Randall-Sundrum Brane-World

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January 7, 2015

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1 Introduction

One of the most important problems in physics is the unification of fundamental forces. The superstring theory is unified theory of everything as the best candidate. In this theory, all particles are replaced by string, and particles are characterized from modes of string. Moreover, there is a critical dimension for anomaly cancelation. Superstring theory requires the spacetime is 10 dimension. There are 5 types superstrings, and there is unifying theory of these called "M-theory". Horava and Witten suggested 11D supergravity as the low energy limit of M-theory.[1]

Recently, in response to the that, the interest over extra dimensions is becoming high. However, even when we live in high dimensional spacetime, it is meaningless if the 4-dimensional space-time which we know is not reproduced. In other words, a certain mechanism where extra dimensions beyond four dimensions cannot be seen, is required. Two approaches are known.

The first one is Kaluza-Klein approach. This idea was suggested by T.Kaluza and O.Klein in 1926 to unify the electromagnetic field and the gravitational field in 5D spacetime.[2] This approach reconstructs the effective 4D spacetime by the compactification of the extra dimension. However, in this approach, the size of extra dimensions are restricted from experiments .

The other one is brane-world. This idea is that we can't see the extra dimensions because the Standard Model fields are confined on the 3D sub-manifold called "brane" embeddded in the higher dimensional manifold. In the brane-world scenario, extra dimension may be large, and even infinite. This reason is no restrictions from corider experiments differ from KK theory since SM particles are confined on the brane.

The prototype of brane model was suggested by V.A. Rubakov, M.E. Shaposhnikov in 1983.[3] This model succeeded in confinement of Inspired from this model and concept of "D-brane" in string theory, some models are suggested from 1998 . The ADD model which suggested by N.Arkani-Hamed,S.Domopoulos and G.Dvali solves the hierarchy problem in particle physics.[4]

Another way to solve the hierarcjy problem using brane idea is suggested by Randall and Sundrum in 1999.[5] This model called RSI is two brane model embedded in the five-dimensional spacetime with a negative cosmological constant, assuming the Z_2 symmetry near the brane inspired from Horava-Witten model. RSI generate the hierarchy between weak scale mass and Planck mass by geometrical factor called "warp factor".

After the RSI, Randall and Sundrum suggested one brane model called RSII for alternative to compactification.[6] RSII is infinitly large size of extra dimension by removing the brane to infinity from RSII, and reconstructing the conventional four-dimensional gravity. Since intoducing extra dimensions causes breaking Newton's law, RSII has attracteved many physists.

This paper contains of 6 sections. First, we introduce the Kaluza-Klien theory and ADD brane model in section 2. In section 3, we introduce the Randall-

Sundrum's idea and five-dimensional geometry. After that, we see how the hierarchy problem is solved. In section 4, we consider the RSII and confirm the reconstructing four-dimensional gravity in infinitly large extra dimension model. In section 5 and section 6, we consider the application of the Randall-Sundrum brane model. We derive the effective four-dimensional Einstein equations and its characteristics in section 5. Finally, we consider the cosmology in brane-world scenario in section 6.

Further the following thesis was consulted in case of writing of a thesis.[7,8,9]

2 Kaluza-Klein theory and ADD model

2.1 Kaluza-Klein theory

The model of extra dimension of space was first put by T.Kaluza and O.Klein to unify the electromagnetic and gravitational fields.[2] In this section,we consider the case of a five-dimensional theory, with periodically identified coordinate:

$$y \sim y + 2\pi r. \tag{2.1}$$

This procedure is called compactification, and r is its radius. Such five-dimensional spacetime is the product of the four dimensional Minkowski space with a circle, noted $M^4 \times S^1$.

Hereafter, the indices $A, B, \cdots$ denote $0, 1, 2, 3, 4$, $\mu, \nu, \cdots$ denote $0, 1, 2, 3$, $i, j, \cdots$ denote $1, 2, 3$. Consider the massless scalar field $\phi(x^\mu, y)$ in the five-dimensional spacetime. The equation of motion of this field is five-dimensional Klein-Gordon equation:

$$\partial_A \partial^A \phi(x^\mu, y) = 0. \tag{2.2}$$

We may then expand the field in Fourier series:

$$\phi(x^\mu, y) = \sum_{n=0}^{\infty} \phi^n(x^\mu) e^{i\frac{n}{r}y}. \tag{2.3}$$

With this decomposition, (2.2) becomes

$$\partial_\mu \partial^\mu \phi^n(x^\mu) = \frac{n^2}{r^2} \phi^n(x^\mu). \tag{2.4}$$

In this way, mass of the field $m^2 = n^2/r^2$ is generated. At energies small compared to r^{-1} , only the y-independent massless zero-mode remains and the physics is effectively four dimensional. At energies above r^{-1}, the tower of Kaluza-Klein(KK) state comes into play. Since KK state is not detected in an experiment, compactification radius r can be restricted. Their masses would thus have to be greater, $n/r >$TeV, which implies a strong constraint on r:

$$r \leq 10^{-21} cm. \tag{2.5}$$

It is difficult to detect the such small extra dimension.

2.2 ADD model

In 1998, N.Arkani-Hamed,S.Dimopoulos and G.Dvali suggested the phenomenological model whose size of extra dimension is large.[4] Also, the motivation of this model is to solve the hierarchy problem in the particle physics. This approach (called ADD model) considers the brane whose tension is neglected after embeding in the higher dimension with flat and compact extra dimensions. The large extra dimension is allowed because of confinement of the matter fields on the brane and only gravity can propagate the extra dimension. Here, the dimension of spacetime is D-dimension($D > 4$),and fundamental mass in D-dimensional spacetime is denoted M to distinguish it from the Plank mass M_{pl} in the four-dimensional spacetime. The gravitational action in the D-dimensional spacetime is:

$$S = \frac{1}{M^{D-2}} \int d^D x \sqrt{^{(D)}g} \left(^{(D)}R\right) \tag{2.6}$$

where

$$M^{D-2} = M^{d+2} \tag{2.7}$$

is the D-dimensional fundamental mass, $d = D-4$ is the number of extra dimensions. The indecies (D) denote D-dimensional geometric quantity. In this model , the long distance four-dimensional gravity is mediated by the graviton zero mode whose wave function is homogeneous over extra dimensions. Hence, the four-dimensional effective action describing long distance gravity is obtained from eq (2.6) by taking the metric to be independent of extra dimensions and integrate over extra coodinate:

$$S_{eff} = \frac{V_d}{M^{D-2}} \int d^4 x \sqrt{^{(4)}g}^{(4)} R \tag{2.8}$$

where $V_D \sim r^d$ is the volume of extra dimensions. So, the four-dimensional Planck mass is determinded by volume of extra dimensions:

$$M_{pl} = M(Mr)^{\frac{2}{d}}. \tag{2.9}$$

If the size of extra dimensions is large compared to the fundamental length M^{-1}, the Planck mass is much larger than the fundamental gravity scale M. Then, the hierarchy between M_{Pl} and M_{EW} is entirely due to the large size of extra dimensions. Assuming that $M \sim 1TeV$, one calculates from (2.9) the value of r:

$$r \sim M^{-1}(\frac{M_{Pl}}{M})^{\frac{2}{d}} \sim 10^{\frac{32}{d}} \cdot 10^{-17} cm. \tag{2.10}$$

For example, if $d = 2$, $r \sim$ 1mm. This large extra dimension is allowed in the brane models. In the KK type extra dimension, the size of extra dimension is restricted from experiments of particle physics hardly. However, in the brane models, there is no restriction from particle physics since SM fields are supposed to be confined on the brane.

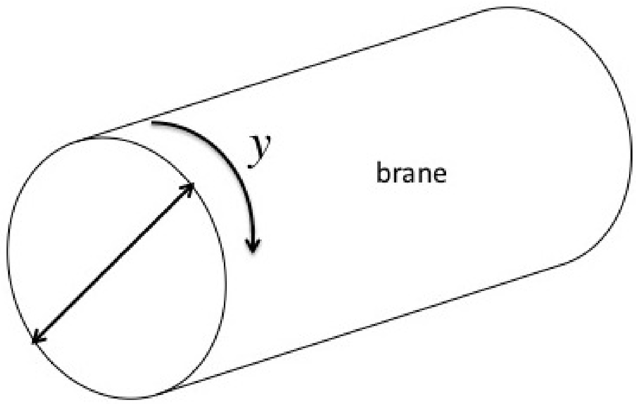
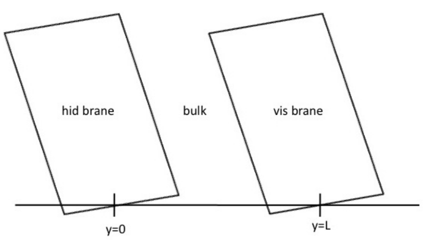


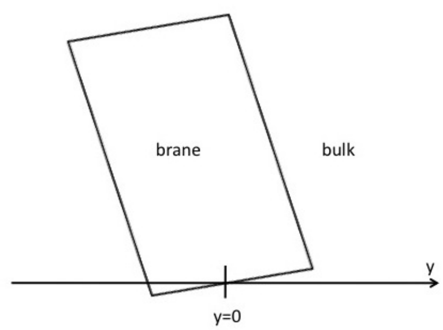
Figure 1: ADD brane model

3 Randall-Sundrum I model

In 1999, L.Randall and R.Sundrum suggested two brane models(Figure 1). The first one called RSI is two brane model to solve the hierarchy problem.[5] The second model called RSII is the model removing one brane from the RSI.[6] We introduce the geometry of RS models, and its basic idea. Hereafter, the number of extra dimensions is one.



[1]RSI brane model



[2]RSII brane model

Figure 2: RS brane models

3.1 Warped gemetry

The RSI assumes the existence of one extra dimension compactified on a circle whose upper and lower halves are identified, called S^1/Z_2 orbifold. The orbifold condition is inspired from Horava-Witten model.[1] This contruction entails two fixed points, $y = 0$ and $y = \pi R \equiv L$. We assumes two branes which have a tension and they locate at those points.

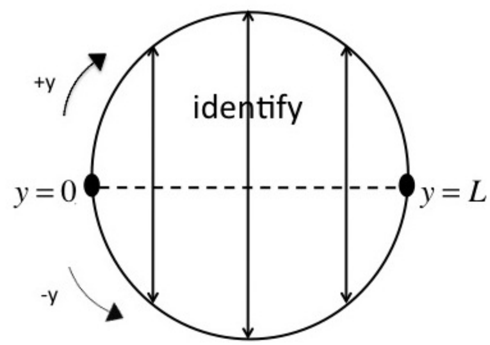


Figure 3: S^1/Z_2 orbifold

Also, there is the five-dimensional cosmological constant Λ_5 in the action. As will be seen later, Λ_5 takes a negative value required in order to make four-dimensional spacetime on the brane flat. The action of this set-up is

$$S = S_{gravity} + S_{hid} + S_{vis} \tag{3.1}$$

$$S_{gravity} = \int d^4x \int_{-L}^{L} dy \sqrt{-^{(5)}g}(-\Lambda_5 + 2M_5^3 \, ^{(5)}R) \tag{3.2}$$

$$S_{hid} = \int d^4x \sqrt{-g_{hid}} \, \lambda_{hid} \tag{3.3}$$

$$S_{vis} = \int d^4x \sqrt{-g_{vis}} \, \lambda_{vis}. \tag{3.4}$$

where, M_5 is fundamental mass in the five-dimensional spacetime, λ is tension of brane,the index "5" means five-dimensional value, "vis" and "hid" mean values on brane. The five-dimensional Einstein equation is

$$G_{AB} =^{(5)} R_{AB} - \frac{1}{2} g_{AB} \, ^{(5)}R = \frac{1}{2M^3}(\Lambda_5 + \lambda_{vis} g_{\mu\nu} \delta^\mu (y-L)_A \delta^\nu (y-L)_B + \lambda_{hid} g_{\mu\nu} \delta^\mu_A(y) \delta^\nu_{B}.(y)) \tag{3.5}$$

Since the solution of this equation should fit the real world, we require that metric should preserve the Poincare invariance. This leads to the following Ansatz:

$$ds^2 = e^{-2A(y)} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2 \tag{3.6}$$

where $\eta_{\mu\nu} = diag(-1, 1, 1, 1)$ is the four-dimensional Minkowski metric. The prefactor $e^{-2A(y)}$, called the "warp factor" is written as an exponential for convenience. Its dependence on the extra dimensional coordinate y causes this metric to be non-factorizable, which means that, unlike the metrics appearing in the usual Kaluza-Klein scenarios, it cannot be expressed as a product of the four-dimensional Minkowski metric and that of a manifold of extra dimensions. To determine the $A(y)$, we must calculate (3.5) with this Ansatz. The 55 component of Einstein equation gives

$$G_{55} = 6A'^2 = \frac{-\Lambda_5}{2M^3} \equiv k^2. \tag{3.7}$$

In order for A to have a real solution, Λ_5 must be negative, and it means that the space between the branes is set to Anti-de Sitter space. Anti-de Sitter spacetime is defined as that of its spatial curvature be negative constant. Since M and Λ_5 are constants, we call the RHS of (3.7) k^2. Also, (3.7) yields l^{-2} as the curvature radius of Anti-de Sitter space. So, l means scale of extra dimension. Integrating over y and considering the orbifold symmetry, we get

$$A(y) = k|y|. \tag{3.8}$$

We get the background metric in the Randall-Sundrum model as

$$ds^2 = e^{-2k|y|}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2 \tag{3.9}$$

with $-L \leq y \leq L$.

Next, we look at the $\mu\nu$ components of Einstein equations. The Einstein tensor is

$$G_{\mu\nu} = (6A'^2 - 3A'')g_{\mu\nu}. \tag{3.10}$$

Also, from (3.8) we have

$$A' = sgn(y)k. \tag{3.11}$$

The term $sgn(y)$ may be written as a combination of Heaviside function as

$$sgn(y) = \Theta(y) - \Theta(-y). \tag{3.12}$$

The heaviside function is defined as

$$\Theta(y) = \begin{cases} 1 & (y > 0) \\ 0 & (y < 0) \end{cases} \tag{3.13}$$

Its derivative is a delta fiction. Let us consider the branes located at $y = 0$ and $y = L$:

$$A'' - 2k(\delta(y) - \delta(y - L)) \tag{3.14}$$

Plugging those results into (3.10) gives

$$G_{\mu\nu} = 6k^2 g_{\mu\nu} - 6k(\delta(y) - \delta(y - L))g_{\mu\nu}. \tag{3.15}$$

Comparing this to the energy-momentum tensor, we get the relations:

$$\frac{-\Lambda_5}{2M^3} = 6k^2 \tag{3.16}$$

$$\lambda_{hid} = -\lambda_{vis} = 12kM^3. \tag{3.17}$$

So, the absolute values of the tension on each brane are coincide, but their signatures do not. Moreover, we need fine-tuning between Λ_5 and λ to reconstruct the four-dimensional Minkowski spacetime.

3.2 Exponential hierarchy

We get the metric in the RS I model, so we look at, how hierarchy problem is solved. In the Standard Model(SM) of particle physics, the mass of matter are generated by Higgs mechanism. The mass of weak boson determines the strength of weak interaction, and its value is determinded by vaccum exceptation value(vev) of Higgs field. So, if the vev of Higgs field is fully suppressed on our brane, the hierarchy is generated.

The RS I model requires our world to have a negative tension brane. In other words, the SM fields are confined on the visible brane whose tension is negative. The action of Higgs field on the visible brane is

$$S_{Higgs} = \int d^4x\sqrt{-g_{vis}}[g^{\mu\nu}_{vis}D_\mu H^\dagger D_\nu H - \lambda(H^\dagger H - v^2)^2], \tag{3.18}$$

where H denotes the Higgs field and v denotes the vev of Higgs field. Using $g_{\mu\nu} = e^{-2kL}\eta_{\mu\nu}$, the action becomes

$$S = \int d^4x e^{-4kL}[e^{2kL}\eta^{\mu\nu}D_\mu H^\dagger D_\nu H - \lambda(H^\dagger H - v^2)^2], \tag{3.19}$$

Redefining the Higgs field as $H = e^{kL}\tilde{H}$, the action becomes

$$S_{Higgs} = \int d^4x[\eta^{\mu\nu}D_\mu \tilde{H}^\dagger D_\nu \tilde{H} - \lambda(\tilde{H}^\dagger \tilde{H} - (e^{-kL}v^2))^2]. \tag{3.20}$$

So, the vacuum expectation value is exponentially suppressed as

$$v_{eff} = e^{-kL}v. \tag{3.21}$$

If the value of the bare Higgs mass is is of order of the Planck scale, the physical Higgs mass could be warped down to the weak scale. Since $M_w \simeq 10^{-16}M_{pl}$, the appropriate value for the size of the extra dimension is given by

$$kL \simeq ln10^{16} \simeq 35. \tag{3.22}$$

It's necessary to know whether the strength of gravity on our brane is affected by this mechanism. To check it, we need to get the four dimensional gravitational action from the five dimensional action,

$$S = M_5^3 \frac{1-e^{-2kL}}{k} \int d^4x \sqrt{-g_0}^{(4)} R(h_{\mu\nu}) \tag{3.23}$$

where

$$M_{pl}^2 = M^3 \frac{1-e^{-2kL}}{k}. \tag{3.24}$$

We see that it weakly depends on L. So, if kL becomes large, M_{pl} is not suppressed.

We see the mechanism of generating hierarchy in the RSI model. The degree of hierarchy depends on the distance between two branes, and we have to stabilize it. There is fine-tunig of that distance to match the real world. This problem is called "radion stabilization" since the degree of freedom of extra dimension is related to scalar field called radion. W.D.Gpldberger and M.B.Wise suggested a solution of this problem by considering the potential of radion mode and its minimum.[10,11]

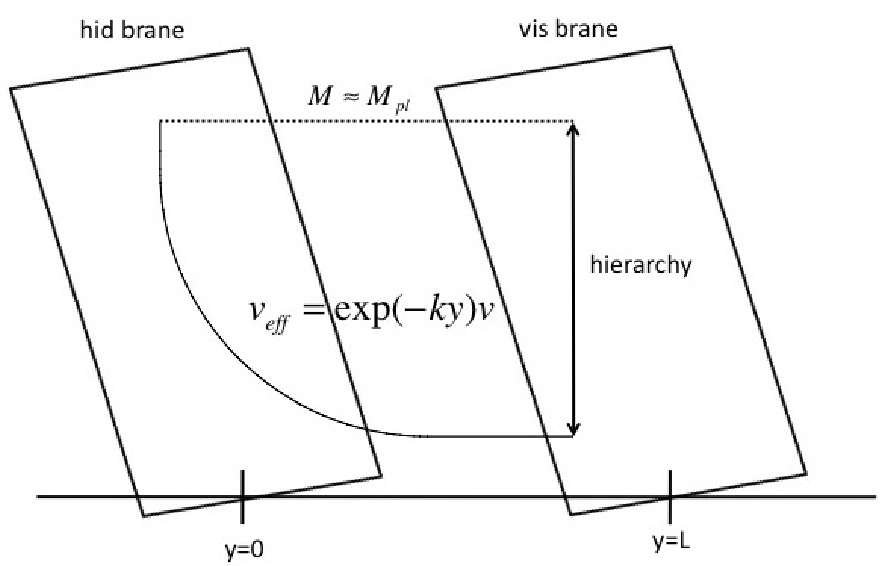


Figure 4: Generating exponential hierarchy.

4 Randall-Sundrum II brane model

The RSII model is single brane model, which is obtained by removing the negative tension brane to $y \to \infty$ in the RSI model.[6] This means we are living in the positive tension brane in the RSII, and this model dose not solve the hierarchy problem. On the other hand, this model succeeds in introducing a infinitely large extra dimension.

In general, introducing the infinitely large extra dimensions breaks Newton's law. In the n dimensional spacetime, Newton's law becomes:

$$F = G_N \frac{mM}{r^{n-2}} \tag{4.1}$$

So, it appears to be forbidden introducing the infinitely large extra dimension. But RSII reconstructs the four-dimensional effective gravity by confinement of graviton zero mode on the brane.

4.1 Graviton mode

In order to understand how gravity works in the RS models, we look at behavior of graviton. So, we consider a small fluctuation $h_{AB}(x, y)$ around the background metric. But we use setup of RS I. After the calculation, the negative tension brane is removed to $y \to \infty$.

It is convenient to work with a conformally flat metric. So, we define a new extra coordinate:

$$dy^2 \equiv e^{-2k|y|} dz^2. \tag{4.2}$$

We set to have the zero value of y to the zero value of z, and metric is given by

$$ds^2 = e^{-2A(z)} \eta_{AB} dx^A dx^B \tag{4.3}$$

where

$$e^{-A(z)} = \frac{1}{(k|z|+1)^2} \tag{4.4}$$

The perturbed metric has the form

$$g_{AB} = e^{-2A}(\eta_{AB} + h_{AB}) \tag{4.5}$$

We solve the Einstein equation with this metric,and choose the gauge:

$$h_{A5} = 0, \tag{4.6}$$

$$\partial_\mu h_{\mu\nu} = 0 \tag{4.7}$$

$$h^\mu_\mu = 0 \tag{4.8}$$

After some calculation, we obtain the wave equation about $h_{\mu\nu}$ as

$$-\frac{1}{2}\partial_C \partial^C h_{\mu\nu} + \frac{3}{2} A' h'_{\mu\nu} = 0 \tag{4.9}$$

In order to get rid of $h'_{\mu\nu}$, we make following rescaling:

$$h_{\mu\nu} \to e^{\frac{3}{2}} h_{\mu\nu}, \tag{4.10}$$

and parform a Kaluza-Klein decomposition,

$$h_{\mu\nu}(x,z)=\sum_{n=0}^{\infty}h^n_{\mu\nu}(x)\psi_n(z). \tag{4.11}$$

with $\partial_\rho\partial^\rho h^n_{\mu\nu}=m_n^2h^n_{\mu\nu}$. We get

$$-\psi''_n(z)+[\frac{9}{4}A'^2(z)-\frac{3}{2}A''(z)]\psi_n(z)=m_n^2\psi_n(z). \tag{4.12}$$

This equation looks like a Schrödinger equation with the potenial

$$V(z)=\frac{15}{4}\frac{k^2}{(k|z|+1)^2}-\frac{3k(\delta(z)-\delta(z-L))}{k|z|+1} \tag{4.13}$$

This potential is called the "volcano potential" due to its form.

Next, we consder the boundary condition. We integrate equation (4.12) over small domains around the boundaries and consider the $z\to-z$ symmetry. For the boundary at $z=0$ we get

$$\psi'_n(0)=-\frac{3k}{2}\psi_n(0) \tag{4.14}$$

Similarly, we get at the boundary L:

$$\psi'_n(L)=-\frac{3k}{2(kL+1)}\psi_n(L) \tag{4.15}$$

The zero mode ($m=0$) solution exsit, satisfying these boundary conditions:

$$\psi_0(z)=e^{-\frac{3}{2}A}=(k|z|+1)^{-\frac{3}{2}} \tag{4.16}$$

This function is peaked around the $y=0$. It means the zero mode graviton is localizing around the positive tension brane. In RS I model, this condition makes weakness of gravity on the negative tension brane. In RS II model, this condition is preserved if the negative tension brane is removed. This means that four dimensional gravity is reconstructed on the positive tension brane.

Between the boundaries, massive Kaluza-Klein modes have to satisfy the following equation:

$$\psi''_n+(m_n^2-\frac{15}{4}\frac{k^2}{(k|z|+1)^2})\psi_n=0. \tag{4.17}$$

Its solution is given by a linear combination of Bessel functions.

$$\psi_n=N_n(|z|+\frac{1}{k})^{\frac{1}{2}}[Y_2(m_n(|z|+\frac{1}{k}))+\frac{4k^2}{\pi m^2}J_2(m_n(|z|+\frac{1}{k}))]. \tag{4.18}$$

Using approximation for large value of $m_n|z|$ and the normalization relation, finally we get the KK states wave functions in the limit of large $m_n|z|$ as

$$\psi_n=\frac{cos(m_n|z|-\frac{5\pi}{4})}{\sqrt{L}}. \tag{4.19}$$

So, the KK mode of graviton can propagate in the extra dimension.

4.2 Newtonian limit

We know that the zero mode graviton is confined on the brane in RS II. Since four-dimensional long distance gravity is governed by zero mode gravitons, it should be possible to reconstruct the four-dimensional gravity in RS II. But, we did not consider corrections from KK states.

Let us consider the contribution of KK graviton exchange into gravitational potential on the brane . Each KK graviton produces the potential of Yukawa type, so the total contribution is

$$V_{KK}(r) = -^{(5)}Gm_1m_2\int_0^\infty dm[h^m_{\mu\nu}(0)]^2\frac{e^{-mr}}{r} \tag{4.20}$$

$$\approx -const\frac{G_Nm_1m_2}{r}\frac{1}{r^2k^2} \tag{4.21}$$

Since, the graviton zero mode construct the conventional Newton potential, the total gravitational potential is

$$V(r) = G_N\frac{m_1m_2}{r}(1+\frac{\alpha}{r^2k^2}) \tag{4.22}$$

where α is constant.

This means, if r is large, the correction term can be neglected. Also, the correction term must be neglected in $r \sim O(mm)$ scale, because Newton's law is confirmed down to the scale of $O(mm)$ from experiments. This fact restricts the fundamental mass. To vanish the correction term, we require

$$k^{-1} \lesssim 0.1mm. \tag{4.23}$$

In RSII, the Planck mass is determined from () to remove the brane $y \to \infty$:

$$M^2_{pl} = \frac{M^3}{k} \tag{4.24}$$

So, using this relation, we get the restriction of the fundamental mass

$$M \gtrsim 10^{10}GeV \tag{4.25}$$

5 Effective four dimensional Einstein equations on the brane

In this section, we derive the four-dimensional effective Einstein equations on the 3-brane in RSII model, and consider its characteristics. This method is suggested by Shiromizu, Maeda and Sasaki. [12]

In order to simplify, there is no matter in five-dimensional bulk, but there exist the cosmological constant Λ_5. Let us impose the Z_2 symmetry near the brane .

We take the Gaussian normal coodinate in a neighborhood of the brane, and we write the metric in the form:

$$ds^2 = q_{\mu\nu} dx^\mu dx^\nu + dy^2 \tag{5.1}$$

where y is the extra coordinate, and the brane locates at $y = 0$. We define the n^A as the unit normal vector vertical to the brane. So, the induced metric on the brane is

$$q_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu \tag{5.2}$$

In such set-up, we derive the effective four-dimensional Einstein equations by using the Gauss-Codacci equation[Appendix A]:

$${}^{(4)}R^\alpha_{\beta\gamma\delta} = {}^{(5)}R^A_{BCD} q^\alpha_A q^B_\beta q^C_\gamma q^D_\delta + K^\alpha_\gamma K_{\beta\delta} - K^\alpha_\delta K_{\beta\gamma} \tag{5.3}$$

$$D_\nu K^\nu_\mu - D_\mu K = {}^{(5)}R_{\rho\sigma} n^\sigma q^\rho_\mu \tag{5.4}$$

where $K_{\mu\nu} = q^A_\mu q^B_\nu \nabla_A n_B$ is the extrinsic curvature, ∇_A is the covariant differentiation with respect to g_{AB} and D_μ is the covariant differentiation with respect to $q_{\mu\nu}$. The indicies (4) and (5) denote the dimension of spacetime. First, we contract the Gauss equation to derive the four-dimensional Ricci tensor:

$${}^{(4)}R_{\mu\nu} = {}^{(5)}R_{AB} q^A_\mu q^B_\nu - {}^{(5)}R^A_{BCD} n_A q^B_\mu n^C q^D_\nu + K K_{\mu\nu} - K^A_\mu K_{\nu A} \tag{5.5}$$

Similary, we can derive the Ricci scalar. So, the four-dimensional Einstein tensor is written in five dimensional geometric quantities as:

$${}^{(4)}G_{\mu\nu} = \left[{}^{(5)}R_{AB} - \frac{1}{2} g_{AB} {}^{(5)}R \right] q^A_\mu q^B_\nu + {}^{(5)}R_{AB} n^A n^B q_{\mu\nu} \tag{5.6}$$

$$+ K K_{\mu\nu} - K^A_\mu K_{\nu A} - \frac{1}{2} q_{\mu\nu} \left(K^2 - K^{AB} K_{AB} \right) - \tilde{E}_{\mu\nu} \tag{5.7}$$

where

$$\tilde{E}_{\mu\nu} \equiv {}^{(5)}R^A_{BCD} n_A n^C q^B_\mu q^D_\nu . \tag{5.8}$$

Using the five-dimensional Einstein equation,

$${}^{(5)}R_{AB} - \frac{1}{2} g_{AB} {}^{(5)}R = \kappa_5^2 T_{AB} \tag{5.9}$$

where $\kappa_5^2 \equiv 1/M^3$, and a decomposition of the Riemann tensor into the Weyl curvature, the Ricci tensor and the Ricci scalar:

$${}^{(5)}R_{\mu\alpha\nu\beta} = \frac{2}{3} \left(g_{\mu[\nu} {}^{(5)}R_{\beta]\alpha} - g_{\alpha[\nu} {}^{(5)}R_{\beta]\mu} \right) - \frac{1}{6} g_{\mu[\nu} g_{\beta]\alpha} {}^{(5)}R + {}^{(5)}C_{\mu\alpha\nu\beta} \tag{5.10}$$

where, [ ] denote antisymmetrization of indicies , $C_{\mu\alpha\nu\beta}$ is Weyl tensor. We obtain the four-dimensional relations:

$$\begin{aligned} {}^{(4)}G_{\mu\nu} = \frac{2\kappa_5^2}{3} \left[T_{AB} q^A_\mu q^B_\nu + \left(T_{AB} n^A n^B - \frac{1}{4} T^A_A \right) q_{\mu\nu} \right] \\ + K K_{\mu\nu} - K^A_\mu K_{\nu A} - \frac{1}{2} q_{\mu\nu} \left(K^2 - K^{AB} K_{AB} \right) - E_{\mu\nu} \end{aligned} \tag{5.11}$$

where

$$E_{\mu\nu} \equiv^{(5)} C^A_{BCD} n_A n^C q^B_\mu q^D_\nu \tag{5.12}$$

Because the matter is confined on the brane, the energy-momentum tensor T_{AB} is written as

$$T_{AB} = -\Lambda_5 g_{AB} + S_{AB}\delta(y) \tag{5.13}$$

$$S_{\mu\nu} = -\lambda q_{\mu\nu} + \tau_{\mu\nu} \tag{5.14}$$

where λ is the tension of brane and $\tau_{\mu\nu}$ is energy-momentum tensor of matter on the brane. As we will see later, the signature of the tension must be positive in the RSII model.

Next, we consider the Israel's junction condition[Appendix B]:

$$[K_{\mu\nu}] = -\kappa_5^2 \left(S_{\mu\nu} - \frac{1}{3} q_{\mu\nu} S \right) \tag{5.15}$$

The bracket [ ] denotes:

$$[X] = \lim_{y \to +0} X - \lim_{y \to -0} X \equiv X^+ - X^- \tag{5.16}$$

. There is Z_2 symmetry near the brane. So, Israel's junction condition becomes

$$K^+_{\mu\nu} = -K^-_{\mu\nu} = -\frac{1}{2}\kappa_5^2 \left(S_{\mu\nu} - \frac{1}{3} q_{\mu\nu} S \right) \tag{5.17}$$

Using this relation, we write $K_{\mu\nu}$ in $\tau_{\mu\nu}$. The final form of the effective four-dimensional Einstein equation is

$$^{(4)}G_{\mu\nu} = -\Lambda_4 q_{\mu\nu} + 8\pi G_N \tau_{\mu\nu} + \kappa_4^2 \pi_{\mu\nu} - E_{\mu\nu} \tag{5.18}$$

where

$$\Lambda_4 = \frac{1}{2}\kappa_5^2 \left(\Lambda_5 + \frac{1}{6}\kappa_5^2 \lambda^2 \right), \tag{5.19}$$

$$G_N = \frac{\kappa_5^4 \lambda^2}{48\pi}, \tag{5.20}$$

$$\pi_{\mu\nu} = -\frac{1}{4}\tau_{\mu\alpha}\tau^\alpha_\nu + \frac{1}{12}\tau\tau_{\mu\nu} + \frac{1}{8}q_{\mu\nu} + \frac{1}{24}q_{\mu\nu}\tau^2. \tag{5.21}$$

So, we can identify G_N with Newton's gravitational constant, and Λ_4 with four dimensional cosmological constant. In the RSI model, we adjusted Λ_4 to 0. In the (5.21) relation, the tension of brane must be positive. If its signature is negative, the gravitational force becomes a repulsive force. But in the RSI model, we are living on the negative tension brane. In this case, caluculating the correction of "radion", the Newton's gravitational constant gets positive.

The effective equation differs from the conventional Einatein equation by the presence of two new terms "$\pi_{\mu\nu}$" and "$E_{\mu\nu}$". The $\pi_{\mu\nu}$ is quadratic in $\tau_{\mu\nu}$, and $E_{\mu\nu}$ is called "dark radiation" to be considered later. $E_{\mu\nu}$ does not close in four dimensional relation. So, this term contains the information of the extra dimension.

We can neglect $\pi_{\mu\nu}$ when the energy density of matter fields is much lower than the tension of brane. For simplicity, we assume the matter is the perfect fluid:

$$\tau_{\mu\nu} = \rho u_\mu u_\nu + p(q_{\mu\nu} + u_{\mu\nu}) \tag{5.22}$$

where, u_μ is four dimensional velocity. The part of the effective equations becomes

$$8\pi G_N \tau_{\mu\nu} + \frac{48\pi G_N}{\lambda}\pi_{\mu\nu} = \rho\left(1 + \frac{\rho}{2\lambda}\right) + \left[p + \frac{\rho}{2\lambda}(\rho + 2p)\right](q_{\mu\nu} + u_\mu u_\nu) \tag{5.23}$$

So, we can neglect the correction of the quadratic part in $\rho/\lambda \ll 1$.

Since $E_{\mu\nu}$ is constructed by Weyl tensor, if Weyl tensor is zero, $E_{\mu\nu}$ becomes zero too.

We derive the conservation equation on the brane by using tje Codacci equation and the junction condition:

$$D^\nu \tau_{\mu\nu} = 0 \tag{5.24}$$

Taking the divergence of both side of (5.19), we get another relation by using $D^{\nu(4)}G_{\mu\nu} = 0$ and (5.25):

$$\begin{aligned} D^\mu E_{\mu\nu} &= \frac{48\pi G_N}{\lambda}\pi_{\mu\nu} & (5.25) \\ &= \frac{1}{4}\kappa_5^4 \left[\tau^{\alpha\mu} D_\nu \tau_{\alpha\mu} -\right] & (5.26) \end{aligned}$$

So, $E_{\mu\nu}$ is on quadratic in $\tau_{\mu\nu}$ too. But this only determines the transverse-traceless components of $E_{\mu\nu}(E_{TT}$'s dependence). The dependence of the orthogonal component of $E_{\mu\nu}(E_L)$ is caluculated by Shiromizu, Maeda and Sasaki.[12]

Then, we neglecte $\pi_{\mu\nu}$ and $E_{\mu\nu}$ in the low energy condition, and reconstruct the conventional Einstein equation. Otherwise, we can't neglect these corrections in the high energy condition like $\rho > \lambda$. So, in the high-energy physics, like black hole and early universe, we expect these corrections to be important and reserch for of such physics may be the evidence of the brane-world scenario.

We assume there is no matter in the bulk for simplicity. However, we can derive the effective four-dimensional Einstein equation in general if there is matter in the bulk. In such case, it is added to the equation(5.18) the correction term from bulk matter as

$$F_{\mu\nu} = T_{AB} q_\mu^A q_\nu^B + \left[T_{AB} n^A n^B - \frac{1}{4}T\right] q_{\mu\nu} \tag{5.27}$$

where T_{AB} denotes the matter in the bulk.

6 Brane-world cosmology

In this section, we consider the expansion of universe in the brane-world scenalio. The effective four dimensional Einstein equations on the brane differ from the conventional Einstein equations (section5). There are two new terms which can be neglected in low energy condition. So, we expect the modified Friedmann equation in brane-world scenario include some corrections . We derive the modified Friedmann equation and compare it to the usual cosmology.

6.1 Friedmann equation on the brane

We get the modified Friedmann equation in brane-world scenario in several ways. Here we derive it by plugging five-dimensional metric and the junction condition into the five dimensional Einstein equations.

We take the Gaussian normal coodinate system near the brane:

$$ds^2 = q_{\mu\nu} dx^\mu dx^\nu + dy^2 \tag{6.1}$$

To consider the expansion of universe, we take the four-dimensional FLRW metric like the conventional cosmology,

$$ds^2 = -n(t,y)dt^2 + a(t,y)\gamma_{ij} dx^i dx^j + dy^2 \tag{6.2}$$

where and $a(t,y)$ is a scale factor, γ_{ij} means a maximally symmetric three-dimensional metric. The 3-brane locates at $y = 0$. If we take $n(t,0) = 1$, t means the time of universe. Also, to consider the homogeneus and isotropic universe, we take a perfect fluid on the brane, and no matter in the five-dimensional bulk for simplicity,

$$T_{AB} = S_{AB}\delta(y) \tag{6.3}$$

with

$$S_{\mu\nu} = diag(-\rho, p, p, p, 0) \tag{6.4}$$

where ρ denotes energy density, p denotes pressure. We derive the Friedmann equation on the brane by plugging that metric into the 5D Einstein equation:

$$R_{AB} - \frac{1}{2} g_{AB} R = \kappa_5^2 T_{AB} - g_{AB}\Lambda_5 \tag{6.5}$$

We get the components of the Einstein tensor:

$$G_{00} = 3\frac{\dot{a}^2}{a^2} - 3n^2 \left(\frac{a''}{a} + \frac{a'^2}{a} \right) + 3k\frac{n^2}{a^2} \tag{6.6}$$

$$G_{ij} = a^2 \left(2\frac{a''}{a} + \frac{n''}{n} + \frac{a'^2}{a^2} + 2\frac{a'n'}{an} \right) \gamma_{ij} \tag{6.7}$$

$$+\frac{a^2}{n^2} \left(-2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} + 2\frac{\dot{a}\dot{n}}{an} \right) \gamma_{ij} - k\gamma_{ij} \tag{6.8}$$

$$G_{0y} = 3\left(\frac{\dot{a}n'}{an} - \frac{\dot{a}'}{a}\right) \tag{6.9}$$

$$G_{yy} = 3\left(\frac{a'^2}{a^2} + \frac{a'n'}{an}\right) - \frac{3}{n^2}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} - \frac{\dot{a}\dot{n}}{an}\right) - 3\frac{k}{a^2} \tag{6.10}$$

where the dots mean differentiation with t, and primes mean differentiation with y. Next, we consider the junction condition. Since the matter is confined as delta functional on the 3-brane, there is the jump at $y = 0$. So, we need the junction condition at $y = 0$. The Israel's junction condition is

$$K_{AB} = -\frac{\kappa_5^2}{2}\left(S_{AB} - \frac{1}{3}q_{AB}S\right) \tag{6.11}$$

On the other hand, we calculate K_{AB} from its difinition. Then we get another relation, by using the junction condition, as

$$\frac{n'}{n}|_{y=0^+} = \frac{\kappa_5^2}{6}(3p + 2\rho - \lambda) \tag{6.12}$$

$$\frac{a'}{a}|_{y=0^+} = -\frac{\kappa_5^2}{6}(\rho + \lambda). \tag{6.13}$$

Comparing it with the Einstein tensor(6.6), we find

$$H^2 = \frac{\dot{a}_0^2}{a_0^2} = \frac{\kappa_5^2}{36}(\rho + \lambda)^2 + \frac{\Lambda_5}{6} - \frac{k}{a_0^2} + \frac{C}{a_0^4}, \tag{6.14}$$

and rewrite this equation to the form

$$H^2 = \frac{\kappa_5^4\lambda}{18}\rho + \frac{\kappa_5^4}{36}\rho^2 + \left(\frac{\kappa_5^4\lambda^2}{36} + \frac{\Lambda_5}{6}\right) - \frac{k}{a_0^2} + \frac{C}{a_0^4}. \tag{6.15}$$

We identify the bracket part of RHS with the four-dimensional cosmological constant:

$$\frac{\Lambda_4}{3} \equiv \frac{\kappa_5^4\lambda^2}{36} + \frac{\Lambda_5}{6} \tag{6.16}$$

and Newton's gravitational constant is defined by:

$$\frac{\kappa_5^4\lambda}{18} \equiv \frac{8\pi G_N}{3} \tag{6.17}$$

Then we get the modified Freidmann equation on the brane

$$H^2 = \frac{8\pi G_N}{3}\rho\left(1 + \frac{\rho}{2\lambda}\right) + \frac{\Lambda_4}{3} - \frac{k}{a_0^2} + \frac{C}{a_0^4} \tag{6.18}$$

The new terms are ρ^2 and constant C. The C is called "dark radiation" since it behaves like radiation. We can get this equation by using the effective four-dimensional Einstein equation and the four-dimensional FLRW metric. In such way, dark radiation term comes from the Weyl tensor in equation(5.19):

$$\frac{C}{r^4} = \frac{8\pi G_N}{3}\rho_\omega = \frac{1}{3}E^0_0 \tag{6.19}$$

Next, we consider the conservation equation. We can derive plugging (6.12), (6.13) into $G_{0y} = 0$ around the brane:

$$\dot{\rho} + 3H(\rho + p) = 0 \tag{6.20}$$

This means no flucs to the bulk.

To reconstract the usual cosmology, which means after the nucleosynthesis, the correction term must be vanished. So, we can constrain the tension of brane from energy scale of nucleosynthesis:

$$\lambda^{\frac{a}{4}} > 1MeV \tag{6.21}$$

And combining this with (6.17), this implies for the fundamental mass scale

$$M > 10^4 GeV \tag{6.22}$$

But, this constrait is weaker than (4.25) to reconstruct Newton's law.

The dark radiation influence the degree of freedam of relativistic radiation. So, we can constrain the correction of dark radiation comparing the inhomogenius of Cosmic Microwave Background (CMB). Such constraint is calculated in [13].

6.2 The brane as domain wall in the Schwarzschild-AdS spacetime

Here, we consider the brane as domain wall in 5D Schwarzschild-AdS spacetime, and we show the motion of domain wall looks like expansion of universe from the observer on the brane.[14] And more, we will show the dark radiation term comes from 5D black hole.

We assume the five dimensional Schwarzschild-AdS metric as backgroud metric:

$$ds^2 = -f(r)dT^2 + \frac{1}{f(r)}dr^2 + r^2\gamma_{ij}dx^i dx^j \tag{6.23}$$

This metric is the vacuum solution of five dimensional Einstein equations in the bulk. where

$$f(r) = k - \frac{\Lambda_5}{6}r^2 - \frac{\mu}{r^2} \tag{6.24}$$

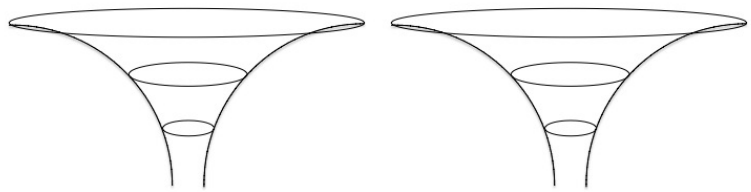


Figure 5: Five dimensional Schwarzschild-AdS spacetime

and γ_{ij} is maximally symmetric three dimensionalmetric with its spatial curvature $k = -1, 0, 1$. r denotes the extra coordinate, and radial coordinate of five dimensional Schwarzschild black hole. μ is the mass of five dimensional Schwarzschild black hole.

We assume the brane is embeding this spacetime, and parametrize the its moton in τ. Because $dT = \dot{T}d\tau$, $dr = \dot{r}d\tau$, indused metric on the brane is

$$ds_B^2 = -\left(f\dot{T}^2 - f^{-1}\dot{r}^2\right)d\tau^2 + r^2(\tau)\gamma_{ij}dx^i dx^j \tag{6.25}$$

where, dot means derivation respect to τ. The verocity of brane can be written as

$$u^A = \left(\dot{T}, 0, 0, 0, \dot{r}\right) \tag{6.26}$$

$$u_A = \left(-f\dot{T}, 0, 0, 0, f^{-1}\dot{r}\right) \tag{6.27}$$

In order to accord τ with proper time on the brane, these verocitys are normalized as

$$u^A u_A = -f\dot{T}^2 + f^{-1}\dot{r}^2 = -1 \tag{6.28}$$

Next, we consider the unit normal vector to u^A and u_A. Such vectors n^A can be expressed

$$n_A = \left(\dot{r}, 0, 0, 0, -\dot{T}\right) \tag{6.29}$$

$$n^A = \left(-f^{-1}\dot{r}, 0, 0, 0, -f\dot{T}\right) \tag{6.30}$$

and these vectors satisfy $u^A n_a = 0$. So, the extrinsic curvature is caluculate from these vectors:

$$\begin{aligned} K_{ij} &= q_i^A q_j^B \nabla_A n_B \\ &= \frac{1}{2} n^A \partial_A g_{ij} \\ &= -f\dot{T} r^{-1} g_{ij} \end{aligned} \quad (6.31)$$

Also, we assume Z_2 symmetry, and Israel's junction condition:

$$K_i^j = -\frac{\kappa_5^2}{6}(\rho + \lambda)\delta_i^j = -\frac{\sqrt{f + \dot{r}^2}}{r}\delta_i^j \quad (6.32)$$

using () and() relation, we get the Freidmann equation

$$\frac{\dot{r}^2}{r^2} = \frac{\kappa_5^4}{36}(\rho + \lambda)^2 + \frac{\Lambda_5}{6} - \frac{k}{r^2} + \frac{\mu}{r^4} \quad (6.33)$$

This equation is equivalent to (6.18). The dark radiation term is determined by mass of five dimensional black hole in this equarion.

We saw the two equaivalent picture about evolution of univerese in brane-world scenario. The first one is that the brane locate at fixed position in Gaussian normal coordinate system and the evolution of universe is expressed as time evolutio of metric components. In this subsection, we consider the second one whose evolution of universe is expressed as motion of brane in static bulk spacetime.

7 Conclusion

We introduce the basic idea and its application of the Randall-Sundrum brane models in this paper. These models have warped geometry, and this geometrical factor generate the exponentialy hierarchy between the weak scale mass and Planck mass in RSI. Also, the RSII succeed in constracting the infinitly large extra dimension by confining the graviton zero mode. This is allowed experimentally since the SM particles are confined on the brane and it has not done the experiment of the gravity at short range.

By the suggestion of infinitly large extra dimension, the higher-dimensional universe started to be studied actively. The effective four-dimensionl Einstein equations and the modified Freidmann equation deffer from convensional equations in high-energy scale. So, there is a possibility that existence of extra dimension can be inspected by checking early universe. Additionally, it may be able to get the information of the extra dimension by reseaching about the CMB. From these facst, ithe inflation and its quantum fluctuation are studied actively and suggested some interesting models of inflation such like the model which is caused by the scalar field(for example, dilaton and radion) in the bulk.[17] Especially, Ekipilotic/Cyclic models are paid much attention recently since they may be alternative to the inflation.[18,19]

Thus Brane model by Randall - Sundrum is permitted experimentally and is model with a possibility of the inspection.

8 Acknowledgement

A Appendix A

We derive the Gauss equation and the Codacci equation.[15] These equations are purely geometric relation satisfying any dimensions , and they make a correlation between n+1 dimensional geometric quantities and n dimensional geometric quantities. We consider n dimensional hypersurface Σ embedded in the n+1 dimensional manifold V. The indices $A, B, \cdots$ denote the quantities on the n+1 dimensional manifold, $a, b, \cdots$denote the the quantities on the n dimensional hypersurface.

First, we need to select the particular hypersurface Σ, for example, putting a restriction on the coordinates,

$$\Phi(x^A) = 0 \tag{A.1}$$

We can define the unit notmal vector n^A orthogonal to Σ. The tangent vector of Σ is defined by

$$e^A_a \equiv \frac{\partial x^A}{\partial y^a} \tag{A.2}$$

and these vectors satisfy

$$n_A e^A_a = 0 \tag{A.3}$$

Considing the metric on the hypersurfaces, we write:

$$\begin{aligned} ds^2_\Sigma &= g_{AB} dx^A dx^B && \text{(A.4)} \\ &= g_{AB}\left(\frac{\partial x^A}{\partial y^a} dy^a\right)\left(\frac{\partial x^B}{\partial y^b} dy^b\right) && \text{(A.5)} \\ &= h_{ab} dy^a dy^b && \text{(A.6)} \end{aligned}$$

where

$$h_{ab} = g_{AB} e^A_a e^B_b \tag{A.7}$$

This metric is the induced metric on the hypersurface. The induced metric h_{ab} has

$$g^{AB} = n^A n^B + h^{ab} e^A_a e^B_b \tag{A.8}$$

The e^A_a work as projectors on the Σ. So, we write a vector on the Σ by using vector in the V:

$$X_a = e^A_a X_A, X^A = e^A_a X^a. \tag{A.9}$$

We consider the covariant derivative on the hypersurfaces of a X^a as the projection of $\nabla_B X_A$ on to the hypersurface:

$$D_b X^a \equiv \nabla_B X_A e_a^A e_b^B \tag{A.10}$$

We show that $D_b X_a$ can be constructed in q_{ab}. To get that, we rewrite the RHS:

$$\nabla_B X_A e_a^A e_b^B = \nabla_B \left(X_A e_a^A \right) e_b^B - X_A \nabla_B e_a^A e_b^B \tag{A.11}$$

Then, using $\nabla_B X_a = \frac{\partial X_a}{\partial x^B}$, $e_b^B = \frac{\partial x^B}{\partial y^b}$, and

$$RHS = \frac{\partial X_a}{\partial^y b} - \Gamma^c_{ab} A_c \tag{A.12}$$

where, we defined

$$\Gamma^c_{ab} = q^{cd} e_c^C e_{aC;B} e_b^B \tag{A.13}$$

So, equation (A.12) looks like a covariant derivative. In fact, the connection term defined in (A.13) is exressed as

$$\Gamma^c_{ab} = \frac{1}{2} q^{cd} \left(q_{ca,b} + q_{cb,a} - h_{ab,c} \right) \tag{A.14}$$

We get this by transforming the (A.13)

We define a covariant derivative on the hypersurfaces as a projection of $\nabla_B X^A e_b^B$. It means $D_b X_a$ is the tangent component of $\nabla_B X^A e_b^B$. How express the notmal components of $\nabla_B X^A e_b^B$? We rewrite the $\nabla_B X^A e_b^B$ as

$$\begin{aligned} \nabla_B X^A e_b^B &= g_C^A \nabla_B X^C e_b^B & \text{(A.15)} \\ &= \epsilon (n^A n_C + h^{ac} e_a^A e_{cC}) \nabla_B X^C e_b^B & \text{(A.16)} \\ &= \epsilon (n_C \nabla_B X^A e_b^B) n^A + h^{ac} (\nabla_B X_C e_c^C e_b^B) e_a^A & \text{(A.17)} \end{aligned}$$

Using (A.10) and $n_C \nabla_B X^C = -\nabla_B n_C X^C$, we find

$$\begin{aligned} \nabla_B X^A e_b^B &= -\epsilon (\nabla_B n_C X^C e_b^B) n^A + h^{ac} D_b X_c e_a^A & \text{(A.18)} \\ &= D_b X^a e_a^A - \epsilon X^a K_{ab} n^A & \text{(A.19)} \end{aligned}$$

where, we difined "extrinsic curvature as

$$K_{ab} \equiv \nabla_B n_A e_a^A e_b^B \tag{A.20}$$

Hence $D_b X^a$ give the tangential part of the vector fields, while $-\epsilon A^a K_{ab}$ represents the normal component,

$$\nabla_B e_a^A e_b^B = \Gamma^c_{ab} e_c^A - \epsilon K_{ab} n^A. \tag{A.21}$$

The Riemann tensor on Σ is defined by h_{ab}:

$$D_b D_a X^c - D_a D_b X^c = -R^c_{dab} A^d \tag{A.22}$$

Also, the Riemann tensor on Σ is the projection of the Riemann tensor in V. So, the n dimensional Reimann tensor can be expressed by using n+1 dimensional geometric quantities. This relation is called the Gauss-Codacci equation. We derive this equation by calculating $R_{ABCD}e_a^A e_b^B e_c^C e_d^D$ directly, and using Gauss-Weigarten equation. We start with the identity:

$$\nabla_C(\nabla_B e_a^A e_b^B)e_c^C = \nabla_C(\Gamma^d_{ab}e_a^A - \epsilon K_{ab}n^A)e_c^C \tag{A.23}$$

After some calculating, we get

$$\nabla_C\nabla_B e_a^A e_b^B e_c^C + \Gamma^d_{bc}(\Gamma^e_{ad}e_e^A - \epsilon K_{ad}n^A) - \epsilon K_{bc}\nabla_B e_a^A b^B \tag{A.24}$$

$$= \partial_c\Gamma^d_{ab}e_d^A + \Gamma^d_{ab}(\Gamma^e_{dc}e_e^A - \epsilon K_{dc}n^A) - \epsilon\partial_c K_{ab}n^A - \epsilon K_{ab}\nabla_C n^A e_c^C \tag{A.25}$$

Then using the difinition of Riemann tensor:

$$\nabla_c\nabla_b e_a^A e_b^B e_c^C - \nabla_B\nabla_C e_a^A e_b^B e_c^C = -R^A_{DBC}e_a^D e_b^B e_c^C \tag{A.26}$$

we obtain the Gauss equation and Coddacci equation

$$R_{ABCD}e_a^A e_b^B e_c^C e_d^D = R_{abcd} + \epsilon\left(K_{ad}K_{bc} - K_{ac}K_{bd}\right), \tag{A.27}$$

$$R_{DABC}n^D e_a^A e_b^B e_c^C = D_c K_{ab} - D_b K_{ac}. \tag{A.28}$$

B Appendix B

Here we derive the Israel's junction conditions.[15,16] We consider the following setup when Σ separates spacetime into two parts V^+ and V^- (fig7). In V^+, the metric and coordinates are denoted g^+_{AB}, x^A_+ ,and in V^-, the metric and coordinates are denoted g^-_{AB}, x^A_- The coordinates on the hypersurface are expressed by y^a too.

We assume that the geodesics intersecting Σ orthogonally. We take l to denote the proper distance along the geodesics, and we adjust the parametrization so that $l = 0$ when the geodesics intersect the hypersurface. l takes the positive value in V^+ in our convention. Choosing the vector n^A, which is unit normal to $Sigma$, to point from V^- to, V^+.

We introduce the Heaviside distribution:

$$\Theta(l) = \begin{cases} +1 & (l > 0) \\ -1 & (l < 0) \end{cases} \tag{B.1}$$

and

$$\frac{d}{dl}\Theta = \delta(l) \tag{B.2}$$

where $\delta(l)$ is the delta distribution defined as

$$\delta(l) = \begin{cases} \infty & (l = 0) \\ 0 & (l \neq 0) \end{cases} \tag{B.3}$$

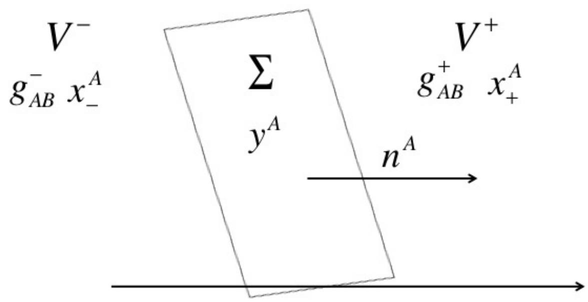


Figure 6: The hypersurface separates spacetime

The following notation is usefull:

$$[X] \equiv X(V^+)|_\Sigma - X(V^-)|_\Sigma \tag{B.4}$$

X is the any tensorial quantity defined on both side of the hypersurface. We note the relation

$$[n^A] = [e^A_a] = 0 \tag{B.5}$$

The metric can be expressed as a distribution valued,

$$g_{AB} = \Theta(l)g^+_{AB} + \Theta(-l)g^-_{AB} \tag{B.6}$$

Differentiating (B.6) yeikds

$$\partial_C g_{AB}\Theta(l)\partial_C g^+_{AB} + \Theta(-l)\partial_C g^-_{AB} + \epsilon\delta(l)[g_{AB}]n_C \tag{B.7}$$

The last term is singular and it makes difficulties to express the Christoffel simbol in distribution forms. So, we imply the continuous to the metric to vanish that term:

$$[g_{AB}] = 0 \tag{B.8}$$

We obtain relation by using (B.8) and this condition

$$[g_{AB}]e^A_a e^B_b = [g_{AB}e^A_a e^B_b] = 0 \tag{B.9}$$

That means

$$[h_{ab}] = 0. \tag{B.10}$$

We calculate the connection in these condition:

$$\Gamma^A_{BC} = \Theta(l)\Gamma^{+A}_{BC} + \Theta(-l)\Gamma^{-A}_{BC} \tag{B.11}$$

Also, Riemann tensor is

$$R^A_{BCD} = \Theta(l)R^{+A}_{BCD} + \Theta(-l)R^{-A}_{BCD} + \delta(l)F^A_{BCD} \tag{B.12}$$

where

$$F^A_{BCD} = \epsilon([\Gamma^A_{BD}]n_C - [\Gamma^A_{BC}]n_D) \tag{B.13}$$

Since the metric is continuous across Σ in the coordinates x^A, the tangrntial derivative of the metric also must be continuous. This means that if $\partial_C g_{AB}$ is to be discontinuous, the discontinuity must be directed along the normal vctor n^A. There must exist s tensor field κ_{AB} such that

$$[\partial_C g_{AB}] = \kappa_{AB}n_C \tag{B.14}$$

and explicitly

$$\kappa_{AB} = \epsilon[\partial_C g_{AB}]n^C \tag{B.15}$$

The equation (B.14) implies

$$[\Gamma^A_{BC}] = \frac{1}{2}\left(\kappa^A_B n_C + \kappa^A_C n_B - \kappa^A_{BC}\right). \tag{B.16}$$

We obtain F^A_{BCD} by usng (B.13), and contracting the first and third indices of it:

$$F_{AB} \equiv \frac{\epsilon}{2}\left(\kappa_{CA}n^C n_B + \kappa_{CB}n^C n_A - \kappa n_A n_B - \epsilon\kappa_{AB}\right) \tag{B.17}$$

Also, we get

$$F \equiv F^A F_A = \epsilon\left(\kappa_{AB}n^A n^B - \epsilon\kappa\right) \tag{B.18}$$

Next, we define the energy-momentum tensor as a distribution form:

$$T_{AB} = \Theta(l)T^+_{AB} + \Theta(-l)T^-_{AB} + \delta(l)S_{AB} \tag{B.19}$$

where S_{AB} is energy-momentum tensor on Σ. Using the Einstein equations, we obtain:

$$8\pi G_N S_{AB} = F_{AB} - \frac{1}{2}F g_{AB} \tag{B.20}$$

This equations are delta functional parts of the Einstein equations.

Using the equations (B.17), (B.18), we obtain S_{AB} explisitly

$$\begin{aligned}16\pi G_N \epsilon S_{AB} = {} & \kappa_{CA}n^C n_B + \kappa_{CB}n^C n_A - \kappa n_A n_B - \epsilon\kappa_{AB} \\ & - \left(\kappa_{CD}n^C n^D - \epsilon\kappa\right) g_{AB}\end{aligned} \tag{B.21}$$

So, S_{AB} is tangent to the hypersurface: $S_{AB}n^B = 0$. It therefore admits the decomposition

$$S^{AB} = S^{ab} e_a^A e_b^B. \tag{B.22}$$

We evaluat as follow:

$$16\pi G_N S_{ab} = -\kappa_{AB} e_a^A e_b^B + h^{cd} \kappa_{CD} e_c^C e_d^D h_{ab} \tag{B.23}$$

On the other hand, we have

$$[\nabla_B n_a] = -[\Gamma^C_{AB}] n_C = \frac{1}{2} \left(\epsilon \kappa_{AB} - \kappa_{CA} n_B n^C - \kappa_{CB} n_A n^C \right) \tag{B.24}$$

This allows us to write

$$[K_{ab}] = [\nabla_B n_A] e_a^A e_b^B = \frac{\epsilon}{2} \kappa_{AB} e_a^A e_b^B \tag{B.25}$$

Collecting these results we obtain:

$$S_{ab} = -\frac{\epsilon}{8\pi G_N} \left([K_{ab}] - [K] h_{ab} \right). \tag{B.26}$$

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