mathcal{E}_{12\,\lambda}(l)X_\lambda(0)\right\},$$

$$\widetilde{F}_{2\upsilon\,\lambda} := \begin{bmatrix}\widetilde{F}_{2\upsilon\,\lambda}(1) & \widetilde{F}_{2\upsilon\,\lambda}(2) & \cdots & \widetilde{F}_{2\upsilon\,\lambda}(l)\end{bmatrix},$$

$$\widetilde{F}_{2\upsilon\,\lambda}(\theta) := F_{2\lambda}\begin{bmatrix}\mathcal{E}_{11\,\lambda}(l) - \mathcal{E}_{12\,\lambda}(l)X_\lambda(0) & O\end{bmatrix} H_{\mathrm{FH}\,\delta\,\lambda}^{-1}\begin{bmatrix}B_{2\,\lambda}^{\mathcal{D}}(\theta)\\ S_{2\,\lambda}^{\mathcal{D}}(\theta)\end{bmatrix}\ (\theta = 1,\,2,\,\ldots,\,l),$$

and $X_\lambda(0)$ *is given by Eq. (3.45) with* $n = 0$. *Moreover, let* $\widetilde{X}_\lambda$ *be the positive semidefinite matrix defining the optimal value of the min-max optimization:*

$$J_\lambda(x(0), \upsilon(0);\, w_*, u_*) = \begin{bmatrix}x(0)\\ \upsilon(0)\end{bmatrix}^* \widetilde{X}_\lambda \begin{bmatrix}x(0)\\ \upsilon(0)\end{bmatrix},$$

and partition it conformably with $x(0)$ *and* $\upsilon(0)$ *as*

$$\widetilde{X}_\lambda = \begin{bmatrix}\widetilde{X}_{x\,\lambda} & \widetilde{X}_{x\upsilon\,\lambda}\\ \widetilde{X}_{\upsilon x\,\lambda} & \widetilde{X}_{\upsilon\,\lambda}\end{bmatrix}. \tag{3.48}$$

Then each of the above blocks is given as follows:

$$\widetilde{X}_{x\,\lambda} := X_\lambda(0),$$

$$\widetilde{X}_{x\upsilon\,\lambda} := \begin{bmatrix}\widetilde{X}_{x\upsilon\,\lambda}(1) & \widetilde{X}_{x\upsilon\,\lambda}(2) & \cdots & \widetilde{X}_{x\upsilon\,\lambda}(l)\end{bmatrix},$$

$$\widetilde{X}_{x\upsilon\,\lambda}(\theta) := \begin{bmatrix}X_\lambda(0) & I\end{bmatrix} H_{\mathrm{FH}\,\delta\,\lambda}^{-1}\begin{bmatrix}B_{2\,\lambda}^{\mathcal{D}}(\theta)\\ S_{2\,\lambda}^{\mathcal{D}}(\theta)\end{bmatrix}\ (\theta = 1,\,2,\,\ldots,\,l),$$

$$\widetilde{X}_{\upsilon x\,\lambda} := \widetilde{X}_{x\upsilon\,\lambda}^*,$$

$$\widetilde{X}_{\upsilon\,\lambda} := \begin{bmatrix}\widetilde{X}_{\upsilon\,\lambda}(1,1) & \widetilde{X}_{\upsilon\,\lambda}(1,2) & \cdots & \widetilde{X}_{\upsilon\,\lambda}(1,l)\\ \widetilde{X}_{\upsilon\,\lambda}(2,1) & \widetilde{X}_{\upsilon\,\lambda}(2,2) & \cdots & \vdots\\ \vdots & \vdots & \ddots & \vdots\\ \widetilde{X}_{\upsilon\,\lambda}(l,1) & \cdots & \cdots & \widetilde{X}_{\upsilon\,\lambda}(l,l)\end{bmatrix},$$

$$\begin{aligned}\widetilde{X}_{\upsilon\,\lambda}(\phi,\theta) := {}& \left(R_2 - S_2^* A^{-1}B_2\right)\delta_{\phi,\theta} + S_{2\,\lambda}^{\mathcal{D}\,*}(\phi)A^{-1}B_{2\,\lambda}^{\mathcal{D}}(\theta)\\ & -\frac{1}{\lambda^2}S_{2\,\lambda}^{\mathcal{D}\,*}(\phi)A^{-1}B_1B_1^*A^{-*}S_{2\,\lambda}^{\mathcal{D}}(\theta) + B_{2\,\lambda}^{\mathcal{D}\,*}(\phi)X_\lambda(0)B_{2\,\lambda}^{\mathcal{D}}(\theta)\ \ (\textit{if}\ 1 \le \theta \le \phi \le l),\end{aligned}$$

$$\widetilde{X}_{\upsilon\,\lambda}(\phi,\theta) := \widetilde{X}_{\upsilon\,\lambda}(\theta,\phi)^*\quad (\textit{if}\ 1 \le \phi < \theta \le l).$$

□

Proof. Fix the time instant $n \geq 0$ and focus on the evolution equations (3.35)-(3.41) on the horizon $[n,\, n+l]$. By the variation of constants formula, the state variables $x(n+t)$ and $p(n+t)$ $(t = 0,\, 1,\, \ldots,\, l)$ in the two-point boundary-value problem defined by Eqs. (3.35), (3.36) and (3.41) are expressed with the state variables $x(n)$ and $\upsilon(n)$ as follows:

$$\begin{aligned}\begin{bmatrix} x(n+t) \\ p(n+t) \end{bmatrix} =& \mathcal{E}_\lambda(t) \begin{bmatrix} I \\ -X_\lambda(0) \end{bmatrix} x(n) \\ &+ \sum_{\theta=0}^{t-1} \mathcal{E}_\lambda(t) \begin{bmatrix} I & O \\ -X_\lambda(0) & O \end{bmatrix} H_{\mathrm{FH}\,\delta\,\lambda}^{-1} \begin{bmatrix} B_{2\,\lambda}^{\mathcal{D}}(\theta+1) \\ S_{2\,\lambda}^{\mathcal{D}}(\theta+1) \end{bmatrix} \upsilon(\theta+1, n) \\ &- \sum_{\theta=t}^{l-1} \mathcal{E}_\lambda(t) \begin{bmatrix} O & O \\ X_\lambda(0) & I \end{bmatrix} H_{\mathrm{FH}\,\delta\,\lambda}^{-1} \begin{bmatrix} B_{2\,\lambda}^{\mathcal{D}}(\theta+1) \\ S_{2\,\lambda}^{\mathcal{D}}(\theta+1) \end{bmatrix} \upsilon(\theta+1, n). \end{aligned} \tag{3.49}$$

Substituting $p(n+t)|_{t=1}$ and $x(n+t)|_{t=l}$ into Eqs. (3.39) and (3.31), respectively, Eqs. (3.46) and (3.47) are derived.

Next let us derive the closed-form expression for $\widetilde{X}_\lambda$ in Eq. (3.48). As noted in the proof of Lemma 7, the conditions for the extremum of $J_{\mathrm{FH}\,\lambda}(x(0), \upsilon(0);\, w)$ are given by Eqs. (3.35)-(3.37), (3.39) and (3.42). Then the summand of $J_{\mathrm{FH}\,\lambda}(x(0), \upsilon(0);\, w)$ is represented by the increment of $\mathrm{Re}\, p^*(n)x(n)$ as follows:

$$\begin{aligned}\|z(n)\|_2^2 - \lambda^2 \|w(n)\|_2^2 =& \{\mathrm{Re}\, p^*(n+1)x(n+1) - \mathrm{Re}\, p^*(n)x(n)\} \\ &+ \mathrm{Re}\, \upsilon^*(n+1,0)B_2^* x(n) - \mathrm{Re}\, \upsilon^*(n+1,0)S_2^* p(n+1) \\ &+ \upsilon^*(n+1,0)R_2 \upsilon(n+1,0). \end{aligned}$$

Summing the above equations for $n = 0,\, 1,\, \ldots,\, l-1$, we have the equation

$$\begin{aligned} J_{\mathrm{FH}\,\lambda}(x(0), \upsilon(0);\, w_*) =& -\mathrm{Re}\, p^*(0)x(0) + \sum_{n=0}^{l-1} \mathrm{Re} \left(H_{\mathrm{FH}\,\delta\,\lambda}^{-1} \begin{bmatrix} B_2 \\ S_2 \end{bmatrix} \upsilon(n+1,0) \right)^* J_{\mathrm{s}} \begin{bmatrix} x(n+1) \\ p(n+1) \end{bmatrix} \\ &+ \sum_{n=0}^{l-1} \mathrm{Re}\, \upsilon^*(n+1,0) \left(R_2 - S_2^* A^{-1} B_2 \right) \upsilon(n+1,0). \end{aligned}$$

Substituting $x(n+t)|_{t=1}$ and $p(n+t)|_{t=1}$ derived from Eq. (3.49) into the above equation, the closed-form expression is determined. □

Lemma 10. *The positive semidefinite matrix $\widetilde{X}_\lambda$ and feedback gain $\widetilde{F}_\lambda$ constructed in Lemma 9 satisfy the augmented KYP equation, and $\widetilde{A} + \widetilde{B}\widetilde{F}_\lambda$ is stable.* □

Proof. Subsection 3.7.5. □

The following theorem characterizes the discrete-time H^∞ disturbance attenuation condition in terms of the Riccati difference equation.

Theorem 5. *Under the conditions (X) and (H), the H^∞ full information problem with the performance bound γ is solvable if and only if the following conditions (DA1)-(DA2) are satisfied.*

(DA1) *For $\lambda = \gamma$, the condition* (R) *is satisfied.*

(DA2) *For $\lambda = \gamma$, $\widetilde{X}_\lambda$ constructed in Lemma 9 satisfies the definiteness conditions in* (FI_+2).

□

Proof. By Lemma 6, the condition (DA1) is necessary. Moreover, by Lemma 10, $\widetilde{X}_\gamma$ is the stabilizing solution. Therefore, by the uniqueness of the stabilizing solution, it should satisfy the definiteness conditions in (FI_+2). The converse is obvious. □

By summarizing Lemma 9 and Theorem 5, the design procedure for the H^∞ state feedback law with the performance bound γ is described as follows.

(Step1) Check the condition (DA1) by iterating the Riccati difference equation in Eq. (3.27). If it is satisfied, apply Lemma 9 to construct $\widetilde{F}_\gamma$ and $\widetilde{X}_\gamma$ in Eqs. (3.46)-(3.48) with $\lambda = \gamma$.

(Step2) Check the condition (DA2) by substituting $\widetilde{X}_\gamma$ into the right-hand side of Eq. (3.24) with $\lambda = \gamma$. If it is satisfied, define $\widetilde{F}_{21\,\gamma}$ by Eq. (3.26) with $\lambda = \gamma$. Then implement the state feedback law in Eq. (3.25).

3.5.2 Interpretation of approaches

In Subsections 3.4.1 and 3.5.1 we presented the two independent approaches for the reduced-order construction of the H^∞ state feedback law. Their feasibility conditions, namely, the H^∞ disturbance attenuation conditions (DA1)-(DA2) and the J-spectral factorizability conditions (J1)-(J2) were seemingly different. In this section, we verify that the resulting control laws are identical by focusing on a common interpretation to the conditions (DA1) and (J1).

Define $\mathcal{G}_\lambda$ as the input-out mapping from $\{w(n)\}_{n=0}^{l-1}$ to $\left(\{z(n)\}_{n=0}^{l-1},\, X_\lambda^{1/2} x(l)\right)$ under the zero initial condition that $x(0) = 0$ and $v(0) = 0$: $\begin{bmatrix} z \\ X_\lambda^{1/2} x(l) \end{bmatrix} = \mathcal{G}_\lambda w$. The operator describes the response of the regulated output and terminal state variable while the control input cannot affect them due to the input-delay.

For the proof of the equivalence, we focus on the fact that the inequalities $\sup_w \dfrac{\|\mathcal{G}_\lambda w\|_2}{\|w\|_2} < \lambda$ or $\lambda^2 I - \mathcal{G}_\lambda^* \mathcal{G}_\lambda > O$ holds if and only if the condition (N) in Lemma 5 is satisfied.

The following lemma characterizes the eigenvalue configuration of $\mathcal{G}_\lambda^* \mathcal{G}_\lambda$ in terms of the symplectic matrix $\mathcal{E}_\lambda(n)$. It can be regarded as a discrete-time counterpart of Theorem 13.5.1 in [6].

Lemma 11. *Let $\mathcal{M}_\lambda := \sigma\left(\mathcal{G}_\lambda^* \mathcal{G}_\lambda\right) \backslash \{0\}$ be the set of the non-zero eigenvalues of $\mathcal{G}_\lambda^* \mathcal{G}_\lambda$. It is characterized as follows:*

$$\mathcal{M}_\lambda = \left\{\mu^2 \;\middle|\; \mu > 0 \textit{ such that } \det\left(X_\lambda \mathcal{E}_{12\,\mu}(l) + \mathcal{E}_{22\,\mu}(l)\right) = 0\right\}.$$

□

Proof. Subsection 3.7.6. □

We further derive the following lemma by interpreting the conditions (DA1) and (J1) as the positive definiteness of the operator $\lambda^2 I - \mathcal{G}_\lambda^* \mathcal{G}_\lambda$.

Lemma 12. *The conditions (DA1) in Theorem 5 and (J1) in Theorem 3 are equivalent.* □

Proof. ((J1) ⇒ (DA1)) Since X_λ is continuous and monotonically non-increasing with respect to $\lambda \geq \gamma$ [17], there exists $c \geq \gamma$ such that for $\forall \lambda > c$ the following inequality holds.

$$\lambda^2 I - \mathcal{G}_\lambda^* \mathcal{G}_\lambda > O. \tag{3.50}$$

We show that the inequality (3.50) also holds for $\gamma \leq \forall \lambda \leq c$ by contradiction. Suppose that the inequality (3.50) does not hold for $\gamma \leq \exists \lambda \leq c$. Since the eigenvalues of $\lambda^2 I - \mathcal{G}_\lambda^* \mathcal{G}_\lambda$ are continuous with respect to λ, there exists $\gamma \leq \lambda_0 \leq c$ such that

$$\sigma\left(\lambda_0^2 I - \mathcal{G}_{\lambda_0}^* \mathcal{G}_{\lambda_0}\right) \ni 0. \tag{3.51}$$

Note that the following inclusion holds by Lemma 11.

$$\sigma\left(\lambda_0^2 I - \mathcal{G}_{\lambda_0}^* \mathcal{G}_{\lambda_0}\right) = \left\{\lambda_0^2\right\} - \sigma\left(\mathcal{G}_{\lambda_0}^* \mathcal{G}_{\lambda_0}\right) \subset \left\{\lambda_0^2\right\} - \{0\} \cup \mathcal{M}_{\lambda_0}. \tag{3.52}$$

By Eqs. (3.51) and (3.52), $\lambda_0^2 \in \mathcal{M}_{\lambda_0}$. Hence, we have

$$\det\left(X_{\lambda_0} \mathcal{E}_{12\,\lambda_0}(l) + \mathcal{E}_{22\,\lambda_0}(l)\right) = 0.$$

This contradicts the condition (E) for $\lambda = \lambda_0$. Therefore, Eq. (3.50) holds for $\gamma \leq \forall \lambda \leq c$. Substituting $\lambda = \gamma$ into Eq. (3.50) and applying Lemma 5, we reach the condition (DA1).
((DA1) ⇒ (J1)) The inequality $\sup_w \dfrac{\|\mathcal{G}_\gamma w\|_2}{\|w\|_2} < \gamma$ holds by Lemma 5. By the non-increasing monotonicity of X_λ, we have the following inequalities for $\forall \lambda \geq \gamma$:

$$\sup_{\substack{w \in \ell^2[0,\, l-1] \\ w \neq 0}} \frac{\|\mathcal{G}_\lambda w\|_2}{\|w\|_2} \leq \sup_{\substack{w \in \ell^2[0,\, l-1] \\ w \neq 0}} \frac{\|\mathcal{G}_\gamma w\|_2}{\|w\|_2} < \gamma \leq \lambda.$$

Again by Lemma 5, the condition (R) is satisfied for $\forall \lambda \geq \gamma$. Furthermore, the condition (R) implies the condition (E) by Lemma 8. □

The following theorem claims the equivalence between the H^∞ disturbance attenuation condition in Theorem 5 and the J-spectral factorizability condition in Theorem 3. It also evinces that the condition (J2), a part of the J-spectral factorizability condition, can be weakened to the following condition ($\mathrm{J_w}$2) for the multi-step delay functions.

($\mathrm{J_w}$2) For $\lambda = \gamma$, the condition (C2) is satisfied.

Theorem 6. *The H^∞ disturbance attenuation conditions (DA1)-(DA2), the J-spectral factorizability conditions (J1)-(J2) and the weaker conditions (J1)-($\mathrm{J_w}$2) are equivalent.* □

Proof. ((DA1)-(DA2) $\Rightarrow$ (J1)-(J2)) By Lemma 12, the condition (J1) is satisfied. Therefore, for $\forall\,\lambda \geq \gamma$, $R^{\mathrm{R}}_{\mathrm{mm\,c}\,\lambda}$ in Eq. (3.7) is constructed by Lemma 3.

Recall that the conditions (DA1)-(DA2) are necessary and sufficient for the H^∞ disturbance attenuation with the performance bound γ. This implies that for $\forall\,\lambda \geq \gamma$, $\widetilde{R}_{\mathrm{mm\,c}\,\lambda}$ in Eq. (3.24) is constructed by Lemma 9, and that the definiteness conditions on it in (FI$_+$2) are satisfied.

By direction calculation, it is verified that the weight matrices $R^{\mathrm{R}}_{\mathrm{mm\,c}\,\lambda}$ and $\widetilde{R}_{\mathrm{mm\,c}\,\lambda}$ constructed as above are equal:

$$R^{\mathrm{R}}_{\mathrm{mm\,c}\,\lambda} = \widetilde{R}_{\mathrm{mm\,c}\,\lambda}. \tag{3.53}$$

This equality and the definiteness conditions on $\widetilde{R}_{\mathrm{mm\,c}\,\lambda}$ in (FI$_+$2) enforce that those on $R^{\mathrm{R}}_{\mathrm{mm\,c}\,\lambda}$ in (C2) be satisfied for $\forall\,\lambda \geq \gamma$.
((J1)-(J2) $\Rightarrow$ (J1)-(J$_\mathrm{w}$2)) This direction is obvious.
((J1)-(J$_\mathrm{w}$2) $\Rightarrow$ (DA1)-(DA2)) By Lemma 12, the condition (DA1) is satisfied. Hence, $X_{+\,\gamma}$ is constructed by Lemma 9, and Eq. (3.53) with $\lambda = \gamma$ holds. Then the condition (J$_\mathrm{w}$2) implies the condition (DA2). □

The following theorem claims that the two approaches yield the identical control law. Note that the representation $z^{-\theta}u(n)$ ($\theta = 1, 2, \ldots, l$) used in the J-spectral factorization approach corresponds to $u(n-\theta) = \upsilon(\theta, n)$ in the augmented state-space approach.

Theorem 7. *The H^∞ state feedback law obtained in Corollary 1 is identical to that constructed following the steps (Step1)-(Step2).* □

Proof. By Lemmas 3 and 9, the state feedback gains of $x(n)$ are identical: $F^{\mathrm{R}}_{1\,\gamma} = \widetilde{F}_{1x\,\gamma}$, $F^{\mathrm{R}}_{2\,\gamma} = \widetilde{F}_{2x\,\gamma}$. By Eqs. (3.26), (3.15) and (3.7) with $\lambda = \gamma$, the disturbance feedforward gains of $w(n)$ are also identical: $F^{\mathrm{R}}_{21\,\gamma} = \widetilde{F}_{21\,\gamma}$.

Since both of $\Pi_{x\,\gamma}(z)$ and $\chi_{z^{-l}\ominus\,\gamma}(z)$ are strictly causal, and of lth-order finite impulse response, the third term on the right-hand side of Eq. (3.18) is rewritten as

$$\left\{\overline{F}_\gamma(1)z^{-1} + \overline{F}_\gamma(2)z^{-2} + \cdots + \overline{F}_\gamma(l)z^{-l}\right\} u(n)$$

for the appropriate matrices $\overline{F}_\gamma(\theta)$ ($\theta = 1, 2, \ldots, l$). Furthermore, it is verified that $\overline{F}_\gamma(\theta)$ is identical to the state feedback gain of $\upsilon(\theta, n)$, namely, $\widetilde{F}_{2\upsilon\,\gamma}(\theta) - \widetilde{F}_{21\,\gamma}\widetilde{F}_{1\upsilon\,\gamma}(\theta)$ constructed by Lemma 9 with $\lambda = \gamma$. □

Remark 9. *Both of the J-spectral factorizability conditions (J1)-(J2) and H^∞ disturbance attenuation conditions (DA1)-(DA2) explicitly indicate the additional requirements for the corresponding delay-free KYP equation in (X) while the LQ reduced-order construction in [43] is always possible if the Riccati equation for the delay-free case only has the positive semidefinite stabilizing solution.*

The J-spectral factorizability conditions (J1)-(J2) involves checking the regularity of $\mathcal{E}_{X\,22\,\lambda}(l)$ for the varying performance bound $\lambda \geq \gamma$ and fixed delay length $\theta = l$. Due to this property the J-spectral factorizability conditions (J1)-(J2) are suitable for finding the achievable performance limits for large input delays. On the other hand, the H^∞ disturbance attenuation conditions (DA1)-(DA2) require the definiteness of $\lambda^2 I - B_1^{\mathrm{T}} X_\lambda(l-\theta) B_1$ for $\theta \in [1, l]$ and *fixed performance bound $\lambda = \gamma$. Therefore, the H^∞ disturbance attenuation conditions (DA1)-(DA2) are suitable for computing the maximal delay length for which a given performance bound γ is achievable.* □

From the viewpoint of numerical accuracy, it seems that the H^∞ disturbance attenuation conditions (DA1)-(DA2) are superior to the J-spectral factorizability conditions (J1)-(J2) because the former require only checking the matrix positive definiteness on finitely many points of the parameter. The correspondence of the state feedback gains in Theorem 7 allows us to implement the output feedback controller in Theorem 4 by replacing the J-spectral factorizability condition with the H^∞ disturbance attenuation condition.

3.6 Example

Consider the input-delayed second order system

$$\begin{aligned} x(n+1) &= \begin{bmatrix} 1.1 & 0.1 \\ 0 & -0.8 \end{bmatrix} x(n) + \begin{bmatrix} 0.3 \\ 1.2 \end{bmatrix} w(n) + \begin{bmatrix} 0.6 \\ 2.3 \end{bmatrix} u(n-l), \\ z(n) &= \begin{bmatrix} 1.2 & 0.3 \end{bmatrix} x(n) + 2.2u(n-l). \end{aligned}$$

The augmented KYP equation for this system has the order of $2 \cdot (2+l)$. The bisection method of solving it iteratively is a straightforward way to compute the achievable full information H^∞ performance $\gamma^{\rm opt}_{+\,\rm FI}(l)$. However, numerical computation of its stabilizing solution requires special effort [4], [49]. The results in this chapter enable to check the solvability of the full information problem by only solving the KYP equations for the delay-free case and checking the eigenvalue configurations. In the following subsections, we illustrate the features of Theorems 3 and 5 by calculating $\gamma^{\rm opt}_{+\,\rm FI}(l)$ based on them.

3.6.1 Performance limit via J-spectral factorization

Let us calculate $\gamma^{\rm opt}_{+\,\rm FI}(l)$ based on Theorem 3. First, we need to find the minimal performance bound $\gamma^{\rm opt}_{+\,\mathcal{E}}(l)$ satisfying the condition (J1). Its concrete expression is given by

$$\gamma^{\rm opt}_{+\,\mathcal{E}}(l) = \inf \left\{ \gamma > \gamma^{\rm opt}_{\rm FI} \;\middle|\; \det \mathcal{E}_{X\,22\,\lambda}(l) \neq 0 \;\; (\forall \lambda \geq \gamma) \right\},$$

where $\gamma^{\rm opt}_{\rm FI} = 1.14$ is the achievable H^∞ performance for the delay-free case. Therefore, $\gamma^{\rm opt}_{+\,\mathcal{E}}(l)$ is determined by the roots of the equation $\det \mathcal{E}_{X\,22\,\lambda}(l) = 0$.

Since an upper bound for λ guaranteeing the non-zeroness of $\det \mathcal{E}_{X\,22\,\lambda}(l)$ is not available, we traced $\det \mathcal{E}_{X\,22\,\lambda}(l)$ up to a sufficiently large λ, and determined the values of $\gamma^{\rm opt}_{+\,\mathcal{E}}(l)$ as depicted in Fig. 3.8.

Next, we check the condition (J2) to determine $\gamma^{\rm opt}_{+\,\rm FI}(l)$ as follows:

$$\gamma^{\rm opt}_{+\,\rm FI}(l) = \inf \left\{ \lambda > \gamma^{\rm opt}_{+\,\rm FI\,(22)}(l) \;\middle|\; \Lambda^{\rm R}_{\rm c\,\lambda} > O \right\},$$

where $\gamma^{\rm opt}_{+\,\rm FI\,(22)}(l)$ is defined using $\gamma^{\rm opt}_{+\,\rm Ric}(l)$:

$$\gamma^{\rm opt}_{+\,\rm FI\,(22)}(l) := \inf \left\{ \lambda > \gamma^{\rm opt}_{+\,\rm Ric}(l) \;\middle|\; R^{\rm R}_{\rm mm\,22\,\lambda} > O \right\}.$$

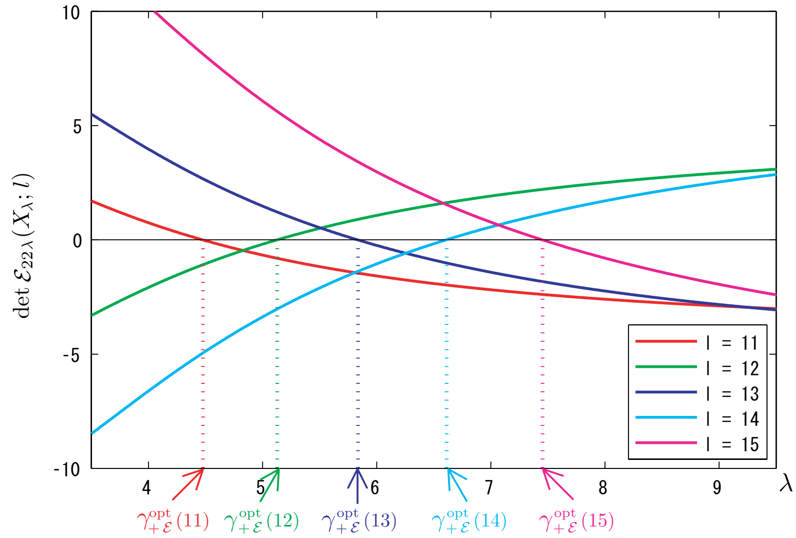

Figure 3.8: Calculation of $\gamma^{\rm opt}_{+\mathcal{E}}(l)$ for $11 \le l \le 15$.

From the graphs of $\lambda_{\rm min}(R^{\rm R}_{{\rm mm}\,22\,\lambda})$ and $\lambda_{\rm min}(\Lambda^{\rm R}_{{\rm c}\,\lambda})$ in Figs. 3.9 and 3.10, the definiteness conditions on $R^{\rm R}_{{\rm mm}\,\lambda}$ are satisfied for $\forall\lambda \ge \gamma^{\rm opt}_{+\mathcal{E}}(l)$. Therefore, in this example, $\gamma^{\rm opt}_{+\,{\rm FI}}(l)$ is found to be equal to $\gamma^{\rm opt}_{+\mathcal{E}}(l)$ (Table 3.1).

The values of $\gamma^{\rm opt}_{+\,{\rm FI}}(l)$ determined as above is depicted in Fig. 3.11. It is observed that the achievable H^∞ performance severely deteriorates as the input-delay length increases.

Table 3.1: Values of $\gamma^{\rm opt}_{+\mathcal{E}}$, $\gamma^{\rm opt}_{+\,{\rm FI}\,(22)}$ and $\gamma^{\rm opt}_{+\,{\rm FI}}$.

λ \ l	11	12	13	14	15
$\gamma^{\rm opt}_{+\mathcal{E}}$	4.48	5.13	5.84	6.61	7.45
$\gamma^{\rm opt}_{+\,{\rm FI}\,(22)}$	4.48	5.13	5.84	6.61	7.45
$\gamma^{\rm opt}_{+\,{\rm FI}}$	4.48	5.13	5.84	6.61	7.45

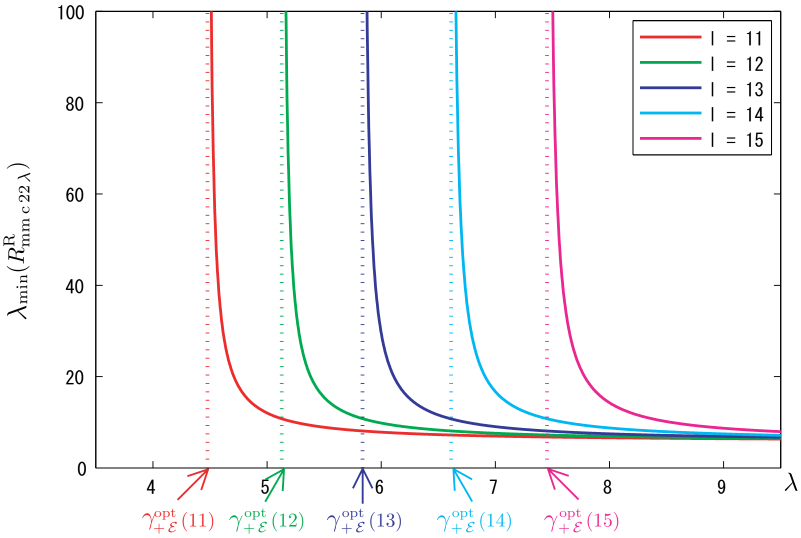

Figure 3.9: Positive definiteness of $R^{\rm R}_{\rm mm\,22\,\lambda}$.

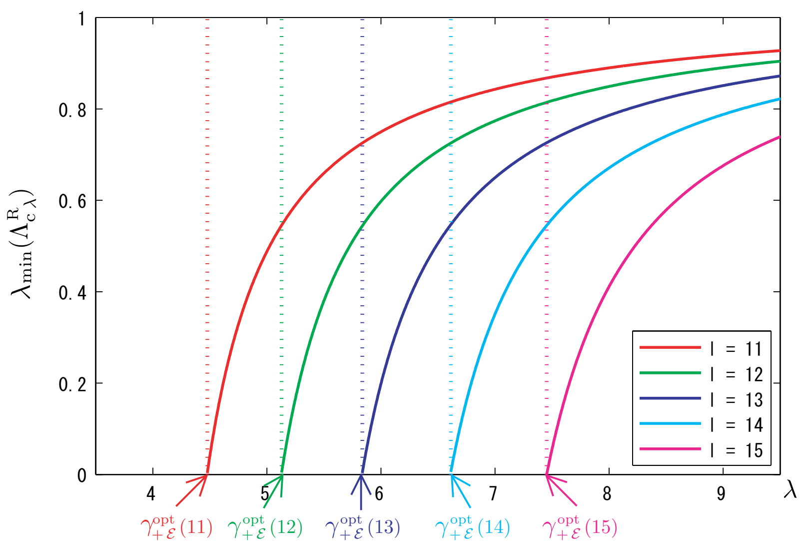

Figure 3.10: Positive definiteness of $\Lambda^{\rm R}_{\rm c\,\lambda}$.

3.6.2 Performance limit via min-max optimization

In this subsection, we calculate $\gamma^{\rm opt}_{+\,\rm FI}(l)$ based on Theorem 5, and explain the advantages of the disturbance attenuation conditions (DA1)-(DA2) over the J-spectral factorizability conditions.

The first step is to find the minimal performance bound $\gamma^{\rm opt}_{+\,\rm Ric}(l)$ satisfying (DA1). Intro-

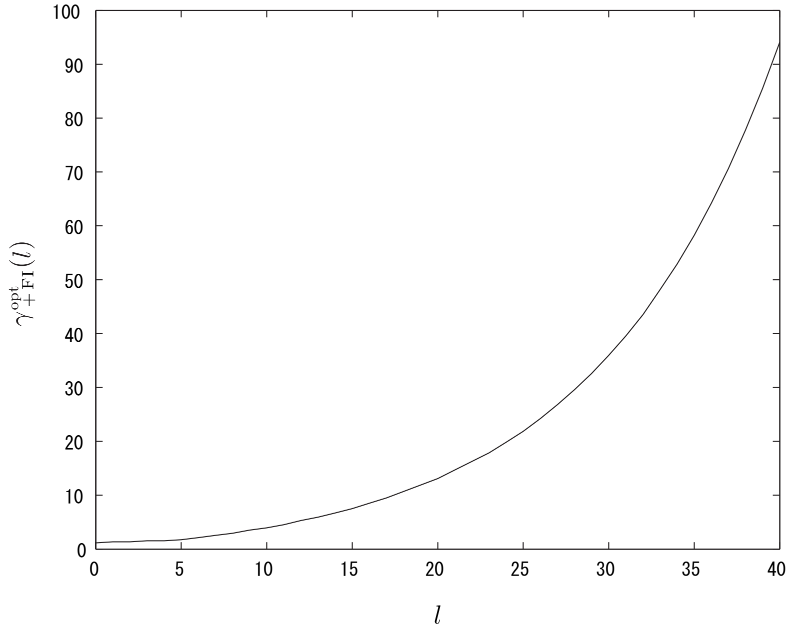

Figure 3.11: Performance deterioration.

duce the following forward Riccati difference equation:

$$\overline{X}_\lambda(\theta+1) = Q + A^* \left(I - \frac{1}{\lambda^2}\overline{X}_\lambda(\theta) B_1 B_1^* \right)^{-1} \overline{X}_\lambda(\theta) A \ \ (\theta = 0,\, 1,\, 2,\, \ldots),\ \overline{X}_\lambda(0) = X_\lambda.$$

For fixed l and λ, the values of $X_\lambda(l-\theta)$ in (DA1) are given by $X_\lambda(l-\theta) = \overline{X}_\lambda(\theta)$ for $\theta = 0,\, 1,\, \ldots,\, l$. Therefore, $\gamma_{+\,\mathrm{Ric}}^{\mathrm{opt}}(l)$ ($\gamma_{+\,\mathrm{Ric}}^{\mathrm{opt}}(0) := \gamma_{\mathrm{FI}}^{\mathrm{opt}}$) is determined as follows:

$$\gamma_{+\,\mathrm{Ric}}^{\mathrm{opt}}(l) = \inf \left\{ \lambda > \gamma_{+\,\mathrm{Ric}}^{\mathrm{opt}}(l-1) \ \middle|\ f(\lambda;\, l) > 0 \right\},$$

where $f(\lambda;\, l) := \lambda_{\min}(\lambda^2 I - B_1^* \overline{X}_\lambda(l-1) B_1)$. We begin with finding $\gamma_{+\,\mathrm{Ric}}^{\mathrm{opt}}(l)$ as the zero of $f(\lambda;\, l)$ for $l = 1$. Then, we continue the same procedure recursively incrementing l as shown in Fig. 3.12.

The recursive nature of the above procedure is suitable for successive examination of the performance deterioration along the increasing input delay. More importantly, if we once find a zero of $f(\lambda;\, l)$, we can stop tracing it at that point. This is because $\lambda^2 I - B_1^{\mathrm{T}} \overline{X}_\lambda(l-1) B_1$ is proved to be non-decreasing with respect to $\lambda > \gamma_{+\,\mathrm{Ric}}^{\mathrm{opt}}(l-1)$ [55].

From Figs. 3.8 and 3.12, we see that $\gamma_{+\,\mathrm{Ric}}^{\mathrm{opt}}(l) = \gamma_{+\,\mathcal{E}}^{\mathrm{opt}}(l)$ for each delay length. This coincidence of their values is consistent with the equivalence claimed in Lemma 12. For $\lambda > \gamma_{+\,\mathrm{Ric}}^{\mathrm{opt}}(l) = \gamma_{+\,\mathcal{E}}^{\mathrm{opt}}(l)$, as noted in the proof of Theorem 6, the equality $\widetilde{R}_{\mathrm{mm}\,\lambda} = R_{\mathrm{mm}\,\lambda}^{\mathrm{R}}$ holds, and hence the definiteness condition (DA2) on $\widetilde{R}_{\mathrm{mm}\,\lambda}$ is guaranteed by the definiteness condition (J2) on $R_{\mathrm{mm}\,\lambda}^{\mathrm{R}}$. Consequently, the calculation of $\gamma_{+\,\mathrm{FI}}^{\mathrm{opt}}(l)$ by checking (DA1)-(DA2) yields the same achievable performance.

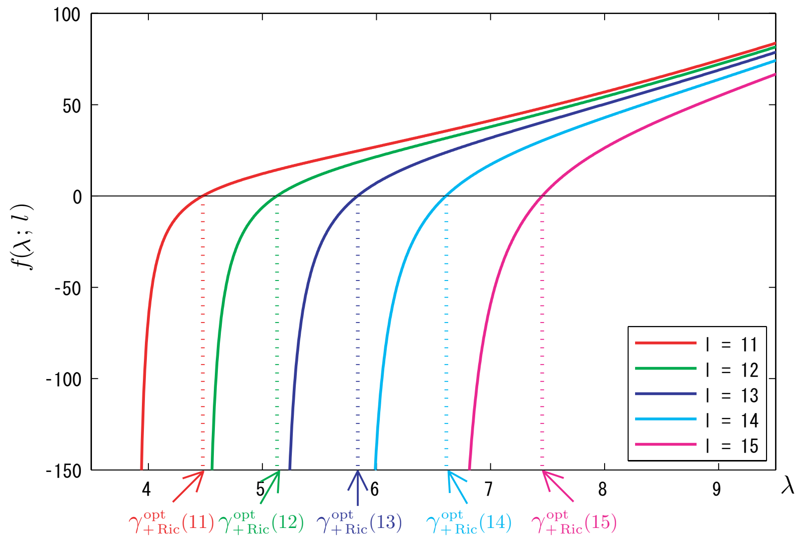

Figure 3.12: Calculation of $\gamma^{\mathrm{opt}}_{+\,\mathrm{Ric}}(l)$ for $11 \leq l \leq 15$.

3.7 Proofs

3.7.1 Proof of Lemma 3

The KYP equation in (C1) is rewritten as the following generalized eigenvalue problem:

$$\Phi^{\mathrm{R}}_{+\,\sigma\,\lambda}\mathcal{B}^{\mathrm{R}}_{\lambda} = \Phi^{\mathrm{R}}_{+\,\delta\,\lambda}\mathcal{B}^{\mathrm{R}}_{\lambda}A^{\mathrm{R}}_{\mathrm{c}\,\lambda}, \tag{3.54}$$

where $\mathcal{B}^{\mathrm{R}}_{\lambda} := \left[\begin{array}{cc|cc} I^{\mathrm{T}} & -X^{\mathrm{R\,T}}_{\lambda} & F^{\mathrm{R\,T}}_{1\,\lambda} & F^{\mathrm{R\,T}}_{2\,\lambda} \end{array}\right]^{\mathrm{T}}$,

$$\Phi^{\mathrm{R}}_{+\,\delta\,\lambda} := \left[\begin{array}{cc|c} I & O & O \\ O & A^{*} & O \\ \hline O & (\dagger 1)^{*} & O \end{array}\right], \ \Phi^{\mathrm{R}}_{+\,\sigma\,\lambda} := \left[\begin{array}{cc|c} A & O & (\dagger 1) \\ Q & I & (\dagger 2) \\ \hline (\dagger 2)^{*} & O & R^{\mathrm{R}}_{\mathrm{mm}\,\lambda} \end{array}\right],$$

$$(\dagger 1) = \left[B_1 \quad B^{\mathcal{D}}_{2\,\lambda}(l) - B_1\chi_{z^{-l}0\,\lambda}\right], \ (\dagger 2) = \left[O \quad S^{\mathcal{D}}_{2\,\lambda}(l)\right].$$

After lengthy equivalence transformations, Eq. (3.54) is rewritten as follows:

$$\Phi_{\sigma\,\lambda}\mathcal{B}_{\lambda} = \Phi_{\delta}\mathcal{B}_{\lambda}A^{\mathrm{R}}_{\mathrm{c}\,\lambda}, \tag{3.55}$$

where $\mathcal{B}_{\lambda} := \left[\begin{array}{cc|cc} \alpha^{\mathrm{T}}_{\lambda} & -(X_{\lambda}\alpha_{\lambda}+\beta_{\lambda})^{\mathrm{T}} & \phi^{\mathrm{T}}_{\lambda} & F^{\mathrm{R\,T}}_{2\,\lambda} \end{array}\right]^{\mathrm{T}}$, $\begin{bmatrix} \alpha_{\lambda} \\ -\beta_{\lambda} \end{bmatrix} := \mathcal{E}_{X\,\lambda}(l)\begin{bmatrix} I \\ X_{\lambda} - X^{\mathrm{R}}_{\lambda} \end{bmatrix}$,

$$\phi_{\lambda} := F^{\mathrm{R}}_{1\,\lambda} + \frac{1}{\lambda^2}B^{*}_{1}A^{-*}\left(Q - X^{\mathrm{R}}_{\lambda}\right) - \frac{1}{\lambda^2}B^{*}_{1}A^{-*}Q\alpha_{\lambda} + \frac{1}{\lambda^2}B^{*}_{1}A^{-*}\left(X_{\lambda}\alpha_{\lambda}+\beta_{\lambda}\right) - \frac{1}{\lambda^2}B^{*}_{1}A^{-*}S_2F^{\mathrm{R}}_{2\,\lambda},$$

$$\Phi_{\delta} := \left[\begin{array}{cc|c} I & O & O \\ O & A^{*} & O \\ \hline O & B^{*} & O \end{array}\right], \ \Phi_{\sigma\,\lambda} := \left[\begin{array}{cc|c} A & O & B \\ Q & I & S \\ \hline S^{*} & O & R_{\mathrm{mm}\,\lambda} \end{array}\right].$$

By the KYP equation in (X), the following identity holds:

$$\left[\begin{array}{cc|c} I & O & O \\ A_{\mathrm{c}\,\lambda}^{*}X_{\lambda} & I & F_{\lambda}^{*} \\ \hline B^{*}F_{\lambda}^{*} & O & I \end{array}\right](\Phi_{\sigma\,\lambda}-z\Phi_{\delta})\left[\begin{array}{cc|c} I & O & O \\ -X_{\lambda} & I & O \\ \hline F_{\lambda} & O & I \end{array}\right]=\Phi_{\sigma\,\lambda}^{\times}-z\Phi_{\delta\,\lambda}^{\times},$$

where

$$\Phi_{\delta\,\lambda}^{\times}:=\left[\begin{array}{cc|c} I & O & O \\ O & A_{\mathrm{c}\,\lambda}^{*} & O \\ \hline O & B^{*} & O \end{array}\right],\ \Phi_{\sigma\,\lambda}^{\times}:=\left[\begin{array}{cc|c} A_{\mathrm{c}\,\lambda} & O & B \\ O & I & O \\ \hline O & O & R_{\mathrm{mm\,c}\,\lambda} \end{array}\right].$$

By the above identity, Eq. (3.55) is further rewritten as follows:

$$\Phi_{\sigma\,\lambda}^{\times}\mathcal{B}_{\lambda}^{\times}=\Phi_{\delta\,\lambda}^{\times}\mathcal{B}_{\lambda}^{\times}A_{\mathrm{c}\,\lambda}^{\mathrm{R}},\tag{3.56}$$

where $\mathcal{B}_{\lambda}^{\times}:=\left[\begin{array}{cc|cc} \alpha_{\lambda}^{\mathrm{T}} & -\beta_{\lambda}^{\mathrm{T}} & (\phi_{\lambda}-F_{1\,\lambda}\alpha_{\lambda})^{\mathrm{T}} & \left(F_{2\,\lambda}^{\mathrm{R}}-F_{2\,\lambda}\alpha_{\lambda}\right)^{\mathrm{T}} \end{array}\right]^{\mathrm{T}}$.

From the second row of Eq. (3.56), we have

$$\beta_{\lambda}-A_{\mathrm{c}\,\lambda}^{*}\beta_{\lambda}A_{\mathrm{c}\,\lambda}^{\mathrm{R}}=O.\tag{3.57}$$

Since both $A_{\mathrm{c}\,\lambda}$ and $A_{\mathrm{c}\,\lambda}^{\mathrm{R}}$ are stable,

$$O=-\beta_{\lambda}=\mathcal{E}_{X\,21\,\lambda}(l)+\mathcal{E}_{X\,22\,\lambda}(l)\left(X_{\lambda}-X_{\lambda}^{\mathrm{R}}\right).\tag{3.58}$$

Suppose that $\mathcal{E}_{X\,22\,\lambda}(l)$ is not regular, then there exists $v\neq 0$ such that $v\mathcal{E}_{X\,22\,\lambda}(l)=0$. Premultiplying v to Eq. (3.58), we have $v\mathcal{E}_{X\,21\,\lambda}(l)=0$, and hence $\begin{bmatrix}0 & v\end{bmatrix}\mathcal{E}_{X\,\lambda}(l)=0$. This contradicts the regularity of $\mathcal{E}_{X\,\lambda}(l)$. Consequently, $\mathcal{E}_{X\,22\,\lambda}(l)$ is regular.

By Eq. (3.58), X_{λ}^{R} is given by Eq. (3.8) and

$$\alpha_{\lambda}=\mathcal{E}_{X\,\lambda\,22}^{-*}(l).\tag{3.59}$$

Substituting Eqs. (3.58) and (3.59) into Eq. (3.56), we have Eq. (3.9). 3rd and 4th rows of Eq. (3.56) yield

$$\phi_{\lambda}-F_{1\,\lambda}\alpha_{\lambda}=O,\ F_{2\,\lambda}^{\mathrm{R}}-F_{2\,\lambda}\alpha_{\lambda}=O.$$

From these equations, we obtain Eqs. (3.10)-(3.11).

3.7.2 Proof of Theorem 3

By Lemma 2, the conditions (C1) and (C2) are necessary, and the J-spectral factor $M_{+\,\lambda}$ is constructed as in Eq. (3.12). By Lemma 3, (C1) is replaced with (Σ).

For sufficiency, we prove that $N_{+\,\lambda}$ defined by Eq. (3.3) is a J-inner function. By Eqs. (3.4) and (3.6), $\Phi_{+\,\lambda}$ is factorized as in Eq. (3.2) with $M_{+\,\lambda}$ in Eq. (3.12). Therefore, $N_{+\,\lambda}$ defined as in (SF2) satisfies the following identity:

$$N_{+\,\lambda}^{\sim}\begin{bmatrix}-\lambda^{2}I & O \\ O & I\end{bmatrix}N_{+\,\lambda}=\begin{bmatrix}-\lambda^{2}\Lambda_{\mathrm{c}\,\lambda}^{\mathrm{R}} & O \\ O & R_{\mathrm{mm\,c\,22}\,\lambda}^{\mathrm{R}}\end{bmatrix}.\tag{3.60}$$

On the other hand, using the transfer function

$$M_\lambda := \left[\begin{array}{c|cc} A_{c\,\lambda} & B_1 & B_2 \\ \hline F_{1\,\lambda} & I & O \\ F_{2\,\lambda} & O & I \end{array}\right] \begin{bmatrix} I & O \\ F_{21\,\lambda} & I \end{bmatrix},$$

we rewrite $N_{+\,\lambda}$ as follows:

$$N_{+\,\lambda} = \mathcal{C}\left(P_{\rm mm}\right) M_\lambda \cdot M_\lambda^{-1} \begin{bmatrix} I & \chi_{z^{-l}\ominus\,\lambda} \\ O & z^{-l} I \end{bmatrix} M_{+\,\lambda}^{\rm R} = N_\lambda \cdot N_{+\,\lambda}^{z^{-l}},$$

where N_λ and $N_{+\,\lambda}^m$ are equal to Eqs. (3.14) and (3.16), respectively. This fact is implied in [63], [64]. Note that N_λ is the J-inner function which appears in the standard H^∞ problem, and satisfies the identity

$$N_\lambda^\sim \begin{bmatrix} -\lambda^2 I & O \\ O & I \end{bmatrix} N_\lambda = \begin{bmatrix} -\lambda^2 \Lambda_{\rm mm\,c\,\lambda} & O \\ O & R_{\rm mm\,c\,22\,\lambda} \end{bmatrix}. \tag{3.61}$$

In the following lemma, $N_{+\,\lambda}^{z^{-l}}$ is shown to be a J-inner function as well. Its proof is done by an argument similar to the proof of Theorem 5.3 in [36]

Lemma 13. *For* $\forall\,\lambda \geq \gamma$, $N_{+\,\lambda}^{z^{-l}}$ *is a* J*-inner function.* □

Proof. From Eqs. (3.60) and (3.61), $N_{+\,\lambda}^m$ is the J-unitary function which satisfies

$$N_{+\,\lambda}^{m\,\sim} \begin{bmatrix} -\lambda^2 \Lambda_{c\,\lambda} & O \\ O & R_{\rm mm\,c\,22\,\lambda} \end{bmatrix} N_{+\,\lambda}^m = \begin{bmatrix} -\lambda^2 \Lambda_{c\,\lambda}^{\rm R} & O \\ O & R_{\rm mm\,c\,22\,\lambda}^{\rm R} \end{bmatrix}. \tag{3.62}$$

To show that it is a J-inner function, it suffices to show that $G_\lambda(z) := N_{+\,11\,\lambda}^{z^{-l}}(z)$ is bi-stable for $\forall\lambda \geq \gamma$ [59]. From Eq. (3.16), G_λ is written as $G_\lambda = I + \widetilde{G}_\lambda$, where $\widetilde{G}_\lambda$ is a stable function such that $\left\|\widetilde{G}_\lambda\right\|_\infty \to 0$ as $\lambda \to \infty$. Therefore, by the small gain theorem, there exists $c > 0$ such that for $\forall\,\lambda > c$, G_λ^{-1} is stable. Next, suppose that for $\gamma \leq \exists\,\lambda_0 \leq c$, $G_{\lambda_0}^{-1}$ is unstable. Then, by Nyquist's theorem, there exists θ_0 such that $\det G_{\lambda_0}(e^{j\theta_0}) = 0$. Hence, for $\exists\,v \neq 0$, we have $G_{\lambda_0}(e^{j\theta_0})v = 0$. On the other hand, from the (1, 1) block in Eq. (3.62), the following equality holds:

$$-\lambda_0^2 \Lambda_{c\,\lambda_0}^{\rm R} = -\lambda_0^2 G_{\lambda_0}^*(e^{j\theta_0}) \Lambda_{c\,\lambda_0} G_{\lambda_0}(e^{j\theta_0}) + N_{+\,21\,\lambda_0}^{z^{-l}\,*}(e^{j\theta_0}) R_{\rm mm\,c\,22\,\lambda_0} N_{+\,21\,\lambda_0}^{z^{-l}}(e^{j\theta_0}).$$

By multiplying v and v^* from the right and left, respectively, we have a contradiction. □

Consequently, $N_{+\,\lambda}$ is a J-inner function because it is the products of the J-inner functions N_λ and $N_{+\,\lambda}^{z^{-l}}$.

3.7.3 Proof of Lemma 4

To find the relationship between the solutions of the KYP equations in (T1) and (Y), we apply the method in [20] of augmenting the pencils associated with them. First, the KYP equation in (T1) is rewritten as the following generalized eigenvalue problem:

$$\acute{\Phi}^{\mathrm{ext\,R}}_{+\,\mathrm{tmp}\,\sigma\,\gamma}\mathcal{B}_{\tau\,\gamma} = \acute{\Phi}^{\mathrm{ext\,R}}_{+\,\mathrm{tmp}\,\delta\,\gamma}\mathcal{B}_{\tau\,\gamma}\acute{A}^{\mathrm{R}\,*}_{\tau\,\mathrm{c}\,\gamma}, \tag{3.63}$$

where $\mathcal{B}_{\tau\,\gamma} := \left[\begin{array}{cccc|c} I^{\mathrm{T}} & -Y^{\mathrm{R\,T}}_{\mathrm{tmp}\,\gamma} & L^{\mathrm{R*\,T}}_{\mathrm{tmp1}\,\gamma} & L^{\mathrm{R*\,T}}_{\mathrm{tmp2}\,\gamma} & \psi^{\mathrm{R*\,T}}_{\mathrm{tmp}\,\gamma}\end{array}\right]^{\mathrm{T}}$,

$$\acute{\Phi}^{\mathrm{ext\,R}}_{+\,\mathrm{tmp}\,\delta\,\gamma} := \left[\begin{array}{cc|c|c} \multicolumn{3}{c|}{\multirow{3}{*}{$\acute{\Phi}^{\mathrm{R}}_{+\,\mathrm{tmp}\,\delta\,\gamma}$}} & O \\ & & & O \\ \cline{4-4} & & & O \\ \hline O & C_1 & O & O \end{array}\right],\ \acute{\Phi}^{\mathrm{ext\,R}}_{+\,\mathrm{tmp}\,\sigma\,\gamma} := \left[\begin{array}{cc|c|c} \multicolumn{3}{c|}{\multirow{3}{*}{$\acute{\Phi}^{\mathrm{R}}_{+\,\mathrm{tmp}\,\sigma\,\gamma}$}} & O \\ & & & O \\ \cline{4-4} & & & O \\ \hline O & O & O & -\gamma^2 I \end{array}\right],$$

$$\acute{\Phi}^{\mathrm{R}}_{+\,\mathrm{tmp}\,\delta\,\gamma} := \left[\begin{array}{cc|c} I & O & O \\ O & \acute{A}^{\mathrm{R}}_{\tau\,\gamma} & O \\ \hline O & \acute{C}^{\mathrm{R}}_{\tau\,\gamma} & O \end{array}\right],\ \acute{\Phi}^{\mathrm{R}}_{+\,\mathrm{tmp}\,\sigma\,\gamma} := \left[\begin{array}{cc|c} \acute{A}^{\mathrm{R}\,*}_{\tau\,\gamma} & O & \acute{C}^{\mathrm{R}\,*}_{\tau\,\gamma} \\ \acute{Q}^{\mathrm{R}}_{\mathrm{tmp}\,\gamma} & I & \acute{S}^{\mathrm{R}\,*}_{\mathrm{tmp}\,\gamma} \\ \hline \acute{S}^{\mathrm{R}}_{\mathrm{tmp}\,\gamma} & O & \acute{R}^{\mathrm{R}}_{\mathrm{tmp}\,\gamma} \end{array}\right].$$

In Eq. (3.63), the unknown variable $\psi^{\mathrm{R}}_{\mathrm{tmp}\,\gamma}$ is introduced to augment the pencil $\acute{\Phi}^{\mathrm{R}}_{+\,\mathrm{tmp}\,\sigma\,\gamma} - z\acute{\Phi}^{\mathrm{R}}_{+\,\mathrm{tmp}\,\delta\,\gamma}$ to the pencil $\acute{\Phi}^{\mathrm{ext\,R}}_{+\,\mathrm{tmp}\,\sigma\,\gamma} - z\acute{\Phi}^{\mathrm{ext\,R}}_{+\,\mathrm{tmp}\,\delta\,\gamma}$. Similarly, the KYP equation in (Y) is rewritten as the following generalized eigenvalue problem:

$$\acute{\Phi}^{\mathrm{ext\,R}}_{\mathrm{mm}\,\sigma\,\gamma}\mathcal{B}_{\mu\,\gamma} = \acute{\Phi}^{\mathrm{ext\,R}}_{\mathrm{mm}\,\delta\,\gamma}\mathcal{B}_{\mu\,\gamma}\acute{A}^{*}_{\mathrm{c}\,\gamma}, \tag{3.64}$$

where $\mathcal{B}_{\mu\,\gamma} := \left[\begin{array}{cccc|c} I^{\mathrm{T}} & -Y^{\mathrm{T}}_{\gamma} & L^{*\,\mathrm{T}}_{1\,\gamma} & L^{*\,\mathrm{T}}_{2\,\gamma} & \psi^{\mathrm{R*\,T}}_{\gamma}\end{array}\right]^{\mathrm{T}}$,

$$\acute{\Phi}^{\mathrm{ext\,R}}_{\mathrm{mm}\,\delta\,\gamma} := \left[\begin{array}{cc|c|c} \multicolumn{3}{c|}{\multirow{3}{*}{$\acute{\Phi}_{\mathrm{mm}\,\delta\,\gamma}$}} & O \\ & & & O \\ \cline{4-4} & & & O \\ \hline O & -\left(F^{\mathrm{R}}_{2\,\gamma} - F^{\mathrm{R}}_{21\,\gamma}F^{\mathrm{R}}_{1\,\gamma}\right) & O & O \end{array}\right],$$

$$\acute{\Phi}^{\mathrm{ext\,R}}_{\mathrm{mm}\,\sigma\,\gamma} := \left[\begin{array}{cc|c|c} \multicolumn{3}{c|}{\multirow{3}{*}{$\acute{\Phi}_{\mathrm{mm}\,\sigma\,\gamma}$}} & O \\ & & & O \\ \cline{4-4} & & & O \\ \hline (\sharp 1) & (\sharp 2) & (\sharp 3) & -\gamma^2\acute{\Lambda}^{\mathrm{R}}_{\mathrm{tmp}\,\gamma} \end{array}\right],$$

$$(\sharp 1) = \gamma^2\acute{\Lambda}^{\mathrm{R}}_{\mathrm{tmp}\,\gamma}\left(B^{\mathcal{D}}_{2\,\gamma}(l) - B_1\chi_{z^{-l_0}\,\gamma}\right)^{*} - F^{\mathrm{R}}_{21\,\gamma}B^{*}_{1},\ (\sharp 2) = \gamma^2\acute{\Lambda}^{\mathrm{R}}_{\mathrm{tmp}\,\gamma}\left(B^{\mathcal{D}}_{2\,\gamma}(l) - B_1\chi_{z^{-l_0}\,\gamma}\right)^{*}X^{\mathrm{R}}_{\gamma},$$
$$(\sharp 3) = \left[\begin{array}{cc} O & -F^{\mathrm{R}}_{21\,\gamma}D^{*}_{21}\end{array}\right],$$

$$\acute{\Phi}_{\mathrm{mm}\,\delta} := \left[\begin{array}{cc|c} I & O & O \\ O & A & O \\ \hline O & C & O \end{array}\right],\ \acute{\Phi}_{\mathrm{mm}\,\sigma\,\gamma} := \left[\begin{array}{cc|c} A^{*} & O & C^{*} \\ \acute{Q} & I & \acute{S}^{*} \\ \hline \acute{S} & O & \acute{R}_{\mathrm{mm}\,\gamma} \end{array}\right].$$

In Eq. (3.64), the unknown variable $\psi^{\mathrm{R}}_{\gamma}$ is introduced to augment the pencil $\acute{\Phi}_{\mathrm{mm}\,\sigma\,\gamma} - z\acute{\Phi}_{\mathrm{mm}\,\delta}$ to the pencil $\acute{\Phi}^{\mathrm{ext\,R}}_{\mathrm{mm}\,\sigma\,\gamma} - z\acute{\Phi}^{\mathrm{ext\,R}}_{\mathrm{mm}\,\delta\,\gamma}$. It is determined from the last column in Eq. (3.64) and given by

$$\psi^{\mathrm{R}}_{\gamma} = B^{\mathcal{D}}_{2\,\gamma}(l) + \frac{1}{\gamma^2}\acute{A}_{\mathrm{c}\,\gamma}Y_{\gamma}S^{\mathrm{D}}_{2\,\gamma}(l) + \left(B_1 + L_{2\,\gamma}D_{21}\right)\chi_{z^{-l_0}\,\gamma}.$$

Using the KYP equation in (C1), it is verified that the pencils $\acute{\Phi}^{\mathrm{ext\,R}}_{+\,\mathrm{tmp}\,\sigma\,\gamma} - z\acute{\Phi}^{\mathrm{ext\,R}}_{+\,\mathrm{tmp}\,\delta\,\gamma}$ and $\acute{\Phi}^{\mathrm{ext\,R}}_{\mathrm{mm}\,\sigma\,\gamma} - z\acute{\Phi}^{\mathrm{ext\,R}}_{\mathrm{mm}\,\delta}$ are related as follows:

$$U_{\mathrm{l}\,\gamma}\left(\acute{\Phi}^{\mathrm{ext\,R}}_{+\,\mathrm{tmp}\,\sigma\,\gamma} - z\acute{\Phi}^{\mathrm{ext\,R}}_{+\,\mathrm{tmp}\,\delta\,\gamma}\right)U_{\mathrm{r}\,\gamma} = \acute{\Phi}^{\mathrm{ext\,R}}_{\mathrm{mm}\,\sigma\,\gamma} - z\acute{\Phi}^{\mathrm{ext\,R}}_{\mathrm{mm}\,\delta}, \tag{3.65}$$

where

$$U_{\mathrm{l}\,\gamma} := \left[\begin{array}{cccc|c} I & -\frac{1}{\gamma^2}A^*X^{\mathrm{R}}_{\gamma} & \frac{1}{\gamma^2}S^{\mathcal{D}}_{2\,\gamma}(l) & O & -\frac{1}{\gamma^2}C_1^* \\ O & I-\frac{1}{\gamma^2}\acute{Q}X^{\mathrm{R}}_{\gamma} & B^{\mathcal{D}}_{2\,\gamma}(l) & O & O \\ O & O & O & O & I \\ O & -\frac{1}{\gamma^2}\acute{S}_2X^{\mathrm{R}}_{\gamma} & D_{21}\chi_{z^{-l}0\,\gamma} & I & O \\ \hline O & \frac{1}{\gamma^2}F^{\mathrm{R}\,*}_{21\,\gamma}B_1^*X^{\mathrm{R}}_{\gamma} & I-F^{\mathrm{R}}_{21}\chi_{z^{-l}0\,\gamma} & O & O \end{array}\right],\ U_{\mathrm{r}\,\gamma} := \left[\begin{array}{cccc|c} I & \frac{1}{\gamma^2}X^{\mathrm{R}}_{\gamma} & O & O & O \\ O & I & O & O & O \\ O & O & O & O & I \\ O & O & O & I & O \\ \hline O & O & I & O & O \end{array}\right].$$

Eq. (5.30) implies that the solutions of KYP equations in (T1) and (Y) are related as follows:

$$\mathcal{B}_{\tau\,\gamma}Z^{\mathrm{R}\,*}_{\gamma} = U_{\mathrm{r}\,\gamma}\mathcal{B}_{\mu\,\gamma}.$$

3.7.4 Proof of Lemma 8

For the proof of the representation (3.45), see Lemma 3.5.2 in [2]. We prove the equivalence of (D1) and (D2) by induction. Let $0 \le k \le l-1$ be an integer, and suppose that Eqs. (3.43) and (3.44) are equivalent and Eq. (3.45) holds for $k \le \forall n \le l-1$. Each block of the identity $\mathcal{E}_\lambda(l-k)H_{\mathrm{FH}\,\lambda} = \mathcal{E}_\lambda(l-(k-1))$ is employed in the subsequent equations.

By Eq. (3.45) for $n = k$, we have the identity

$$\begin{aligned}
&\det\left(I - \frac{1}{\lambda^2}X_\lambda(k)B_1B_1^*\right)\cdot\det A^{-*} \\
&= \det\left(X_\lambda\mathcal{E}_{12\,\lambda}(l-k) + \mathcal{E}_{22\,\lambda}(l-k)\right)^{-1}\det\left(X_\lambda\cdot(*1) + (*2)\right) \\
&= \det\left(X_\lambda\mathcal{E}_{12\,\lambda}(l-k) + \mathcal{E}_{22\,\lambda}(l-k)\right)^{-1}\det\left(X_\lambda\mathcal{E}_{12\,\lambda}(l-(k-1)) + \mathcal{E}_{22\,\lambda}(l-(k-1))\right),
\end{aligned}$$

$$\begin{aligned}
(*1) &= \mathcal{E}_{11\,\lambda}(l-k)H_{\mathrm{FH}\,12\,\lambda} + \mathcal{E}_{12\,\lambda}(l-k)H_{\mathrm{FH}\,22\,\lambda}, \\
(*2) &= \mathcal{E}_{21\,\lambda}(l-k)H_{\mathrm{FH}\,12\,\lambda} + \mathcal{E}_{22\,\lambda}(l-k)H_{\mathrm{FH}\,22\,\lambda},
\end{aligned}$$

which means that Eqs. (3.43) and (3.44) are also equivalent for $n = k-1$.

If either of Eqs. (3.43) and (3.44) is satisfied for $n = k-1$, by the definition of $X_\lambda(k-1)$,

$$\begin{aligned}
X_\lambda(k-1) &= \left(X_\lambda\cdot(*1) + (*2)\right)^{-1}\left(X_\lambda\cdot(*3) + (*4)\right) \\
&= \left(X_\lambda\mathcal{E}_{12\,\lambda}(l-(k-1)) + \mathcal{E}_{22\,\lambda}(l-(k-1))\right)^{-1} \\
&\quad\cdot\left(X_\lambda\mathcal{E}_{11\,\lambda}(l-(k-1)) + \mathcal{E}_{21\,\lambda}(l-(k-1))\right),
\end{aligned}$$

$$\begin{aligned}
(*3) &= \mathcal{E}_{11\,\lambda}(l-k)H_{\mathrm{FH}\,11\,\lambda} + \mathcal{E}_{12\,\lambda}(l-k)H_{\mathrm{FH}\,21\,\lambda}, \\
(*4) &= \mathcal{E}_{21\,\lambda}(l-k)H_{\mathrm{FH}\,11\,\lambda} + \mathcal{E}_{22\,\lambda}(l-k)H_{\mathrm{FH}\,21\,\lambda},
\end{aligned}$$

which means that Eq. (3.45) also holds for $n = k-1$.

3.7.5 Proof of Lemma 10

Direct substitution shows that $\widetilde{X}_\lambda$ and $\widetilde{F}_\lambda$ constructed in Lemma 9 satisfy the augmented KYP equation.

To prove the stability of $\widetilde{A}+\widetilde{B}\widetilde{F}_\lambda$, we use the block matrix $U_\lambda = \begin{bmatrix} U_\lambda(1) & U_\lambda(2) & \cdots & U_\lambda(l) \end{bmatrix}$,

$$U_\lambda(\theta) := H_{\mathrm{FH}\,\delta\,\lambda}^{-1} \begin{bmatrix} B_{2\,\lambda}^{\mathcal{D}}(\theta) \\ S_{2\,\lambda}^{\mathcal{D}}(\theta) \end{bmatrix} \ (\theta = 1,\, 2,\, \ldots,\, l)\,.$$

Let us partition U_λ conformably with $x(n)$ and $p(n)$:

$$U_\lambda = \begin{bmatrix} U_{x\,\lambda} \\ U_{p\,\lambda} \end{bmatrix} \in \mathbb{R}^{(\dim x + \dim x)\times(l\cdot\dim u)},$$

and transform $\widetilde{A}+\widetilde{B}\widetilde{F}_\lambda$ by the equation

$$\overline{A}_{\mathrm{c}\,\lambda} := \begin{bmatrix} I & -U_{x\,\lambda} \\ O & I \end{bmatrix}^{-1} \left(\widetilde{A}+\widetilde{B}\widetilde{F}_\lambda\right) \begin{bmatrix} I & -U_{x\,\lambda} \\ O & I \end{bmatrix}. \tag{3.66}$$

The (1, 2)-block of $\overline{A}_{\mathrm{c}\,\lambda}$ is calculated as

$$\underline{(U_{x\,\lambda}\mathcal{S} - AU_{x\,\lambda} + B_2\Gamma_1) + B_1\left(\widetilde{F}_{1\upsilon\,\lambda} - \widetilde{F}_{1x\,\lambda}U_{x\,\lambda}\right)} + U_{x\,\lambda}\Delta_l \underline{\left(\widetilde{F}_{2\upsilon\,\lambda} - \widetilde{F}_{2x\,\lambda}U_{x\,\lambda}\right)}.$$

This is zero since both of the underlined parts vanish. Therefore, we find that $\widetilde{A}_{+\,\mathrm{c}\,\lambda}$ is a lower triangular matrix as shown below and that it is stable:

$$\overline{A}_{\mathrm{c}\,\lambda} = \begin{bmatrix} \mathcal{E}_{X\,22\,\lambda}^{*}(l) A_{\mathrm{c}\,\lambda} \mathcal{E}_{X\,22\,\lambda}^{-*}(l) & O \\ \Delta_l F_{2x\,\lambda} & \mathcal{S} \end{bmatrix}.$$

Remark 10. *In the J-spectral factorization approach, the state variables $x(z)$ and $p(z)$ are decomposed as in Eq. (3.5). Through the correspondence $z^{-\theta}u(n) = u(n-\theta) = \upsilon(\theta, n)$, the decomposition is interpreted as the state transformation*

$$\begin{bmatrix} x^{\mathrm{R}}(n) \\ p^{\mathrm{R}}(n) \end{bmatrix} = \begin{bmatrix} x(n) \\ p(n) \end{bmatrix} + \begin{bmatrix} U_{x\,\lambda} \\ U_{p\,\lambda} \end{bmatrix} \upsilon(n).$$

This means that $\overline{A}_{\mathrm{c}\,\lambda}$ defined by Eq. (3.66) describes the state transition of $x^{\mathrm{R}}(n)$ and $\upsilon(n)$ under the optimal strategies $w_*(n)$ and $u_*(n)$ given in Lemma 9. □

3.7.6 Proof of Lemma 11

We give a simpler proof than that of the continuous-time formula in [6] by the argument used in the proof of Theorem 6 in [29]. For a given parameter $\mu > 0$, we determine whether there exists a non-zero w for the eigenequation $\left(\mu^2 I - \mathcal{G}_\lambda^{*}\,\mathcal{G}_\lambda\right) w = 0$. Let $\widetilde{z}$ be the output of $\mathcal{G}_\lambda$: $\widetilde{z} := \mathcal{G}_\lambda w$, then the existence of $w \neq 0$ is equivalent to that of $\begin{bmatrix} w^{\mathrm{T}} & \widetilde{z}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \neq 0$ to the following equation: $\begin{bmatrix} -\mu^2 I & \mathcal{G}_\lambda^* \\ \mathcal{G}_\lambda & -I \end{bmatrix} \begin{bmatrix} w \\ \widetilde{z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Furthermore, this equation is equivalently rewritten as

$$\begin{bmatrix} x(n+1) \\ p(n+1) \end{bmatrix} = H_{\mathrm{FH}\,\mu} \begin{bmatrix} x(n) \\ p(n) \end{bmatrix}, \tag{3.67}$$

$$x(0) = 0,\ p(l) = -X_\lambda x(l), \tag{3.68}$$

$$w(n) = -\frac{1}{\mu^2} B_1^* p(n+1),\ z(n) = C_1 x(n)\ \ (0 \le n \le l-1)\,.$$

In the above equations, w and $\widetilde{z}$ are regarded as the outputs of the linear system given by Eqs. (3.67)-(3.68). One of the important techniques in [29] is to characterize the existence of a non-zero output in terms of the internal state variables.

Lemma 14. *The existence of a non-zero* $\begin{bmatrix} w^{\mathrm{T}} & \widetilde{z}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$ *is equivalent to that of a non-zero terminal state* $x(l)$*.* □

Proof. We prove the lemma by contraposition. Suppose that $\begin{bmatrix} w^{\mathrm{T}} & \widetilde{z}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$ is zero. Then, considering the realization of $\mathcal{G}_\lambda$, we have $x(l) = 0$. Conversely, suppose that $x(l)$ is zero. Then, solving Eq. (3.67) backwards in the time instant n, we have $\begin{bmatrix} w^{\mathrm{T}} & \widetilde{z}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} = 0$. □

It is seen that the existence of a non-zero terminal state $x(l)$ is equivalent to the condition that $\det\left(X_\lambda \mathcal{E}_{12\,\mu}(l) + \mathcal{E}_{22\,\mu}(l)\right) = 0$ by substituting the equality $x(0) = 0$ into the following equation:

$$\begin{bmatrix} x(0) \\ p(0) \end{bmatrix} = \mathcal{E}_\mu^{-1}(l) \begin{bmatrix} x(l) \\ p(l) \end{bmatrix} = \begin{bmatrix} \mathcal{E}_{22\,\mu}^*(l) & -\mathcal{E}_{12\,\mu}^*(l) \\ -\mathcal{E}_{21\,\mu}^*(l) & \mathcal{E}_{11\,\mu}^*(l) \end{bmatrix} \begin{bmatrix} I \\ -X_\lambda \end{bmatrix} x(l).$$

3.8 Conclusion

This chapter addressed the H^∞ control problem for the discrete-time input-delay system. The proposed solution method is based on the reductions of closed-loop systems, where the one-sided model matching and output estimation problems are successively formulated. It is revealed that the results similar to those in the continuous-time setting [38] hold: The parameterization of the H^∞ controllers is obtained only by solving the KYP equations for the delay-free case and checking the matrix eigenvalues. They are implemented using the past history of the control input.

As a supplementary result, the min-max optimization is adopted for the H^∞ disturbance attenuation in the full information problem. The stabilizing solution of the augmented KYP equation and another characterization of the solvability are provided using the KYP equation for the delay-free case. The J-spectral factorizability condition is proved to be equivalent to the H^∞ disturbance attenuation condition by analyzing the initial finite-time response of the input-delay system.

Chapter 4 Continuous-time H^2 preview control

4.1 Introduction

In Chapters 2 and 3, the explicit H^2 and H^∞ optimal controllers are obtained in the Smith predictor form by the state decomposition approach. In Chapters 4 and 5, we extend the state decomposition approach for the continuous-time H^2 and H^∞ preview controller syntheses in a continuous-time output feedback setting. The preview output feedback problems are solved through the full information and output estimation ones in a way consistent with the input-delayed problems. While the H^2 and H^∞ input-delayed controllers involved the internal feedback of the past history of the control input, it is revealed that the H^2 and H^∞ preview controllers are realized based on the observers incorporating the future information of the exogenous disturbance.

In [32], [57], the continuous-time H^2 preview controller designs are reported under the settings where only the partial information of the state variables is available. Both of the designs require the preliminary steps before applying the principal theories [27], [31], and do not guarantee the exact H^2 optimality of the overall closed-loop systems; 1) At the preliminary step of [32], the Youla parameterization technique is employed to modify the output feedback configuration to the two-sided model matching configuration, and the choice of the stabilizing gains involves arbitrariness. The Youla parameter optimal for the model matching configuration is determined by the orthogonality principle in H^2 space [31]; 2) At the preliminary step of [57], the standard finite-dimensional H^2 controller for the non-preview case is constructed ignoring the advantage that the control input can act in advance of the disturbance. The preview information is incorporated as the additional input compensation based on the technique for the full information case [27].

Contrary to [32], [57], our design method exploits the available preview information at both the full information and output estimation problems, and yields the output feedback controller achieving the exact optimal performance. In the full information problem, we construct the optimal state feedback law via the one-sided model matching problem. To solve the model matching problem based on the spectral factorization theory, alternative state transformations are introduced referring to the infinite-dimensional state-space representation of the spectral density. In the output estimation problem, we focus on the state transformations defining the state decomposition parallel to that in Chapter 2. The generalized plant in the output estimation form is shown to have the structure amenable to the explicit solution, if it is described with the newly introduced state variable.

This chapter is organized as follows. In Section 4.2, the problem formulation and assumptions are stated. In Section 4.3, the optimal state feedback law and corresponding optimal cost is obtained via the one-sided model matching problem. In Section 4.4, the optimal output feedback controller is constructed based on the results of Section 4.4. In Section 4.5, a numerical example is presented to illustrate the H^2 preview control performance in the time and frequency domains. In Section 4.6, the proofs left in the previous sections are given.

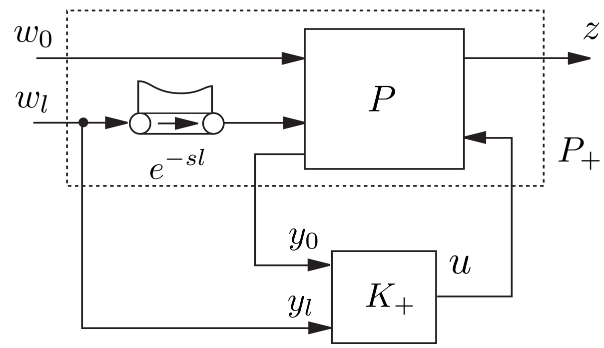

Figure 4.1: Preview control system.

4.2 Problem formulation

Let us formulate the preview control system as shown in Fig. 4.1. The exogenous disturbance $w(t)$ is partitioned as $w(t) =: \begin{bmatrix} w_0(t)^{\mathrm{T}} & w_l(t)^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$, where $w_l(t)$ is previewable and $w_0(t)$ is not. The controller K_+ can act on the controlled plant P without delay while the previewable disturbance $w_l(t)$ is delayed by the preview length l. The generalized plant $P : (w,\, u) \to (z,\, y_0)$ is partitioned conformably with $w_0(t)$ and $w_l(t)$:

$$P := \left[\begin{array}{c|cc} A & \begin{bmatrix} B_{1/0} & B_{1/l} \end{bmatrix} & B_2 \\ \hline C_1 & \begin{bmatrix} O & O \end{bmatrix} & D_{12} \\ C_{2/0} & \begin{bmatrix} D_{21/00} & D_{21/0l} \end{bmatrix} & O \end{array}\right].$$

The overall system $P_+ : (w,\, u) \to (z,\, y)$ $(y(t) := \begin{bmatrix} y_0(t)^{\mathrm{T}} & y_l(t)^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}})$ including P and the delay element e^{-sl} is described as follows:

$$\dot{x}(t) = Ax(t) + B_{1/0}w_0(t) + B_{1/l}w_l(t-l) + B_2u(t), \tag{4.1}$$

$$z(t) = C_1x(t) + D_{12}u(t), \tag{4.2}$$

$$\begin{bmatrix} y_0(t) \\ y_l(t) \end{bmatrix} = \begin{bmatrix} C_{2/0}x(t) + D_{21/00}w_0(t) + D_{21/0l}w_l(t-l) \\ w_l(t) \end{bmatrix}.$$

It is noted that the delayed-disturbance is given by $w_l(t-(l-\theta)) = \omega(\theta, t)$ $(0 \le \theta \le l)$, where $\omega(\theta, t)$ follows the PDE

$$\frac{\partial \omega}{\partial t}(\theta, t) = \frac{\partial \omega}{\partial \theta}(\theta, t)\ (0 < \theta < l),\ \omega(l, t) = w_l(t).$$

We derives the stabilizing controller K_+ minimizing the H^2 norm of the transfer function T_{zw} from w to z. The following conditions (A1)-(A3) are assumed throughout this chapter.

(A1) $(A,\, B_2)$ and $(A,\, C_{2/0})$ are stabilizable and detectable, respectively.

(A2) For $\forall\, \omega \in \mathbb{R}$, $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ and $\begin{bmatrix} A - j\omega I & B_{1/0} \\ C_{2/0} & D_{21/00} \end{bmatrix}$ are of full column rank and of full row rank, respectively.

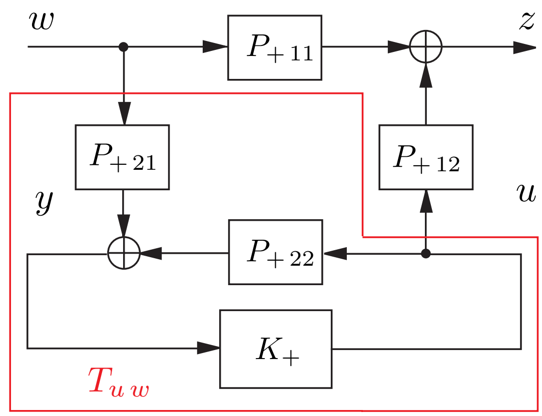

Figure 4.2: Model matching problem.

(A3) D_{12} and $D_{21/00}$ are of full column rank and of full row rank, respectively.

The above assumptions ensure the existence of the positive semidefinite stabilizing solutions X and Y^{R} to the following control- and filtering-type Riccati equations.

$$Q + A^*X + XA - (S_2 + XB_2)\, R_2^{-1}\, (S_2 + XB_2)^* = O, \tag{4.3}$$

$$\acute{Q}_0 + AY^{\mathrm{R}} + Y^{\mathrm{R}}A^* - \left(\acute{S}_{2/0}^* + Y^{\mathrm{R}}C_{2/0}^*\right) \acute{R}_{2/0}^{-1} \left(\acute{S}_{2/0}^* + Y^{\mathrm{R}}C_{2/0}^*\right)^* = O, \tag{4.4}$$

where the following definitions are used for simplicity.

$$\begin{bmatrix} Q & S_2 \\ S_2^* & R_2 \end{bmatrix} := \begin{bmatrix} C_1 & D_{12} \end{bmatrix}^* \begin{bmatrix} C_1 & D_{12} \end{bmatrix}, \quad \begin{bmatrix} \acute{Q}_0 & \acute{S}_{2/0}^* \\ \acute{S}_{2/0} & \acute{R}_{2/0} \end{bmatrix} := \begin{bmatrix} B_{1/0} \\ D_{21/00} \end{bmatrix} \begin{bmatrix} B_{1/0} \\ D_{21/00} \end{bmatrix}^* .$$

The following Hamiltonian matrix is associated with the Riccati equation (4.3).

$$H := \begin{bmatrix} A - B_2R_2^{-1}S_2^* & B_2R_2^{-1}B_2^* \\ Q - S_2R_2^{-1}S_2^* & -\left(A - B_2R_2^{-1}S_2^*\right)^* \end{bmatrix}.$$

4.3 Model matching and spectral factorization

Let us focus on the following model matching problem [MM]. The problem is to optimize the transfer function of the boxed part in Fig. 4.2.

[MM] Find the transfer function $T_{u\,w}$ from w to u which is a solution of the following optimization problem.

$$\text{Minimize } \|P_{+11} + P_{+12}T_{u\,w}\|_2^2 \text{ with respect to } T_{u\,w}.$$
$$\text{Subject to } T_{u\,w},\ P_{+11} + P_{+12}T_{u\,w} \in H^2.$$

One of the solution methods, which is amenable to transfer function representation, is based on the following spectral factorization of $\Phi_{+22} := P_{+12}^{\sim} P_{+12}$ [17]:

$$\Phi_{+22} = M_{+22}^{-\sim} R_2 M_{+22}^{-1}, \tag{4.5}$$

where M_{+22}^{-1} is the spectral factor and $M_{+22}^{-\sim}$ is its adjoint factor. If the above spectral factor M_{+22}^{-1} is obtained, the solution of [MM] is given as follows:

$$T_{u\,w} = -M_{+\,22} \left\{ R_2^{-1} M_{+\,22}^{\sim} P_{+\,12}^{\sim} P_{+\,11} \right\}_+, \tag{4.6}$$

where $\{\,\cdot\,\}_+$ denotes the casual and stable part of the transfer function.

In this section, we introduce a series of state transformations for the spectral factorization of Φ_{+22}. by explicitly considering the state-space dynamics of the delay element. We choose the L^2 space as the state space of the delay element, and denote the adjoint variable of $\omega(\theta, t)$ $(0 \le \theta \le l)$ by $\alpha(\phi, t)$ $(0 \le \phi \le l)$. Then, the realization of $\Phi_{+22}^{-1} : k \to u$ is given as follows:

$$\begin{aligned}
\begin{bmatrix} \dot{x}(t) \\ \dot{p}(t) \end{bmatrix} &= H \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} + \begin{bmatrix} B_{1/l} \\ O \end{bmatrix} \Gamma_0 \omega(t) + \begin{bmatrix} B_2 \\ S_2 \end{bmatrix} R_2^{-1} k(t), \\
E_\omega \dot{\omega}(t) &= A_\omega \omega(t), \\
E_\alpha \dot{\alpha}(t) &= -A_\alpha \alpha(t) + \begin{bmatrix} O \\ \begin{bmatrix} O & -B_{1/l}^* \end{bmatrix} \end{bmatrix} \begin{bmatrix} x(t) \\ p(t) \end{bmatrix}, \\
u(t) &= -R_2^{-1} \begin{bmatrix} S_2^* & -B_2^* \end{bmatrix} \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} + R_2^{-1} k(t).
\end{aligned}$$

In the above state-space representation, the following operators on $L^2([0,\, l], \mathbb{R}^{\dim w_l})$ are used. Each of their domains is $W^{2,1}([0,\, l], \mathbb{R}^{\dim w_l})$.

$$E_\omega := \begin{bmatrix} I \\ O \end{bmatrix},\; A_\omega := \begin{bmatrix} \frac{\partial}{\partial \theta} \\ -\Gamma_l \end{bmatrix},\; E_\alpha := \begin{bmatrix} I \\ O \end{bmatrix},\; A_\alpha := \begin{bmatrix} -\frac{\partial}{\partial \phi} \\ -\Gamma_0 \end{bmatrix}.$$

We introduce the following state transformations (4.7)-(4.9) to perform the spectral factorization:

$$\begin{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} \\ \alpha \end{bmatrix} =: \begin{bmatrix} I & O \\ V & I \end{bmatrix} \begin{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} \\ \alpha^{\mathrm{R}} \end{bmatrix}, \tag{4.7}$$

$$\begin{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} \\ \omega \end{bmatrix} =: \begin{bmatrix} I & -U E_\omega \\ O & I \end{bmatrix} \begin{bmatrix} \begin{bmatrix} x^{\mathrm{R}} \\ p^{\mathrm{R}} \end{bmatrix} \\ \omega \end{bmatrix}, \tag{4.8}$$

$$\alpha^{\mathrm{R}\times} := \alpha^{\mathrm{R}} + \Xi\, \omega. \tag{4.9}$$

The operators $V := \begin{bmatrix} V_x & V_p \end{bmatrix}$, U and Ξ in Eqs. (4.7)-(4.9) are defined as follows:

$$\begin{bmatrix} V_x & V_p \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} (\phi) := \begin{bmatrix} O & B_{1/l}^* \end{bmatrix} e^{H\phi} \begin{bmatrix} x \\ p \end{bmatrix} \;\; (0 < \phi < l) \;\text{ for } (x,\, p) \in \mathbb{R}^{\dim x} \times \mathbb{R}^{\dim x}, \tag{4.10}$$

$$U\begin{bmatrix} f \\ g \end{bmatrix} := \begin{bmatrix} B_{1/l} \\ O \end{bmatrix} g + \int_{\theta=0}^{l} e^{-H\theta} \begin{bmatrix} B_{1/l} \\ O \end{bmatrix} f(\theta)d\theta \text{ for } (f,\, g) \in L^2([0,\, l], \mathbb{R}^{\dim w_l}) \times \mathbb{R}^{\dim w_l}, \quad (4.11)$$

$$\Xi\, \omega(\phi) := -\int_{\theta=0}^{\phi} B_{1/l}^* \left\{ e^{H\theta} \right\}_{21} B_{1/l}\omega(\phi-\theta)d\theta \;\; (0<\phi<l) \text{ for } \omega \in L^2([0,\, l], \mathbb{R}^{\dim w_l}). \quad (4.12)$$

Note that $UE_\omega\omega(t) =: \begin{bmatrix} \{UE_\omega\}_x^{\mathrm{T}} & \{UE_\omega\}_p^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \omega(t)$ is given by

$$\begin{bmatrix} \{UE_\omega\}_x \\ \{UE_\omega\}_p \end{bmatrix} \omega(t) = \int_{\theta=0}^{l} e^{-H\theta} \begin{bmatrix} B_{1/l} \\ O \end{bmatrix} \omega(\theta, t)d\theta.$$

The transformations in Eqs. (4.7)-(4.9) enable us to perform the spectral factorization explicitly considering the state-space representation of the delay element.

Lemma 15. *Let us choose (x, ω) as the state variable of the causal and stable spectral factor M_{+22}^{-1} in Eq. (4.5). Then, we can choose the following state variable $(p^\times, \alpha^\times)$ defined by*

$$\begin{bmatrix} p^\times \\ \alpha^\times \end{bmatrix} := \begin{bmatrix} p^{\mathrm{R}\times} \\ \alpha^{\mathrm{R}\times} + V_p p^{\mathrm{R}\times} \end{bmatrix}$$

as that of the anticausal and antistable spectral factor $M_{+22}^{-\sim}$ in Eq. (4.5). Furthermore, the state variable $(p^\times,\, \alpha^\times)$ is obtained by transforming the adjoint variable $(p,\, \alpha)$ as follows:

$$\begin{bmatrix} p^\times \\ \alpha^\times \end{bmatrix} := \begin{bmatrix} p \\ \alpha \end{bmatrix} + \widetilde{X} \begin{bmatrix} x \\ \omega \end{bmatrix}, \quad (4.13)$$

where the operator $\widetilde{X}$ is constructed by the equation

$$\widetilde{X} := \begin{bmatrix} X & X\{UE_\omega\}_x + \{UE_\omega\}_p \\ V_pX - V_x & V_pX\{UE_\omega\}_x + V_p\{UE_\omega\}_p + \Xi \end{bmatrix}. \quad (4.14)$$

□

Proof. Subsection 4.6.1. □

4.4 Solution via closed-loop reduction

4.4.1 Full information problem

The disturbance-delayed system P_+ is in the class of the Pritchard-Salamon system, and therefore the H^2 optimal state feedback law is constructed from the positive semidefinite stabilizing solution $\widetilde{X}$ of the associated operator Riccati equation. Specifically, the optimal cost is given by the equation

$$E_{+\mathrm{FI}}^2 := \min \|P_{+11} + P_{+12}T_{u\,w}\|_2^2 = \mathrm{tr}\, \widetilde{B}_1^* \widetilde{X} \widetilde{B}_1, \quad (4.15)$$

with the appropriately defined disturbance input operator $\widetilde{B}_1$, and the solution in Eq. (4.6) is represented by the following state feedback law:

$$u_*(t) = \widetilde{F}_{2x}x(t) + \widetilde{F}_{2\omega}\omega(t), \tag{4.16}$$

where $\widetilde{F}_{2x}$ and $\widetilde{F}_{2\omega}$ are determined from $\widetilde{X}$. In this section, we exploit the above facts to obtain a solution of the full information problem, and construct the optimal feedback law based on the results in the previous section.

First, let us rewrite the delayed equations (4.1)-(4.2) as the delay-free form by noting that $w_l(t-(l-\theta))$ $(0 \le \theta \le l)$ is given as the weak solution of the partial differential equation

$$\frac{\partial \omega}{\partial t}(\theta, t) = \frac{\partial \omega}{\partial \theta}(\theta, t)\ (0 < \theta < l),\ \omega(l, t) = w_l(t).$$

Let $\mathcal{X}_0$ be $\mathbb{R}^{\dim x} \times L^2([0,\, l], \mathbb{R}^{\dim w_l})$ and $\mathcal{X}_1$ be $\mathbb{R}^{\dim x} \times W^{2,1}([0,\, l], \mathbb{R}^{\dim w_l})$. In the homogeneous boundary condition case, i.e., when $w_l(t) = 0$, the infinitesimal generator $\widetilde{A}_{\mathrm{H}} : \mathcal{D}(\widetilde{A}_{\mathrm{H}}) := \mathcal{X}_1 \to \mathcal{X}_0$ of the system P_+ is given as follows:

$$\widetilde{A}_{\mathrm{H}} \begin{bmatrix} x \\ \omega(\cdot) \end{bmatrix} = \begin{bmatrix} Ax(t) + B_{1/l}\omega(0) \\ \dfrac{\partial \omega}{\partial \theta}(\cdot) \end{bmatrix} \text{ for } \begin{bmatrix} x \\ \omega(\cdot) \end{bmatrix} \in \mathbb{R}^{\dim x} \times W^{2,1}([0,\, l], \mathbb{R}^{\dim w_l}),$$

where $\widetilde{A}_{\mathrm{H}}$ is an unbounded operator on $\mathcal{X}_0 := \mathbb{R}^{\dim x} \times L^2([0,\, l], \mathbb{R}^{\dim w_l})$, and its domain is given by

$$\mathcal{D}\left(\widetilde{A}_{\mathrm{H}}\right) = \left\{ \begin{bmatrix} x \\ \omega(\cdot) \end{bmatrix} \in \mathbb{R}^{\dim x} \times W^{2,1}([0,\, l], \mathbb{R}^{\dim w_l}) \,\middle|\, \omega(l) = 0 \right\}.$$

The operator $\widetilde{A}_{\mathrm{H}}$ can be extended to that from $\mathcal{D}(\widetilde{A}) := \mathcal{X}_0$ to $\mathcal{X}_{-1} := \mathcal{D}(\widetilde{A}_{\mathrm{H}}^*)^*$. Using the extension $\widetilde{A}$, the state equations (4.1)-(4.2) can be rewritten as follows:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\omega}_l(t) \end{bmatrix} = \widetilde{A} \begin{bmatrix} x(t) \\ \omega(t) \end{bmatrix} + \widetilde{B}_1 w(t) + \widetilde{B}_2 u(t),$$

$$z(t) = \widetilde{C}_1 \begin{bmatrix} x(t) \\ \omega(t) \end{bmatrix} + D_{12}u(t).$$

$$\widetilde{B}_1 := \left[\widetilde{B}_{1/0}\, \widetilde{B}_{1/l} \right],\ \widetilde{B}_{1/0} := \begin{bmatrix} B_{1/0} \\ O \end{bmatrix},\ \widetilde{B}_{1/l} := \begin{bmatrix} O \\ \delta_l(\cdot) I_{\dim w_l} \end{bmatrix},$$

$$\widetilde{B}_2 := \begin{bmatrix} B_2 \\ O \end{bmatrix},\ \widetilde{C}_1 := \begin{bmatrix} C_1 & O \end{bmatrix}.$$

The following operator Riccati equation corresponds to the LQ optimal control problem for the the above-represented system.

$$\widetilde{Q} + \widetilde{A}^*\widetilde{X} + \widetilde{X}\widetilde{A} - \left(\widetilde{S}_2 + \widetilde{X}\widetilde{B}_2\right) R_2^{-1} \left(\widetilde{S}_2 + \widetilde{X}\widetilde{B}_2\right)^* = O, \tag{4.17}$$

$$\widetilde{Q} := \widetilde{C}_1^*\widetilde{C}_1,\ \widetilde{S}_2 := \widetilde{C}_1^* D_{12}.$$

The positive semidefinite stabilizing solution $\widetilde{X} : \mathcal{X}_{-1} \to \mathcal{X}_1$ yields the optimal state feedback gain $\widetilde{F}_2$ as follows:

$$\widetilde{F}_2 = -R_2^{-1}\left(\widetilde{S}_2^* + \widetilde{B}_2^* \widetilde{X}\right). \tag{4.18}$$

If $\widetilde{F}_2$ is partitioned conformably with x and ω: $\widetilde{F}_2 =: \begin{bmatrix}\widetilde{F}_{2x} & \widetilde{F}_{2\omega}\end{bmatrix}$, then the optimal state feedback law $u_*(t)$ in Eq. (4.16) is obtained.

The following lemma assures that the operator $\widetilde{X}$ used for the spectral factorization in Lemma 15 is the positive semidefinite stabilizing solution of the Riccati equation (4.17).

Lemma 16. *The operator $\widetilde{X}$ constructed in Eq. (4.14) is the positive semidefinite stabilizing solution of the Riccati equation (4.17).* □

Proof. Subsection 4.6.2 □

By Lemma 16, we obtain the following proposition which gives the explicit representation of the optimal state feedback gain and corresponding optimal cost.

Proposition 1. *The optimal state feedback gain $\widetilde{F}_2 = \begin{bmatrix}\widetilde{F}_{2x} & \widetilde{F}_{2\omega}\end{bmatrix}$ in Eq. (4.18) and the corresponding optimal cost $E_{+\mathrm{FI}}$ in Eq. (4.15) are given as follows:*

$$\widetilde{F}_{2x} = -R_2^{-1}\left(S_2^* + B_2^* X\right), \tag{4.19}$$

$$\widetilde{F}_{2\omega} = \widetilde{F}_{2x}\left\{UE_\omega\right\}_x + R_2^{-1}\begin{bmatrix}S_2^* & -B_2^*\end{bmatrix}UE_\omega, \tag{4.20}$$

$$E_{+\mathrm{FI}}^2 = \mathrm{tr}\, B_{1/0}^* X B_{1/0} + \mathrm{tr}\, B_{1/l}^*\left\{X - X\left(G - e^{\widetilde{A}_{\mathrm{c}\,x} l} G e^{\widetilde{A}_{\mathrm{c}\,x}^* l}\right)X\right\}B_{1/l}, \tag{4.21}$$

where $\widetilde{A}_{\mathrm{c}\,x}^{\mathrm{R}} := A + B_2\widetilde{F}_{2x}$ is stable, and G is the positive semidefinite solution of the Lyapunov equation

$$\widetilde{A}_{\mathrm{c}\,x}^{\mathrm{R}} G + G\widetilde{A}_{\mathrm{c}\,x}^{\mathrm{R}\,*} + B_2 R_2^{-1} B_2^* = O.$$

□

Proof. Substituting Eq. (4.14) into the right-hand side of Eq. (4.18), we have Eqs. (4.19)-(4.20). Similarly, we substitute Eq. (4.14) into the right-hand side of Eq. (4.15), and use the following identities to obtain Eq. (4.21).

$$T_X^{-1} H T_X = \begin{bmatrix}\widetilde{A}_{\mathrm{c}\,x}^{\mathrm{R}} & B_2 R_2^{-1} B_2^* \\ O & -\widetilde{A}_{\mathrm{c}\,x}^{\mathrm{R}\,*}\end{bmatrix},\ T_X := \begin{bmatrix}I & O \\ -X & I\end{bmatrix},$$

$$S_G^{-1} T_X^{-1} H T_X S_G = \begin{bmatrix}\widetilde{A}_{\mathrm{c}\,x}^{\mathrm{R}} & O \\ O & -\widetilde{A}_{\mathrm{c}\,x}^{\mathrm{R}\,*}\end{bmatrix},\ S_G := \begin{bmatrix}I & G \\ O & I\end{bmatrix}.$$ □

Remark 11. *By integrating the identity*

$$\frac{d}{d\theta} e^{\widetilde{A}_{\mathrm{c}\,x}^{\mathrm{R}}\theta} G e^{\widetilde{A}_{\mathrm{c}\,x}^{\mathrm{R}\,*}\theta} + e^{\widetilde{A}_{\mathrm{c}\,x}^{\mathrm{R}}\theta} B_2 R_2^{-1} B_2^* e^{\widetilde{A}_{\mathrm{c}\,x}^{\mathrm{R}\,*}\theta} = O,$$

we see that G satisfies the equation

$$G - e^{\widetilde{A}^{\mathrm{R}}_{\mathrm{c}\,x} l} G e^{\widetilde{A}^{\mathrm{R}*}_{\mathrm{c}\,x} l} = \int_{\theta=0}^{l} e^{\widetilde{A}^{\mathrm{R}}_{\mathrm{c}\,x}\theta} B_2 R_2^{-1} B_2^* e^{\widetilde{A}^{\mathrm{R}*}_{\mathrm{c}\,x}\theta}\, d\theta.$$

Therefore, the optimal cost E_+ in Eq. (4.21) is monotonically nonincreasing with respect to the preview length l.

The expression (4.21) is identical to that derived in [45]. Since the matrix $\widetilde{A}^{\mathrm{R}}_{\mathrm{c}\,x}$ is stable, the asymptotic value of $E_{+\mathrm{FI}}$ is given by

$$E^2_{+\mathrm{FI}}\big|_{l\to\infty} = \mathrm{tr}\, B^*_{1/0} X B_{1/0} + \mathrm{tr}\, B^*_{1/l}\,(X - XGX)\, B_{1/l}.$$

□

4.4.2 Output estimation problem

Let us introduce the following difference of the actual control input and the optimal state feedback control law:

$$\widetilde{u}(t) := u(t) - \left(\widetilde{F}_{2x} x(t) + \widetilde{F}_{2\omega}\omega(t)\right). \tag{4.22}$$

Since the state feedback gains are determined from the positive semidefinite stabilizing solution of the Riccati equation (4.17), the following equality holds:

$$\|T_{z\,w}\|_2^2 = E^2_{+\mathrm{FI}} + \left\| R_2^{1/2} T_{\widetilde{u}\,w}\right\|_2^2,$$

where $T_{\widetilde{u}\,w}$ denotes the transfer function from w to $\widetilde{u}$.

In the output feedback setting, the optimal state feedback law in Eq. (4.16) is not implementable. In this section, we obtain the output feedback controller as the minimizer of the scaled H^2 norm of $T_{\widetilde{u}\,w}$. Let us define the generalized plant $P_{+\,\mathrm{tmp}} : (w,\, u) \to (\widetilde{u},\, y)$ by replacing the regulated output $z(t)$ of $P_+ : (w,\, u) \to (z,\, y)$ with $\widetilde{u}(t)$ in Eq. (4.22).

Recall that the state transformation in Eq. (4.7) introduced the new state variable $x^{\mathrm{R}}(t)$. The transformation can be seen decomposing the state-variable $x(t)$ as follows:

$$x(t) = x^{\mathrm{R}}(t) - U E_\omega \omega(t),$$

where $x^{\mathrm{R}}(t)$ follows the finite-dimensional dynamics and the second term is given as the output of the PDE.

In the similar manner to the discrete-time H^2 control problem in Chapter 2, we describe $P_{+\,\mathrm{tmp}}$ using th state-variable $x^{\mathrm{R}}(t)$ instead of $x(t)$, and obtain the following state-space realization of $P_{+\,\mathrm{tmp}}$:

$$\begin{aligned}
\dot{x}^{\mathrm{R}}(t) &= A x^{\mathrm{R}}(t) - B_2 R_2^{-1} \begin{bmatrix} S_2^* & -B_2^* \end{bmatrix} U E_\omega \omega(t) + B_{1/0} w_0(t) + B^{\mathrm{R}}_{1/l} w_l(t) + B_2 u(t),\\
\frac{\partial \omega}{\partial t}(\theta,t) &= \frac{\partial \omega}{\partial \theta}(\theta,t)\ (0<\theta<l),\ \omega(l,t) = w_l(t),\\
\widetilde{u}(t) &= -\widetilde{F}_{2/x} x^{\mathrm{R}}(t) - R_2^{-1}\begin{bmatrix} S_2^* & -B_2^* \end{bmatrix} U E_\omega \omega(t) + u(t),\\
y_0(t) &= C_{2/0} x^{\mathrm{R}}(t) - C_2 \left\{U E_{\omega l}\right\}_x \omega(t) + D_{21/00} w_0(t) + D_{21/0l} w_l(t-l),\\
y_l(t) &= w_l(t),
\end{aligned}$$

where the following definition is used: $\begin{bmatrix} B^{\mathrm{R}}_{1/l} \\ S^{\mathrm{R}}_{1/l} \end{bmatrix} := e^{-Hl} \begin{bmatrix} B_{1/l} \\ O \end{bmatrix}$.

Referring to the above state-space realization of $P_{+\,\mathrm{tmp}}$, we define the new input $u^{\mathrm{R}}(t)$ and output $\begin{bmatrix} y^{\mathrm{R}}_0(t)^{\mathrm{T}} & y^{\mathrm{R}}_l(t)^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$ from $u(t)$ and $\begin{bmatrix} y_0(t)^{\mathrm{T}} & y_l(t)^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$ as follows:

$$\begin{aligned} u^{\mathrm{R}}(t) &:= u(t) - R_2^{-1} \begin{bmatrix} S_2^* & -B_2^* \end{bmatrix} U E_\omega \omega(t), \\ y^{\mathrm{R}}_0(t) &:= y_0(t) + C_2 \{U E_{\omega l}\}_x \omega(t), \\ y^{\mathrm{R}}_l(t) &:= y_l(t). \end{aligned}$$

We observe that $P_{+\,\mathrm{tmp}}$ has the structure shown in Fig. 4.3. The delay-free generalized plant $P^{\mathrm{R}}_{+\,\mathrm{tmp}} : (w,\, u^{\mathrm{R}}) \to (\widetilde{u},\, y^{\mathrm{R}})$ ($y^{\mathrm{R}} := \begin{bmatrix} y^{\mathrm{R\,T}}_0 & y^{\mathrm{R\,T}}_l \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{\dim y_0} \times \mathbb{R}^{\dim y_l}$) is described with the state variable x^{R}:

$$P^{\mathrm{R}}_{+\,\mathrm{tmp}} := \left[\begin{array}{c|cc} A & \begin{bmatrix} B_{1/0} & B^{\mathrm{R}}_{1/l} \end{bmatrix} & B_2 \\ \hline -\widetilde{F}_{2x} & \begin{bmatrix} O & O \end{bmatrix} & I \\ \begin{bmatrix} C_{2/0} \\ O \end{bmatrix} & \begin{bmatrix} D_{21/00} & O \\ O & I \end{bmatrix} & \begin{bmatrix} O \\ O \end{bmatrix} \end{array}\right].$$

Moreover, Π_{0l} and Δ_l in Fig. 4.3 are the FIR systems described with the infinite-dimensional state variable ω, and their time-domain input-output characteristics are represented as follows:

$$\Pi_{0l} y_l(t) := -D_{21/0l} y_l(t-l) + \begin{bmatrix} C_{2/0} & O \end{bmatrix} \int_{\theta=0}^{l} e^{-H\theta} \begin{bmatrix} B_{1/l} \\ O \end{bmatrix} y_l(t-(l-\theta)) d\theta, \tag{4.23}$$

$$\Delta_l y_l(t) := R_2^{-1} \begin{bmatrix} S_2^* & -B_2^* \end{bmatrix} \int_{\theta=0}^{l} e^{-H\theta} \begin{bmatrix} B_{1/l} \\ O \end{bmatrix} y_l(t-(l-\theta)) d\theta. \tag{4.24}$$

The above structure of $P_{+\,\mathrm{tmp}}$ exhibits the relationship between the stabilizing control law and the value of the cost function.

Theorem 8 (H^2 preview output feedback law)**.** *Under the assumptions (A1)-(A3), there exists a parameterization of the stabilizing controller $K_+ : y \to u$ shown in Fig. 4.4. The preview feedforward elements Π_{0l} and Δ_l are given by Eqs. (5.8)-(5.9). $K^{\mathrm{R}}_+ : y^{\mathrm{R}} \to u^{\mathrm{R}}$ is parameterized with the* Youla *parameter $Q_+ \in H^2 : \nu \to \mu$ $\left(\nu := \begin{bmatrix} \nu_0^{\mathrm{T}} & \nu_l^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{\dim y_0} \times \mathbb{R}^{\dim y_l}\right)$ as follows:*

$$K^{\mathrm{R}}_+ = \mathcal{F}_l(J^{\mathrm{R}}_+,\, Q_+), \tag{4.25}$$

$$J^{\mathrm{R}}_+ := \left[\begin{array}{c|cc} \widetilde{A}^{\mathrm{R}}_{\mathrm{c}\,x} + L^{\mathrm{R}}_2 \begin{bmatrix} C_{2/0} \\ O \end{bmatrix} & -L^{\mathrm{R}}_2 & B_2 \\ \hline \widetilde{F}_{2x} & \begin{bmatrix} O & O \end{bmatrix} & I \\ -\begin{bmatrix} C_{2/0} \\ O \end{bmatrix} & \begin{bmatrix} I & O \\ O & I \end{bmatrix} & \begin{bmatrix} O \\ O \end{bmatrix} \end{array}\right],$$

$$L^{\mathrm{R}}_2 := \begin{bmatrix} L^{\mathrm{R}}_{2/0} & L^{\mathrm{R}}_{2/l} \end{bmatrix} := \begin{bmatrix} -\left(\acute{S}^*_{2/0} + Y^{\mathrm{R}} C^*_{2/0}\right) \acute{R}^{-1}_{2/0} & -B^{\mathrm{R}}_{1/l} \end{bmatrix}. \tag{4.26}$$

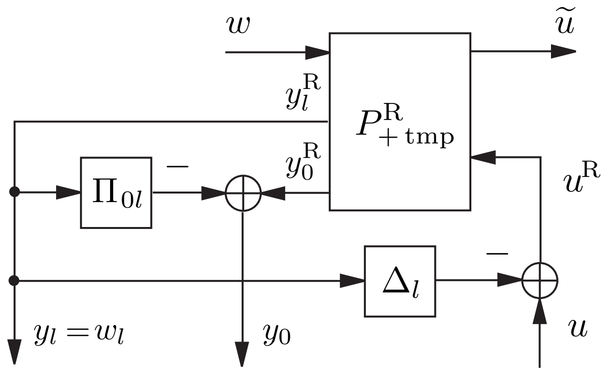

Figure 4.3: Structure of $P_{+\,\mathrm{tmp}}$.

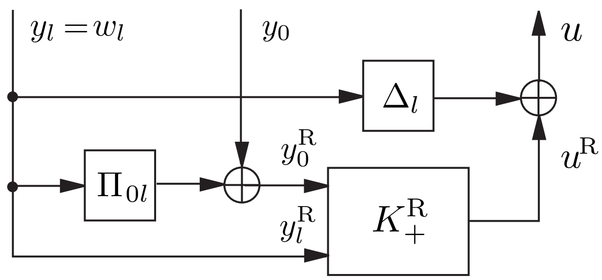

Figure 4.4: Structure of K_+.

Furthermore, the optimal controller is obtained when $Q_+ = O$ and the corresponding optimal cost E_+ is given by

$$E_+^2 = \min \|T_{u\,w}\|_2^2 = E_{+\mathrm{FI}}^2 + \mathrm{tr}\, R_2 \widetilde{F}_{2x} Y^{\mathrm{R}} \widetilde{F}_{2x}^*, \tag{4.27}$$

where $E_{+\,\mathrm{FI}}^2$ is determined in Eq. (4.21). □

Proof. Referring to the structure of $P_{+\,\mathrm{tmp}}$ in Fig. 4.3, we introduce K_+^{R}, namely, the undetermined part of the controller K_+ as shown in Fig. 4.4. Noting that Π_{0l} and Δ_l are causal and stable systems, we can cancel them out of the interconnection between $P_{+\,\mathrm{tmp}}$ and K_+. Therefore, K_+^{R} is determined as the observer-based H^2 controller for the generalized plant $P_{+\,\mathrm{tmp}}^{\mathrm{R}}$ in the output estimation form.

The optimal observer K_+^{R} is constructed from the positive semidefinite solution Y^{R} of the following filtering-type Riccati equation:

$$\acute{Q}_{\mathrm{tmp}} + AY^{\mathrm{R}} + Y^{\mathrm{R}}A^* - \left(\acute{S}_{\mathrm{tmp2}}^* + Y^{\mathrm{R}} \begin{bmatrix} C_{2/0} \\ O \end{bmatrix}^*\right) \acute{R}_{\mathrm{tmp2}}^{-1} \left(\acute{S}_{\mathrm{tmp2}}^* + Y^{\mathrm{R}} \begin{bmatrix} C_{2/0} \\ O \end{bmatrix}^*\right)^* = O,$$

where

$$\begin{bmatrix} \acute{Q}_{\rm tmp} & \acute{S}_{\rm tmp2}^* \\ \acute{S}_{\rm tmp2} & \acute{R}_{\rm tmp2} \end{bmatrix} := \begin{bmatrix} \begin{bmatrix} B_{1/0} & B_{1/l}^{\rm R} \end{bmatrix} \\ \begin{bmatrix} D_{21/00} & O \\ O & I \end{bmatrix} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} B_{1/0} & B_{1/l}^{\rm R} \end{bmatrix} \\ \begin{bmatrix} D_{21/00} & O \\ O & I \end{bmatrix} \end{bmatrix}^* .$$

It is verified that the above Riccati equation coincides with the Riccati equation (4.4), and hence the standard calculation yields the optimal observer gain $L_2^{\rm R}$ in Eq. (4.26) and following closed-loop performance expression:

$$\|T_{z\,w}\|_2^2 = E_{+\rm FI}^2 + {\rm tr}\, R_2 \widetilde{F}_{2x} Y^{\rm R} \widetilde{F}_{2x}^* + \left\| R_2^{1/2} Q_+ \begin{bmatrix} \acute{R}_{2/0}^{1/2} & O \\ O & I \end{bmatrix} \right\|_2^2 . \tag{4.28}$$

□

4.5 Example

This section examines the relationship between the preview length and the H^2 control performance by active suspension control of a quarter-car model. The model of the active suspension [14] is shown in Fig. 4.5. The state variables $\eta_{\rm b}(t)$, $\eta_{\rm w}(t)$ and $\eta_{\rm r}(t)$ represent the longitudinal displacements of the load, wheel and road profile, respectively. We use the following values for their mass, damping coefficient and stiffness constant:

$$m_{\rm b} = 400\,[\text{kg}],\ c_{\rm b} = 1000\,[\text{Ns/m}],\ k_{\rm b} = 5000\,[\text{N/m}],$$
$$m_{\rm w} = 400\,[\text{kg}],\ k_{\rm w} = 50000\,[\text{N/m}].$$

The controller K_+ reads the incoming information of the road profile $y_l(t) = w_l(t) := \eta_{\rm r}(t+l)$, and exert the control input or force $u(t)$ based on the following measurement output:

$$y_0(t) = (\dot{\eta}_{\rm b}(t) - \dot{\eta}_{\rm w}(t)) + w_0(t), \tag{4.29}$$

where $w_0(t)$ is considered as a measurement noise. The first term on the right-hand side of Eq. (4.29) is the relative velocity between the load and wheel, and the second term is a measurement noise. Since the acceleration of the load $\ddot{\eta}_{\rm b}(t)$ reflects driving comfort, the regulated output $z(t)$ is defined so that its L^2 norm incorporates the square integral of $\ddot{\eta}_{\rm b}(t)$:

$$\|z\|_2^2 = \int_0^\infty \ddot{\eta}_{\rm b}(t)^2 + \rho^2 u(t)^2\, dt,\ \rho := 0.001\,[\text{m/Ns}^2].$$

Under the above setup, the values of the optimal cost E_+ with respect to the preview length l is determined by Eqs. (4.21) and (4.27). From Fig. 4.6, it is seen that E_+^2 sufficiently reaches its asymptotic value $7.3 \cdot 10^4$ within the preview length $l = 1\,[\text{s}]$.

The frequency response of the transfer function $T_{\ddot{\eta}_{\rm b}\,\eta_{\rm r}}$ from $\eta_{\rm r}$ to $\ddot{\eta}_{\rm b}$ is depicted in Fig. 4.7. The gain tends to lower along the increasing preview length. It is also observed that the increase of the preview length does not contribute to the decrease of the peak around $\omega = 35\,[\text{s}^{-1}]$. The phase plot is set within the range $[-\pi,\ \pi]$. If not set inside it, the phase decreases monotonically as the band increases.

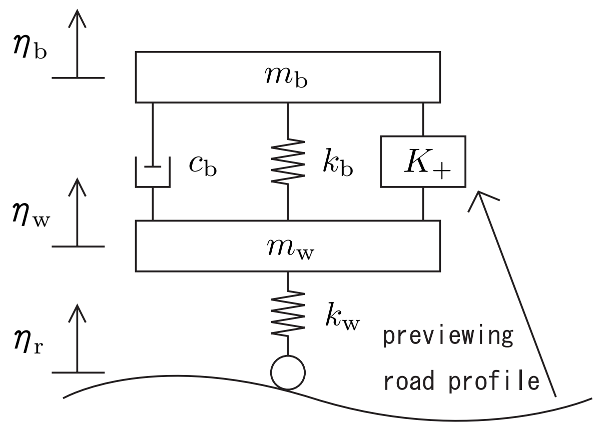

Figure 4.5: Quarter-car model.

The time responses of $\ddot{\eta}_b(t)$ and $u(t)$ is depicted in Fig. 4.8, where the road profile $\eta_r(t)$ is set as follows:

$$\eta_r(t) = \begin{cases} 0 & (0 \le t < 1) \\ 0.05\left(1 - \cos\dfrac{\pi}{4}(t-1)\right) & (1 \le t < 5) \\ 0 & (t \ge 5) \end{cases} .$$

Although the discontinuity of $\eta_r(t)$ at $t = 5$ [s] causes the large oscillations, the preview action enables to produce the less peak responses (Tab. 4.1).

Table 4.1: Maximum peak values of time response.

l	0 [s]	0.2 [s]	0.4 [s]	0.6 [s]
$\max_t \lvert\ddot{\eta}_b(t)\rvert$	2.78 [m/s^2]	2.32 [m/s^2]	2.05 [m/s^2]	1.71 [m/s^2]
$\max_t \lvert u(t)\rvert$	1.99 [N]	1.75 [N]	1.51 [N]	1.39 [N]

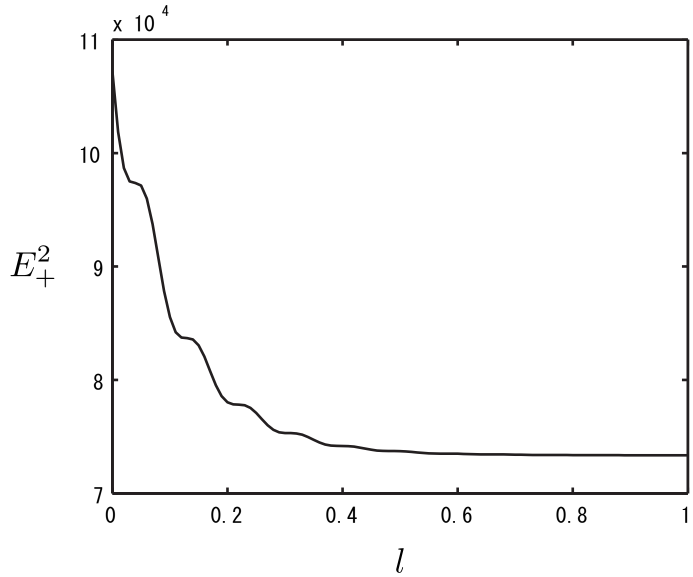

Figure 4.6: H^2 performance improvement with preview.

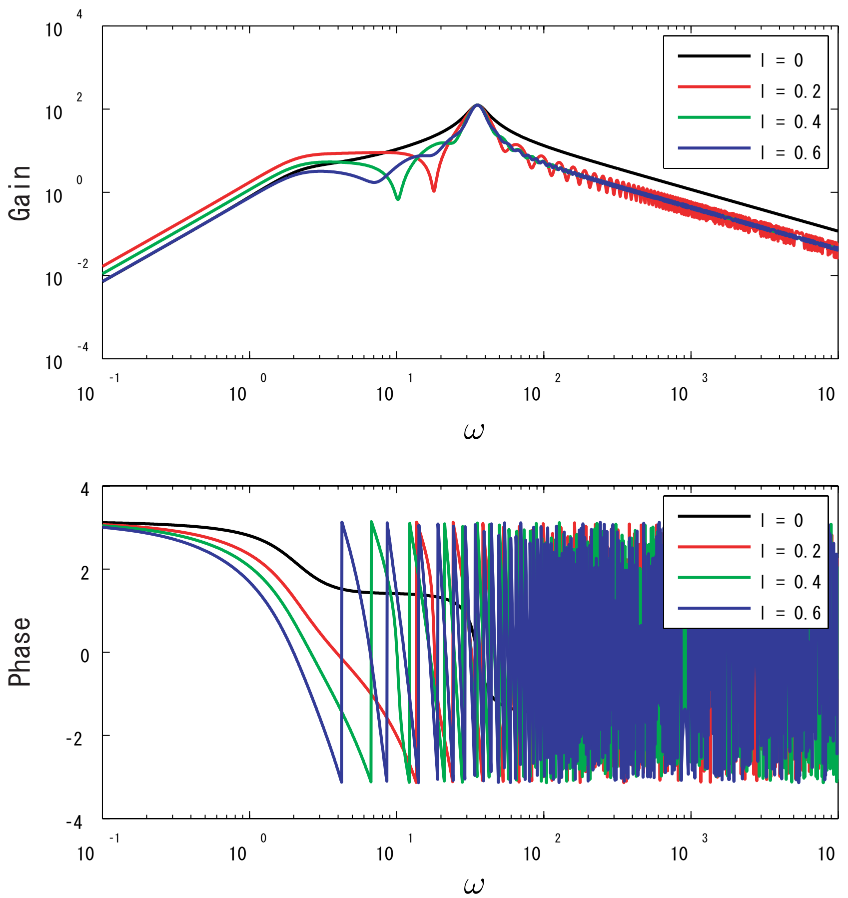

Figure 4.7: Frequency response of $T_{\ddot{\eta}_b \eta_r}$.

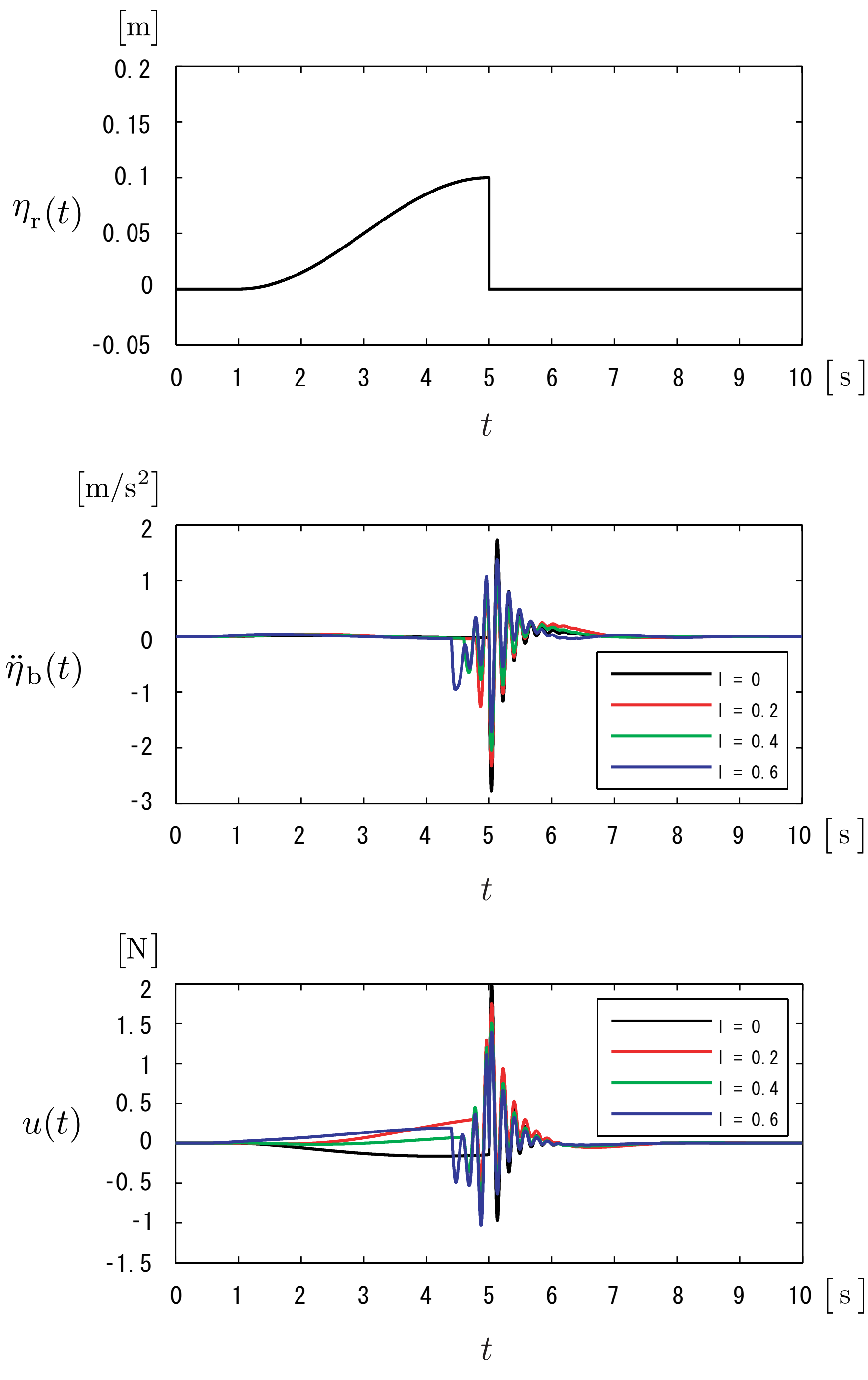

Figure 4.8: Reduction of maximum peaks with preview.

4.6 Proofs

4.6.1 Proof of Lemma 15

After applying the state transformations (4.7)-(4.9), the state-space realization of Φ_{+22}^{-1} : $k \to u$ is given as follows:

$$\begin{aligned}
\begin{bmatrix} \dot{x}^{\mathrm{R}}(t) \\ \dot{p}^{\mathrm{R}}(t) \end{bmatrix} &= H \begin{bmatrix} x^{\mathrm{R}}(t) \\ p^{\mathrm{R}}(t) \end{bmatrix} + \begin{bmatrix} B_2 \\ S_2 \end{bmatrix} R_2^{-1} k(t), \\
E_\omega \dot{\omega}(t) &= A_\omega \omega(t), \\
E_\alpha \dot{\alpha}^{\mathrm{R}\times}(t) &= -A_\alpha \alpha^{\mathrm{R}\times}(t) - E_\alpha V \begin{bmatrix} B_2 \\ S_2 \end{bmatrix} R_2^{-1} k(t), \\
u(t) &= -R_2^{-1} \begin{bmatrix} S_2^* & -B_2^* \end{bmatrix} \begin{bmatrix} x^{\mathrm{R}}(t) \\ p^{\mathrm{R}}(t) \end{bmatrix} + R_2^{-1} \begin{bmatrix} S_2^* & -B_2^* \end{bmatrix} U E_\omega \omega + R_2^{-1} k(t).
\end{aligned}$$

The dynamics of $\left(x^{\mathrm{R}},\, p^{\mathrm{R}}\right)$ is decoupled from that of $\left(\omega,\, \alpha^{\mathrm{R}\times}\right)$ in the above realization. Noting that the solution X of the Riccati equation in Eq. (4.3) triangularize the Hamiltonian matrix H, the spectral factorization of Φ_{+22} is completed by the state transformation

$$\begin{bmatrix} x^{\mathrm{R}}(t) \\ p^{\mathrm{R}}(t) \end{bmatrix} =: \begin{bmatrix} I & O \\ -X & I \end{bmatrix} \begin{bmatrix} x^{\mathrm{R}}(t) \\ p^{\mathrm{R}\times}(t) \end{bmatrix}.$$

4.6.2 Proof of Lemma 16

Partition $\widetilde{X}$ in Lemma 15 conformably with $\begin{bmatrix} x^{\mathrm{T}} & \omega^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$: $\widetilde{X} =: \begin{bmatrix} \widetilde{X}_{xx} & \widetilde{X}_{x\omega} \\ \widetilde{X}_{\omega x} & \widetilde{X}_{\omega\omega} \end{bmatrix}$. Then each of the subblocks are represented by

$$\begin{aligned}
\widetilde{X}_{xx} x &= \widetilde{X}_{xx}^{\mathrm{K}} x, \\
\widetilde{X}_{x\omega} \omega = \int_{\theta=0}^{l} X_{x\omega}^{\mathrm{K}}(\theta)\omega(\theta)\, d\theta,\ \widetilde{X}_{\omega x} x(\phi) &= \widetilde{X}_{\omega x}^{\mathrm{K}}(\phi) x, \\
\widetilde{X}_{\omega\omega}\omega(\phi) &= \int_{\theta=0}^{l} X_{\omega\omega}^{\mathrm{K}}(\phi,\theta)\omega(\theta)\, d\theta \ \ (0 < \phi < l),
\end{aligned}$$

where $\widetilde{X}_{ij}^{\mathrm{K}}$ $(i, j = x,\, \omega)$ are defined as follows:

$$\begin{aligned}
\widetilde{X}_{xx}^{\mathrm{K}} := X,\ \widetilde{X}_{x\omega}^{\mathrm{K}}(\theta) &:= \begin{bmatrix} X & I \end{bmatrix} e^{-H\theta} \begin{bmatrix} B_{1/l} \\ O \end{bmatrix},\ \widetilde{X}_{\omega x}^{\mathrm{K}}(\phi) := \widetilde{X}_{x\omega}^{\mathrm{K}}(\phi)^*, \\
\widetilde{X}_{\omega\omega}^{\mathrm{K}}(\phi,\theta) &:= \begin{bmatrix} B_{1/l}^* & O \end{bmatrix} e^{-H^*\phi} \begin{bmatrix} X & H(\theta-\phi)I \\ H(\phi-\theta)I & O \end{bmatrix} e^{-H\theta} \begin{bmatrix} B_{1/l} \\ O \end{bmatrix} \ \ (0 < \theta,\, \phi < l).
\end{aligned}$$

Since the above defined $\widetilde{X}_{ij}^{\mathrm{K}}$ $(i, j = x,\, \omega)$ satisfy the following system of PDEs, $\widetilde{X}$ in Eq. (4.14) is a solution of the operator Riccati equation (4.17).

$$
\begin{aligned}
&Q + A^*\widetilde{X}^{\mathrm{K}}_{xx} + \widetilde{X}^{\mathrm{K}}_{xx}A - \left(S_2 + \widetilde{X}^{\mathrm{K}}_{xx}B_2\right) R_2^{-1} \left(S_2 + \widetilde{X}^{\mathrm{K}}_{xx}B_2\right)^* = O,\\
&-\frac{\partial}{\partial\theta}\widetilde{X}^{\mathrm{K}}_{x\omega}(\theta) + A^*\widetilde{X}^{\mathrm{K}}_{x\omega}(\theta) - \left(S_2 + \widetilde{X}^{\mathrm{K}}_{xx}B_2\right) R_2^{-1}B_2^*\widetilde{X}^{\mathrm{K}}_{x\omega}(\theta) = O,\\
&-\frac{\partial}{\partial\phi}\widetilde{X}^{\mathrm{K}}_{\omega x}(\phi) + \widetilde{X}^{\mathrm{K}}_{\omega x}(\phi)A - \widetilde{X}^{\mathrm{K}}_{\omega x}(\phi)B_2R_2^{-1}\left(S_2 + \widetilde{X}^{\mathrm{K}}_{xx}B_2\right)^* = O,\\
&-\frac{\partial}{\partial\phi}\widetilde{X}^{\mathrm{K}}_{\omega\omega}(\phi,\theta) - \frac{\partial}{\partial\theta}\widetilde{X}^{\mathrm{K}}_{\omega\omega}(\phi,\theta) - \widetilde{X}^{\mathrm{K}}_{\omega x}(\phi)B_2R_2^{-1}B_2^*\widetilde{X}^{\mathrm{K}}_{x\omega}(\theta) = O,\\
&\qquad \widetilde{X}^{\mathrm{K}}_{x\omega}(0) = \widetilde{X}^{\mathrm{K}}_{xx}B_{1/l},\ \widetilde{X}^{\mathrm{K}}_{\omega\omega}(\phi,0) = \widetilde{X}^{\mathrm{K}}_{\omega x}(\phi)B_{1/l},\\
&\qquad \widetilde{X}^{\mathrm{K}}_{\omega x}(0) = B_{1/l}^*\widetilde{X}^{\mathrm{K}}_{xx},\ \widetilde{X}^{\mathrm{K}}_{\omega\omega}(0,\theta) = B_{1/l}^*\widetilde{X}^{\mathrm{K}}_{x\omega}(\theta)\\
&\qquad\qquad (0 < \theta,\, \phi < l).
\end{aligned}
$$

To show the stability of $\widetilde{A}+\widetilde{B}_2\widetilde{F}_2$, we transform its domain and range spaces by Eq. (4.8). The realization of $\widetilde{A}+\widetilde{B}_2\widetilde{F}_2$ after the transformation is described as follows:

$$
\begin{aligned}
\dot{x}^{\mathrm{R}}(t) &= \left(A + B_2\widetilde{F}_{2x}\right) x^{\mathrm{R}}(t) + B_2R_2^{-1}\left[S_2^* \ \ -B_2^*\right] UE_\omega\omega(t),\\
\frac{\partial\omega}{\partial t}(\theta,t) &= \frac{\partial\omega}{\partial\theta}(\theta,t)\ (0<\theta<l),\ \omega(l,t) = 0.
\end{aligned}
$$

This system is stable as $A + B_2\widetilde{F}_{2x}$ is a stable matrix.

To show the positive semidefiniteness of $\widetilde{X}$, we rewrite the Riccati equation (4.17) as the Lyapunov equation

$$
\left(\widetilde{Q} - \widetilde{S}_2^*R_2^{-1}\widetilde{S}_2 + \widetilde{X}\widetilde{B}_2R_2^{-1}\widetilde{B}_2^*\widetilde{X}\right) + \left(\widetilde{A} + \widetilde{B}_2\widetilde{F}_2\right)^*\widetilde{X} + \widetilde{X}\left(A + \widetilde{B}_2\widetilde{F}_2\right) = O.
$$

Since $\widetilde{A}+\widetilde{B}_2\widetilde{F}_2$ is a stable, $\widetilde{X}$ is positive semidefinite by the results on operator Lyapunov equations [24].

4.7 Conclusion

The design method of the continuous-time H^2 preview controller is derived along the lines of Chapter 2. We solved the output feedback problem by simplifying it to the full information and output estimation ones, and guaranteed the exact optimal performance by the orthogonality principle in H^2 space.

In the full information problem, we introduced the state transformations to perform the spectral factorization considering the state variable of the delay element. By the spectral factorization, the optimal state feedback law is constructed as a solution of the one-sided model matching problem. One of the state transformations defines the decomposition of the state variable $x(t)$ of the finite-dimensional generalized plant P. In the output estimation problem, we employed the state decomposition to describe the infinite-dimensional generalized plant in the form amenable to the explicit solution. The essential part of the output feedback controller, which is estimating the possibly unstable state variable $x^{\mathrm{R}}(t)$, is determined.

Chapter 5 Continuous-time H^∞ preview control

5.1 Introduction

This chapter extends the design method in Chapter 4 to the H^∞ preview output feedback control problem. The output feedback controller is constructed through the full information and output estimation problems. The technique of reducing an infinite-dimensional J-spectral density to finite-dimensional one is employed to deal with the full information problem. It is initially proposed for a continuous-time input-delayed mixed sensitivity problem in [36] and extended to the fixed-lag smoothing problems in [31], [38]. We introduce the state transformations, which are parallel to the one proposed in Chapter 4, to find the relationship between the state variables of the infinite-dimensional and reduced finite-dimensional J-spectral densities. The explicit stabilizing solution of the control-type operator Riccati equation is constructed by combining the proposed state transformations. One of the state transformations defines the decomposition of the state variable of the controlled plant. The subsequent output estimation problem is reduced to finite-dimensional one along the lines of Chapter 4.

In the recent study [30], the H^∞ output feedback controller for the more general class of input-delayed and preview systems are obtained by solving the operator Riccati equations via the other kind of state transformations. However, the proposed state transformation allows us to reveal the following aspects of the preview control problems; 1) The H^∞ output feedback preview controller is implemented as the combination of the finite-dimensional observer and preview-feedforward compensation; 2) There exists the direct relationship between the J-spectral factorization techniques in [31], [38] and the stabilizing solution of the operator Riccati equation.

This chapter is organized as follows. In Section 5.2, the problem formulation and assumptions are stated. In Section 5.3, the original output feedback problem is first restricted to the full information problem where the state and disturbance are available for the control. After extracting the J-lossless factor of the generalized plant associated with the one-sided model matching problem, the full information setting is extended to the output feedback setting. In Section 5.4, the positive semidefinite stabilizing solution of the control-type operator Riccati equation is constructed via the proposed state transformations. In Section 5.5, a design example of a H^∞ preview tracking system is presented. In Section 5.6, the proofs left in the previous sections are given.

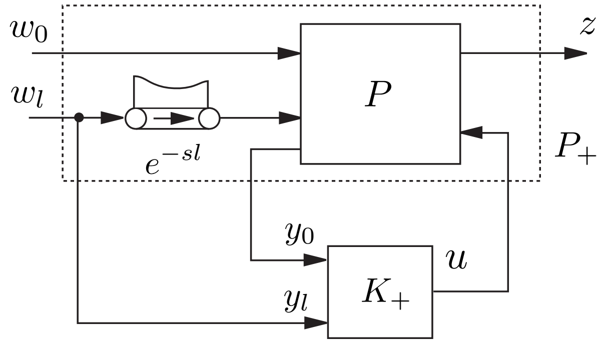

Figure 5.1: Preview control system.

5.2 Problem formulation

Let us formulate the H^∞ preview control problem for the following generalized plant P_+:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B_{1/0}w_0(t) + B_{1/l}w_l(t-l) + B_2u(t),\\ z(t) &= C_1x(t) + D_{12}u(t),\\ \begin{bmatrix} y_0(t) \\ y_l(t) \end{bmatrix} &= \begin{bmatrix} C_{2/0}x(t) + D_{21/00}w_0(t) + D_{21/0l}w_l(t-l) \\ w_l(t) \end{bmatrix}.\end{aligned}$$

The overall control system is depicted as in Fig. 5.1, where P is the finite-dimensional generalized plant defined by

$$P := \left[\begin{array}{c|cc} A & \begin{bmatrix} B_{1/0} & B_{1/l} \end{bmatrix} & B_2 \\ \hline C_1 & \begin{bmatrix} O & O \end{bmatrix} & D_{12} \\ C_{2/0} & \begin{bmatrix} D_{21/00} & D_{21/0l} \end{bmatrix} & O \end{array}\right],$$

and K_+ is the controller to design. In the setup, $w_l(t)$ is regarded as preview information and l is a preview length. The information on $x(t)$ is assumed to be partially available through $y_0(t)$, and the preview information is assumed to be fully available through $y_l(t)$.

The problems are to obtain a tractable solvability condition on the H^∞ control problem for P_+, and to reveal the clearly interpretable structure of the controller K_+ which archives the following pefromance based on the partial information $y(t) := \begin{bmatrix} y_0(t)^{\mathrm{T}} & y_l(t)^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$:

$$\sup_{w \in L^2,\, w \neq 0} \frac{\|z\|_2}{\|w\|_2} < \gamma, \; w(t) := \begin{bmatrix} w_0(t)^{\mathrm{T}} & w_1(t)^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}.$$

The following conditions (A1)-(A3) are assumed throughout this chapter.

(A1) (A, B_2) and $\left(A, C_{2/0}\right)$ are stabilizable and detectable, respectively.

(A2) For $\forall\, \omega \in \mathbb{R}$, $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ and $\begin{bmatrix} A - j\omega I & B_{1/0} \\ C_{2/0} & D_{21/00} \end{bmatrix}$ are of full column rank and of full row rank, respectively.

(A3) D_{12} and $D_{21/00}$ are of full column rank and of full row rank, respectively.

For simplicity, the following definitions are used.

$$\begin{bmatrix} Q & S_2 \\ S_2^* & R_2 \end{bmatrix} := \begin{bmatrix} C_1 & D_{12} \end{bmatrix}^* \begin{bmatrix} C_1 & D_{12} \end{bmatrix}, \quad \begin{bmatrix} \acute{Q}_0 & \acute{S}_{2/0}^* \\ \acute{S}_{2/0} & \acute{R}_{2/0} \end{bmatrix} := \begin{bmatrix} B_{1/0} \\ D_{21/00} \end{bmatrix} \begin{bmatrix} B_{1/0} \\ D_{21/00} \end{bmatrix}^*.$$

5.3 Solution via closed-loop reduction

5.3.1 Full information problem

We begin with simplifying the original problem to the full information problem, where all the information on the state variables and exogenous disturbances can be utilized to construct the control law. We apply the J-spectral factorization techniques in [31], [38] to this simplified problem. The possible straightforward approach to the full information problem is to dualize the results in [31], [38]. In [31], [38], however, the relationship between the state variables of the infinite-dimensional and reduced finite-dimensional J spectral densities are not clearly mentioned. In contrast to [31], [38], we clarify the relationship by introducing the new state transformations, which are along the lines in Chapter 2.

Denote the available disturbances by $y_{\mathrm{mm}} := \begin{bmatrix} y_{\mathrm{mm}\,0}^{\mathrm{T}} & y_{\mathrm{mm}\,l}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} := \begin{bmatrix} w_0^{\mathrm{T}} & w_l^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$ and let $P_{+\,\mathrm{mm}}$ be the transfer function from $(w,\, u)$ to $(z,\, y_{\mathrm{mm}})$. Note that $P_{+\,\mathrm{mm}}$ is the generalized plant corresponding to the following model matching problem [MM].

[MM] Find the causal transfer function $T_{u\,w}$ which satisfies the following conditions:

$$P_{+\,11} + P_{+\,12} T_{u\,w} \in H^\infty \text{ and } \|P_{+\,11} + P_{+\,12} T_{u\,w}\|_\infty < \gamma.$$

As in Subsection 2.5.1, the model matching problem [MM] can be recast as the following J-spectral factorization problem [SF] [17], and the solution of [MM] is parameterized as follows:

$$T_{u\,w} = \mathcal{F}_l(\mathcal{C}^{-1}(M_+),\, T_{\widetilde{u}\,\widetilde{w}}), \quad \forall\, T_{\widetilde{u}\,\widetilde{w}} \in H^\infty \text{ such that } \left\| R_2^{1/2} T_{\widetilde{u}\,\widetilde{w}} \right\|_\infty < \gamma,$$

where M_+ is the J-spectral factor of the J-spectral density and $T_{\widetilde{u}\,\widetilde{w}}$ denotes the family of the transfer functions from $\widetilde{w}$ to $\widetilde{u}$.

[SF] Define the J-spectral density $\Phi_+ : (y_{\mathrm{mm}}, u) \to (h_{\mathrm{mm}}, k)$ by

$$\Phi_+ := \mathcal{C}\,(P_{+\,\mathrm{mm}})^\sim \begin{bmatrix} -\gamma^2 I & O \\ O & I \end{bmatrix} \mathcal{C}\,(P_{+\,\mathrm{mm}}).$$

Then, find the J-spectral factorization of it which satisfies the following conditions.

(SF1) There exists a stable J-spectral factor $M_+ : (\widetilde{w},\, \widetilde{u}) \to (y_{\mathrm{mm}},\, u)$ such that

$$\Phi_+ = M_+^{-\sim} \begin{bmatrix} -\gamma^2 I & O \\ O & R_2 \end{bmatrix} M_+^{-1}. \tag{5.1}$$

(SF2) The transfer function $N_+ : (\widetilde{w},\, \widetilde{u}) \to (w,\, z)$ defined by $N_+ := \mathcal{C}\,(P_{+\,\mathrm{mm}})\, M_+$ is a J-inner function.

Before starting to solve the problem [SF], define the following Hamiltonian matrices

$$H_0 := \begin{bmatrix} A_\nu & P_{1/0} + P_2 \\ Q_\nu & -A_\nu^* \end{bmatrix},\ H_l := \begin{bmatrix} A_\nu & P_{1/0} + P_{1/l} + P_2 \\ Q_\nu & -A_\nu^* \end{bmatrix},$$

where

$$A_\nu := A - B_2 R_2^{-1} S_2^*,\ Q_\nu := Q - S_2 R_2^{-1} S_2^*$$
$$P_{1/0} := -\frac{1}{\gamma^2} B_{1/0} B_{1/0}^*,\ P_{1/l} := -\frac{1}{\gamma^2} B_{1/l} B_{1/l}^*,\ P_2 := B_2 R_2^{-1} B_2^*.$$

Furthermore, let $p(t)$ be the adjoint variable of $x(t)$, and introduce the new state variables $x^{\mathrm{R}}(t)$ and $p^{\mathrm{R}}(t)$ as follows:

$$\begin{bmatrix} x^{\mathrm{R}}(t) \\ p^{\mathrm{R}}(t) \end{bmatrix} := \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} + U E_{\omega l} \omega_l(t). \tag{5.2}$$

In Eq. (5.2), the operators U and E_ω are defined as follows:

$$U \begin{bmatrix} f \\ g \end{bmatrix} := \begin{bmatrix} B_{1/l} \\ O \end{bmatrix} g + \int_{\theta=0}^{l} e^{-H_0 \theta} \begin{bmatrix} B_{1/l} \\ O \end{bmatrix} f(\theta) d\theta \text{ for } (f,\, g) \in L^2([0,\, l], \mathbb{R}^{\dim w_l}) \times \mathbb{R}^{\dim w_l}, \tag{5.3}$$

and $E_\omega := \begin{bmatrix} I \\ O \end{bmatrix} : L^2([0,\, l], \mathbb{R}^{\dim w_l}) \to L^2([0,\, l], \mathbb{R}^{\dim w_l}) \times \mathbb{R}^{\dim w_l}$. Hence, $U E_\omega \omega(t) =: \begin{bmatrix} \{U E_\omega\}_x^{\mathrm{T}} & \{U E_\omega\}_p^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \omega(t)$ is given by

$$\begin{bmatrix} \{U E_\omega\}_x \\ \{U E_\omega\}_p \end{bmatrix} \omega(t) = \int_{\theta=0}^{l} e^{-H_0 \theta} \begin{bmatrix} B_{1/l} \\ O \end{bmatrix} \omega(\theta, t)\, d\theta.$$

The following Lemma 17 and Theorem 9 are the dual results to those in [31], [38]. In our proofs, the new state-variable transformations are introduced using the explicit solutions of the Sylvester equations, and it is made clear that the state variable $x(t)$ should be changed to the above-introduced $x^{\mathrm{R}}(t)$ to solve [SF]. This fact is essential to extend the full information setting to the output feedback setting in Subsection 5.3.2.

Lemma 17. *The delayed J-spectral density Φ_+ is linked with the delay-free Φ_+^{R} by the transformation*

$$\Phi_+^{\mathrm{R}} = M_+^{\mathrm{FIR}\,\sim} \Phi_+ M_+^{\mathrm{FIR}}. \tag{5.4}$$

The transfer function $M_+^{\mathrm{FIR}} : (y_{\mathrm{mm}}^{\mathrm{R}},\, u^{\mathrm{R}}) \longrightarrow (y_{\mathrm{mm}},\, u)$ $\left(y_{\mathrm{mm}}^{\mathrm{R}} := \begin{bmatrix} y_{\mathrm{mm}\,0}^{\mathrm{R\,T}} & y_{\mathrm{mm}\,l}^{\mathrm{R\,T}} \end{bmatrix}^{\mathrm{T}} \right)$ is given by the following state-space realization:

$$\frac{\partial \omega}{\partial t}(\theta, t) = \frac{\partial \omega}{\partial \theta}(\theta, t)\ (0 < \theta < l),\ \omega(l, t) = y_{\mathrm{mm}\,l}^{\mathrm{R}}(t),$$
$$y_{\mathrm{mm}\,0}(t) = \widetilde{F}_{1/0l}^{\mathrm{R}} \omega(t) + y_{\mathrm{mm}\,0}^{\mathrm{R}}(t),\ y_{\mathrm{mm}\,l}(t) = y_{\mathrm{mm}\,l}^{\mathrm{R}}(t),$$
$$u(t) = \widetilde{F}_{2/l}^{\mathrm{R}} \omega(t) + u^{\mathrm{R}}(t),$$

where the state feedback gains are defined by

$$\widetilde{F}^{\mathrm{R}}_{1/0l} := -\frac{1}{\gamma^2}\begin{bmatrix} O & -B^*_{1/0}\end{bmatrix} UE_\omega,\ \ \widetilde{F}^{\mathrm{R}}_{2/l} := R_2^{-1}\begin{bmatrix} S_2^* & -B_2^*\end{bmatrix} UE_\omega.$$

Moreover, $\Phi_+^{\mathrm{R}\,(-1)} : (h^{\mathrm{R}}_{\mathrm{mm}},\, k^{\mathrm{R}}) \longrightarrow (y^{\mathrm{R}}_{\mathrm{mm}},\, u^{\mathrm{R}})$ $\left(h^{\mathrm{R}}_{\mathrm{mm}} := \begin{bmatrix} h^{\mathrm{R\,T}}_{\mathrm{mm}\,0} & h^{\mathrm{R\,T}}_{\mathrm{mm}\,l}\end{bmatrix}^{\mathrm{T}}\right)$ *has the following state-space realization:*

$$\begin{aligned}
\begin{bmatrix}\dot{x}^{\mathrm{R}}(t)\\ \dot{p}^{\mathrm{R}}(t)\end{bmatrix} &= e^{-H_0 l} H_l e^{H_0 l}\begin{bmatrix} x^{\mathrm{R}}(t)\\ p^{\mathrm{R}}(t)\end{bmatrix}\\
&\qquad - \frac{1}{\gamma^2}\begin{bmatrix} B_{1/0}\\ O\end{bmatrix} h^{\mathrm{R}}_{\mathrm{mm}\,0}(t) - \frac{1}{\gamma^2}\begin{bmatrix} B^{\mathrm{R}}_{1/l}\\ S^{\mathrm{R}}_{1/l}\end{bmatrix} h^{\mathrm{R}}_{\mathrm{mm}\,l}(t) + \begin{bmatrix} B_2\\ S_2\end{bmatrix} R_2^{-1} k^{\mathrm{R}}(t),\\
y^{\mathrm{R}}_{\mathrm{mm}\,0}(t) &= \frac{1}{\gamma^2}\begin{bmatrix} O & -B^*_{1/0}\end{bmatrix}\begin{bmatrix} x^{\mathrm{R}}(t)\\ p^{\mathrm{R}}(t)\end{bmatrix} - \frac{1}{\gamma^2} h^{\mathrm{R}}_{\mathrm{mm}\,0}(t),\\
y^{\mathrm{R}}_{\mathrm{mm}\,l}(t) &= \frac{1}{\gamma^2}\begin{bmatrix} S^{\mathrm{R}\,*}_{1/l} & -B^{\mathrm{R}\,*}_{1/l}\end{bmatrix}\begin{bmatrix} x^{\mathrm{R}}(t)\\ p^{\mathrm{R}}(t)\end{bmatrix} - \frac{1}{\gamma^2} h^{\mathrm{R}}_{\mathrm{mm}\,l}(t),\\
u^{\mathrm{R}}(t) &= -R_2^{-1}\begin{bmatrix} S_2^* & -B_2^*\end{bmatrix}\begin{bmatrix} x^{\mathrm{R}}(t)\\ p^{\mathrm{R}}(t)\end{bmatrix} + R_2^{-1} k^{\mathrm{R}}(t),
\end{aligned}$$

where $\begin{bmatrix} B^{\mathrm{R}}_{1/l}\\ S^{\mathrm{R}}_{1/l}\end{bmatrix} := e^{-H_0 l}\begin{bmatrix} B_{1/l}\\ O\end{bmatrix}$. □

Proof. Subsection 5.6.1. □

From the realization of M_+^{FIR}, the impulse responses from $y^{\mathrm{R}}_{\mathrm{mm}\,l}(t)$ to $y_{\mathrm{mm}\,0}(t)$ and $u(t)$ are of finite-time duration. The following theorem gives a tractable solvability condition to [SF], and constructs the J-spectral factor explicitly.

Theorem 9 (J-spectral factorization)**.** *The problem [SF] is solvable if and only if the following condition (X) is satisfied.*

(X) The Riccati equation associated with the Hamiltonian matrix $e^{-H_0 l} H_l e^{H_0 l}$ *has the positive semidefinite stabilizing solution* X^{R} *such that the following matrix is stable:*

$$\widetilde{A}^{\mathrm{R}}_{\mathrm{c}\,x} := A + B_{1/0}\widetilde{F}^{\mathrm{R}}_{1/0x} + B^{\mathrm{R}}_{1/l}\widetilde{F}^{\mathrm{R}}_{1/lx} + B_2\widetilde{F}^{\mathrm{R}}_{2/x},$$

where $\widetilde{F}^{\mathrm{R}}_{1/0x}$, $\widetilde{F}^{\mathrm{R}}_{1/lx}$ *and* $\widetilde{F}^{\mathrm{R}}_{2/x}$ *are the state feedback gains defined by*

$$\widetilde{F}^{\mathrm{R}}_{1/0x} := \frac{1}{\gamma^2} B^*_{1/0} X^{\mathrm{R}},\ \ \widetilde{F}^{\mathrm{R}}_{1/lx} := \frac{1}{\gamma^2}\left(S^{\mathrm{R}\,*}_{1/l} + B^{\mathrm{R}\,*}_{1/l} X^{\mathrm{R}}\right),\ \ \widetilde{F}^{\mathrm{R}}_{2/x} := -R_2^{-1}\left(S_2^* + B_2^* X^{\mathrm{R}}\right).$$

Suppose that the above condition is satisfied. Introduce the intermediate signals $\widetilde{w} := \begin{bmatrix}\widetilde{w}_0^{\mathrm{T}} & \widetilde{w}_l^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}$ *and* $\widetilde{u}$ *as the outputs from* M_+^{-1} *or the inputs to* N_+*. Then, the inverse of the* J*-spectral factor* $M_+ : (\widetilde{w},\, \widetilde{u}) \longrightarrow (y_{\mathrm{mm}},\, u)$ *is given by*

$$M_+ = M_+^{\mathrm{FIR}} M_+^{\mathrm{R}},$$

where $M_+^{\mathrm{R}} : (\widetilde{w}, \widetilde{u}) \longrightarrow (y_{\mathrm{mm}}^{\mathrm{R}}, u^{\mathrm{R}})$ *is given by the following state-space realization:*

$$\begin{aligned}
\dot{x}^{\mathrm{R}}(t) &= \widetilde{A}_{\mathrm{c}\,x}^{\mathrm{R}} x^{\mathrm{R}}(t) + B_{1/0}\widetilde{w}_0(t) + B_{1/l}^{\mathrm{R}}\widetilde{w}_l(t) + B_2\widetilde{u}(t),\\
y_{\mathrm{mm}\,0}^{\mathrm{R}}(t) &= \widetilde{F}_{1/0x}^{\mathrm{R}} x^{\mathrm{R}}(t) + \widetilde{w}_0(t),\\
y_{\mathrm{mm}\,l}^{\mathrm{R}}(t) &= \widetilde{F}_{1/lx}^{\mathrm{R}} x^{\mathrm{R}}(t) + \widetilde{w}_l(t),\\
u^{\mathrm{R}}(t) &= \widetilde{F}_{2/x}^{\mathrm{R}} x^{\mathrm{R}}(t) + \widetilde{u}(t).
\end{aligned}$$

Moreover, $N_+ : (\widetilde{w}, \widetilde{u}) \longrightarrow (w, z)$ *is given by the state-space following realization:*

$$\begin{aligned}
\dot{x}^{\mathrm{R}}(t) &= \widetilde{A}_{\mathrm{c}\,x}^{\mathrm{R}} x^{\mathrm{R}}(t) + B_{1/0}\widetilde{w}_0(t) + B_{1/l}^{\mathrm{R}}\widetilde{w}_l(t) + B_2\widetilde{u}(t),\\
\frac{\partial \omega}{\partial t}(\theta, t) &= \frac{\partial \omega}{\partial \theta}(\theta, t)\ (0 < \theta < l),\ \omega(l, t) = \widetilde{F}_{1/lx}^{\mathrm{R}} x^{\mathrm{R}}(t) + \widetilde{w}_l(t),\\
w_0(t) &= \widetilde{F}_{1/0x}^{\mathrm{R}} x^{\mathrm{R}}(t) + \widetilde{F}_{1/0l}^{\mathrm{R}} \omega(t) + \widetilde{w}_0(t),\\
w_l(t) &= \widetilde{F}_{1/lx}^{\mathrm{R}} x^{\mathrm{R}}(t) + \widetilde{w}_l(t),\\
z(t) &= \left(C_1 + D_{12}\widetilde{F}_{2/x}^{\mathrm{R}}\right) x^{\mathrm{R}}(t) + \left(-C_1\{UE_\omega\}_x + D_{12}\widetilde{F}_{2/l}^{\mathrm{R}}\right)\omega(t) + D_{12}\widetilde{u}(t).
\end{aligned}$$

□

Proof. Since Φ_+^{R} in Eq. (5.4) is multiplied by the bi-stable transfer function and its para-conjugate from the right and left, respectively, it is necessary and sufficient for (SF1) that the Riccati equation associated with the state-transition matrix $e^{-H_0 l} H_l e^{H_0 l}$ of $\Phi_+^{\mathrm{R}\,(-1)}$ has the stabilizing solution. By the state transformation:

$$\begin{bmatrix} x^{\mathrm{R}} \\ p^{\mathrm{R}} \end{bmatrix} =: \begin{bmatrix} I & O \\ -X^{\mathrm{R}} & I \end{bmatrix} \begin{bmatrix} x^{\mathrm{R}} \\ p^{\mathrm{R}\,\times} \end{bmatrix}, \tag{5.5}$$

the following factorization of $\Phi_+^{\mathrm{R}\,(-1)}$ is obtained.

$$\Phi_+^{\mathrm{R}\,(-1)} = M_+^{\mathrm{R}} \begin{bmatrix} -\dfrac{1}{\gamma^2} I & O \\ O & R_2^{-1} \end{bmatrix} M_+^{\mathrm{R}\sim}. \tag{5.6}$$

By Eqs. (5.4) and (5.6), Eq. (5.1) is obtained and N_+ is a J-unitary system:

$$N_+^{\sim} \begin{bmatrix} -\gamma^2 I & O \\ O & I \end{bmatrix} N_+ = \begin{bmatrix} -\gamma^2 I & O \\ O & R_2 \end{bmatrix}.$$

Next, it is shown to be necessary and sufficient for (SF2) that X^{R} is positive semidefinite. By the J-unitariness of N_+, the J-inner condition on N_+ is equivalent to the stability of $N_{+\,11}^{-1}$. We see that the latter is equivalent to the stability of $A + B_2\widetilde{F}_{2/x}^{\mathrm{R}}$ from the following state-space realization of $N_{+\,11}^{-1} : w \longrightarrow \widetilde{w}$:

$$\begin{aligned}
\dot{x}(t) &= \left(A + B_2\widetilde{F}_{2/x}^{\mathrm{R}}\right) x^{\mathrm{R}}(t) - B_{1/0}\widetilde{F}_{1/0l}^{\mathrm{R}}\omega(t) + B_{1/0} w_0(t) + B_{1/l}^{\mathrm{R}} w_l(t),\\
\frac{\partial \omega}{\partial t}(\theta, t) &= \frac{\partial \omega}{\partial \theta}(\theta, t)\ (0 < \theta < l),\ \omega(l, t) = w_l(t),\\
\widetilde{w}_0(t) &= -\widetilde{F}_{1/0x}^{\mathrm{R}} x^{\mathrm{R}}(t) - \widetilde{F}_{1/0l}^{\mathrm{R}}\omega(t) + w_0(t),\\
\widetilde{w}_l(t) &= -\widetilde{F}_{1/lx}^{\mathrm{R}} x^{\mathrm{R}}(t) + w_l(t).
\end{aligned}$$

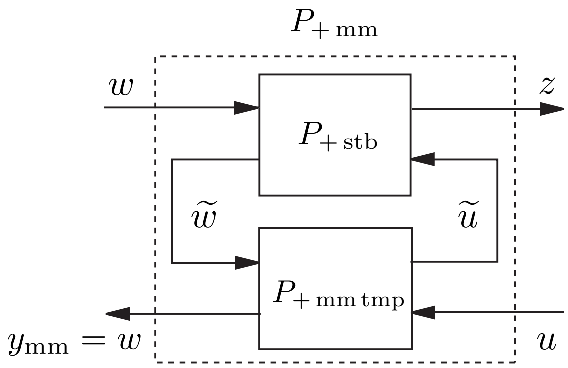

Figure 5.2: Factorization of $P_{+\,\mathrm{mm}}$.

By expanding $e^{-H_0 l} H_l e^{H_0 l}$ as

$$e^{-H_0 l} H_l e^{H_0 l} = H_0 - \frac{1}{\gamma^2} e^{-H_0 l} \begin{bmatrix} B_{1/l} \\ O \end{bmatrix} \begin{bmatrix} B_{1/l}^* & O \end{bmatrix} e^{-H_0^* l} J_{\mathrm{s}}^{-1}, \tag{5.7}$$

we have the identity

$$\begin{bmatrix} I & O \\ X^{\mathrm{R}} & I \end{bmatrix} e^{-H_0 l} H_l e^{H_0 l} \begin{bmatrix} I & O \\ -X^{\mathrm{R}} & I \end{bmatrix} = \begin{bmatrix} \widetilde{A}_{\mathrm{c}\,x}^{\mathrm{R}} & -\frac{1}{\gamma^2} B_{1/0} B_{1/0}^* - \frac{1}{\gamma^2} B_{1/l}^{\mathrm{R}} B_{1/l}^{\mathrm{R}\,*} + B_2 R_2^{-1} B_2^* \\ O & -\widetilde{A}_{\mathrm{c}\,x}^{\mathrm{R}\,*} \end{bmatrix}.$$

The (2, 1) block in the above matrix is zero, and X^{R} satisfies the following Lyapunov equation.

$$\begin{aligned} &\left(Q_\nu + X^{\mathrm{R}} P_2 X^{\mathrm{R}} + \gamma^2 \widetilde{F}_{1/0x}^{\mathrm{R}\,*} \widetilde{F}_{1/0x}^{\mathrm{R}} + \gamma^2 \widetilde{F}_{1/lx}^{\mathrm{R}\,*} \widetilde{F}_{1/lx}^{\mathrm{R}} \right) \\ &\quad + \left(A + B_2 \widetilde{F}_{2/x}^{\mathrm{R}} \right)^* X^{\mathrm{R}} + X^{\mathrm{R}} \left(A + B_2 \widetilde{F}_{2/x}^{\mathrm{R}} \right) = O. \end{aligned}$$

Moreover, $\left(A + B_2 \widetilde{F}_{2/x}^{\mathrm{R}}, \begin{bmatrix} \widetilde{F}_{1/0x}^{\mathrm{R}\,\mathrm{T}} & \widetilde{F}_{1/lx}^{\mathrm{R}\,\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \right)$ is detectable since $A_{\mathrm{c}}^{\mathrm{R}}$ is stable. Therefore, by the results on Lyapunov equations, $A + B_2 \widetilde{F}_{2/x}^{\mathrm{R}}$ is stable if and only if X^{R} is positive semidefinite. □

By (SF2), the chain scattering representation $\mathcal{C}(P_{+\,\mathrm{mm}})$ is factorized as $\mathcal{C}(P_{+\,\mathrm{mm}}) = N_+ M_+^{-1}$. Taking the inverse chain scattering representation of this identity, we have $P_{+\,\mathrm{mm}} = P_{+\,\mathrm{stb}} \star P_{+\,\mathrm{mm\,tmp}}$, where $P_{+\,\mathrm{stb}} := \mathcal{C}^{-1}(N_+)$ and $P_{+\,\mathrm{mm\,tmp}} := \mathcal{C}^{-1}\left(M_+^{-1}\right)$ (Fig. 5.2).

5.3.2 Output estimation problem

In this subsection, we derive the H^∞ preview output feedback law based on the results in Subsection 5.3.1. Recall that we factorized the generalized plant $P_{+\,\mathrm{mm}}$ as shown in Fig. 5.2. The significance of this factorization lies in that the upper part of the star-product interconnection is a lossless system. Conveniently, P_+ is also factorized in such a way. Let $P_{+\,\mathrm{tmp}}$ be

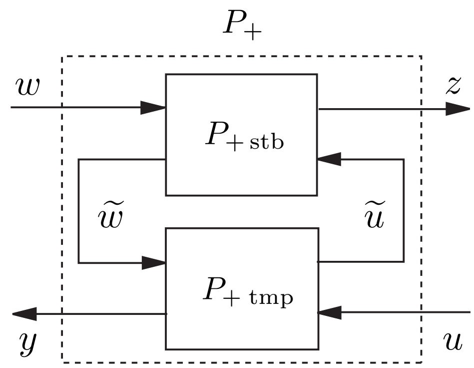

Figure 5.3: Factorization of P_+.

the generalized plant defined by replacing the measured output $y_{\mathrm{mm}}(t)$ of $P_{+\,\mathrm{mm\,tmp}}$ with the measured output $y(t)$ of P_+. Then, we have the factorization $P_+ = P_{+\,\mathrm{stb}} \star P_{+\,\mathrm{tmp}}$. This is because the original measured output $y(t)$ is represented by the state variables $(x^{\mathrm{R}}(t),\, \omega_l(\cdot, t))$ and the exogenous disturbance $\widetilde{w}(t)$ to $P_{+\,\mathrm{mm\,tmp}}$. See the following state-space realization of $P_{+\,\mathrm{tmp}}$ and Fig. 5.3:

$$\begin{aligned}
\dot{x}^{\mathrm{R}}(t) &= \left(A + B_{1/0}\widetilde{F}^{\mathrm{R}}_{1/0x} + B^{\mathrm{R}}_{1/l}\widetilde{F}^{\mathrm{R}}_{1/lx}\right) x^{\mathrm{R}}(t) \\
&\qquad - B_2\widetilde{F}^{\mathrm{R}}_{2/l}\omega_l(t) + B_{1/0}\widetilde{w}_0(t) + B^{\mathrm{R}}_{1/l}\widetilde{w}_l(t) + B_2 u(t), \\
\frac{\partial \omega}{\partial t}(\theta, t) &= \frac{\partial \omega}{\partial \theta}(\theta, t)\ (0 < \theta < l),\ \omega(l, t) = \widetilde{F}^{\mathrm{R}}_{1/lx} x^{\mathrm{R}}(t) + \widetilde{w}_l(t), \\
\widetilde{u}(t) &= -\widetilde{F}^{\mathrm{R}}_{2/x} x^{\mathrm{R}}(t) - \widetilde{F}^{\mathrm{R}}_{2/l}\omega(t) + u(t), \\
y_0(t) &= \left(C_2 + D_{21/00}\widetilde{F}^{\mathrm{R}}_{1/0x}\right) x^{\mathrm{R}}(t) + D_{21/0l}\omega(0, t) \\
&\qquad + \left(-C_2 \left\{U E_\omega\right\}_x + D_{21/00}\widetilde{F}^{\mathrm{R}}_{1/0l}\right)\omega(t) + D_{21/00}\widetilde{w}_0(t), \\
y_l(t) &= \widetilde{F}^{\mathrm{R}}_{1/lx} x^{\mathrm{R}}(t) + \widetilde{w}_l(t).
\end{aligned}$$

We used the state variable $x^{\mathrm{R}}(t)$ instead of $x(t)$ to describe $P_{+\,\mathrm{tmp}}$, which is in output estimation form.

Since $P_{+\,\mathrm{stb}}$ is J-lossless, the H^∞ problem for P_+ is reduced to that for $P_{+\,\mathrm{tmp}}$ by Redheffer's lemma. From the above state-space realization of $P_{+\,\mathrm{tmp}}$, we observe that $P_{+\,\mathrm{tmp}}$ has the structure shown in Fig. 5.4. The finite-dimensional generalized plant $P^{\mathrm{R}}_{+\,\mathrm{tmp}} : (\widetilde{w},\, u^{\mathrm{R}}) \to (\widetilde{u},\, y^{\mathrm{R}})$ is described with the state variable $x^{\mathrm{R}}(t)$:

$$P^{\mathrm{R}}_{+\,\mathrm{tmp}} := \left[\begin{array}{c|cc}
(*) & \begin{bmatrix} B_{1/0} & B^{\mathrm{R}}_{1/l} \end{bmatrix} & B_2 \\ \hline
-\widetilde{F}^{\mathrm{R}}_{2/x} & \begin{bmatrix} O & O \end{bmatrix} & I \\
\begin{bmatrix} C_2 + D_{21/00}\widetilde{F}^{\mathrm{R}}_{1/0x} \\ \widetilde{F}^{\mathrm{R}}_{1/lx} \end{bmatrix} & \begin{bmatrix} D_{21/00} & O \\ O & I \end{bmatrix} & O
\end{array}\right],$$

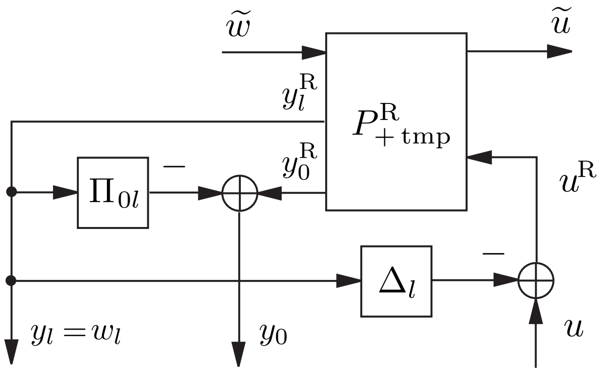

Figure 5.4: Structure of $P_{+\,\mathrm{tmp}}$.

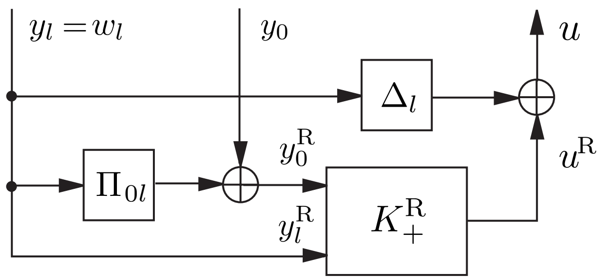

Figure 5.5: Structure of K_+.

$$(*) = A + B_{1/0}\widetilde{F}_{1/0x}^{\mathrm{R}} + B_{1/l}\widetilde{F}_{1/lx}^{\mathrm{R}}.$$

The FIR systems Π_{0l} and Δ_l are described by the following input-output relationship:

$$\Pi_{0l}y_l(t) := -D_{21/0l}y_l(t-l) + \begin{bmatrix} C_{2/0} & -\frac{1}{\gamma^2}\acute{S}_{2/0} \end{bmatrix} \int_{\theta=0}^{l} e^{-H_0\theta} \begin{bmatrix} B_{1/l} \\ O \end{bmatrix} y_l(t-(l-\theta))\,d\theta, \quad (5.8)$$

$$\Delta_l y_l(t) := R_2^{-1} \begin{bmatrix} S_2^* & -B_2^* \end{bmatrix} \int_{\theta=0}^{l} e^{-H_0\theta} \begin{bmatrix} B_{1/l} \\ O \end{bmatrix} y_l(t-(l-\theta))\,d\theta. \quad (5.9)$$

Referring to the structure of $P_{+\,\mathrm{tmp}}$, we introduce K_+^{R}, namely, the undetermined part of the controller K_+ as shown in Fig. 5.5. Noting that Π_{0l} and Δ_l are causal and stable systems, we can cancel them out of the interconnection between $P_{+\,\mathrm{tmp}}$ and K_+. Then, the problem of parameterizing the H^∞ controller K_+ for $P_{+\,\mathrm{tmp}}$ is reduced to that of parameterizing the H^∞ controller K_+^{R} for $P_{+\mathrm{tmp}}^{\mathrm{R}}$. The explicit form of H^∞ controller is obtained using the following KYP equation:

$$\begin{bmatrix} L_1^{\mathrm{R}*} \\ L_{2/0}^{\mathrm{R}*} \end{bmatrix}^* \begin{bmatrix} -\gamma^2 I & O \\ O & \acute{R}_{2/0} \end{bmatrix} \begin{bmatrix} L_1^{\mathrm{R}*} \\ L_{2/0}^{\mathrm{R}*} \end{bmatrix} = \acute{Q}_0 + AY^{\mathrm{R}} + Y^{\mathrm{R}}A^*, \quad (5.10)$$

$$-\begin{bmatrix} -\gamma^2 I & O \\ O & \acute{R}_{2/0} \end{bmatrix} \begin{bmatrix} L_1^{\mathrm{R}*} \\ L_{2/0}^{\mathrm{R}*} \end{bmatrix} = \begin{bmatrix} O \\ \acute{S}_{2/0} \end{bmatrix} + \begin{bmatrix} C_1 \\ C_{2/0} \end{bmatrix} Y^{\mathrm{R}}. \quad (5.11)$$

The KYP equation is associated with the H^∞ control problem for the following generalized plant in full control form.

$$\acute{P}^{\mathrm{R}}_{+\mathrm{FC}} := \left[\begin{array}{c|cc} A & B_{1/0} & \begin{bmatrix} I & O \end{bmatrix} \\ \hline C_1 & O & \begin{bmatrix} O & I \end{bmatrix} \\ C_{2/0} & D_{21/00} & \begin{bmatrix} O & O \end{bmatrix} \end{array}\right]. \tag{5.12}$$

Theorem 10 (H^∞ preview output feedback law)**.** *Under the assumptions (A1)-(A3), the H^∞ control problem for P_+ is solvable if and only if the conditions (X) in Theorem 9, (Y) and (Z) below are satisfied.*

(Y) The KYP equation in Eqs. (5.10)-(5.11) has the positive semidefinite stabilizing solution Y^{R} such that $\acute{A}^{\mathrm{R}}_{\mathrm{c}} := A + L^{\mathrm{R}}_1 C_1 + L^{\mathrm{R}}_{2/0} C_{2/0}$ is stable.

(Z) The spectral radius condition $\rho\left(Y^{\mathrm{R}} X^{\mathrm{R}}\right) < \gamma^2$ is satisfied, and hence the inverse of $Z^{\mathrm{R}} := I - \frac{1}{\gamma^2} Y^{\mathrm{R}} X^{\mathrm{R}}$ is well-defined.

If the above solvability conditions are satisfied, the H^∞ preview output feedback controller $K_+ : (y, \mu) \to (u, \nu)$ ($\nu := \begin{bmatrix} \nu_0^{\mathrm{T}} & \nu_l^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{\dim y_0} \times \mathbb{R}^{\dim y_l}$) is implemented as shown in Fig. 5.5. It consists of the FIR systems Π_{0l} and Δ_l in Eqs. (5.8)-(5.9), and the following finite-dimensional system K^{R}_+:

$$K^{\mathrm{R}}_+ = \mathcal{F}_l(J^{\mathrm{R}}_+, Q_+(s)),$$

where $\forall\, Q_+ \in H^\infty$ is the Youla parameter satisfying the condition

$$\left\| R_2^{1/2} Q_+ \begin{bmatrix} \acute{R}_{2/0} & O \\ O & I \end{bmatrix}^{1/2} \right\|_\infty < \gamma,$$

and J^{R}_+ is defined by

$$J^{\mathrm{R}}_+ := \left[\begin{array}{c|cc} A^{\mathrm{R}}_{\mathrm{c}} + L^{\mathrm{R}}_{\mathrm{tmp}\,2} \cdot (\dagger) & -L^{\mathrm{R}}_{\mathrm{tmp}\,2} & L^{\mathrm{R}}_{\mathrm{tmp}\,1} \\ \hline \widetilde{F}^{\mathrm{R}}_2 & O & I \\ -(\dagger) & I & O \end{array}\right], \ (\dagger) = \begin{bmatrix} C_{2/0} + D_{21/00} \widetilde{F}^{\mathrm{R}}_{1/0x} \\ \widetilde{F}^{\mathrm{R}}_{1/lx} \end{bmatrix},$$

$$L^{\mathrm{R}}_{\mathrm{tmp}\,1} := Z^{\mathrm{R}\,(-1)} \left(B_2 + L^{\mathrm{R}}_1 D_{12}\right), \ L^{\mathrm{R}}_{\mathrm{tmp}\,2} := \begin{bmatrix} L^{\mathrm{R}}_{\mathrm{tmp}\,2/0} & L^{\mathrm{R}}_{\mathrm{tmp}\,2/l} \end{bmatrix},$$

$$L^{\mathrm{R}}_{\mathrm{tmp}\,2/0} := Z^{\mathrm{R}\,(-1)} L^{\mathrm{R}}_{2/0}, \ L^{\mathrm{R}}_{\mathrm{tmp}\,2/l} := -Z^{\mathrm{R}\,(-1)} \left(B^{\mathrm{R}}_{1/l} + \frac{1}{\gamma^2} Y^{\mathrm{R}} S^{\mathrm{R}}_{1/l}\right).$$

□

Proof. Subsection 5.6.2. □

The finite-dimensional system K^{R}_+ is interpreted as the H^∞-type observer estimating the state variable $x^{\mathrm{R}}(t)$. The FIR systems Π_{0l} and Δ_l shape the measured output $y_0(t)$ and control input $u^{\mathrm{R}}(t)$ using the preview information $y_l(t) = w_l(t)$.

Remark 12. *Theorem 10 claims that the H^∞ preview output feedback controller exists if and only if both of the full information and output estimation problems are solvable. The solvability of the full information and output estimation problems are equivalent to the J-spectral factorizability condition (X), and the couple of the conditions (Y) and (Z), respectively.*

The remarkable difference between the existence conditions for the input-delayed and preview H^∞ controllers in Theorems 4 and 10 lies in the J-spectral factorizability conditions. While the J-spectral factorizability condition for the input-delayed controller should be checked for infinitely many points of the performance bound $\lambda \geq \gamma$, the H^∞ preview controller achieving the performance bound γ is characterized independently of the larger performance bound parameter $\lambda > \gamma$. □

5.4 Solution of operator Riccati equation

In Subsection 5.3.1, the H^∞ state feedback law is obtained as a solution of the model matching problem by the J-spectral factorization technique. The disturbance-delayed system P_+ is in the class of the Pritchard-Salamon system [24], and therefore the H^∞-state feedback law can be also constructed from the positive semidefinite stabilizing solution of the associated operator Riccati equation. In this section, it is noted that the explicit solution of the operator Riccati equation is found from the proposed state transformations.

We prepare the following lemma along the proofs of Lemma 17 and Theorem 9.

Lemma 18. *Suppose that the condition (X) in Theorem 9 is satisfied. Let the adjoint variables of x and ω be p and α, respectively. Then, the state variables $(p^\times, \alpha^\times)$ of the adjoint of the J-spectral factor M_+^{-1} is obtained by the following transformation:*

$$\begin{bmatrix} p^\times \\ \alpha^\times \end{bmatrix} = \begin{bmatrix} p \\ \alpha \end{bmatrix} + \widetilde{X} \begin{bmatrix} x \\ \omega \end{bmatrix}. \tag{5.13}$$

The operator $\widetilde{X}$ in Eq. (5.13) is constructed by

$$\widetilde{X} := \begin{bmatrix} X^{\mathrm{R}} & X^{\mathrm{R}}\{UE_{\omega l}\}_x + \{UE_{\omega l}\}_p \\ V_p X^{\mathrm{R}} - V_x & V_p X^{\mathrm{R}}\{UE_{\omega l}\}_x + V_p\{UE_{\omega l}\}_p + \Xi \end{bmatrix}, \tag{5.14}$$

where $E_\omega U$ is given by Eq. (5.3), and $V := \begin{bmatrix} V_x & V_p \end{bmatrix}$ is the multiplication operator defined by

$$\begin{bmatrix} V_x & V_p \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} (\phi) := \begin{bmatrix} O & B_{1/l}^{\mathrm{T}} \end{bmatrix} e^{H_0 \phi} \begin{bmatrix} x \\ p \end{bmatrix} \;\; (0 < \phi < l) \;\; \textit{for } (x,\, p) \in \mathbb{R}^{\dim x} \times \mathbb{R}^{\dim x}, \tag{5.15}$$

and Ξ is the integral operator defined by

$$\Xi\,\omega(\phi) := -\int_{\theta=0}^{\phi} B_{1/l}^{\mathrm{T}} \left\{ e^{H_0 \theta} \right\}_{21} B_{1/l}\, \omega(\phi - \theta) d\theta \;\; (0 < \phi < l) \;\; \textit{for } \omega \in L^2([0,\, l], \mathbb{R}^{\dim w_l}). \tag{5.16}$$

□

Proof. By Eqs. (5.5) and (5.24), we have

$$\begin{bmatrix} p^{\mathrm{R}\times} \\ \alpha^{\mathrm{R}\times} \end{bmatrix} = \begin{bmatrix} p^{\mathrm{R}} \\ \alpha^{\mathrm{R}} \end{bmatrix} + \begin{bmatrix} X^{\mathrm{R}} & O \\ O & \Xi \end{bmatrix} \begin{bmatrix} x^{\mathrm{R}} \\ \omega \end{bmatrix}.$$

Replacing x^{R}, p^{R} and α^{R} in the above equation with x, p and α by Eqs. (5.2) and (5.22), we have the equation

$$\begin{bmatrix} p^{\mathrm{R}\times} \\ \alpha^{\mathrm{R}\times} + V_p p^{\mathrm{R}\times} \end{bmatrix} = \begin{bmatrix} p \\ \alpha \end{bmatrix} + \widetilde{X} \begin{bmatrix} x \\ \omega \end{bmatrix}.$$

Defining $(p^{\times}, \alpha^{\times})$ as the left side of the above equation, we obtain Eq. (5.13). □

Let $\mathcal{X}_0$ be $\mathbb{R}^{\dim x} \times L^2([0,\, l], \mathbb{R}^{\dim w_l})$ and $\mathcal{X}_1$ be $\mathbb{R}^{\dim x} \times W^{2,1}([0,\, l], \mathbb{R}^{\dim w_l})$. In the homogeneous boundary condition case, i.e., when $w_l(t) = 0$, the infinitesimal generator $\widetilde{A}_{\mathrm{H}} : \mathcal{D}(\widetilde{A}_{\mathrm{H}}) := \mathcal{X}_1 \to \mathcal{X}_0$ of the disturbance-delayed system $P_{+\,\mathrm{mm}}$ is given as follows:

$$\widetilde{A}_{\mathrm{H}} \begin{bmatrix} x \\ \omega(\cdot) \end{bmatrix} = \begin{bmatrix} Ax(t) + B_{1/l}\omega(0) \\ \dfrac{\partial \omega}{\partial \theta}(\cdot) \end{bmatrix} \text{ for } \begin{bmatrix} x \\ \omega(\cdot) \end{bmatrix} \in \mathbb{R}^{\dim x} \times W^{2,1}([0,\, l], \mathbb{R}^{\dim w_l})$$

where $\widetilde{A}_{\mathrm{H}}$ is an unbounded operator on $\mathcal{X}_0 := \mathbb{R}^{\dim x} \times L^2([0,\, l], \mathbb{R}^{\dim w_l})$, and its domain is given by

$$\mathcal{D}\left(\widetilde{A}_{\mathrm{H}}\right) = \left\{ \begin{bmatrix} x \\ \omega(\cdot) \end{bmatrix} \in \mathbb{R}^{\dim x} \times W^{2,1}([0,\, l], \mathbb{R}^{\dim w_l}) \,\middle|\, \omega(l) = 0 \right\}.$$

By defining $\mathcal{X}_{-1} := \mathcal{D}(A^*_{+\,\mathrm{H}})^*$, we have the following dense and continuous inclusions: $\mathcal{X}_1 \subset \mathcal{X}_0 \subset \mathcal{X}_{-1}$. Then, $P_{+\,\mathrm{mm}}$ is described in the framework of [24] as follows:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\omega}_l(t) \end{bmatrix} = \widetilde{A} \begin{bmatrix} x(t) \\ \omega(t) \end{bmatrix} + \widetilde{B}_1 w(t) + \widetilde{B}_2 u(t),$$
$$z(t) = \widetilde{C}_+ \begin{bmatrix} x(t) \\ \omega(t) \end{bmatrix} + D_{12} u(t),\ y_{\mathrm{mm}}(t) = w(t),$$

where the infinitesimal generator $\widetilde{A} : \mathcal{D}(\widetilde{A}) := \mathcal{X}_0 \to \mathcal{X}_{-1}$ is the extension of $\widetilde{A}_{\mathrm{H}}$ and

$$\widetilde{B}_1 := \begin{bmatrix} \widetilde{B}_{1/0} & \widetilde{B}_{1/l} \end{bmatrix},\ \widetilde{B}_{1/0} := \begin{bmatrix} B_{1/0} \\ O \end{bmatrix},\ \widetilde{B}_{1/l} := \begin{bmatrix} O \\ \delta_l(\cdot) I_{\dim w_l} \end{bmatrix},$$
$$\widetilde{B}_2 := \begin{bmatrix} B_2 \\ O \end{bmatrix},\ \widetilde{C}_1 := \begin{bmatrix} C_1 & O \end{bmatrix}.$$

The following Riccati equation in Eq. (5.17) is associated with the above representation. In Theorem 11, the operator $\widetilde{X}$ constructed in Lemma 18 is shown to be the positive semidefinite stabilizing solution of it.

$$\widetilde{Q} + \widetilde{A}^* \widetilde{X} + \widetilde{X} \widetilde{A} + \frac{1}{\gamma^2} \widetilde{X} \widetilde{B}_1 \widetilde{B}_1^* \widetilde{X} - \left(\widetilde{X} \widetilde{B}_2 + \widetilde{S}_2 \right) R_2^{-1} \left(\widetilde{X} \widetilde{B}_2 + \widetilde{S}_2 \right)^* = O, \tag{5.17}$$
$$\widetilde{Q} := \widetilde{C}_1^* \widetilde{C}_1,\ \widetilde{S}_2 := \widetilde{C}_1^* D_{12}.$$

Theorem 11. *Suppose that the condition (X) in Theorem 9 is satisfied. Then, the operator $\widetilde{X}$ constructed by Eq. (5.14) is the positive semidefinite solution of the operator Riccati equation (5.17) such that $\widetilde{A}_{\mathrm{c}} := \widetilde{A} + \widetilde{B}_1\widetilde{F}_1 + \widetilde{B}_2\widetilde{F}_2$ is stable, where $\widetilde{F}_1$ and $\widetilde{F}_2$ are the state feedback gains given by*

$$\widetilde{F}_1 := \frac{1}{\gamma^2}\widetilde{B}_1^*\widetilde{X}, \;\; \widetilde{F}_2 := -R_2^{-1}\left(\widetilde{B}_2^*\widetilde{X} + \widetilde{S}_2^*\right).$$

□

Proof. Subsection 5.6.3. □

Theorem 11 exhibits that the operator Riccati allows the representation in the form of the composite of the state transformation operators (Eq. (5.14)), and more interestingly that its positive definiteness is assured by the matrix condition (X). Recalling that the state transformation operators are introduced for the J-spectral factorization, Lemma 18 and Theorem 11 shows a direct link between the J-spectral factorization technique and the stabilizing solution of the operator Riccati equation.

5.5 Example

It is reported that the discrete-time LQ preview controller significantly reduces the phase-lag of the tracking response in [60]. As an example, we apply the formula in Theorem 10 to designing a preview tracking system, and demonstrate that the resulting controller also improves the phase characteristics.

Suppose that the transfer function $G(s) = \dfrac{0.2\,(s-0.8)}{s^2+s+0.34}$ from the control input $u(t)$ to the controlled output $v(t)$ be given. The control objective is to make $v(t)$ track the delayed reference $r(t-l)\,(r(t) = w_l(t))$ in the bandwidth less than the frequency $\omega = 1$. The tracking system configuration is shown in Fig. 5.6. The weighting function $F(s) = \dfrac{3}{2s+1}$ emphasizes the tracking error $e(t)$ in the low bandwidth, and the parameter $\epsilon = 0.1$ adjusts the magnitude of the measurement noise $n(t)\,(= w_0(t))$.

For each value of the preview length l, the achievable performance $\gamma_{\mathrm{opt}}(l)$ is calculated by the bisection method, and the central controller is determined for the upper bound $\gamma(l) := 1.01 \cdot \gamma_{\mathrm{opt}}(l)$. The frequency response of $T_{v\,r}(s)\,e^{sl}$ is depicted in Fig. 5.7. Note that e^{sl} is multiplied to $T_{v\,r}(s)$ to reflect the purpose to suppress the tracking error after the time $t = l$.

From Fig. 5.7, the gain of the previewed response approaches 1 as the preview length increases. Although the phase of the previewed response leads excessively in the high bandwidth, its lag is suppressed around the frequency $\omega = 1$. Therefore, the designed preview H^∞ controller yields better tracking performance compared to the standard H^∞ controller.

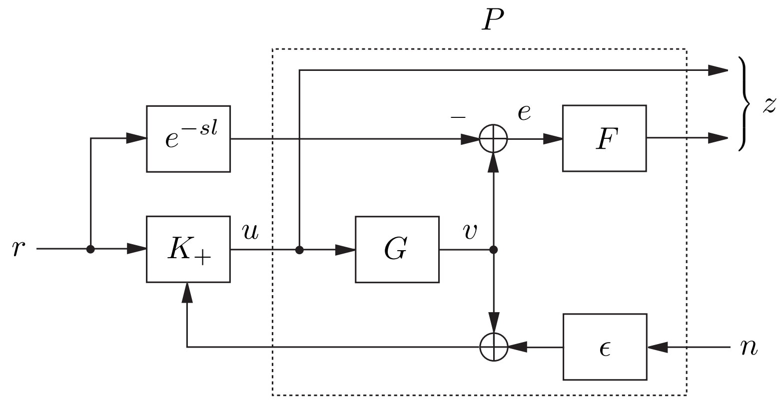

Figure 5.6: Preview tracking system.

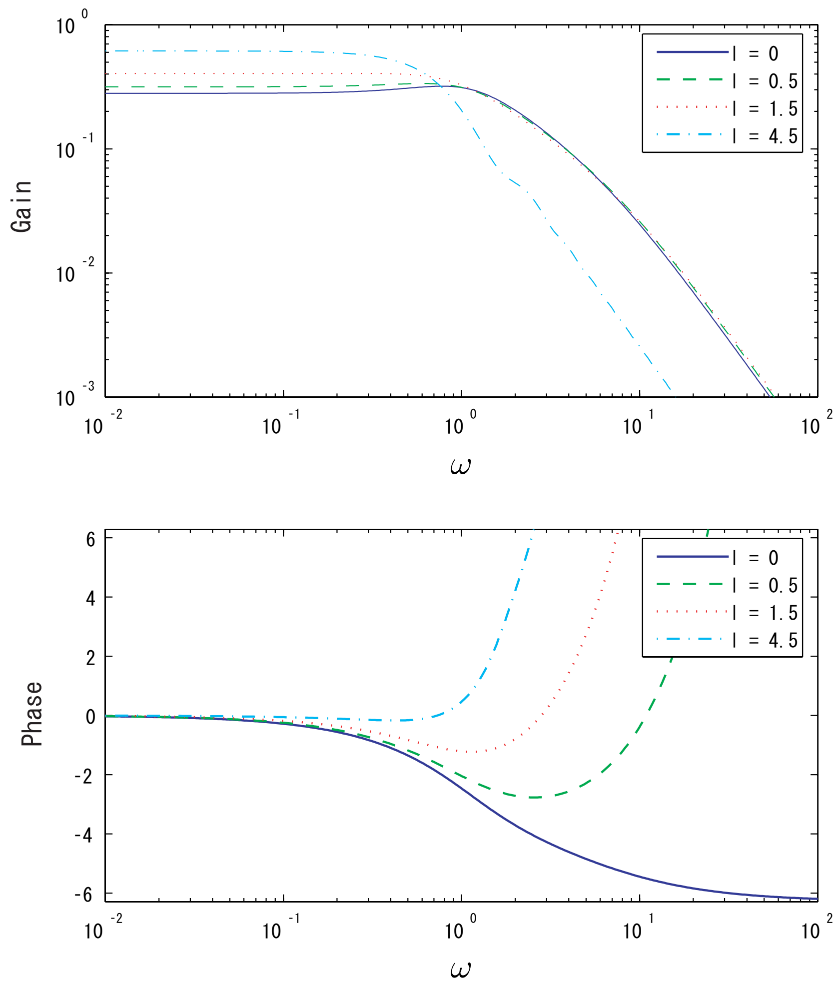

Figure 5.7: Frequency response of $T_{vr}(s)\,e^{sl}$.

5.6 Proofs

5.6.1 Proof of Lemma 17

By changing the input-output signals of $\Phi_+ : ((h_{\mathrm{mm}\,0},\, y_{\mathrm{mm}\,l}),\, k) \to ((y_{\mathrm{mm}\,0},\, -h_{\mathrm{mm}\,l}),\, u)$, define $\Omega_+ : ((h_{\mathrm{mm}\,0},\, y_{\mathrm{mm}\,l}),\, k) \to ((y_{\mathrm{mm}\,0},\, -h_{\mathrm{mm}\,l}),\, u)$. Note that the state-space realization of $\mathcal{C}\,(P_{+\,\mathrm{mm}})^{\sim} : (a,\, b) \to (h_{\mathrm{mm}},\, k)$ $\left(a = \begin{bmatrix} a_0^{\mathrm{T}} & a_l^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}\right)$ is given by

$$
\begin{aligned}
\dot{p}(t) &= -A^* p(t) + C_1^* b(t), \\
\frac{\partial \alpha}{\partial t}(\phi, t) &= \frac{\partial \alpha}{\partial \phi}(\phi, t)\ (0 < \phi < l),\ \alpha(0, t) = B_{1/l}^* p(t), \\
h_{\mathrm{mm}\,0}(t) &= -B_{1/0}^* p(t) + a_0(t),\ h_{\mathrm{mm}\,l}(t) = -\alpha(l, t) + a_l(t), \\
k(t) &= -B_2^* p(t) + D_{12}^* b(t).
\end{aligned}
$$

Therefore, the state-space realization of Ω_+ is calculated as follows:

$$
\begin{bmatrix} \dot{x}(t) \\ \dot{p}(t) \end{bmatrix} = H_0 \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} + \begin{bmatrix} B_{1/l} \\ O \end{bmatrix} \Gamma_0 \omega(t) - \frac{1}{\gamma^2} \begin{bmatrix} B_{1/0} \\ O \end{bmatrix} h_{\mathrm{mm}\,0}(t) + \begin{bmatrix} B_2 \\ S_2 \end{bmatrix} R_2^{-1} k(t), \tag{5.18}
$$

$$
E_\omega \dot{\omega}(t) = A_\omega \omega(t) + \begin{bmatrix} O \\ I \end{bmatrix} y_{\mathrm{mm}\,l}(t), \tag{5.19}
$$

$$
E_\alpha \dot{\alpha}_l(t) = -A_\alpha \alpha(t) - \begin{bmatrix} O & B_{1/l}^* \end{bmatrix} \begin{bmatrix} x(t) \\ p(t) \end{bmatrix}, \tag{5.20}
$$

$$
\begin{aligned}
y_{\mathrm{mm}\,0}(t) &= -\frac{1}{\gamma^2} \begin{bmatrix} O & B_{1/0}^* \end{bmatrix} \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} - \frac{1}{\gamma^2} h_{\mathrm{mm}\,0}(t), \\
-h_{\mathrm{mm}\,l}(t) &= \Gamma_l \alpha(t) + \gamma^2 y_{\mathrm{mm}\,l}(t), \\
u(t) &= R_2^{-1} \begin{bmatrix} -S_2^* & B_2^* \end{bmatrix} \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} + R_2^{-1} k(t).
\end{aligned}
$$

The following operators on $L^2([0,\, l], \mathbb{R}^{\dim w_l})$ are used. Each of their domains is $W^{2,1}([0,\, l], \mathbb{R}^{\dim w_l})$.

$$
E_\omega := \begin{bmatrix} I \\ O \end{bmatrix},\ A_\omega := \begin{bmatrix} \frac{\partial}{\partial \theta} \\ -\Gamma_l \end{bmatrix},\ E_\alpha := \begin{bmatrix} I \\ O \end{bmatrix},\ A_\alpha := \begin{bmatrix} -\frac{\partial}{\partial \phi} \\ -\Gamma_0 \end{bmatrix}.
$$

First, referring to Eqs. (5.18) and (5.20), we consider the following transformation.

$$
\begin{aligned}
&\begin{bmatrix} I & O \\ -E_\alpha V & I \end{bmatrix} \begin{bmatrix} H_0 - sI & O \\ -\begin{bmatrix} O & B_{1/l}^{\mathrm{T}} \end{bmatrix} & -A_\alpha - sE_\alpha \end{bmatrix} \begin{bmatrix} I & O \\ V & I \end{bmatrix} \\
&= \begin{bmatrix} H_0 - sI & O \\ -\begin{bmatrix} O & B_{1/l}^{\mathrm{T}} \end{bmatrix} - E_\alpha V H_0 - A_\alpha V & -A_\alpha - sE_\alpha \end{bmatrix}.
\end{aligned} \tag{5.21}
$$

Using Krein's formula [13], we determine V which makes the (2, 1) block of Eq. (5.21) zero by the following complex integration.

$$
V \begin{bmatrix} x \\ p \end{bmatrix} (\phi) = -\frac{1}{2\pi j} \oint_{\sigma(H_0)} (sE_\alpha + A_\alpha)^{-1} \begin{bmatrix} O & B_{1/l}^* \end{bmatrix} (sI - H_0)^{-1} \begin{bmatrix} x \\ p \end{bmatrix} ds\ (0 < \phi < l).
$$

This is calculated as in Eq. (5.15). Using it, we introduce the new state variable α^{R} by the equation

$$\begin{bmatrix}\begin{bmatrix}x\\p\end{bmatrix}\\ \alpha\end{bmatrix} =: \begin{bmatrix}I & O\\ V & I\end{bmatrix}\begin{bmatrix}\begin{bmatrix}x\\p\end{bmatrix}\\ \alpha^{\mathrm{R}}\end{bmatrix}. \tag{5.22}$$

Similarly, referring to Eqs. (5.18)-(5.19), we consider the following transformation.

$$\begin{bmatrix}I & U\\ O & I\end{bmatrix}\begin{bmatrix}H_0 - sI & \begin{bmatrix}B_{1/l}\\ O\end{bmatrix}\Gamma_l\\ O & A_\omega - sE_\omega\end{bmatrix}\begin{bmatrix}I & -UE_\omega\\ O & I\end{bmatrix} = \begin{bmatrix}H_0 - sI & \begin{bmatrix}B_{1/l}\\ O\end{bmatrix}\Gamma_l - H_0 U E_\omega + U A_\omega \cdot\\ O & A_\omega - sE_\omega\end{bmatrix}. \tag{5.23}$$

Using Krein's formula again, we determine U which makes the (1, 2) block in Eq. (5.23) zero by the following complex integration. For $(f,\, g) \in L^2([0,\, l], \mathbb{R}^{\dim w_l}) \times \mathbb{R}^{\dim w_l}$,

$$U\begin{bmatrix}f\\ g\end{bmatrix} = \frac{1}{2\pi j}\oint_{\sigma(H_0)} (sI - H_0)^{-1}\begin{bmatrix}B_{1/l}\\ O\end{bmatrix}\Gamma_l\,(sE_\omega - A_\omega)^{-1}\begin{bmatrix}f\\ g\end{bmatrix} ds.$$

This is calculated as in Eq. (5.3). Using it, we have introduced the new state variables x^{R} and p^{R} in Eq. (5.2). Finally, we make the following state variable transformation using Ξ defined by Eq. (5.16).

$$\alpha^{\mathrm{R}\times} := \alpha^{\mathrm{R}} + \Xi\,\omega. \tag{5.24}$$

After the state transformations in Eqs. (5.22), (5.2) and (5.24), the state-space realization of Ω_+ is given as follows:

$$\begin{aligned}
\begin{bmatrix}\dot{x}^{\mathrm{R}}(t)\\ \dot{p}^{\mathrm{R}}(t)\end{bmatrix} &= H_0\begin{bmatrix}x^{\mathrm{R}}(t)\\ p^{\mathrm{R}}(t)\end{bmatrix} - \frac{1}{\gamma^2}\begin{bmatrix}B_{1/0}\\ O\end{bmatrix}h_{\mathrm{mm}\,0}(t) + \begin{bmatrix}B_2\\ S_2\end{bmatrix}R_2^{-1}k(t) + U\begin{bmatrix}O\\ I\end{bmatrix}y_{\mathrm{mm}\,l}(t),\\
\frac{\partial\omega}{\partial t}(\theta,t) &= \frac{\partial\omega}{\partial\theta}(\theta,t)\ (0<\theta<l),\ \omega(l,t) = y_{\mathrm{mm}\,l}(t),\\
\frac{\partial\alpha^{\mathrm{R}\times}}{\partial t}(\phi,t) &= \frac{\partial\alpha^{\mathrm{R}\times}}{\partial\phi}(\phi,t) - V\left(-\frac{1}{\gamma^2}\begin{bmatrix}B_{1/0}\\ O\end{bmatrix}h_{\mathrm{mm}\,0}(t) + \begin{bmatrix}B_2\\ S_2\end{bmatrix}R_2^{-1}k(t)\right)\ (0<\phi<l),\\
\alpha^{\mathrm{R}\times}(0,t) &= 0,
\end{aligned} \tag{5.25}$$

$$y_{\mathrm{mm}\,0}(t) = \frac{1}{\gamma^2}\begin{bmatrix}O & -B_{1/0}^*\end{bmatrix}\begin{bmatrix}x^{\mathrm{R}}(t)\\ p^{\mathrm{R}}(t)\end{bmatrix} - \frac{1}{\gamma^2}h_{\mathrm{mm}\,0}(t) - \frac{1}{\gamma^2}\begin{bmatrix}O & -B_{1/0}^*\end{bmatrix}UE_\omega\omega(t), \tag{5.26}$$

$$-h_{\mathrm{mm}\,l}(t) = \alpha^{\mathrm{R}\times}(l,t) + V\begin{bmatrix}x^{\mathrm{R}}(t)\\ p(t)\end{bmatrix}(l) + \gamma^2 y_{\mathrm{mm}\,l}(t), \tag{5.27}$$

$$u(t) = -R_2^{-1}\begin{bmatrix}S_2^* & -B_2^*\end{bmatrix}\begin{bmatrix}x^{\mathrm{R}}(t)\\ p^{\mathrm{R}}(t)\end{bmatrix} + R_2^{-1}k(t) + R_2^{-1}\begin{bmatrix}S_2^* & -B_2^*\end{bmatrix}UE_\omega\omega(t). \tag{5.28}$$

Referring to Eqs. (5.26), (5.27) and (5.28), we define the input-output signals by the equations

$$u^{\mathrm{R}}(t) := u(t) - \widetilde{F}^{\mathrm{R}}_{2/l}\omega(t),\ k^{\mathrm{R}}(t) := k(t),$$

$$y^{\mathrm{R}}_{\mathrm{mm}\,0}(t) := y_{\mathrm{mm}\,0}(t) - \widetilde{F}^{\mathrm{R}}_{1/0l}\omega(t),\ h^{\mathrm{R}}_{\mathrm{mm}\,0}(t) := h_{\mathrm{mm}\,0}(t),$$
$$y^{\mathrm{R}}_{\mathrm{mm}\,l}(t) := y_{\mathrm{mm}\,l}(t),\ h^{\mathrm{R}}_{\mathrm{mm}\,l}(t) := h_{\mathrm{mm}\,l}(t) + \alpha^{\mathrm{R}\,\times}(l,t).$$

Then, the input-output relationship between the above signals are represented by M_{+}^{FIR} as follows:

$$\begin{bmatrix} y_{\mathrm{mm}} \\ u \end{bmatrix} = M_{+}^{\mathrm{FIR}} \begin{bmatrix} y^{\mathrm{R}}_{\mathrm{mm}} \\ u^{\mathrm{R}} \end{bmatrix},\ \begin{bmatrix} h^{\mathrm{R}}_{\mathrm{mm}} \\ k^{\mathrm{R}} \end{bmatrix} = M_{+}^{\mathrm{FIR}\sim} \begin{bmatrix} h_{\mathrm{mm}} \\ k \end{bmatrix}, \tag{5.29}$$
$$h^{\mathrm{R}}_{\mathrm{mm}} := \begin{bmatrix} h^{\mathrm{R}\,\mathrm{T}}_{\mathrm{mm}\,0} & h^{\mathrm{R}\,\mathrm{T}}_{\mathrm{mm}\,l} \end{bmatrix}^{\mathrm{T}}.$$

Moreover, let Ω^{R}_{+} be the input-output system from $((h^{\mathrm{R}}_{\mathrm{mm}\,0},\ y^{\mathrm{R}}_{\mathrm{mm}\,l}),\ k^{\mathrm{R}})$ to $((y^{\mathrm{R}}_{\mathrm{mm}\,0},\ -h^{\mathrm{R}}_{\mathrm{mm}\,l}),\ u^{\mathrm{R}})$, and define the input-out system $\Phi_{+}^{\mathrm{R}\,(-1)} : (h^{\mathrm{R}}_{\mathrm{mm}},\ k^{\mathrm{R}}) \to (y^{\mathrm{R}}_{\mathrm{mm}},\ u^{\mathrm{R}})$ by changing the input-output signals of Ω^{R}_{+}. Then, we obtain Eq. (5.4) by Eq. (5.29).

5.6.2 Proof of Theorem 10

First, we show the necessity of the condition (Y). Note that the original problem in Section 5.2 is solvable only if the H^{∞} control problem for the following generalized plant $\acute{P}_{+\,\mathrm{FC}}$ in full control form is solvable.

$$\dot{x}(t) = Ax(t) + B_{1/0}w_0(t) + B_{1/l}w_l(t-l) + u_{\mathrm{FC}\,x}(t),$$
$$z(t) = C_1 x(t) + u_{\mathrm{FC}\,z}(t),$$
$$\begin{bmatrix} y_0(t) \\ y_l(t) \end{bmatrix} = \begin{bmatrix} C_{2/0}x(t) + D_{21/00}w_0(t) + D_{21/0l}w_l(t-l) \\ w_l(t) \end{bmatrix}.$$

The problem here is to determine $(u_{\mathrm{FC}\,x},\ u_{\mathrm{FC}\,z})$ based on y to render the L^2 gain from w to z less than γ. Since the previewed disturbance w_l is available, we cancel its effect on x and y_0 in the above realization, and get to the H^{∞} control problem for $\acute{P}^{\mathrm{R}}_{+\,\mathrm{FC}}$ defined in Eq. (5.12). Therefore, the condition (Y) is necessary.

Next, recall that we have reduced the H^{∞} control problem for $P_{+\,\mathrm{tmp}}$ to that for $P^{\mathrm{R}}_{+\,\mathrm{tmp}}$. The latter problem is solved via the dual J-spectral factorization of

$$\acute{\Phi}^{\mathrm{R}}_{+\,\tau} := \acute{P}^{\mathrm{R}}_{+\,\tau} \begin{bmatrix} -\gamma^2 R_2^{-1} & O \\ O & I \end{bmatrix} \acute{P}^{\mathrm{R}\sim}_{+\,\tau},\ \acute{P}^{\mathrm{R}}_{+\,\tau} := \left[\begin{array}{c|c} \acute{A}^{\mathrm{R}}_{+\,\tau} & \acute{B}^{\mathrm{R}}_{+\,\tau} \\ \hline \acute{C}^{\mathrm{R}}_{+\,\tau} & \acute{D}^{\mathrm{R}}_{+\,\tau} \end{array}\right],$$

$$\acute{A}^{\mathrm{R}}_{+\,\tau} := \widetilde{A}^{\mathrm{R}}_{\mathrm{c}\,x},\ \acute{B}^{\mathrm{R}}_{+\,\tau} := \begin{bmatrix} -B_2 & \begin{bmatrix} B_{1/0} & B^{\mathrm{R}}_{1/l} \end{bmatrix} \end{bmatrix},$$
$$\acute{C}^{\mathrm{R}}_{+\,\tau} := \begin{bmatrix} -\widetilde{F}^{\mathrm{R}}_{2x} \\ \begin{bmatrix} C_{2/0} + D_{21/00}\widetilde{F}^{\mathrm{R}}_{1/0x} \\ \widetilde{F}^{\mathrm{R}}_{1/lx} \end{bmatrix} \end{bmatrix},\ \acute{D}^{\mathrm{R}}_{+\,\tau} := \begin{bmatrix} I & \begin{bmatrix} O & O \end{bmatrix} \\ \begin{bmatrix} O \\ O \end{bmatrix} & \begin{bmatrix} D_{21/00} & O \\ O & I \end{bmatrix} \end{bmatrix}.$$

The factorizability condition is the following condition (T).

(T) The following KYP equation has the positive semidefinite stabilizing solution $Y_{\mathrm{tmp}}^{\mathrm{R}}$ such that $\acute{A}_{+\tau\,\mathrm{c}}^{\mathrm{R}} := \acute{A}_{+\tau}^{\mathrm{R}} + L_{+\,\mathrm{tmp}}^{\mathrm{R}} \acute{C}_{+\tau}^{\mathrm{R}}$ is stable.

$$\begin{aligned} L_{+\,\mathrm{tmp}}^{\mathrm{R}} \acute{R}_{+\tau}^{\mathrm{R}} L_{+\,\mathrm{tmp}}^{\mathrm{R}\,*} &= \acute{Q}_{+\tau}^{\mathrm{R}} + \acute{A}_{+\tau}^{\mathrm{R}} Y_{\mathrm{tmp}}^{\mathrm{R}} + Y_{\mathrm{tmp}}^{\mathrm{R}} \acute{A}_{+\tau}^{\mathrm{R}\,*}, \\ -\acute{R}_{+\tau}^{\mathrm{R}} L_{+\,\mathrm{tmp}}^{\mathrm{R}\,*} &= \acute{S}_{+\tau}^{\mathrm{R}} + \acute{C}_{+\tau}^{\mathrm{R}} Y_{\mathrm{tmp}}^{\mathrm{R}}, \end{aligned}$$

$$\begin{bmatrix} \acute{Q}_{+\tau}^{\mathrm{R}} & \acute{S}_{+\tau}^{\mathrm{R}\,*} \\ \acute{S}_{+\tau}^{\mathrm{R}} & \acute{R}_{+\tau}^{\mathrm{R}} \end{bmatrix} := \begin{bmatrix} \acute{B}_{+\tau}^{\mathrm{R}} \\ \acute{D}_{+\tau}^{\mathrm{R}} \end{bmatrix} \begin{bmatrix} -\gamma^2 R_2^{-1} & O \\ O & I \end{bmatrix} \begin{bmatrix} \acute{B}_{+\tau}^{\mathrm{R}} \\ \acute{D}_{+\tau}^{\mathrm{R}} \end{bmatrix}^* .$$

To show that (T) is equivalent to (Y) and (Z), we apply the idea in [20] of relating the pencils associated with the KYP equations. First, the KYP equation in (T) is rewritten as the generalized eigenvalue problem:

$$\acute{\Phi}_{+\tau\,\sigma}^{\mathrm{R\,ext}} \mathcal{B}_\tau = \acute{\Phi}_{+\tau\,\delta}^{\mathrm{R\,ext}} \mathcal{B}_\tau \acute{A}_{+\tau\,\mathrm{c}}^{\mathrm{R}\,*},$$

where $\mathcal{B}_\tau := \left[\begin{array}{cccc|c} Y_{\mathrm{tmp}}^{\mathrm{R\,T}} & I^{\mathrm{T}} & L_{\mathrm{tmp1}}^{\mathrm{R}*\,\mathrm{T}} & L_{\mathrm{tmp2}}^{\mathrm{R}*\,\mathrm{T}} & O^{\mathrm{T}} \end{array}\right]^{\mathrm{T}}$,

$$\acute{\Phi}_{+\tau\,\sigma}^{\mathrm{R\,ext}} := \left[\begin{array}{c|c} \acute{\Phi}_{+\tau\,\sigma}^{\mathrm{R}} & O \\ \hline O & -\gamma^2 I \end{array}\right], \ \acute{\Phi}_{+\tau\,\delta}^{\mathrm{R\,ext}} := \left[\begin{array}{c|c} \acute{\Phi}_{+\tau\,\delta}^{\mathrm{R}} & O \\ \hline O & O \end{array}\right],$$

$$\acute{\Phi}_{+\tau\,\sigma}^{\mathrm{R}} := \left[\begin{array}{cc|c} \acute{A}_{+\tau}^{\mathrm{R}} & \acute{Q}_{+\tau}^{\mathrm{R}} & \acute{S}_{+\tau}^{\mathrm{R}\,*} \\ O & -\acute{A}_{+\tau}^{\mathrm{R}\,*} & -\acute{C}_{+\tau}^{\mathrm{R}\,*} \\ \hline \acute{C}_{+\tau}^{\mathrm{R}} & \acute{S}_{+\tau}^{\mathrm{R}} & \acute{R}_{+\tau}^{\mathrm{R}} \end{array}\right], \ \acute{\Phi}_{+\tau\,\delta}^{\mathrm{R}} := \left[\begin{array}{cc|c} I & O & O \\ O & I & O \\ \hline O & O & O \end{array}\right].$$

Similarly, the KYP equation in (Y) is rewritten as follows:

$$\acute{\Phi}_{+\mu\,\sigma}^{\mathrm{R\,ext}} \mathcal{B}_\mu = \acute{\Phi}_{+\mu\,\delta}^{\mathrm{R\,ext}} \mathcal{B}_\mu \acute{A}_{\mathrm{c}}^{\mathrm{R}\,*},$$

where $\mathcal{B}_\mu := \left[\begin{array}{cccc|c} Y^{\mathrm{R\,T}} & I^{\mathrm{T}} & L_1^{\mathrm{R}*\,\mathrm{T}} & \begin{bmatrix} L_{2/0}^{\mathrm{R}*\,\mathrm{T}} & O^{\mathrm{T}} \end{bmatrix} & O^{\mathrm{T}} \end{array}\right]^{\mathrm{T}}$,

$$\acute{\Phi}_{+\mu\,\sigma}^{\mathrm{R\,ext}} := \left[\begin{array}{c|c} \acute{\Phi}_{+\mu\,\sigma}^{\mathrm{R}} & O \\ \hline O & -\gamma^2 R_2^{-1} \end{array}\right], \acute{\Phi}_{+\mu\,\delta}^{\mathrm{R\,ext}} := \left[\begin{array}{c|c} \acute{\Phi}_{+\mu\,\delta}^{\mathrm{R}} & O \\ \hline O & O \end{array}\right],$$

$$\acute{\Phi}_{+\mu\,\sigma}^{\mathrm{R}} := \left[\begin{array}{cc|cc} A & \acute{Q}_0 & O & \begin{bmatrix} \acute{S}_{2/0} & O \end{bmatrix} \\ O & -A^* & -C_1^* & \begin{bmatrix} -C_{2/0}^* & O \end{bmatrix} \\ \hline C_1 & O & -\gamma^2 I & \begin{bmatrix} O & O \end{bmatrix} \\ \begin{bmatrix} C_{2/0} \\ O \end{bmatrix} & \begin{bmatrix} \acute{S}_{2/0}^* \\ O \end{bmatrix} & \begin{bmatrix} O \\ O \end{bmatrix} & \begin{bmatrix} \acute{R}_{2/0} & \\ & I \end{bmatrix} \end{array}\right], \ \acute{\Phi}_{+\mu\,\delta}^{\mathrm{R}} := \left[\begin{array}{cc|cc} I & O & O & \begin{bmatrix} O & O \end{bmatrix} \\ O & I & O & \begin{bmatrix} O & O \end{bmatrix} \\ \hline O & O & O & \begin{bmatrix} O & O \end{bmatrix} \\ \begin{bmatrix} O \\ O \end{bmatrix} & \begin{bmatrix} O \\ O \end{bmatrix} & \begin{bmatrix} O \\ O \end{bmatrix} & \begin{bmatrix} O & \\ & O \end{bmatrix} \end{array}\right].$$

Using the Riccati equation associated with the Hamiltonian matrix $e^{-H_0 l} H_l e^{H_0 l}$, it is verified that the pencils $\acute{\Phi}_{+\tau\,\sigma}^{\mathrm{R\,ext}} - s\,\acute{\Phi}_{+\tau\,\delta}^{\mathrm{R\,ext}}$ and $\acute{\Phi}_{+\mu\,\sigma}^{\mathrm{R\,ext}} - s\,\acute{\Phi}_{+\mu\,\delta}^{\mathrm{R\,ext}}$ are related as follows:

$$T_{\mathrm{l}} \left(\acute{\Phi}_{+\tau\,\sigma}^{\mathrm{R\,ext}} - s\,\acute{\Phi}_{+\tau\,\delta}^{\mathrm{R\,ext}} \right) T_{\mathrm{r}} = \acute{\Phi}_{+\mu\,\sigma}^{\mathrm{R\,ext}} - s\,\acute{\Phi}_{+\mu\,\delta}^{\mathrm{R\,ext}}, \tag{5.30}$$

where

$$
T_{\mathrm{l}} := \left[\begin{array}{cccc|c}
I & O & B_2 & \begin{bmatrix} O & -B_{1/l}^{\mathrm{R}} \end{bmatrix} & O \\
\frac{1}{\gamma^2} X^{\mathrm{R}} & I & -\frac{1}{\gamma^2} S_2 & \begin{bmatrix} O & \frac{1}{\gamma^2} S_{1/l}^{\mathrm{R}} \end{bmatrix} & \frac{1}{\gamma^2} C_1^* \\
O & O & O & \begin{bmatrix} O & O \end{bmatrix} & I \\
\begin{bmatrix} O \\ O \end{bmatrix} & \begin{bmatrix} O \\ O \end{bmatrix} & \begin{bmatrix} O \\ O \end{bmatrix} & \begin{bmatrix} I & \\ & I \end{bmatrix} & \begin{bmatrix} O \\ O \end{bmatrix} \\
\hline
O & O & I & \begin{bmatrix} O & O \end{bmatrix} & O
\end{array}\right],
$$

$$
T_{\mathrm{r}} := \left[\begin{array}{cccc|c}
I & O & O & \begin{bmatrix} O & O \end{bmatrix} & O \\
-\frac{1}{\gamma^2} X^{\mathrm{R}} & I & O & \begin{bmatrix} O & O \end{bmatrix} & O \\
\frac{1}{\gamma^2} S_2^* & B_2^* & O & \begin{bmatrix} O & O \end{bmatrix} & I \\
\begin{bmatrix} O \\ -\frac{1}{\gamma^2} S_{1/l}^{\mathrm{R}\,*} \end{bmatrix} & \begin{bmatrix} O \\ -B_{1/l}^{\mathrm{R}\,*} \end{bmatrix} & \begin{bmatrix} O \\ O \end{bmatrix} & \begin{bmatrix} I & \\ & I \end{bmatrix} & \begin{bmatrix} O \\ O \end{bmatrix} \\
\hline
-\frac{1}{\gamma^2} C_1 & O & I & \begin{bmatrix} O & O \end{bmatrix} & O
\end{array}\right].
$$

Eq. (5.30) implies that the bases of the generalized eigenvalue problems derived from (T) and (Y) are related as follows:

$$
\mathcal{B}_\tau Z^{\mathrm{R}\,*} = T_{\mathrm{r}} \mathcal{B}_\mu.
$$

5.6.3 Proof of Theorem 11

Partition $\widetilde{X}$ in Lemma 18 conformably with $\begin{bmatrix} x^{\mathrm{T}} & \omega^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$: $\widetilde{X} =: \begin{bmatrix} \widetilde{X}_{xx} & \widetilde{X}_{x\omega} \\ \widetilde{X}_{\omega x} & \widetilde{X}_{\omega\omega} \end{bmatrix}$. By the representations of U, V and Ξ in Eqs. (5.3), (5.15) and (5.16), $\widetilde{X}_{ij}$ $(i, j = x, \omega)$ are given by

$$
\begin{gathered}
\widetilde{X}_{xx} x = \widetilde{X}_{xx}^{\mathrm{K}} x, \\
\widetilde{X}_{x\omega} \omega = \int_{\theta=0}^{l} X_{x\omega}^{\mathrm{K}}(\theta) \omega(\theta)\, d\theta, \ \widetilde{X}_{\omega x} x(\phi) = \widetilde{X}_{\omega x}^{\mathrm{K}}(\phi) x, \\
\widetilde{X}_{\omega\omega} \omega(\phi) = \int_{\theta=0}^{l} X_{\omega\omega}^{\mathrm{K}}(\phi, \theta) \omega(\theta)\, d\theta \ \ (0 < \phi < l),
\end{gathered}
$$

where $\widetilde{X}_{ij}^{\mathrm{K}}$ $(i, j = x, \omega)$ are defined as follows:

$$
\begin{gathered}
\widetilde{X}_{xx}^{\mathrm{K}} := X^{\mathrm{R}}, \ \widetilde{X}_{x\omega}^{\mathrm{K}}(\theta) := \begin{bmatrix} X^{\mathrm{R}} & I \end{bmatrix} e^{-H_0 \theta} \begin{bmatrix} B_{1/l} \\ O \end{bmatrix}, \ \widetilde{X}_{\omega x}^{\mathrm{K}}(\phi) := \widetilde{X}_{x\omega}^{\mathrm{K}}(\phi)^*, \\
\widetilde{X}_{\omega\omega}^{\mathrm{K}}(\phi, \theta) := \begin{bmatrix} B_{1/l}^* & O \end{bmatrix} e^{-H_0^* \phi} \begin{bmatrix} X^{\mathrm{R}} & H(\theta - \phi) I \\ H(\phi - \theta) I & O \end{bmatrix} e^{-H_0 \theta} \begin{bmatrix} B_{1/l} \\ O \end{bmatrix} \ (0 < \theta, \phi < l).
\end{gathered}
$$

Since the above defined $\widetilde{X}_{ij}^{\mathrm{K}}$ $(i, j = x, \omega)$ satisfy the following system of PDEs, $\widetilde{X}$ is a solution of the operator Riccati equation (5.17).

$$
\begin{aligned}
&Q_\nu + A_\nu^* \widetilde{X}_{xx}^{\mathrm{K}} + \widetilde{X}_{xx}^{\mathrm{K}} A_\nu - \widetilde{X}_{xx}^{\mathrm{K}} P_{1/0} \widetilde{X}_{xx}^{\mathrm{K}} + \frac{1}{\gamma^2} \widetilde{X}_{x\omega}^{\mathrm{K}}(l) \widetilde{X}_{lx}^{\mathrm{K}}(l) - \widetilde{X}_{xx}^{\mathrm{K}} P_2 \widetilde{X}_{xx}^{\mathrm{K}} = O, \\
&-\frac{\partial}{\partial \phi} \widetilde{X}_{\omega x}^{\mathrm{K}}(\phi) + \widetilde{X}_{\omega x}^{\mathrm{K}}(\phi) A_\nu - \widetilde{X}_{\omega x}^{\mathrm{K}}(\phi) P_{1/0} \widetilde{X}_{xx}^{\mathrm{K}} + \frac{1}{\gamma^2} \widetilde{X}_{xx}^{\mathrm{K}}(\phi, l) \widetilde{X}_{\omega x}^{\mathrm{K}}(l) - \widetilde{X}_{\omega x}^{\mathrm{K}}(\phi) P_2 \widetilde{X}_{xx}^{\mathrm{K}} = O, \\
&-\frac{\partial}{\partial \theta} \widetilde{X}_{x\omega}^{\mathrm{K}}(\theta) + A_\nu^* \widetilde{X}_{x\omega}^{\mathrm{K}}(\theta) - \widetilde{X}_{xx}^{\mathrm{K}} P_{1/0} \widetilde{X}_{x\omega}^{\mathrm{K}}(\theta) + \frac{1}{\gamma^2} \widetilde{X}_{x\omega}^{\mathrm{K}}(l) \widetilde{X}_{\omega\omega}^{\mathrm{K}}(l, \theta) - \widetilde{X}_{xx}^{\mathrm{K}} P_2 \widetilde{X}_{x\omega}^{\mathrm{K}}(\theta) = O, \\
&-\frac{\partial}{\partial \phi} \widetilde{X}_{\omega\omega}^{\mathrm{K}}(\phi, \theta) - \frac{\partial}{\partial \theta} \widetilde{X}_{\omega\omega}^{\mathrm{K}}(\phi, \theta) \\
&\quad - \widetilde{X}_{\omega x}^{\mathrm{K}}(\phi) P_{1/0} \widetilde{X}_{x\omega}^{\mathrm{K}}(\theta) + \frac{1}{\gamma^2} \widetilde{X}_{\omega\omega}^{\mathrm{K}}(\phi, l) \widetilde{X}_{\omega\omega}^{\mathrm{K}}(l, \theta) - \widetilde{X}_{\omega x}^{\mathrm{K}}(\phi) P_2 \widetilde{X}_{x\omega}^{\mathrm{K}}(\theta) = O,
\end{aligned}
$$

$$
\begin{gathered}
\widetilde{X}_{x\omega}^{\mathrm{K}}(0) = \widetilde{X}_{xx}^{\mathrm{K}} B_{1/l}, \ \widetilde{X}_{\omega\omega}^{\mathrm{K}}(\phi, 0) = \widetilde{X}_{\omega x}^{\mathrm{K}}(\phi) B_{1/l}, \\
\widetilde{X}_{\omega x}^{\mathrm{K}}(0) = B_{1/l}^* \widetilde{X}_{xx}^{\mathrm{K}}, \ \widetilde{X}_{\omega\omega}^{\mathrm{K}}(0, \theta) = B_{1/l}^* \widetilde{X}_{x\omega}^{\mathrm{K}}(\theta) \\
(0 < \theta, \, \phi < l).
\end{gathered}
$$

We can verify the above system of PDEs using Eq. (5.7) and the identities such that

$$
\frac{\partial}{\partial \phi} e^{H_0 \phi} = e^{H_0 \phi} H_0, \ e^{-H_0 \theta} = J_{\mathrm{s}}^{-1} \left(e^{H_0 \theta} \right)^* J_{\mathrm{s}}.
$$

To show the stability of $\widetilde{A}_{\mathrm{c}}$ and $\widetilde{A} + \widetilde{B}_2 \widetilde{F}_2$, we transform their domain and range spaces by Eq. (5.2). Then, $\widetilde{A}_{\mathrm{c}}$ and $\widetilde{A} + \widetilde{B}_2 \widetilde{F}_2$ are transformed to the infinitesimal generators of N_+ in Theorem 9 and N_{+11}^{-1} in the proof of Theorem 9, respectively. Those transformed $\widetilde{A}_{\mathrm{c}}$ and $\widetilde{A} + \widetilde{B}_2 \widetilde{F}_2$ are stable by the stability of $\widetilde{A}_{\mathrm{c}x}^{\mathrm{R}}$ and $A + B_2 \widetilde{F}_{2/x}^{\mathrm{R}}$, respectively.

To show the positive semidefiniteness of $\widetilde{X}$, we rewrite the Riccati equation (5.17) as the following Lyapunov equation:

$$
\left(\widetilde{Q} - \widetilde{S}_2^* R_2^{-1} \widetilde{S}_2 + \widetilde{X} \widetilde{B}_2 R_2^{-1} \widetilde{B}_2^* \widetilde{X} + \gamma^2 \widetilde{F}_1^* \widetilde{F}_1 \right) + \left(\widetilde{A} + \widetilde{B}_2 \widetilde{F}_2 \right)^* \widetilde{X} + \widetilde{X} \left(\widetilde{A} + \widetilde{B}_2 \widetilde{F}_2 \right) = O.
$$

Since $\widetilde{A} + \widetilde{B}_2 \widetilde{F}_2$ is stable, the operator $\widetilde{X}$ is positive semidefinite by the results on operator Lyapunov equations [24].

5.7 Conclusion

The H^∞ preview control design method is derived in parallel with the H^2 control case in Chapter 4. The state decomposition can be seen as the state transformation, however is different from those employed in [26], [28] in that it gives a one-to-one correspondence between the original and tranformed state variables. Furthermore, it is verified that the operator $\widetilde{X}$ defining the state variable of the adjoint spectral factor satisfies the control-type operator Riccati equation. Therefore, the J-spectral factorization techniques in [31], [38] can be alternative ways to [26], [28] for studying the structure of the operator Riccati equation.

Chapter 6 Conclusion

6.1 Summary of the thesis

We proposed the state decomposition approach for the input-delayed and preview H^2/H^∞ control problems. It has the strength in treating the output estimation problems, and yields the optimal output feedback controllers in a consistent manner. The reduced-order construction methods of the discrete-time input-delayed H^∞ and continuous-time preview H^2 controllers were newly derived in the output feedback settings.

In Chapter 2, we solved the H^2 control problem for the discrete-time input-delay system by focusing on the internal state dynamics. The fundamental idea of the state decomposition was described by deriving a Smith predictor for a simple input-delay system. Motivated by it, the optimal controller in the Smith predictor form was obtained from the Riccati equations for the delay-free system.

In Chapter 3, we considered the input-delayed H^∞ control problem in the discrete-time setting. The J-spectral factorization technique is introduced into the discrete-time setting. It yielded the solvability condition involves only solving the KYP equations for the delay-free system and checking the matrix eigenvalues. The optimal controller was implemented in the Smith predictor form. The solvability condition and control law were further investigated via the min-max optimization approach. An alternative solvability condition was provided from the perspective of the finite horizon optimization. As a supplementary result, the min-max optimization approach was adopted to derive another solvability condition. The equivalence between the J-spectral factorization and min-max optimization approaches was provided.

In Chapter 4, we applied the state decomposition approach to the continuous-time H^2 preview control problem. A clear structure of the optimal controller was identified as the combination of the finite-dimensional observer and preview-feedforward compensation. The optimality of the output feedback controller was guaranteed by exploiting available preview information at both of the full information and output estimation problems.

In Chapter 5, we further extended the design method to the continuous-time preview H^∞ control problem. The H^∞ output feedback controller was realized in the form parallel to the H^2 control case. Moreover, the relationship between the J-spectral factorization technique and the stabilizing solution of the operator Riccati equation was clarified by considering the state-space representation of the J-spectral density.

We established the unified reduced-order construction methods of H^2/H^∞ control laws for the input-delay and preview systems. The above-mentioned controllers share a common interpretation that the observer-based controllers for the delay-free systems are compensated by the past history of the control input or the preview information. Moreover, they are implemented by solving the reduced-order or finite-dimensional Riccati/KYP equations for the delay-free systems.

6.2 Subjects of future research

We extended the classes of the input-delay and preview systems for which the reduced-order construction of the optimal H^2/H^∞ laws are tractable. However, the design methods need further extentions for practical applications in mechatronic systems.

Recall that we employed the so-called internal model control technique [46] for solving the output estimation problems. In the proposed approach, the exact models of th delayed dynamics are included in the controllers to cancel the delay elements out of the closed-loop interconnection. The crucial assumption for the internal model control technique is that the controllers can access the perfect information on the delay elements holding the past history of control input or disturbance. The assumption is not realistic in teleoperation operation or active suspension systems. In the former system, there often exist input and output delays between the interaction of a master and slave [3], and the past history of the output signal is not directly available for the controller. In the latter system, the preview information of road profiles is usually corrupted by measurement noise or is given as the outputs of the disturbance models [58]. We will enhance the proposed design methods based on the optimal estimation of the delay elements, and investigate their features through experimental studies.

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