Bott Towers and Torus Actions

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Chapter 1

Introduction

A manifold M is called a real Bott manifold if there is a sequence of $\mathbb{R}\mathsf{P}^1$-bundle

$$M = M_n \xrightarrow{\mathbb{R}\mathsf{P}^1} M_{n-1} \xrightarrow{\mathbb{R}\mathsf{P}^1} \cdots \xrightarrow{\mathbb{R}\mathsf{P}^1} M_1 \xrightarrow{\mathbb{R}\mathsf{P}^1} \{\text{pt}\} \tag{1.1}$$

such that for each $i \in \{1, \cdots, n\}$, $M_i \xrightarrow{\mathbb{R}\mathsf{P}^1} M_{i-1}$ is the projective bundle of the Whitney sum of a real line bundle and the trivial real line bundle over M_{i-1}. The sequence (1.1) is called a *real Bott tower* of depth n and it is a real analogue of a Bott tower introduced by Grossberg and Karshon in [12].

Among several characterizations by group actions, the Halperin-Carlsson conjecture is true for all real Bott manifold. The Halperin-Carlsson torus conjecture says that if there is an almost free torus action T^k on a closed n-manifold M, the following inequality holds:

$$2^k \leq \sum_{j=0}^{n} b_j. \tag{1.2}$$

Here $b_j = \operatorname{rank} H_j(M;\mathbb{Z})$ is the j-th Betti number of M. See [31] for details and the references therein, see also [14]. Another characterization of real Bott towers is that each $\mathbb{R}\mathsf{P}^1$-bundle is the Seifert fibration which is introduced by Conner-Raymond and any real Bott manifold M is diffeomorphic to a euclidean space form (Riemannian flat manifold).

In this thesis, we study a generalization of the real Bott tower from the viewpoint of fiberation. We shall construct the geometric fiber bundles in the sense of Bott tower.

In Chapter 2, we review the theory of Seifert fibration. Especially we state two fundamental theorems of the Seifert fibration. Namely they are *Realization theorem* end *Rigidity theorem* of Seifert manifold. See Theorem2.9 and subsection2.7.

In Chapter 3, we revisit the classical results of the Calabi construction of euclidean space forms with nonzero $b_1 = \operatorname{rank} H_1(M;\mathbb{Z})$ [4] and the Conner-Raymond injective torus actions [8]. Let T^k be a k-dimensional torus ($k \geq 1$).

Given an *effective* T^k-action on a closed manifold M, the orbit map at $x \in M$ is defined to be $\mathrm{ev}(t) = tx$ ($^\forall\, t \in T^k$). Put $\pi_1(T^k) = H_1(T^k;\mathbb{Z}) = \mathbb{Z}^k$ and $\pi_1(M) = \pi$. The map ev induces a homomorphism $\mathrm{ev}_\# : \mathbb{Z}^k \to \pi$ and a homomorphism $\mathrm{ev}_* : \mathbb{Z}^k \to H_1(M;\mathbb{Z})$. According to the definition of Conner-Raymond [8], if $\mathrm{ev}_\#$ is *injective*, the action (T^k, M) is said to be *injective*. (Refer to [23, Theorem 2.4.2, also Subsection 11.1] for the definition to be independent of the choice of the base point $x \in M$.) Classically it is known that $\mathrm{ev}_\#$ is injective for closed *aspherical* manifolds [7]. On the other hand, if $\mathrm{ev}_* : \mathbb{Z}^k \to H_1(M;\mathbb{Z})$ is injective, the T^k-action is said to be *homologically injective* [8]. In Section 3.1, we shall prove the following theorem to show that if $\mathrm{ev}_* : H_1(T^k;\mathbb{Z}) \to H_1(M;\mathbb{Z})$ is injective, then $\mathrm{ev}_* : H_i(T^k;\mathbb{Z}) \to H_i(M;\mathbb{Z})$ is also injective for $i \le k$.

Theorem 1.1. *If (T^k, M) is a homologically injective action on a closed n-manifold M, then*

$$_kC_j \le b_j \quad (j = 0, \ldots, k). \tag{1.3}$$

In particular the Halperin-Carlsson conjecture (3.1) *is true.*

The torus actions are known as homological injective actions on the following closed manifolds:

(1) Every effective T^k-action on a compact euclidean space form.

(2) Every holomorphic action of the complex torus $T^k_{\mathbb{C}}$ on a compact Kähler manifold.

As a consequence the Halperin-Carlsson conjectue is true.

(1) is true more generally for effective T^k-actions on compact *nonpositively curved manifolds.* (See [11].) For (2) this characterization for *holomorphic* torus actions is originally observed by Carrell [5]. (See also [23, Theorem, p.244].) In Section 3.2 we shall give a proof concerning the existence of torus actions common to both the Calabi theorem and the Conner-Raymond theorem as our motivation (cf. Theorem 1.2).

Theorem 1.2. *Let M be an n-dimensional compact euclidean space form. Suppose that* $\operatorname{rank} H_1(M) = k > 0$. *Then M admits a homologically injective T^k-action. Moreover* $\operatorname{rank} C(\pi) = k$.

Corollary 1.3. *There is no torus action on a compact Riemannian flat manifold with $b_1 = 0$.*

In Chapter 4, from the view point of the fibration, we introduce the generalized notion of real Bott tower, namely S^1*-fibred nilBott tower.* It is a sequence of an iterated *Seifert fiber bundle* with fiber a circle which terminates at a point.

$$M = M_n \xrightarrow{S^1} M_{n-1} \xrightarrow{S^1} \cdots \xrightarrow{S^1} M_1 \xrightarrow{S^1} \{\mathrm{pt}\}. \tag{1.4}$$

The top space M of (1.4) is called an S^1*-fibred nilBott manifold* of dimension n. We see easily that M turns out to be a closed aspherical manifold and each fiber bundle $M_i \xrightarrow{S^1} M_{i-1}$ induces a group extension of fundamental groups;

$$1 \longrightarrow \mathbb{Z} \to \pi_i \longrightarrow \pi_{i-1} \longrightarrow 1. \tag{1.5}$$

Associated to each group extension (1.5) there is an equivariant principal bundle:

$$\mathbb{R}{\to}X_i \xrightarrow{p_i} X_{i-1}. \tag{1.6}$$

Here X_i is the universal covering of M_i and put $\pi_1(M_i)=\pi_i$ and $\pi_1(M)=\pi$. In particular, *Seifert* fiber bundle $M_i \xrightarrow{S^1} M_{i-1}$ means that each π_i normalizes $\mathbb{R}$. Then we prove the following results.

Theorem 1.4. *Suppose that M is an S^1-fibred nilBott manifold.*

(I) If every cocycle of $H^2_\phi(\pi_{i-1},\mathbb{Z})$ which represents a group extension (1.5) *is* of finite order*, then M is diffeomorphic to a Riemannian flat manifold $\mathbb{R}^n/\Gamma$ which has a Seifert fibration $S^1{\to}\mathbb{R}^n/\Gamma{\longrightarrow}\hat{\mathbb{R}}^n/\hat{\Gamma}$.*

(II) If there exists a cocycle of $H^2_\phi(\pi_{i-1},\mathbb{Z})$ which represents a group extension (1.5) *is* of infinite order*, then M is diffeomorphic to an infranilmanifold N/Γ which has a Seifert fibration $S^1{\to}N/\Gamma{\longrightarrow}\hat{N}/\hat{\Gamma}$. In addition, M cannot be diffeomorphic to any Riemannian flat manifold.*

Up to 3 dimension, S^1-fibred nilBott manifold is classified.

Proposition 1.5. *The* 3 *dimensional S^1-fibred nilBott manifolds of finite type are those of $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$, $\mathcal{B}_4$.*

(Masuda and Lee [22] also proved the similar results.)

Proposition 1.6. *Any* 3*-dimensional S^1-fibred nilBott manifold of infinite type is either a Heisenberg nilmanifold $\mathsf{N}/\Delta(k)$ or an Heisenberg infranilmanifold $\mathsf{N}/\Gamma(k)$.*

Real Bott manifolds consist of $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{B}_1$, $\mathcal{B}_3$ among these $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$, $\mathcal{B}_4$. (Refer to the classification of 3-dimensional Riemannian flat manifolds by Wolf [36]. We quote the notations $\mathcal{G}_i$, $\mathcal{B}_i$ there.)

As in (1.5), a 3 dimensional S^1-fibred nilBott manifold M gives a group extension:

$$1{\longrightarrow}\mathbb{Z}{\to}\pi_3{\longrightarrow}\pi_2{\longrightarrow}1$$

where π_2 is the fundamental group of a Klein Bottle K or a torus T^2. Then this group extension gives a 2-cocycle in the group cohomology $H^2_\phi(\pi_2,\mathbb{Z})$ with a homomorphism $\phi:\pi_2{\to}\mathrm{Aut}(\mathbb{Z})=\{\pm1\}$. Conversely we have shown

Theorem 1.7. *Every cocycle of $H^2_\phi(\pi_2,\mathbb{Z})$ can be realized as a diffeomorphism class of a* 3*-dimensional S^1-fibred nilBott manifold.*

In Chapter 5 we shall introduce a notion of holomorphic torus-Bott tower to complex manifolds. It is thought of a complex version of S^1-fibred nilBott tower. Namely, a holomorphic torus-Bott tower is a sequence of holomorphic Seifert fiber bundles by complex torus fiber $T^1_{\mathbb{C}}$:

$$M = M_n \to M_{n-1} \to \ldots \to M_1 \to \{\mathrm{pt}\}. \tag{1.7}$$

The top space M of the tower (1.7) is said to be a *holomorphic torus-Bott manifold* of dimension $2n$. See Definition 5.1 of Section 5.1 more precisely. Inductively from (1.7), M turns out to be a closed aspherical manifold. Then it is shown that the fundamental group Γ of M is virtually nilpotent. Let $\mathrm{E}(N) = N \rtimes K$ be the semidirect product of a simply connected nilpotent Lie group N with a compact group K in which K is a maximal compact subgroup of the automorphism group $\mathrm{Aut}(N)$. When we forget a complex structure on M, it is proved that M is *diffeomorphic* to an infranilmanifold $N/\rho(\Gamma)$ where $\rho : \Gamma \to \mathrm{E}(N)$ is a discrete faithful representation. In particular, using *Seifert rigidity*, two holomorphic torus-Bott manifolds with isomorphic fundamental groups are *diffeomorphic*.

In Section 5.1 we introduce a notion of holomorphic torus-Bott tower and prove some topological results.

By a *holomorphic nilmanifold* we shall mean a complex nilmanifold with left invariant complex structure. Refer to [32] for the recent results of deformation of left invariant nilpotent Lie algebras. On the other hand, denote by $T^k_{\mathbb{C}}$ a complex k-dimensional torus. Recall the structure theorem from S. Murakami's classical result [30].

Theorem 1.8. *Let $T^1_{\mathbb{C}} \to Y \longrightarrow T^k_{\mathbb{C}}$ be a principal holomorphic torus bundle. Then Y is biholomorphic to a holomorphic nilmanifold N/Δ where N is a 2-step nilpotent Lie group with left invariant complex structure containing a discrete uniform subgroup Δ.*

To study the holomorphic rigidity of our holomorphic torus-Bott manifolds, we need to generalize this result to the case of holomorphic torus bundles more generally orbibundles over holomorphic infranilmanifolds (infranilorbifolds). We refer to [23], [5] for *holomorphic Seifert fibration.*We shall prove the following.

Theorem 1.9. *Let M be a $2n$-dimensional holomorphic torus-Bott manifold which is a holomorphic fiber bundle over $\hat{M}$ with fiber $T^1_{\mathbb{C}}$. Then M is biholomorphic to a holomorphic infranilmanifold N/Γ in which N/Γ has a holomorphic Seifert fibration $T^1_{\mathbb{C}} \to N/\Gamma \longrightarrow \hat{N}/\hat{\Gamma}$ such that $\hat{M}$ is biholomorphic to a holomorphic infranilmanifold $\hat{N}/\hat{\Gamma}$.*

The proof of this theorem is organized as follows: As the fundamental group of M is virtually nilpotent, the smooth classification implies that M is diffeomorphic to an infranilmanifold N/Γ. Even if N/Γ supports a complex structure, it does not follow that M is *biholomorphic* to N/Γ. However N has a central extension: $1 \to \mathbb{C} \to N \longrightarrow \hat{N} \to 1$ in this case. Assume inductively that $\hat{M}$ is *biholomorphic* to a holomorphic infranilmanifold $\hat{N}/\hat{\Gamma}$. Then we can find a nilpotent Lie group N' isomorphic to N which has the following properties. N' admits a $\mathrm{E}(N')$-invariant complex structure J for which the central extension

$1 \to \mathbb{C} \to N' \longrightarrow \hat{N} \to 1$ becomes a principal holomorphic bundle. Moreover N' is biholomorphic to the complex space $\mathbb{C}^n$, indeed this fact is due to Oka's principle which says that the universal covering (N', J) is biholomorphic as a principal holomorphic bundle with the product $(\mathbb{C} \times \hat{N}, J_0 \times \hat{J})$ inductively. Speculating on the cohomology exact sequence induced from a short exact sequence $1 \to \mathbb{Z}^2 \xrightarrow{i} \mathbb{C} \xrightarrow{j} T^1_{\mathbb{C}} \to 1$;

$$\cdots H^1_{\phi}(\hat{\Gamma}; \mathrm{hol}(\hat{N}, \mathbb{C})) \xrightarrow{j} H^1_{\phi}(\hat{\Gamma}; \mathrm{hol}(\hat{N}, T^1_{\mathbb{C}})) \xrightarrow{\delta} H^2_{\phi}(\hat{\Gamma}; \mathbb{Z}^2) \to \cdots,$$

we can show that M is biholomorohic to a holomorphic infranilmanifold N'/Γ' where $\Gamma' \leq \mathrm{E}_J(N')$ which is the semidirect product $N' \rtimes K'$ invariant under the complex structure J. There we construct a deformation N'/Γ' of N/Γ, see Theorem 1.1 below. Of course, N'/Γ' is nothing but N/Γ topologically.

In Section 5.1, we prove some topological results. In Section 5.2, we construct complex structures on holomorphic infranilmanifolds. In Section 5.3, we study holomorphic infranil-actions and holomorphic Seifert actions on holomorphic torus-Bott manifold M. In Section 5.4, we prove the following theorem which is a key tool to prove Theorem 1.9.

Theorem 1.10. *Let $(\Gamma, N,)$ be a holomorphic Seifert action as above. Then there exist a nilpotent Lie group N' and a discrete subgroup $\Gamma' \leq \mathrm{E}_J(N')$ for which the quotient N/Γ is biholomorphic to the holomorphic infranilmanifold N'/Γ'.*

In Section 5.6, we apply Theorem 1.9 to show the following.

Theorem 1.11. *A holomorphic torus-Bott manifold M of finite type is biholomorphic to a complex euclidean space form $\mathbb{C}^n/\Gamma$ with holonomy group $L(\Gamma)$ lying in $\prod_{i=1}^{n} H_i$ where H_i is either one of $\{1\}$, $\mathbb{Z}_2$, $\mathbb{Z}_4$, $\mathbb{Z}_6$.*

An example of finite type is a *Kähler Bott tower*, that is each M_i is a Kähler manifold such that $T^1_{\mathbb{C}} \to M_i \to M_{i-1}$ is a Kähler submersion. (See Section 5.6.2.) It is shown in Theorem 5.14 that every Kähler Bott manifold M is biholomorphic to a complex euclidean space form $\mathbb{C}^n/\Gamma$ of Theorem 5.11. In Section 5.7 we study holomorphic torus-Bott manifolds of infinite type. As the fundamental group of such a manifold is virtually nilpotent (but never virtually abelian), it is a non-Kähler manifold. It would be difficult to obtain a *holomorphic classification* of holomorphic torus-Bott manifolds of *infinite type*. We shall consider which non-Kähler geometric structure exists on holomorphic torus-Bott manifolds of infinite type. In Theorem 5.19, we provide two classes of geometric structure; (i) A $2n+2$-dimensional *locally homogeneous locally conformal Kähler* manifold $M = \mathbb{R} \times \mathcal{N}/\Gamma$ where $\mathcal{N}$ is the Heisenberg nilpotent Lie group and $\Gamma \leq \mathbb{R} \times (\mathcal{N} \rtimes \mathrm{U}(n))$ is a discrete uniform subgroup. (ii) A complex $2n+1$-dimensional locally homogeneous *complex contact* manifold $\mathcal{L}/\Gamma$ where $\mathcal{L} = \mathcal{L}_{2n+1}$ is a complex $2n+1$-dimensional complex nilpotent Lie group and

Γ is a discrete uniform subgroup of $\mathcal{L} \rtimes (\mathrm{Sp}(n) \cdot S^1)$. In particular, $\mathcal{L}_3$ is the Iwasawa nilpotent Lie group. Up to this stage we found the above two geometric structures on non-Kähler holomorphic torus-Bott manifolds of infinite type. In the future we propose to find other geometric structures.

Chapter 2

Seifert fiberation

2.1 Infrahomogeneous space

Let G be a simply connected Lie group, and $\mathrm{Aut}(G)$ denote the group of automorphisms of G onto itself. Put $\mathrm{A}(G) = G \rtimes \mathrm{Aut}(G)$. $\mathrm{A}(G)$ becomes a group;

$$(g, \alpha) \cdot (h, \beta) = (g \cdot \alpha(h), \alpha \cdot \beta)$$

$(g, h \in G, \alpha, \beta \in \mathrm{Aut}(G))$. $A(G)$ is called the affine group of G. Here, letting $X = G$, an affine action $(\mathrm{A}(G), X)$ is obtained as follows:

$$((g, \alpha), x) = g \cdot \alpha(x).$$

Let $H \subset \mathrm{Aut}(G)$ be a compact subgroup (for example, maximal compact subgroup, finite groups). Form a subgroup $\mathrm{E}(G) = G \rtimes H \subset \mathrm{A}(G)$. Consider the action $(\mathrm{E}(G), X)$. We note that if H is compact, then it is easy to check the following.

Lemma 2.1 (Proper action)**.** $(\mathrm{E}(G), X)$ *is a proper action.*

By Lemma 2.1, if $\pi \subset \mathrm{E}(G)$ is a discrete subgroup, we obtain a properly discontinuous action (π, X).

Definition 2.2. The quotient space X/π is said to be an infrahomogeneous orbifold. When π has no elements of finite order, π is said to be torsionfree, and X/π is called an infrahomogeneous manifold.

Example 2.3. (1) Taking the vector space $\mathbb{R}^n$ as G it gives the usual affine group $\mathrm{A}(n) = \mathbb{R}^n \rtimes \mathrm{GL}(n, \mathbb{R})$. If H is a maximal compact subgroup $\mathrm{O}(n)$ of $\mathrm{GL}(n, \mathbb{R})$, we have the euclidean group $\mathrm{E}(\mathbb{R}^n) = \mathbb{R}^n \rtimes \mathrm{O}(n)$. A discrete uniform subgroup π of $\mathrm{E}(n)$ is called a crystallographic group. If $\pi \subset \mathrm{E}(n)$ is a torsionfree crystallographic group, π is called a Bieberbach group. Moreover, the infrahomogeneous space $\mathbb{R}^n/\pi$ is an euclidean space form, that is a Riemannian flat manifold.

(2) When G is a simply connected nilpotent Lie group $\mathcal{N}$, for any torsionfree discrete uniform subgroup $\pi \subset \mathrm{E}(\mathcal{N})$, $\mathcal{N}/\pi$ is called an infranilmanifold.

We have the fundamental classical result for crystallographic groups.

Theorem 2.4 (Bieberbach first theorem)**.** *Let $\pi \subset \mathrm{E}(n)$ be a crystallographic group, then $\mathbb{R}^n \cap \pi \cong \mathbb{Z}^n$ and $\pi/\mathbb{R}^n \cap \pi$ is a finite group.*

The above theorem is extended to the almost crystallographic groups. See [10] for instance.

Theorem 2.5 (Auslander-Bieberbach theorem)**.** *Let π be a torsionfree discrete uniform subgroup of $\mathrm{E}(\mathcal{N})$, then $\mathcal{N} \cap \pi$ is a maximal normal nilpotent subgroup of π and $\pi/\mathcal{N} \cap \pi$ is a finite group.*

2.2 Nil Geometry

Let

$$1 \to \Delta \to \pi \to F \to 1 \tag{2.1}$$

be a group extension where π is a torsionfree group, Δ is a torsionfree finitely generated nilpotent group, and F is a finite group. By Mal'cev's *existence* theorem, there is a (simply connected) nilpotent Lie group $\mathcal{N}$ containing Δ as a discrete uniform subgroup. The rest of this section is to review the following realization theorem obtained in [18].

Theorem 2.6 (Realization)**.** *There exists a discrete faithful representation $\rho : \pi \to \mathrm{E}(\mathcal{N})$ such that $\rho|\Delta = \mathrm{id}$. In particular, $\mathcal{N}/\rho(\pi)$ is an infranil-manifold.*

In order to prove this theorem, we need several facts. So we shall prepare them in turn.

2.3 2-cocycle

(i) $\phi(\alpha)(\phi(\beta)(n)) = f(\alpha, \beta)\phi(\alpha\beta)(n)f(\alpha, \beta)^{-1}$

(ii) $f(\alpha, 1) = f(1, \alpha) = 1,$

(iii) $\phi(\alpha)(f(\beta, \gamma))f(\alpha, \beta\gamma) = f(\alpha, \beta)f(\alpha\beta, \gamma),$

where $n \in G$ and $\alpha, \beta, \gamma \in Q$. Then f defines a group E which is the product $G \times Q$ with the group law:

$$(n, \alpha)(m, \beta) = (n \cdot \phi(\alpha)(m) \cdot f(\alpha, \beta), \alpha\beta). \tag{2.1}$$

Then there is a ϕ-group extension $1 \to G \to E \xrightarrow{\nu} Q \to 1$ where $\nu(n, \alpha) = \alpha$ and the group E is denoted by $G \times_{(f,\phi)} Q$.

Conversely, given a group extension $1 \to G \to E \stackrel{\nu}{\longrightarrow} Q \to 1$, we can associate E with a ϕ- group extension. Choose a section $q : Q \to E$ ($\nu \circ q = \mathrm{id}$), and $q(1) = 1$. A function $\phi : Q \to \mathrm{Aut}(G)$ is defined to be

$$\phi(\alpha)(n) = q(\alpha) n q(\alpha)^{-1} \quad (\forall\, \alpha \in Q, \forall\, n \in G).$$

Both $q(\alpha\beta)$, $q(\alpha)q(\beta)$ are mapped to $\alpha\beta \in Q$, so there is an element $f(\alpha, \beta) \in G$ such that $f(\alpha, \beta) \cdot q(\alpha\beta) = q(\alpha)q(\beta)$. Then it is easily checked that $f : Q \times Q \to G$ satisfies the above (i) (ii) (iii).

Let $\mathrm{Opext}(Q, G, \phi)$ be the set of all congruence classes of ϕ- group extensions. Then an element $[f] \in \mathrm{Opext}(Q, G, \phi)$ is represented by an extension $1 \to G \to E \to Q \to 1$ with $E = G \times_{(f,\phi)} Q$. It is easy to check that $[f_1] = [f_2] \in \mathrm{Opext}(Q, A, \phi)$ if and only if there is a function $\lambda : Q \to \mathcal{C}(G)$ such that

$$f_1(\alpha, \beta) = \delta^1 \lambda(\alpha, \beta) \cdot f_2(\alpha, \beta) \quad (\forall\ \alpha, \beta \in Q). \tag{2.2}$$

Here $\mathcal{C}(G)$ is the center of G and δ^1 is defined by

$$\delta^1 \lambda(\alpha, \beta) = \phi(\alpha)(\lambda(\beta))\lambda(\alpha)\lambda(\alpha\beta)^{-1}.$$

For simplicity, we write it as $f_1 = \delta^1 \lambda \cdot f_2$.

In particular, when G is an abelian group A, $\phi : Q \to \mathrm{Aut}(A)$ is a homomorphism and hence A is a Q-module. So there is the group cohomology $H^2_\phi(Q, A)$ and f is a 2-cocycle by (iii), i.e. $[f] \in H^2_\phi(Q, A)$. Therefore any extension $1 \to A \to E \to Q \to 1$ corresponds to a cocycle $[f] \in H^2_\phi(Q, A)$. It is easy to check the following.

Proposition 2.7. *Suppose that A is an abelian group. Then there is a one-to-one correspondence between $H^2_\phi(Q, A)$ and $\mathrm{Opext}(Q, A, \phi)$.*

Remark 2.8. *Suppose $Q = F$ is a finite group and $f : F \times F \to \mathbb{R}^n$ is a 2-cocycle relative to $\phi : F \to \mathrm{Aut}(\mathbb{R}^n)$. Put $h : F \to \mathbb{R}^n$;*

$$h(\alpha) = \sum_{\tau \in F} f(\alpha, \tau). \tag{2.3}$$

Then

$$\begin{aligned} \delta^1 h(\alpha, \beta) &= \phi(\alpha)(h(\beta)) - h(\alpha\beta) + h(\alpha) \\ &= \sum_{\tau \in F} \phi(\alpha)(f(\beta, \tau)) - \sum_{\tau \in F} f(\alpha\beta, \tau) + \sum_{\tau \in F} f(\alpha, \tau) \\ &= \sum_{\tau \in F} (f(\alpha\beta, \tau) - f(\alpha, \beta\tau) + f(\alpha, \beta)) - \sum_{\tau \in F} f(\alpha\beta, \tau) + \sum_{\tau \in F} f(\alpha, \tau) \\ &= |F| f(\alpha, \beta) \end{aligned}$$

Thus $\delta^1 \frac{1}{|F|} h = f$. It implies that

$$H^2_\phi(F; \mathbb{R}^n) = 0. \tag{2.4}$$

2.4 Pushout

Let π, Δ and $\mathcal{N}$ be as before and $1 \to \Delta \to \pi \to Q \to 1$ a group extension which is represented by $[f] \in \mathrm{Opext}(Q, \Delta, \phi)$. Given a function $\phi : Q \to \mathrm{Aut}(\Delta)$, Mal'cev's unique extension theorem implies that each automorphism $\phi(\alpha) : \Delta \to \Delta$ extends uniquely to an automorphism $\bar{\phi}(\alpha) : \mathcal{N} \to \mathcal{N}$. In particular, this gives a correspondence $\bar{\phi} : Q \to \mathrm{Aut}(\mathcal{N})$. Note that it is not necessarily a homomorphism. In general it satisfies

$$\bar{\phi}(\alpha)(\bar{\phi}(\beta)(x)) = f(\alpha, \beta)\bar{\phi}(\alpha\beta)(x)f(\alpha, \beta)^{-1} \ (x \in \mathcal{N}). \tag{2.1}$$

Then the "pushout" $\pi\mathcal{N} = \{(x, \alpha) \mid x \in \mathcal{N}, \alpha \in Q\}$ can be constructed. Its group law is defined by $(x, \alpha) \cdot (y, \beta) = (x\bar{\phi}(\alpha)(y)f(\alpha, \beta), \alpha\beta)$;

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{N} & \longrightarrow & \pi\mathcal{N} & \longrightarrow & Q & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \| & & \\ 1 & \longrightarrow & \Delta & \longrightarrow & \pi & \longrightarrow & Q & \longrightarrow & 1. \end{array} \tag{2.2}$$

This group (extension) $\pi\mathcal{N}$ is also represented by $[f] \in \mathrm{Opext}(Q, \mathcal{N}, \bar{\phi})$.

2.5 Existence of the Seifert construction

Let W be a contractible smooth manifold. Suppose that a group Q acts properly discontinuously on W such that the quotient space W/Q is compact. Given a group extension:

$$1 \longrightarrow \Delta \longrightarrow \pi \stackrel{\nu}{\longrightarrow} Q \longrightarrow 1, \tag{2.1}$$

we shall show that there is an action of π on $\mathcal{N} \times W$ which is compatible with the left translations of $\mathcal{N}$. Let $\mathrm{Diff}(\mathcal{N} \times W)$ be the group of all diffeomorphisms of $\mathcal{N} \times W$ onto itself. $\mathcal{N}$ is a subgroup of $\mathrm{Diff}(\mathcal{N} \times W)$ via an embedding: $l(n)(m, \alpha) = (nm, \alpha)$.

We denote $\mathrm{Diff}^{\mathrm{F}}(\mathcal{N} \times W)$ the normalizer of $l(\mathcal{N})$ in $\mathrm{Diff}(\mathcal{N} \times W)$. Let $\mathrm{Map}(W, \mathcal{N})$ be the set of smooth maps from W into $\mathcal{N}$. Then $\mathrm{Diff}^{\mathrm{F}}(\mathcal{N} \times W)$ coincides with the group $\mathrm{Map}(W, \mathcal{N}) \rtimes (\mathrm{Aut}(\mathcal{N}) \times \mathrm{Diff}(W))$ with the group law:

$$(\lambda_1, g_1, h_1)(\lambda, g, h) = ((g_1 \circ \lambda \circ h_1^{-1}) \cdot \lambda_1, g_1 g, h_1 h)$$

and

$$(\lambda, g, h)(x, w) = (g(x) \cdot \lambda(hw), hw) \tag{2.2}$$

for $(x, w) \in \mathcal{N} \times W$, defines an action on $\mathcal{N} \times W$. See [18].

We call the set (Δ, π, Q, W) a smooth data for the group extension (2.1). The following theorem is obtained in [18].

Theorem 2.9. *For any smooth data (Δ, π, Q, W), there exists a continuous homomorphism $\Psi : \pi \to \mathrm{Diff}^{\mathrm{F}}(\mathcal{N} \times W)$ such that $\Psi|_\Delta = l$.*

Ψ is called the Seifert construction of the smooth data (Δ, π, Q, W). We shall review the proof of [18].

Proof. Using the pushout (2.1) in § 2.4, if we show that there exists a continuous homomorphism $\bar{\Psi} : \pi\mathcal{N} \to \mathrm{Diff}^{\mathrm{F}}(\mathcal{N} \times W)$ such that $\bar{\Psi}|_{\mathcal{N}} = l$, then a Seifert construction $\Psi : \pi \to \mathrm{Diff}^{\mathrm{F}}(\mathcal{N} \times W)$ is obtained as a restriction. Suppose there exists a $\bar{\Psi}$. For $(n, \alpha) \in \pi\mathcal{N}$, if we put $\bar{\Psi}(1, \alpha) = (\lambda, g, h) \in \mathrm{Map}(W, \mathcal{N}) \rtimes (\mathrm{Aut}(\mathcal{N}) \times \mathrm{Diff}(W))$, then $\bar{\Psi}(n, \alpha) = \ell(n)\bar{\Psi}(1, \alpha) = (n \cdot \lambda, g, h)$. Then it is easy to check that

$$\bar{\Psi}(n, \alpha) = (n \cdot \lambda(\alpha), \mu(n) \circ \bar{\phi}(\alpha), \alpha)$$

where $\lambda : Q \to \mathrm{Map}(W, \mathcal{N})$ satisfies

$$f(\alpha, \beta) = (\bar{\phi}(\alpha) \circ \lambda(\beta) \circ \alpha^{-1}) \cdot \lambda(\alpha) \cdot \lambda(\alpha\beta)^{-1} \quad (\alpha, \beta \in Q), \tag{2.3}$$

where f be a function representing the group extension (2.1). Therefore to guarantee the existence of such $\bar{\Psi}$, we have only to find a map λ satisfying the condition (2.3). Remark that if $\mathcal{N}$ is a vector space V then $\mathrm{Map}(W, V)$ is a topological group with Q-action by

$$\alpha \cdot \lambda(w) = \bar{\phi}(\alpha)(\lambda(\alpha^{-1}w)). \tag{2.4}$$

So we have a group cohomology $H^2_{\bar{\phi}}(Q, \mathrm{Map}(W, V))$. Then note that

$$H^2_{\bar{\phi}}(Q, \mathrm{Map}(W, V)) = 0$$

for any vector space V. This vanishing is obtained by using Shapiro's lemma. (See [23], page 251, Lemma 8.4.)

By induction, we suppose that the statement is true for any nilpotent Lie group whose dimension is less than dim $\mathcal{N}$. Let $\mathcal{C}$ be the center of $\mathcal{N}$ and put $\mathcal{N}_1 = \mathcal{N}/\mathcal{C}$, $\pi\mathcal{N}_1 = \pi\mathcal{N}/\mathcal{C}$. Consider the group extension

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{N} & \longrightarrow & \pi\mathcal{N} & \xrightarrow{\nu} & Q & \longrightarrow & 1 \\ & & \downarrow p & & \downarrow p & & \| & & \\ 1 & \longrightarrow & \mathcal{N}_1 & \longrightarrow & \pi\mathcal{N}_1 & \xrightarrow{\nu_1} & Q & \longrightarrow & 1, \end{array} \tag{2.5}$$

with a section $q_1 = p \circ q$ of ν_1 where q is a section to ν. The section q_1 determines $f_1 : Q \times Q \to \mathcal{N}_1$ and $\bar{\phi}_1 : Q \to \mathrm{Aut}(\mathcal{N}_1)$ as in §2.3. We suppose by induction on the dimension of $\mathcal{N}$ that there exists $\lambda_1 : Q \to \mathrm{Map}(W, \mathcal{N}_1)$ such that

$$f_1(\alpha, \beta) = (\bar{\phi}_1(\alpha) \circ \lambda_1(\beta) \circ \alpha^{-1}) \cdot \lambda_1(\alpha) \cdot \lambda_1(\alpha\beta)^{-1}$$

Choose any lift $\lambda' : Q \to \mathrm{Map}(W, \mathcal{N})$ of λ_1 so that $\lambda_1 = p \circ \lambda'$. Put

$$g(\alpha, \beta) = (\bar{\phi}(\alpha) \circ \lambda'(\beta) \circ \alpha^{-1}) \cdot \lambda'(\alpha) \cdot \lambda'(\alpha\beta)^{-1},$$

then there exists an element $c(\alpha, \beta) \in \mathrm{Map}(W, \mathcal{C})$ such that

$$f(\alpha, \beta) = c(\alpha, \beta) \cdot g(\alpha, \beta).$$

Since both f and g satisfy (iii) in §2.3, c is also a 2-cocycle. That is $[c] \in H^2_{\bar{\phi}}(Q, \mathrm{Map}(W, \mathcal{C}))$ which vanishes because $\mathcal{C}$ is a vector space. So there is a function $\eta : Q \rightarrow \mathrm{Map}(W, \mathcal{C})$ such that

$$c(\alpha, \beta) = (\bar{\phi}_1(\alpha) \circ \eta(\beta) \circ \alpha^{-1}) \cdot \eta(\alpha) \cdot \eta(\alpha\beta)^{-1}.$$

Put $\lambda = \eta \cdot \lambda' : Q \rightarrow \mathrm{Map}(W, \mathcal{N})$, then λ satisfies (2.3). □

Remark 2.10. *Let $1 \rightarrow \mathbb{Z} \rightarrow \pi_i \longrightarrow \pi_{i-1} \rightarrow 1$ be a group extension as in (1.5). Then π_i acts on the universal cover X_i of M_i as freely. Assume that $\Psi_i : \pi_i \rightarrow \mathrm{Diff}(X_i)$ is the representation homomorphism for this action (π_i, X_i), then $\Psi_i : \pi_i \rightarrow \Psi_i(\pi_i)$ is the Seifert construction of the smooth data $(\mathbb{Z}, \pi_i, \pi_{i-1}, X_{i-1})$.*

2.6 Infranilmanifold

Let $(\Delta, \pi, F, \{pt\})$ be a smooth data with finite group F and f a function representing the given group extension $1 \rightarrow \Delta \rightarrow \pi \rightarrow F \rightarrow 1$. In the same way as the proof of Theorem 2.9, we can obtain a 1-chain $\chi : F \rightarrow \mathcal{N}$ such that $f = \delta^1 \chi$;

$$f(\alpha, \beta) = \bar{\phi}(\alpha)(\chi(\beta))\chi(\alpha)\chi(\alpha\beta)^{-1} \ (\alpha, \beta \in F). \tag{2.1}$$

We shall repeat the construction of χ for our use. Let $\bar{f} : F \times F \rightarrow \mathcal{N}/\mathcal{C}$ be a function which represents $1 \rightarrow \mathcal{N}_1 \rightarrow \pi\mathcal{N}_1 \rightarrow F \rightarrow 1$, then we suppose $\bar{f} = \delta^1 \bar{\lambda}$ for some function $\bar{\lambda} : F \rightarrow \mathcal{N}/\mathcal{C}$ by induction. Choose a lift $\lambda : F \rightarrow \mathcal{N}$ of $\bar{\lambda}$. It is easy to see the function $g = f \cdot (\delta^1 \lambda)^{-1}$ is a cocycle lying in $\mathcal{C}$, that is $[g] \in H^2_{\bar{\phi}}(F, \mathcal{C})$. As $H^2_{\bar{\phi}}(F, \mathcal{C}) = 0$ from (2.4), there is a map $\mu : F \rightarrow \mathcal{C}$ such that $\delta^1 \mu = g$. Then $f = \delta^1(\mu \cdot \lambda)$ and the 1-chain χ denoted by $\mu \cdot \lambda$.

Now define an automorphism of $\mathcal{N}$ $h(\alpha) : \mathcal{N} \rightarrow \mathcal{N}$ for each $\alpha \in F$ to be

$$h(\alpha)(x) = \chi(\alpha)^{-1} \cdot \bar{\phi}(\alpha)(x) \cdot \chi(\alpha) \ (x \in \mathcal{N}).$$

Using (2.1), we can prove that $h(\alpha\beta) = h(\alpha)h(\beta)$ for $\alpha, \beta \in F$. Therefore $h : F \rightarrow \mathrm{Aut}(\mathcal{N})$ is a homomorphism. Since $\mathrm{Aut}(\mathcal{N})$ is a noncompact Lie group, it has a maximal compact group $\mathcal{K}$. Then the finite subgroup $h(F)$ is conjugate to a subgroup of $\mathcal{K}$. We can assume that $h(F) \subset \mathcal{K}$.

Define $\rho : \pi \rightarrow E(\mathcal{N})$ to be

$$\rho((n, \alpha)) = (n\chi(\alpha), h(\alpha)) \ (n \in \Delta, \alpha \in F). \tag{2.2}$$

It is easy to check that ρ is a homomorphism. We define an action of π on $\mathcal{N}$ to be

$$((n, \alpha), x) = \rho(n, \alpha)(x) = n\bar{\phi}(\alpha)(x)\chi(\alpha) \ \ ((n, \alpha) \in \pi). \tag{2.3}$$

Theorem 2.6 is obtained by the following proposition.

Proposition 2.11. *The action $(\pi, \mathcal{N})$ is a properly discontinuous free action. In particular, ρ is a faithful representation.*

Proof. First note that $\rho|_\Delta = id$, so Δ is contained in $\rho(\pi)$. Since Δ acts as left translations of $\mathcal{N}$ from (2.2), it acts properly discontinuously and freely. Moreover since Δ is a finite index subgroup of $\rho(\pi)$ from (2.1), $\rho(\pi)$ acts properly discontinuously on $\mathcal{N}$.

Let $(n,\alpha) \in \operatorname{Ker}\rho$ be an element of π. Then $((n,\alpha),x) = x$ $(\forall x \in \mathcal{N})$ by (2.3). As π acts properly discontinuously, (n,α) is of finite order. On the other hand, π is torsionfree, we obtain $(n,\alpha) = 1$ and so ρ is faithful. □

The following remark shows that ρ is a Seifert construction (cf. Theorem 2.9).

Remark 2.12. *Let* $\mathrm{A}(\mathcal{N})^*$ *be a group which is the product* $\mathcal{N} \times \mathrm{Aut}(\mathcal{N})$ *with the group law:*

$$(n,\alpha)\cdot(m,\beta) = (\alpha(m)\cdot n, \alpha\cdot\beta)$$

for $n, m \in \mathcal{N}$, *and* $\alpha, \beta \in \mathrm{Aut}(\mathcal{N})$. *The action* $(\mathrm{A}(\mathcal{N})^*, \mathcal{N})$ *is obtained as follows:*

$$((n,\alpha),x) = \alpha(x)\cdot n$$

for $x \in \mathcal{N}$. *Then there is an isomorphism* $\delta : \mathrm{A}(\mathcal{N})^* \to \mathrm{A}(\mathcal{N})$ *defined by* $\delta(n,\alpha) = (n, \mu(n^{-1})(\alpha))$. *Here* $\mu : \mathcal{N} \to \mathrm{Aut}(\mathcal{N})$ *denote the conjugation homomorphism:*

$$\mu(n)(x) = nxn^{-1}.$$

It is easily checked that

$$((n,\alpha),x) = (\delta(n,\alpha),x)$$

This shows that the affine action $(\mathrm{A}(\mathcal{N}), \mathcal{N})$ *coincides with the above action* $(\mathrm{A}(\mathcal{N})^*, \mathcal{N})$.

Remark 2.13. *There is a commutative diagram.*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathcal{N} & \longrightarrow & \mathrm{E}(\mathcal{N}) & \longrightarrow & \mathcal{K} & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \cup & & \\
1 & \longrightarrow & \mathcal{N}\cap\rho(\pi) & \longrightarrow & \rho(\pi) & \longrightarrow & H & \longrightarrow & 1.
\end{array} \qquad (2.4)$$

By the theorem of Auslander-Bieberbach, $\mathcal{N}\cap\rho(\pi)$ *is a maximal normal nilpotent subgroup of* $\rho(\pi)$. *Note that* $\Delta \subset \mathcal{N}\cap\rho(\pi)$, *so if* Δ *is maximal, then* $\Delta = \mathcal{N}\cap\rho(\pi)$.

2.7 Seifert rigidity

Let Δ_i be a discrete uniform subgroup of a simply connected nilpotent Lie group $\mathcal{N}_i$ $(i = 1,2)$ respectively. Let Ψ_1, Ψ_2 be Seifert constructions for smooth data $(\Delta_1, \pi_1, Q_1, W_1)$, $(\Delta_2, \pi_2, Q_2, W_2)$ respectively. Suppose there exists an isomorphism $\theta : \pi_1 \to \pi_2$ inducing isomorphisms $\bar\theta : \Delta_1 \to \Delta_2$, $\hat\theta : Q_1 \to Q_2$. Furthermore (Q_1, W_1) is equivariantly diffeomorphic to (Q_2, W_2). with respect to $\hat\theta$. Then *Seifert rigidity* shows that $(\Psi_2(\pi_1), \mathcal{N}_1 \times W_1)$ is equivariantly diffeomorphic to $(\Psi_1(\pi_2), \mathcal{N}_2 \times W_2)$. See [18], page 441.

Chapter 3

Injective Torus actions

3.1 The Halperin-Carlsson conjecture on homologically injective actions

The Halperin-Carlsson torus conjecture says that if there is an almost free torus action T^k on a closed n-manifold M, the following inequality holds:

$$2^k \leq \sum_{j=0}^{n} b_j. \tag{3.1}$$

Here $b_j = \operatorname{rank} H_j(M;\mathbb{Z})$ is the j-th Betti number of M. See [31] for details and the references therein, see also [14].

Let T^k be a k-dimensional torus ($k \geq 1$). Given an *effective* T^k-action on a closed manifold M, the orbit map at $x \in M$ is defined to be $\mathrm{ev}(t) = tx$ ($^\forall\, t \in T^k$). Put $\pi_1(T^k) = H_1(T^k;\mathbb{Z}) = \mathbb{Z}^k$ and $\pi_1(M) = \pi$. The map ev induces a homomorphism $\mathrm{ev}_\# : \mathbb{Z}^k \to \pi$ and a homomorphism $\mathrm{ev}_* : \mathbb{Z}^k \to H_1(M;\mathbb{Z})$. According to the definition of Conner-Raymond [8], if $\mathrm{ev}_\#$ is *injective*, the action (T^k, M) is said to be *injective*. (Refer to [23, Theorem 2.4.2, also Subsection 11.1] for the definition to be independent of the choice of the base point $x \in M$.) Classically it is known that $\mathrm{ev}_\#$ is injective for closed *aspherical* manifolds [7]. On the other hand, if $\mathrm{ev}_* : \mathbb{Z}^k \to H_1(M;\mathbb{Z})$ is injective, the T^k-action is said to be *homologically injective* [8].

Theorem 3.1. *If (T^k, M) is a homologically injective action on a closed n-manifold M, then*

$$_kC_j \leq b_j \quad (j = 0, \ldots, k). \tag{3.2}$$

In particular the Halperin-Carlsson conjecture (3.1) *is true.*

Proof. A homologically injective T^k-action on M induces a central group extension:

$$1 \to \mathbb{Z}^k \to \pi \longrightarrow Q \to 1. \tag{3.3}$$

As the cocycle $[f] \in H^2(Q;\mathbb{Z}^k)$ representing (3.3) has finite order (cf. [23, Lemma11.6.5.]), there exists an integer ℓ such that $\ell \cdot f = \delta^1 \tilde{\lambda}$ for some function $\tilde{\lambda} : Q \to \mathbb{Z}^k$. If we put

$$\lambda = \frac{\tilde{\lambda}}{\ell} : Q \to \mathbb{R}^k, \tag{3.4}$$

then $f = \delta^1 \lambda$. Let $\mathbb{R}^k$ be the universal covering of T^k. Then $\mathbb{R}^k$ acts properly and freely on the universal covering $\tilde{M}$ such that $\tilde{M} = \mathbb{R}^k \times W$ where $W = \tilde{M}/\mathbb{R}^k$ is a simply connected smooth manifold. Then it follows from [7] that the $\pi-$action on $\mathbb{R}^k \times W$ is equivalent with

$$(n,\alpha)(x,w) = (n + x + \lambda(\alpha), \alpha w) \ \ ({}^{\forall}\,(n,\alpha) \in \pi, {}^{\forall}\,(x,w) \in \mathbb{R}^k \times W). \tag{3.5}$$

By [23, Lemma11.6.6], π has a splitting subgroup $\pi' = \mathbb{Z}^k \times Q'$. There exists an element $h \in \mathrm{Map}(W, \mathbb{R}^k)$ such that $\tilde{G} : \mathbb{R}^k \times W \to \mathbb{R}^k \times W$ defined by $\tilde{G}(x,w) = (x + h(w), w)$ is an equivariant diffeomorphism with respect to the π'-action of (3.5) and the product action of $\mathbb{Z}^k \times Q'$ (cf. [23, Theorem 7.3.2]).

Putting $\mathbb{R}^k \times W/\pi' = T^k \underset{Q'}{\times} W$ as a quotient space, $\tilde{G}$ induces a diffeomorphism $G : T^k \underset{Q'}{\times} W \to T^k \times W/Q'$. Let $q : T^k \times W \to T^k \underset{Q'}{\times} W$ be the covering map $(q(t,w) = [t,w])$. Then

$$G \circ q(t,w) = G([t,w]) = (t \exp 2\pi \mathbf{i} h(w), [w]). \tag{3.6}$$

Noting (3.5), π induces an action of Q on $\tilde{M}/\mathbb{Z}^k = T^k \times W$ such that

$$\alpha(t,w) = (t \exp 2\pi \mathbf{i} \lambda(\alpha), \alpha w) \ \ ({}^{\forall}\,\alpha \in Q). \tag{3.7}$$

$F = Q/Q'$ has an induced action on $T^k \underset{Q'}{\times} W$ by $\hat{\alpha}[t,w] = [t \exp 2\pi \mathbf{i} \lambda(\alpha), \alpha w]$ $({}^{\forall}\,\hat{\alpha} \in F)$ which gives rise to a covering map:

$$F \to T^k \underset{Q'}{\times} W \xrightarrow{\nu} T^k \underset{Q}{\times} W = M. \tag{3.8}$$

For any $\alpha \in Q$, consider the commutative diagram:

$$\begin{array}{ccc} H_j(T^k \times W) & \xrightarrow{\alpha_*} & H_j(T^k \times W) \\ \downarrow{\scriptstyle q_*} & & \downarrow{\scriptstyle q_*} \\ H_j(T^k \times_{Q'} W) & \xrightarrow{\hat{\alpha}_*} & H_j(T^k \times_{Q'} W) \end{array} \tag{3.9}$$

in which $H_j(T^k) \otimes H_0(W) \leq H_j(T^k \times W)$. By the formula (3.7), the Q-action on the T^k-summand is a translation by $\exp 2\pi \mathbf{i} \lambda(\alpha) \in T^k$ so the homology action α_* on $H_j(T^k) \otimes H_0(W)$ is trivial. If $H_j(T^k \underset{Q'}{\times} W)^F$ denotes the subgroup left fixed under the homology action for every element $\hat{\alpha} \in F$, it follows

$$q_*(H_j(T^k) \otimes H_0(W)) \leq H_j(T^k \underset{Q'}{\times} W)^F. \tag{3.10}$$

Using the transfer homomorphism (see [3]), ν of (3.8) induces an isomorphism:

$$\nu_* : H_j(T^k \underset{Q'}{\times} W; \mathbb{Q})^F \longrightarrow H_j(M; \mathbb{Q}).$$

Specifically, $\nu_* : q_*(H_j(T^k; \mathbb{Q}) \otimes H_0(W; \mathbb{Q})) \to H_j(M; \mathbb{Q})$ is injective.

On the other hand, let $q' : W \to W/Q'$ be the projection $q'(w) = [w]$. Define a homotopy $\Psi_\theta : T^k \times W \to T^k \times W/Q'$ ($\theta \in [0, 1]$) to be

$$\Psi_\theta(t, w) = (t \exp 2\pi\mathbf{i}(\theta \cdot h(w)), [w]).$$

Then $\Psi_0 = \mathrm{id} \times q' \simeq G \circ q$ from (3.6). As $G_* \circ q_* = \mathrm{id} \times q'_* : H_j(T^k; \mathbb{Q}) \otimes H_0(W; \mathbb{Q}) \to H_j(T^k; \mathbb{Q}) \otimes H_0(W/Q'; \mathbb{Q})$ is obviously isomorphic, it implies that $q_* : H_j(T^k; \mathbb{Q}) \otimes H_0(W; \mathbb{Q}) \longrightarrow H_j(T^k \underset{Q'}{\times} W; \mathbb{Q})$ is injective. If $p = \nu \circ q : T^k \times W \to M$ is the projection, then $p_* : H_j(T^k; \mathbb{Q}) \otimes H_0(W; \mathbb{Q}) \longrightarrow H_j(M; \mathbb{Q})$ is also injective. This implies

$$_kC_j \leq b_j \ \ (j = 0, \dots, k).$$

□

3.2 Calabi construction and torus actions

In [8, § 7], Conner and Raymond have stated that Calabi's theorem [4] shows the existence of a T^k-action with $k = \operatorname{rank} H_1(M; \mathbb{Z}) > 0$. We agree that the Calabi construction induces such actions. However when we look at a proof of Calabi's theorem ([36, p.125]), it is not easy to find such T^k-actions for a given compact euclidean space form with nonzero b_1. Regarding the proof, let $\nu : \pi \to \mathbb{Z}^k$ be the projection onto the direct summand $\mathbb{Z}^k$ of $H_1(M; \mathbb{Z})$. Then there is a group extension $1 \to \Gamma \to \pi \to \mathbb{Z}^k \to 1$ in which Γ is the fundamental group of a euclidean space form $M^{n-k} = \mathbb{R}^{n-k}/\Gamma$. In general an element $\gamma \in \pi$ has the form

$$\left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{pmatrix} A & B \\ 0 & I \end{pmatrix} \right) \quad (a \in \mathbb{R}^{n-k}, b \in \mathbb{R}^k).$$

The holonomy group $L(\pi) = \{\begin{pmatrix} A & B \\ 0 & I \end{pmatrix}\}$ does not necessarily leave the subspace $0 \times \mathbb{R}^k$ invariant. (In particular, $\mathbb{Z}^n \cap (0 \times \mathbb{R}^k)$ is not necessarily uniform in $0 \times \mathbb{R}^k$.) So we have to find another decomposition to get a T^k-action on M.

Lemma 3.2. *Let π be an n-dimensional Bieberbach group such that the rank of $\pi/[\pi, \pi]$ is positive $k > 0$. Then there exists a faithful representation $\rho : \pi \to \mathrm{E}(n)$ such that the euclidean space form $\mathbb{R}^n/\rho(\pi)$ admits an effective T^k-action.*

Proof. By the hypothesis, there is a group extension $1 \to \Gamma \to \pi \xrightarrow{\nu} \mathbb{Z}^k \to 1$. Since π is a Bieberbach group, it admits a maximal normal finite index abelian subgroup $\mathbb{Z}^n$. Put $\nu(\mathbb{Z}^n) = B$. Consider the commutative diagram of the group extensions:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Gamma & \xrightarrow{\iota} & \pi & \xrightarrow{\nu} & \mathbb{Z}^k & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & \Gamma\cap\mathbb{Z}^n & \xrightarrow{\iota} & \mathbb{Z}^n & \xrightarrow{\nu} & B & \longrightarrow & 1. \end{array} \tag{3.1}$$

Since $\pi/\mathbb{Z}^n \stackrel{\hat{\nu}}{\to} \mathbb{Z}^k/B$ is surjective, B is a free abelian subgroup of rank k. By the embedding $\hat{\iota} : \Gamma/\Gamma\cap\mathbb{Z}^n \leq \pi/\mathbb{Z}^n$, $\Gamma\cap\mathbb{Z}^n$ is a finite index subgroup of Γ. It follows easily that $\Gamma\cap\mathbb{Z}^n$ is a maximal normal abelian subgroup of Γ. We may put $\Gamma\cap\mathbb{Z}^n = \mathbb{Z}^{n-k}$ so that $\mathbb{Z}^n = \mathbb{Z}^{n-k}\times B$. Putting $Q = \pi/\mathbb{Z}^{n-k}$ and $F = \Gamma/\mathbb{Z}^{n-k}$ is a finite group, we have the group extensions:

$$1 \longrightarrow \mathbb{Z}^{n-k} \xrightarrow{i} \pi \xrightarrow{\mu} Q \longrightarrow 1, \tag{3.2}$$

where

$$1 \longrightarrow F \xrightarrow{\hat{\iota}} Q \xrightarrow{\hat{\nu}} \mathbb{Z}^k \longrightarrow 1 \tag{3.3}$$

is also a group extension. As $\mu|_{\mathbb{Z}^n} = \nu$ from (3.1), (3.2) has the commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z}^{n-k} & \xrightarrow{\iota} & \pi & \xrightarrow{\mu} & Q & \longrightarrow & 1 \\ & & || & & \uparrow & & \uparrow\iota' & & \\ 1 & \longrightarrow & \mathbb{Z}^{n-k} & \xrightarrow{\iota} & \mathbb{Z}^n & \xrightarrow{\mu} & B & \longrightarrow & 1 \end{array} \tag{3.4}$$

Since (3.2) is not necessarily central, let $\phi : Q \to \mathrm{Aut}(\mathbb{Z}^{n-k})$ be the conjugation homomorphism. As $\mathbb{Z}^n = \mathbb{Z}^{n-k}\times B$ and $\phi|_B = \mathrm{id}$, if $[f] \in H^2_\phi(Q;\mathbb{Z}^{n-k})$ is the representative cocycle of (3.2), then $\iota'^*[f] = 0$ in $H^2(B;\mathbb{Z}^{n-k})$. Then $[f]$ is a torsion because $\tau\circ\iota'^* = (Q:B) : H^2_\phi(Q;\mathbb{Z}^k)\to H^2_\phi(Q;\mathbb{Z}^k)$ still holds for the transfer homomorphism $\tau : H^2(B;\mathbb{Z}^{n-k})\to H^2_\phi(Q;\mathbb{Z}^{n-k})$. (Compare [3].) Therefore there is a function $\lambda : Q\to\mathbb{R}^{n-k}$ similarly as in (3.4) such that

$$\ell\cdot\lambda(Q) \leq \mathbb{Z}^{n-k}. \tag{3.5}$$

Let $\mathbb{Z}^k$ act on $\mathbb{R}^k$ by translations and by (3.3) $\hat{\nu} : Q\to\mathbb{Z}^k \leq \mathrm{E}(k)$ defines a properly discontinuous action of Q on $\mathbb{R}^k$;

$$\alpha(w) = \hat{\nu}(\alpha) + w \ \ ({}^\forall\, \alpha\in Q, {}^\forall\, w\in\mathbb{R}^k).$$

Let $\gamma = (n,\alpha)\in\pi$. By [7] (cf. [23]), we have a properly discontinuous action of π on $\mathbb{R}^n = \mathbb{R}^{n-k}\times\mathbb{R}^k$:

$$\gamma\begin{bmatrix} x \\ w \end{bmatrix} = \begin{bmatrix} n+\bar{\phi}(\alpha)(x)+\lambda(\alpha) \\ \hat{\nu}(\alpha)+w \end{bmatrix} = \left(\begin{bmatrix} n+\lambda(\alpha) \\ \hat{\nu}(\alpha) \end{bmatrix}, \begin{pmatrix} \bar{\phi}(\alpha) & \\ & I_k \end{pmatrix}\right)\begin{bmatrix} x \\ w \end{bmatrix}.$$

Noting $\phi|_B = \mathrm{id}$, the image $\phi(Q)$ is finite in $\mathrm{Aut}(\mathbb{Z}^{n-k})$ which implies $\phi(Q) \leq \mathrm{O}(n-k)$ up to conjugacy. As π is torsionfree and acts properly discontinuously, we obtain a faithful homomorphism $\rho : \pi\to\rho(\pi) \leq \mathrm{E}(n)$ defined by

$$\rho(n,\alpha) = \left(\begin{bmatrix} n+\lambda(\alpha) \\ \hat{\nu}(\alpha) \end{bmatrix}, \begin{pmatrix} \bar{\phi}(\alpha) & \\ & I_k \end{pmatrix}\right). \tag{3.6}$$

Therefore we have a compact euclidean space form $\mathbb{R}^n/\rho(\pi)$.

We prove that $\mathbb{R}^n/\rho(\pi)$ admits a T^k-action. Noting (3.4), (3.5), we define a subgroup of $\mathbb{Z}^n$ by

$$\tilde{B} = \{(-\ell \cdot \lambda(\beta), \ell \cdot \beta) \in \mathbb{Z}^n \mid \beta \in B\}. \tag{3.7}$$

It is isomorphic to $B \cong \mathbb{Z}^k$. As $\phi|_B = \mathrm{id}$,

$$\rho(-\ell \cdot \lambda(\beta), \ell \cdot \beta) = \left(\begin{bmatrix} 0 \\ \ell \cdot \hat{\nu}(\beta) \end{bmatrix}, I_n \right) \in 0 \times \mathbb{R}^k. \tag{3.8}$$

Thus $\rho(\tilde{B})$ is a translation subgroup with rank k:

$$\rho(\tilde{B}) \le (0 \times \mathbb{R}^k) \cap \rho(\pi). \tag{3.9}$$

Since $(0 \times \mathbb{R}^k)/\rho(\tilde{B})$ is compact, so is $(0 \times \mathbb{R}^k)/(0 \times \mathbb{R}^k) \cap \rho(\pi)$. We may put

$$(0 \times \mathbb{R}^k)/(0 \times \mathbb{R}^k) \cap \rho(\pi) = T^k.$$

Moreover, from (3.6) a calculation shows that

$$\rho(n, \alpha) \cdot \left(\begin{bmatrix} 0 \\ y \end{bmatrix}, I_n \right) = \left(\begin{bmatrix} 0 \\ y \end{bmatrix}, I_n \right) \cdot \rho(n, \alpha), \tag{3.10}$$

i.e. each $y \in \mathbb{R}^k$ centralizes $\rho(\pi)$;

$$0 \times \mathbb{R}^k \le C_{\mathrm{E}(n)}(\rho(\pi)). \tag{3.11}$$

We denote that $\mathrm{Isom}(\mathbb{R}^n/\rho(\pi))^0$ is the identity component of euclidean isometries of $\mathbb{R}^n/\rho(\pi)$. From (3.11) we have the following covering groups (cf. [24], [23, §11.7]):

$$\begin{array}{ccccccc} 1 & \longrightarrow & C(\rho(\pi)) & \longrightarrow & C_{\mathrm{E}(n)}(\rho(\pi)) & \longrightarrow & \mathrm{Isom}(\mathbb{R}^n/\rho(\pi))^0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 1 & \longrightarrow & (0 \times \mathbb{R}^k) \cap \rho(\pi) & \longrightarrow & (0 \times \mathbb{R}^k) & \longrightarrow & T^k \end{array}$$

Hence $\mathbb{R}^n/\rho(\pi)$ admits a T^k-action. □

We prove Theorem 1.2. The Bieberbach theorem implies that M is affinely diffeomorphic to $\mathbb{R}^n/\rho(\pi)$. By Lemma 3.2, M admits a T^k-action where $k = b_1$. Let $\mathrm{rank}\, C(\pi) = \ell$. As every effective torus action on a closed aspherical manifold is injective, $b_1 \le \ell$ (cf. [7], [23]). It is well known that if the fundamental group of a compact euclidean space form contains a nontrivial center $\mathbb{Z}^\ell$, then it admits a T^ℓ-action. By Theorem 1.1 and (1)(below Theorem 1.1), $\ell \le b_1$. This shows that $\mathrm{rank}\, H_1(M) = \mathrm{rank}\, C(\pi)$.

Chapter 4

S^1-fibred nilBott tower

4.1 S^1-fibred nilBott tower

Let M be a closed aspherical manifold which is the top space of an iterated S^1-bundle over a point:

$$M = M_n \to M_{n-1} \to \ldots \to M_1 \to \{\mathrm{pt}\}. \tag{4.1}$$

Suppose X is the universal covering of M and each X_i is the universal covering of M_i and put $\pi_1(M_i) = \pi_i$ $(i = 1, \ldots, n-1)$ and $\pi_1(M) = \pi$.

Definition 4.1. An S^1*-fibred nilBott tower* is a sequence (4.1) which satisfies I, II and III below. The top space M is said to be an S^1*-fibred nilBott manifold (of depth n).*

I. Each M_i is a fiber space over M_{i-1} with fiber S^1.

II. For the group extension

$$1 \to \mathbb{Z} \to \pi_i \longrightarrow \pi_{i-1} \to 1 \tag{4.2}$$

associated to the fiber space I, there is an equivariant principal bundle:

$$\mathbb{R} \to X_i \xrightarrow{p_i} X_{i-1}. \tag{4.3}$$

III. Each π_i normalizes $\mathbb{R}$.

The purpose of this section is to prove the following results.

Theorem 4.2. *Any S^1-fibred nilBott manifold M is diffeomorphic to an infranil-manifold.*

Proof. Given a group extension (4.2), we suppose by induction that there exists a torsionfree finitely generated nilpotent normal subgroup Δ_{i-1} of finite index in π_{i-1} such that the induced extension $\tilde{\Delta}_i$ is a central extension:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_i & \longrightarrow & \pi_{i-1} & \longrightarrow & 1 \\ & & || & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \tilde{\Delta}_i & \longrightarrow & \Delta_{i-1} & \longrightarrow & 1. \end{array} \tag{4.4}$$

It is easy to see that $\tilde{\Delta}_i$ is a torsionfree finitely generated normal nilpotent subgroup of finite index in π_i. Then π_i is a virtually nilpotent subgroup, that is $1\to\tilde{\Delta}_i\to\pi_i\longrightarrow F_i\to 1$ where $F_i = \pi_i/\tilde{\Delta}_i$ is a finite group. Let $\tilde{N}_i$, N_{i-1} be a nilpotent Lie group containing $\tilde{\Delta}_i$, Δ_{i-1} as a discrete cocompact subgroup respectively. Let $\mathrm{A}(\tilde{N}_i) = \tilde{N}_i \rtimes \mathrm{Aut}(\tilde{N}_i)$ be the affine group. If $\tilde{K}_i$ is a maximal compact subgroup of $\mathrm{Aut}(\tilde{N}_i)$, then the subgroup $\mathrm{E}(\tilde{N}_i) = \tilde{N}_i \rtimes \tilde{K}_i$ is called the euclidean group of $\tilde{N}_i$. Then there exists a faithful homomorphism (see Theorem2.6):

$$\rho_i : \pi_i \longrightarrow \mathrm{E}(\tilde{N}_i) \tag{4.5}$$

for which $\rho_i|_{\tilde{\Delta}_i} = \mathrm{id}$ and the quotient $\tilde{N}_i/\rho_i(\pi_i)$ is an infranilmanifold. The explicit formula is given by the following

$$\rho_i((n,\alpha)) = (n\cdot\chi(\alpha), \mu(\chi(\alpha)^{-1})\circ\bar{\phi}(\alpha)) \tag{4.6}$$

for $n\in\tilde{\Delta}_i, \alpha\in F_i$ where $\chi : F_i\to\tilde{N}_i$, $\bar{\phi} : F_i\to\mathrm{Aut}(\tilde{N}_i)$. As $\tilde{\Delta}_i \leq \tilde{N}_i$, there is a 1-dimensional vector space R containing $\mathbb{Z}$ as a discrete uniform subgroup which has a central group extension (cf. [34]):

$$1\to\mathsf{R}\to\tilde{N}_i\longrightarrow N_{i-1}\to 1$$

where $N_{i-1} = \tilde{N}_i/\mathsf{R}$ is a simply connected nilpotent Lie group. As $\mathbb{Z}\leq \mathsf{R}\cap\tilde{\Delta}_i$ is discrete cocompact in R and $\mathsf{R}\cap\tilde{\Delta}_i/\mathbb{Z}\to\tilde{\Delta}_i/\mathbb{Z}\cong\Delta_{i-1}$ is an inclusion, noting that Δ_{i-1} is torsionfree, it follows that $\mathsf{R}\cap\tilde{\Delta}_i = \mathbb{Z}$. We obtain the commutative diagram in which the vertical maps are inclusions:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \tilde{\Delta}_i & \longrightarrow & \Delta_{i-1} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathsf{R} & \longrightarrow & \tilde{N}_i & \longrightarrow & N_{i-1} & \longrightarrow & 1. \end{array} \tag{4.7}$$

On the other hand, (4.5) induces the following group extension:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_i & \xrightarrow{p_i} & \pi_{i-1} & \longrightarrow & 1 \\ & & \| & & \rho_i\downarrow & & \hat{\rho}_i\downarrow & & \\ 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \rho_i(\pi_i) & \longrightarrow & \hat{\rho}_i(\pi_{i-1}) & \longrightarrow & 1. \end{array} \tag{4.8}$$

Since $\tilde{\Delta}_i$ and $\tilde{N}_i$ centralizes $\mathbb{Z}$ and R respectively, $\hat{\rho}_i$ is a homomorphism from π_{i-1} into $\mathrm{E}(N_{i-1})$. The explicit formula is given by the following:

$$\hat{\rho}_i((\bar{n},\alpha)) = (\bar{n}\cdot\bar{\chi}(\alpha), \mu(\bar{\chi}(\alpha)^{-1})\circ\hat{\phi}(\alpha)) \tag{4.9}$$

for $\bar{n} \in \Delta_{i-1}$, $\alpha \in F_i$ where $\bar{\chi} = p_i \circ \chi : F_i \to N_{i-1}$, $\hat{\phi} : F_i \to \mathrm{Aut}(N_{i-1})$;

$$\hat{\phi}(\alpha)(\bar{x}) = \overline{\phi(\alpha)(x)}.$$

Using (4.3) and Mal'cev's unique extension property (compare [34]), it is easy to check that the above $\hat{\phi} : F_i \to \mathrm{Aut}(N_{i-1})$ is a well-defined homomorphism. Thus we obtain an equivariant fibration:

$$(\mathbb{Z}, \mathsf{R}) \longrightarrow (\rho_i(\pi_i), \tilde{N}_i) \xrightarrow{\nu_i} (\hat{\rho}_i(\pi_{i-1}), N_{i-1}). \tag{4.10}$$

Suppose by induction that (π_{i-1}, X_{i-1}) is equivariantly diffeomorphic to the infranil-action $(\hat{\rho}_i(\pi_{i-1}), N_{i-1})$ as above. We have two Seifert fibrations from (4.3):

$$(\mathbb{Z}, \mathsf{R}) \to (\pi_i, X_i) \xrightarrow{p_i} (\pi_{i-1}, X_{i-1})$$

and (4.10):

$$(\mathbb{Z}, \mathsf{R}) \to (\rho_i(\pi_i), \tilde{N}_i) \xrightarrow{\nu_i} (\hat{\rho}_i(\pi_{i-1}), N_{i-1}).$$

As $\rho_i : \pi_i \to \rho_i(\pi_i)$ is isomorphic such that $\rho_i|_{\mathbb{Z}} = \mathrm{id}$, the Seifert rigidity implies that (π_i, X_i) is equivariantly diffeomorphic to $(\rho_i(\pi_i), \tilde{N}_i)$. This shows the induction step. If $M = X/\pi$, then (π, X) is equivariantly diffeomorphic to an infranil-action $(\rho(\pi), \tilde{N})$ for which $\rho : \pi \to E(\tilde{N})$ is a faithful representation. □

We have shown that M is diffeomorphic to an infranilmanifold $\tilde{N}/\rho(\pi)$.

4.2 The S^1-fibred nilBott manifold of finite type and infinit type

According to the following case (I), (II), we prove that $\tilde{N}$ is isomorphic to a vector space for (I) or $\tilde{N}$ is a nilpotent Lie group but not a vector space for (II) respectively.

Definition 4.3. Suppose M is an S^1-fibred nilBott manifold M.

(I) If every cocycle of $H^2_\phi(\pi_{i-1}, \mathbb{Z})$ which represents a group extension (4.2) is *of finite order*, then M is said to be of finite type.

(II) If there exists a cocycle of $H^2_\phi(\pi_{i-1}, \mathbb{Z})$ which represents a group extension (4.2) is *of infinite order*, then M is of infinite type.

Theorem 4.4. *S^1-fibred nilBott manifold M is of finite type, if and only if M is diffeomorphic to a Riemannian flat manifold.*

Proof. Let M be of finite type. As every cocycle of $H^2_\phi(\pi_{i-1}, \mathbb{Z})$ representing a group extension (4.2) is finite, the cocycle in $H^2(\Delta_{i-1}, \mathbb{Z})$ for the induced extension of (4.4) that $1 \to \mathbb{Z} \to \tilde{\Delta}_i \longrightarrow \Delta_{i-1} \to 1$ is also finite. By induction, suppose that Δ_{i-1} is isomorphic to a free abelian group $\mathbb{Z}^{i-1}$. Then the cocycle in $H^2(\mathbb{Z}^{i-1},\mathbb{Z})$ is zero, so $\tilde{\Delta}_i$ is isomorphic to a free abelian group $\mathbb{Z}^i$. Hence the nilpotent Lie group N_i is isomorphic to the vector space $\mathbb{R}^i$. This shows the induction step. In particular, π_i is isomorphic to a Bieberbach group $\rho_i(\pi_i) \leq \mathrm{E}(\mathbb{R}^i)$. As a consequence X/π is diffeomorphic to a Riemannian flat manifold $\mathbb{R}^n/\rho(\pi)$.

On the other hand, suppose that π_j is virtually free abelian for any $j \in \{1,\cdots,i-1\}$ and the cocycle $[f] \in H^2_\phi(\pi_j,\mathbb{Z})$ is of infinite order in $H^2_\phi(\pi_j,\mathbb{Z})$. Note that π_j contains a torsionfree normal free abelian subgroup $\mathbb{Z}^j$. As in (4.4), there is a central group extension of $\tilde{\Delta}_i$:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_i & \longrightarrow & \pi_{i-1} & \longrightarrow & 1 \\ & & \| & & \uparrow & & \uparrow \mathrm{i} & & \\ 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \tilde{\Delta}_i & \longrightarrow & \mathbb{Z}^{i-1} & \longrightarrow & 1 \end{array} \tag{4.1}$$

where $[\pi_{i-1} : \mathbb{Z}^{i-1}] < \infty$. Recall that there is a transfer homomorphism $\tau : H^2(\mathbb{Z}^{i-1},\mathbb{Z}) \to H^2_\phi(\pi_{i-1},\mathbb{Z})$ such that

$$\tau \circ \mathrm{i}^* = [\pi_{i-1} : \mathbb{Z}^{i-1}] : H^2_\phi(\pi_{i-1},\mathbb{Z}) \to H^2_\phi(\pi_{i-1},\mathbb{Z}),$$

see [3, (9.5) Proposition p.82] for example. The restriction $\mathrm{i}^*[f]$ gives the bottom extension sequence of (4.1). If $\mathrm{i}^*[f] = 0 \in H^2(\mathbb{Z}^2,\mathbb{Z})$, then $0 = \tau \circ \mathrm{i}^*[f] = [\pi_{i-1} : \mathbb{Z}^{i-1}][f] \in H^2_\phi(\pi_{i-1},\mathbb{Z})$. So $\mathrm{i}^*[f] \neq 0$. Therefore $\tilde{\Delta}_i$ (respectively $\tilde{N}_i$) is not abelian (respectively not isomorphic to a vector space). As a consequence, $\tilde{N}$ is a simply connected (non-abelian) nilpotent Lie group. □

Apparently there is no inter between finite type and infinite type. And S^1-fibred nilBott manifolds are of finite type until dimension 2.

Remark 4.5. *Let M be an S^1-fibred nilBott manifold of finite type, then $\rho(\pi)$ is a Bieberbach group (cf. Theorem 4.3). By the Bieberbach Theorem, $\rho(\pi)$ satisfies a group extension*

$$1 \to \mathbb{Z}^n \to \rho(\pi) \longrightarrow H \to 1 \tag{4.2}$$

where $\mathbb{Z}^n = \rho(\pi) \cap \mathbb{R}^n$, and H is the holonomy group of $\rho(\pi)$. We may identify $\rho(\pi)$ with π whenever π is torsionfree.

Proposition 4.6. *Suppose M is an S^1-fibred nilBott manifold of finite type. Then the holonomy group of π is isomorphic to the power of cyclic group of order two $(\mathbb{Z}_2)^s$ in $\mathrm{O}(n)$ $(0 \leq s \leq n)$.*

Proof. Let M be an S^1-fibred nilBott manifold of finite type. Recall an equivariant fibration:

$$(\mathbb{Z},\mathsf{R}) \to (\pi_i, \tilde{N}_i) \xrightarrow{p_i} (\pi_{i-1}, N_{i-1}).$$

If f is a cocycle in $H^2_\phi(\pi_{i-1},\mathbb{Z})$ for Case I representing (4.2), then there exists a map $\lambda : \pi_{i-1} \to \mathsf{R}$ such that

$$f(\alpha,\beta) = \bar{\phi}(\alpha)(\lambda(\beta)) + \lambda(\alpha) - \lambda(\alpha\beta) \ \ (\alpha,\beta \in \pi_{i-1}) \tag{4.3}$$

(see [7]). Moreover let $(n, \alpha) \in \pi_i$ and $(x, w) \in \tilde{N}_i = \mathsf{R} \times N_{i-1}$, then the action of π_i is given by

$$(n, \alpha)(x, w) = (n + \bar{\phi}(\alpha)(x) + \lambda(\alpha), \alpha w) \tag{4.4}$$

($n \in \mathbb{Z}$, $\alpha \in \pi_{i-1}$). As we have shown in Case I of Theorem 4.3, N_{i-1}/π_{i-1} is a Riemannian flat manifold $\mathbb{R}^{i-1}/\pi_{i-1}$, we may assume that

$$\alpha w = b_\alpha + A_\alpha w \ (w \in \mathbb{R}^{i-1})$$

($b_\alpha \in \mathbb{R}^i$, $A_\alpha \in \mathrm{O}(i-1)$) in the above action of (4.4). Then the above action (4.4) has the formula:

$$(n, \alpha) \begin{bmatrix} x \\ w \end{bmatrix} = \left(\begin{pmatrix} n + \lambda(\alpha) \\ b_\alpha \end{pmatrix}, \begin{pmatrix} \bar{\phi}(\alpha) & 0 \\ 0 & A_\alpha \end{pmatrix} \right) \begin{bmatrix} x \\ w \end{bmatrix}, \tag{4.5}$$

where $\begin{bmatrix} x \\ w \end{bmatrix} \in \tilde{N}_i = \mathsf{R} \times \mathbb{R}^{i-1} = \mathbb{R}^i$. Suppose inductively that $\{A_\alpha \,|\, \alpha \in \pi_{i-1}\} \leq (\mathbb{Z}_2)^{i-1}$. Here

$$(\mathbb{Z}_2)^{i-1} = \{ \begin{pmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{pmatrix} \} \leq \mathrm{O}(i-1). \tag{4.6}$$

Since $\bar{\phi}(\pi_{i-1}) \leq \{\pm 1\}$, the holonomy group H_i of π_i is isomorphic to $(\mathbb{Z}_2)^s$, ($0 \leq s \leq i$). This proves the induction step. □

4.3 3-dimensional S^1-fibred nilBott towers

By the definition of S^1-fibred nilBott manifold M_n of depth n, M_2 is either a torus or a Klein bottle. In particular, M_2 is a Riemannian flat manifold. A 3-dimensional S^1-fibred nilBott manifold M_3 is either a Riemannian flat manifold or an infranil-Heisenberg manifold in accordance with the finite type or infinite type.

On the other hand, there are 10-isomorphism classes $\mathcal{G}_1, \ldots, \mathcal{G}_6$, $\mathcal{B}_1, \ldots, \mathcal{B}_4$ of 3-dimensional Riemannian flat manifolds. (Refer to Wolf [36] for the classification of 3-dimensional Riemannian flat manifolds.) Among these, real Bott manifolds consist of 4; $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{B}_1$, $\mathcal{B}_3$. (See [26].) We shall show that $\mathcal{B}_2$, $\mathcal{B}_4$ are S^1-fibred nilBott manifolds.

4.3.1 3-dimensional S^1-fibred nilBott manifolds of finite type

$\mathcal{G}_1$: T^3. Holonomy group $\{1\}$. The identity Bott matrix I_3.

$\mathcal{G}_2$: $T^3/\mathbb{Z}_2$. Holonomy group $\mathbb{Z}_2 = \langle\alpha\rangle$.

$$\alpha(z_1, z_2, z_3) = (-z_1, \bar{z}_2, \bar{z}_3)$$

The Bott matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

$\mathcal{B}_1$: $T^3/\mathbb{Z}_2 = S^1 \times K$. Holonomy group $\mathbb{Z}_2 = \langle\alpha\rangle$.

$$\alpha(z_1, z_2, z_3) = (-z_1, z_2, \bar{z}_3)$$

The Bott matrix $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

$\mathcal{B}_3$: $T^3/(\mathbb{Z}_2)^2$. Holonomy group $(\mathbb{Z}_2)^2 = \langle\alpha, \beta\rangle$.

$$\alpha(z_1, z_2, z_3) = (-z_1, \bar{z}_2, \bar{z}_3)$$
$$\beta(z_1, z_2, z_3) = (z_1, -z_2, \bar{z}_3)$$

The Bott matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

$\mathcal{B}_2$: Let $T^3/\mathbb{Z}_2$ whose holonomy group $\mathbb{Z}_2 = \langle\alpha\rangle$ acts on T^3;

$$\alpha(z_1, z_2, z_3) = (-z_1 z_3, z_2 z_3, \bar{z}_3)$$

There is no such description of Bott matrix in this case. Define S^1-action on T^3 by

$$t(z_1, z_2, z_3) = (tz_1, tz_2, z_3).$$

Then it is easy to see that this S^1-action induces an S^1-action on $T^3/\mathbb{Z}_2$. This gives a principal bundle

$$S^1 \to T^3/\mathbb{Z}_2 \longrightarrow K$$

where K is a Klein bottle. Letting $\pi_1(K) = G$, there is an central extension

$$1 \to \mathbb{Z} \to \pi \to G \to 1.$$

More precisely, if $\langle t_1, t_2, t_3, \tilde{\alpha}\rangle$ is a set of generators, then

$$1 \to \langle t_1, t_2\rangle \to \pi \longrightarrow \langle \bar{\alpha}\bar{t}_3\rangle \to 1.$$

Here the quotient group $G = \langle \bar{t}_1, \bar{t}_2, \bar{t}_3, \bar{\alpha}\rangle$. Moreover, it is easy to see that But $\bar{t}_1 = \bar{\alpha}^2$, $\bar{t}_2 = \bar{t}_1^{-1}$. Moreover, it follows

$$\bar{\alpha}(\bar{t}_1, \bar{t}_2)\bar{\alpha}^{-1} = (\bar{t}_1, \bar{t}_2^{-1}),$$

that is $\bar{\alpha}$ induces an action of $\mathbb{Z}_2$ on T^2 by $\alpha(z_1, z_2) = (-z_1, \bar{z}_2)$ so $\mathbb{R}^2/G = T^2/\langle\alpha\rangle = K$.

$\mathcal{B}_4$: $T^3/(\mathbb{Z}_2)^2$. Holonomy group $(\mathbb{Z}_2)^2 = \langle \alpha, \beta \rangle$.

$$\begin{aligned} \alpha(z_1, z_2, z_3) &= (-z_1, \bar{z}_2, \bar{z}_3) \\ \beta(z_1, z_2, z_3) &= (z_1, -z_2, -\bar{z}_3) \end{aligned}$$

If we define -1-action to be $z \mapsto -\bar{z}$, then the nil-Bott matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$ but not a Bott matrix.

Denote the $(\mathbb{Z}_2)^2$-action on T^2 by

$$\begin{aligned} \hat{\alpha}(z_1, z_2) &= (-z_1, \bar{z}_2) \\ \hat{\beta}(z_1, z_2) &= (z_1, -z_2). \end{aligned}$$

The quotient manifold is the Klein bottle $T^2/(\mathbb{Z}_2)^2 = (S^1 \times \mathbb{RP}^1)/\mathbb{Z}_2 = K$. The projection $P(z_1, z_2, z_3) = (z_1, z_2)$ is equivariant with respect to the $(\mathbb{Z}_2)^2$-action so the quotient

$$T^3/(\mathbb{Z}_2)^2 \to K \to S^1 \to \{\mathrm{pt}\}$$

is a nil-Bott tower with S^1-fiber.

Proposition 4.7. *The 3-dimensional S^1-fibred nilBott manifold of finite type are those of $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$, $\mathcal{B}_4$.*

Proof. Since the holonomy group are the product of $\mathbb{Z}_2$ by Proposition 4.6, the remaining cases are either $\mathcal{G}_6$ or $\mathcal{B}_2$ from the list [36]. Moreover, by Corollary **??**, an S^1-fibred nilBott manifold M of finite type admits a homologically injective T^k-action for $k = \mathrm{Rank}\, H_1(M)$ $(k \geq 1)$. In particular, $\mathbb{Z}^k$ is a direct summand of $H_1(M)$. By the classification of the first homology (cf. [36]), $H_1(M; \mathbb{Z}) = \mathbb{Z}_4 + \mathbb{Z}_4$ for $\mathcal{G}_6$. So it cannot admit a structure of S^1-fibred nilBott manifold. For the Riemannian flat 3-manifold corresponding to $\mathcal{B}_2$, it follows $H_1(M; \mathbb{Z}) = \mathbb{Z} + \mathbb{Z}$ (cf. [36]). We have shown that there is a S^1-fibred nilBott tower: $M \to K \to S^1$. □

4.3.2 3-dimensional S^1-fibred nilBott manifolds of infinite type

Any 3-dimensional S^1-fibred nilBott manifold M_3 of infinite type is an infranil-Heisenberg manifold. The 3-dimensional simply connected nilpotent Lie group N_3 is isomorphic to the Heisenberg Lie group N which is the product $\mathsf{R} \times \mathbb{C}$ with group law:

$$(x, z) \cdot (y, w) = (x + y - \mathrm{Im} \bar{z} w, z + w).$$

Then a maximal compact Lie subgroup of $\mathrm{Aut}(\mathsf{N})$ is $\mathrm{U}(1) \rtimes \langle \tau \rangle$ which acts on N

$$\begin{aligned} e^{\mathbf{i}\theta}(x, z) &= (x, e^{\mathbf{i}\theta} z), \ \ (e^{\mathbf{i}\theta} \in \mathrm{U}(1)). \\ \tau(x, z) &= (-x, \bar{z}). \end{aligned} \tag{4.1}$$

A 3-dimensional compact infranilmanifold is obtained as a quotient N/Γ where Γ is a torsionfree discrete uniform subgroup of $\mathrm{E}(\mathsf{N}) = \mathsf{N} \rtimes (\mathrm{U}(1) \rtimes \langle\tau\rangle)$. See [10].
Let

$$S^1 \to M_3 \to M_2$$

be an S^1-fibred nilBott manifold of infinite type which has a group extension $1 \to \mathbb{Z} \to \pi_3 \to \pi_2 \to 1$. As before this group extension contains a central group extension $1 \to \mathbb{Z} \to \tilde{\Delta}_3 \longrightarrow \Delta_2 \to 1$. Since $\mathsf{R} \subset \mathsf{N}$ is the center, this induces the commutative diagram of central extensions (cf. (4.7)):

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \tilde{\Delta}_3 & \longrightarrow & \Delta_2 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathsf{R} & \longrightarrow & \mathsf{N} & \longrightarrow & \mathbb{C} & \longrightarrow & 1. \end{array} \tag{4.2}$$

Using this, we obtain an embedding:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_3 & \longrightarrow & \pi_2 & \longrightarrow & 1 \\ & & \downarrow & & \rho\downarrow & & \hat{\rho}\downarrow & & \\ 1 & \longrightarrow & \mathsf{R} & \longrightarrow & \mathrm{E}(\mathsf{N}) & \longrightarrow & \mathbb{C} \rtimes (\mathrm{U}(1) \rtimes \langle\tau\rangle) & \longrightarrow & 1. \end{array} \tag{4.3}$$

Note that $\mathbb{C} \rtimes (\mathrm{U}(1) \rtimes \langle\tau\rangle) = \mathbb{R}^2 \rtimes \mathsf{O}(2) = \mathrm{E}(2)$. Since $\mathsf{R} \cap \pi_3 = \mathbb{Z}$ from (4.3), $\hat{\rho}(\pi_2)$ is a Bieberbach group in $\mathrm{E}(2)$ so that $\mathbb{R}^2/\hat{\rho}(\pi_2)$ is either T^2 or K.

Define $L : \mathrm{E}(\mathsf{N}) \to \mathrm{U}(1) \rtimes \langle\tau\rangle$ to be the canonical projection.

Case (i). Suppose $L(\pi_3) = \{1\}$. Then $\hat{\rho}(\pi_2) \leq \mathbb{C}$. So we may assume $\pi_3 = \tilde{\Delta}_3$ from (4.2). For each $k \in \mathbb{Z}$, we introduce the nilpotent group $\Delta(k)$ which is a subgroup of N generated by

$$c = (2k, 0),\ a = (0, k),\ b = (0, k\mathbf{i}).$$

Put $\mathsf{Z} = \langle c\rangle$ which is a central subgroup of $\Delta(k)$. It is easy to see that

$$[a, b] = c^{-k}. \tag{4.4}$$

Then $\tilde{\Delta}_3 \leq \mathsf{N}$ is isomorphic to $\Delta(k)$ for some $k \in \mathbb{Z}$. Since R is the center of N, we have a principal bundle

$$S^1 = \mathsf{R}/\mathsf{Z} \to \mathsf{N}/\Delta(k) \longrightarrow \mathbb{C}/\mathbb{Z}^2.$$

Then the euler number of the fibration is $\pm k$. (See [29] for example.)

Case (ii). Suppose that the holonomy group of π_3 is nontrivial. Then we note that $L(\pi_3) = \mathbb{Z}_2 \leq \mathrm{U}(1) \rtimes \langle\tau\rangle$, but not in $\mathrm{U}(1)$. By (4.7) $L(\pi_3) = L(\pi_2)$, first remark that $L(\pi_2)$ is not contained in $\mathrm{U}(1)$. For this, suppose that (b, A) is an element of $\pi_2 \leq \mathbb{R}^2 \rtimes \mathrm{O}(2)$. Then for any $x \in \mathbb{R}^2$, $(b, A)x \neq x$, because the action of π_2 on $\mathbb{R}^2$ is free. Therefore $\det(A - I) = 0$. This implies that if $A \in \mathrm{SO}(2) = \mathrm{U}(1)$, then $A = I$. So $L(\pi_2) = L(\pi_3)$ is not contained in $\mathrm{U}(1)$.

Suppose that there exists an element $g \in \pi_3$ such that $L(g) = (e^{\mathbf{i}\theta}, \tau) \in \mathrm{U}(1) \rtimes \langle\tau\rangle$. Noting (4.1), it follows $L(g)^2 = 1$. Then $L(\pi_3) = (\mathrm{U}(1) \cap L(\pi_3)) \cdot \langle L(g)\rangle$. Let $\pi_3' = L^{-1}(\mathrm{U}(1) \cap L(\pi_3)) \leq \pi_3$ which has the commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_3 & \xrightarrow{p_3} & \pi_2 & \longrightarrow & 1 \\ & & \| & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_3' & \longrightarrow & \pi_2' & \longrightarrow & 1. \end{array} \tag{4.5}$$

Here $\pi_2' = p_3(\pi_3')$. Since π_2' also acts on $\mathbb{R}^2$ freely, it follows $L(\pi_2') = L(\pi_3') = \mathrm{U}(1) \cap L(\pi_3) = \{1\}$. Hence $L(\pi_2) = L(\pi_3) = \mathbb{Z}_2 = \langle L(g)\rangle$. In particular M_2 is the Klein bottle K.

Let $n = (x, 0)$ be a generator of $\mathbb{Z} \leq \mathsf{N}$. Choose $h \in \pi_3$ with $L(h) = 1$ such that the subgroup $\langle p_3(g), p_3(h)\rangle$ is the fundamental group of K. It has a relation $p_3(g)p_3(h)p_3(g)^{-1} = p_3(h)^{-1}$. Then $\langle n, g, h\rangle$ is isomorphic to π_3. In particular, those generators satisfy

$$\begin{aligned} &ghg^{-1} = n^k h^{-1} \ (^\exists\, k \in \mathbb{Z}), \\ &gng^{-1} = L(g)n = \tau n = n^{-1},\ hnh^{-1} = L(h)n = n. \end{aligned} \tag{4.6}$$

On the other hand, fix a non-zero integer k. Let $\Gamma(k)$ be a subgroup of $\mathrm{E}(\mathsf{N})$ generated by

$$n = ((k, 0), I)\,,\ \alpha = \left((0, \frac{k}{2}), \tau\right)\,,\ \beta = ((0, k\mathbf{i}), I)\,, \tag{4.7}$$

where $(a, x) \in \mathsf{N} = \mathsf{R} \times \mathbb{C} \leq \mathrm{E}(\mathsf{N})$.

Note that $\alpha^2 = ((0, k), I)$. Then it is easily checked that

$$\alpha\beta\alpha^{-1} = n^k\beta^{-1},\ \alpha n\alpha^{-1} = n^{-1},\ \beta n\beta^{-1} = n. \tag{4.8}$$

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathsf{R} & \longrightarrow & \mathrm{E}(\mathsf{N}) & \longrightarrow & \mathbb{C} \rtimes (\mathrm{U}(1) \rtimes \langle\tau\rangle) & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & \langle n\rangle & \longrightarrow & \Gamma(k) & \longrightarrow & \langle\hat{\alpha}, \hat{\beta}\rangle & \longrightarrow & 1. \end{array} \tag{4.9}$$

Then the subgroup generated by $\hat{\alpha}^2, \hat{\beta}$ is isomorphic to the subgroup of translations of $\mathbb{R}^2$; $t_1 = \begin{bmatrix} k \\ 0 \end{bmatrix}, t_2 = \begin{bmatrix} 0 \\ k \end{bmatrix}$. Let $T^2 = \mathbb{R}^2/\langle t_1, t_2\rangle$. Then it is easy to see that the quotient $\gamma = [\hat{\alpha}]$ of order 2 acts on T^2 as

$$\gamma(z_1, z_2) = (-z_1, \bar{z}_2). \tag{4.10}$$

As a consequence, $\mathbb{R}^2/\langle\hat{\alpha},\hat{\beta}\rangle = T^2/\langle\gamma\rangle$ turns out to be K. So $M_3 = \mathsf{N}/\Gamma(k)$ is an S^1-fibred nilBott manifold:

$$S^1 \to \mathsf{N}/\Gamma(k) \to K$$

where $S^1 = \mathsf{R}/\langle n\rangle$ is the fiber (but not an action).

Compared (4.6) with $\Gamma(k)$, π_3 is isomorphic to $\Gamma(k)$ with the following commutative arrows of isomorphisms:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_3 & \longrightarrow & \pi_2 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \langle n\rangle & \longrightarrow & \Gamma(k) & \longrightarrow & \langle\hat{\alpha},\hat{\beta}\rangle & \longrightarrow & 1. \end{array} \tag{4.11}$$

As both (π_3, X_3) and $(\Gamma(k), \mathsf{N})$ are Seifert actions, the isomorphism of (4.11) implies that they are equivariantly diffeomorphic, that is, $M_3 = X_3/\pi_3 \cong \mathsf{N}/\Gamma(k)$. This shows the following.

Proposition 4.8. *A 3-dimensional S^1-fibred nilBott manifold M_3 of infinite type is either a Heisenberg nilmanifold $\mathsf{N}/\Delta(k)$ or a Heisenberg infranilmanifold $\mathsf{N}/\Gamma(k)$.*

4.3.3 Realization of S^1-fibration over a Klein Bottle K

Let Q be a fundamental group of a Klein Bottle K, then Q has a presentation:

$$\{g, h \mid ghg^{-1} = h^{-1}\}. \tag{4.12}$$

A group extension $1\to\mathbb{Z}\to\pi\to Q\to 1$ for any 3-dimensional S^1-fibred nilBott manifold over K represents a 2-cocycle in $H^2_\phi(Q,\mathbb{Z})$ for some representation ϕ. Conversely, given any representation $\phi : Q\to\mathrm{Aut}(\mathbb{Z}) = \{\pm 1\}$, we shall prove that any element of $H^2_\phi(Q,\mathbb{Z})$ can be realized as an S^1-fibred nilBott manifold.

We must consider following cases of a representation ϕ:

Case 1. $\phi(g) = 1,\ \phi(h) = 1,$
Case 2. $\phi(g) = 1,\ \phi(h) = -1,$
Case 3. $\phi(g) = -1,\ \phi(h) = 1,$
Case 4. $\phi(g) = -1,\ \phi(h) = -1.$

Suppose ϕ_i $(i = 1, 2, 3, 4)$ is the representation ϕ for **Case i.** Any element of $H^2_{\phi_i}(Q,\mathbb{Z})$ gives rise to a group extension

$$1\to\mathbb{Z}\to\pi \xrightarrow{p} Q\to 1.$$

Then π is generated by $\tilde{g}$, $\tilde{h}$, n such that $\langle n\rangle = \mathbb{Z}$ and $p(\tilde{g}) = g$, $p(\tilde{h}) = h$. There exists $k\in\mathbb{Z}$ which satisfies

$$\tilde{g}\tilde{h}\tilde{g}^{-1} = n^k\tilde{h}^{-1}. \tag{4.13}$$

Put $\pi = {}_i\pi(k)$ for each $k \in \mathbb{Z}$ and $[f_k]$ denotes the 2-cocycle of $H^2_{\phi_i}(Q, \mathbb{Z})$ representing ${}_i\pi(k)$. Note that $[f_0] = 0$.

Case 1: Since ϕ_1 is trivial, $H^2_{\phi_1}(Q, \mathbb{Z}) = H^2(Q, \mathbb{Z}) \approx H^2(K, \mathbb{Z}) \approx \mathbb{Z}_2$, and the group ${}_1\pi(k)$ satisfies the following presentation:

$$\tilde{g}n\tilde{g}^{-1} = n,\ \tilde{h}n\tilde{h}^{-1} = n,\ \tilde{g}\tilde{h}\tilde{g}^{-1} = n^k\tilde{h}^{-1}. \tag{4.14}$$

Lemma 4.9. *The groups ${}_1\pi(0)$, ${}_1\pi(1)$ are isomorphic to $\mathcal{B}_1$, $\mathcal{B}_2$ respectively.*

Proof. First we shall discuss ${}_1\pi(0)$. Let $\tilde{g}, \tilde{h}, n \in {}_1\pi(0)$ be as above. Put $\varepsilon = \tilde{g}$, $t_1 = \tilde{g}^2$, $t_2 = n$ and $t_3 = \tilde{h}$. Remark that a group generated by $\varepsilon, t_1, t_2, t_3$ coincides with ${}_1\pi(0)$. Using the relation (4.14),

$$\begin{aligned} \varepsilon^2 &= t_1, \\ \varepsilon t_2 \varepsilon^{-1} &= \tilde{g}\tilde{h}\tilde{g}^{-1} = \tilde{h}^{-1} = t_2^{-1}, \\ \varepsilon t_3 \varepsilon^{-1} &= \tilde{g}n\tilde{g}^{-1} = n = t_3. \end{aligned}$$

Compared these relations with those of $\mathcal{B}_1$, ${}_1\pi(0)$ is isomorphic to $\mathcal{B}_1$ (due to the Wolf's notation [36]).

Second, we shall discuss ${}_1\pi(1)$. Let $\tilde{g}, \tilde{h}, n \in {}_1\pi(1)$ be as above. Put $\varepsilon = \tilde{g}$, $t_1 = \tilde{g}^2$, $t_2 = \tilde{g}^{-2}n$ and $t_3 = \tilde{h}$. A group generated by $\varepsilon, t_1, t_2, t_3$ coincides with ${}_1\pi(1)$. By using the relation (4.14),

$$\begin{aligned} \varepsilon^2 &= t_1, \\ \varepsilon t_2 \varepsilon^{-1} &= \tilde{g}\tilde{g}^{-2}n\tilde{g}^{-1} = \tilde{g}^{-1}n\tilde{g}^{-1} = \tilde{g}^{-2}n = t_1, \\ \varepsilon t_3 \varepsilon^{-1} &= \tilde{g}h\tilde{g}^{-1} = \tilde{g}^2\tilde{g}^{-2}n\tilde{h}^{-1} = t_1t_2t_3^{-1}. \end{aligned}$$

This implies that ${}_1\pi(1)$ is isomorphic to $\mathcal{B}_2$. (See [36].) □

For arbitrary $k \in \mathbb{Z}$, we have the following.

Proposition 4.10. *The group extension ${}_1\pi(k)$ is isomorphic to $\mathcal{B}_1$, or $\mathcal{B}_2$ in accodance with k is even or odd.*

Proof. Take $[f_1] \in H^2_{\phi_1}(Q, \mathbb{Z}) \approx \mathbb{Z}_2$ by Lemma 4.9, then

$$\begin{aligned} n = \tilde{g}\tilde{h}\tilde{g}^{-1}\tilde{h} &= (0, g)(0, h)(-f_1(g^{-1}, g), g^{-1})(0, h) \\ &= f_1(g, h) - f_1(g^{-1}, g) + f_1(gh, g^{-1}) + f_1(h^{-1}, h), \end{aligned} \tag{4.15}$$

and so

$$n^k = kf_1(g, h) - kf_1(g^{-1}, g) + kf_1(gh, g^{-1}) + kf_1(h^{-1}, h). \tag{4.16}$$

Since $[kf_1] \in H^2_{\phi_1}(Q, \mathbb{Z})$, we can construct a group H_k which is represented by (kf_1, ϕ_1). Then H_k is generated by the elements n and $g' = (0, g)$, $h' = (0, h)$ satisfying that

$$(n, \alpha)(m, \beta) = (n + \phi_1(\alpha)(m) + kf_1(\alpha, \beta), \alpha\beta)$$
$$(\forall n, m \in \mathbb{Z}, \forall \alpha, \beta \in Q).$$

It follows

$$\begin{aligned}g'h'g'^{-1}h' &= (0,g)(0,h)(-kf_1(g^{-1},g),g^{-1})(0,h)\\ &= kf_1(g,h)-kf_1(g^{-1},g)+kf_1(gh,g^{-1})+kf_1(h^{-1},h)\\ &= n^k \text{ (from (4.16)).}\end{aligned}$$

Thus we obtain $g'h'g'^{-1} = n^k h'^{-1}$. In view of (4.13), a correspondence $g' \mapsto \tilde{g}$, $h' \mapsto \tilde{h}$ gives an isomorphism Ψ of the group extensions:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & H_k & \longrightarrow & Q & \longrightarrow & 1 \\ & & {\scriptstyle id}\downarrow & & {\scriptstyle \Psi}\downarrow & & {\scriptstyle id}\downarrow & & \\ 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & {}_1\pi(k) & \longrightarrow & Q & \longrightarrow & 1. \end{array} \tag{4.17}$$

If we recall that $[f_k]$ (resp. $[k\cdot f_1]$) represents ${}_1\pi(k)$ (resp. H_k), then it follows $[f_k] = k\cdot[f_1]$. Noting that $[f_1]$ is a two torsion element, the result follows. □

Case 2: Let $\phi_2(g)=1, \phi_2(h)=-1$, then ${}_2\pi(k)$ has the following presentation.

$$\tilde{g}n\tilde{g}^{-1}=n,\ \tilde{h}n\tilde{h}^{-1}=n^{-1},\ \tilde{g}\tilde{h}\tilde{g}^{-1}=n^k\tilde{h}^{-1}, \tag{4.18}$$

for some $k\in\mathbb{Z}$.

Proposition 4.11. *The groups ${}_2\pi(0)$, ${}_2\pi(1)$ are isomorphic to $\mathcal{B}_3$, $\mathcal{B}_4$ respectively.*

Proof. Let $\tilde{g},\tilde{h},n\in{}_2\pi(0)$ be as before. Put $\alpha=\tilde{h}\tilde{g}$, $\varepsilon=\tilde{h}^{-1}$, $t_1=\tilde{g}^2$, $t_2=\tilde{h}^{-2}$ and $t_3=n$. Note that the group generated by $\alpha,\varepsilon,t_1,t_2,t_3$ coincides with ${}_2\pi(0)$. Using the relation (4.18),

$$\begin{aligned}\tilde{\alpha}^2 &= (\tilde{h}\tilde{g})^2=\tilde{h}\tilde{h}^{-1}\tilde{g}\tilde{g}=\tilde{g}^2=t_1,\\ \varepsilon^2 &= t_2,\\ \varepsilon\alpha\varepsilon^{-1} &= \tilde{h}^{-1}\tilde{h}\tilde{g}\tilde{h}=\tilde{h}^{-1}\tilde{g}=t_2\alpha,\\ \alpha t_2\alpha^{-1} &= \tilde{h}\tilde{g}\tilde{h}^{-2}\tilde{g}^{-1}\tilde{h}^{-1}=\tilde{h}^{-2}=t_2^{-1},\\ alt_3\alpha^{-1} &= \tilde{h}\tilde{g}n\tilde{g}^{-1}\tilde{h}^{-1}=n^{-1}=t_3^{-1},\\ \varepsilon t_1\varepsilon^{-1} &= \tilde{h}^{-1}\tilde{g}^2\tilde{h}=\tilde{h}^{-1}\tilde{g}\tilde{h}^{-1}\tilde{g}=\tilde{h}^{-1}\tilde{h}\tilde{g}\tilde{g}=\tilde{g}^2=t_1,\\ \varepsilon t_3\varepsilon^{-1} &= \tilde{h}^{-1}n\tilde{h}=n^{-1}=t_3^{-1}.\end{aligned}$$

Since these relations correspond to those of $\mathcal{B}_3$ (cf. [36]), ${}_2\pi(0)$ is isomorphic to $\mathcal{B}_3$.

Let $\tilde{g},\tilde{h},n\in{}_2\pi(1)$ be as above. Put $\alpha=\tilde{h}\tilde{g}$, $\varepsilon=n^{-1}\tilde{h}^{-1}$, $t_1=n^{-1}\tilde{g}^2$, $t_2=\tilde{h}^{-2}$, and $t_3=n^{-1}$. Using the relation (4.18), we obtain the following presentation:

$$
\begin{aligned}
\tilde{\alpha}^2 &= (\tilde{h}\tilde{g})^2 = \tilde{h}n\tilde{h}^{-1}\tilde{g}\tilde{g} = n^{-1}\tilde{g}^2 = t_1,\\
\varepsilon^2 &= t_2,\\
\varepsilon\alpha\varepsilon^{-1} &= n^{-1}\tilde{h}^{-1}\tilde{h}\tilde{g}\tilde{h}n = \tilde{h}^{-1}\tilde{g}n = t_2t_3\alpha,\\
\alpha t_2\alpha^{-1} &= \tilde{h}\tilde{g}\tilde{h}^{-2}\tilde{g}^{-1}\tilde{h}^{-1} = \tilde{h}^{-2} = t_2^{-1},\\
\alpha t_3\alpha^{-1} &= \tilde{h}\tilde{g}n^{-1}\tilde{g}^{-1}\tilde{h}^{-1} = n = t_3^{-1},\\
\varepsilon t_1\varepsilon^{-1} &= n^{-1}\tilde{h}^{-1}n^{-1}\tilde{g}^2\tilde{h}n = n^{-1}\tilde{g}^2 = t_1,\\
\varepsilon t_3\varepsilon^{-1} &= n^{-1}\tilde{h}^{-1}n^{-1}\tilde{h}n = n = t_3^{-1}.
\end{aligned}
$$

This implies that ${}_2\pi(1)$ is isomorphic to $\mathcal{B}_4$. (See [36]) □

Proposition 4.12. *$H^2_{\phi_2}(Q,\mathbb{Z})$ is isomorphic to $\mathbb{Z}_2$.*

Proof. We first show that $H^2_{\phi_2}(Q,\mathbb{Z})$ is a 2-torsion group. Let Q' be the subgroup of Q generated by $g, h^2 \in Q$ satisfying that $gh^2g^{-1} = (ghg^{-1})^2 = h^{-2}$. We have a commutative diagram:

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & {}_2\pi(k) & \xrightarrow{p} & Q & \longrightarrow & 1\\
 & & \| & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi' & \xrightarrow{p} & Q' & \longrightarrow & 1
\end{array}
\tag{4.19}
$$

where π' is the subgroup of ${}_2\pi(k)$ generated by $n, \tilde{g}, \tilde{h}^2$. Note that

$$\tilde{g}\tilde{h}^2\tilde{g}^{-1} = n^k\tilde{h}^{-1}n^k\tilde{h}^{-1} = \tilde{h}^{-2}.$$

Since the subgroup $\langle \tilde{g}, \tilde{h}^2\rangle$ of π' maps isomorphically onto Q' and a restriction $\phi_2\big|Q' = \mathrm{id}$, it follows $\pi' = \mathbb{Z}\times Q'$. This shows that the restriction homomorphism $\iota^* : H^2_{\phi_2}(Q,\mathbb{Z})\to H^2(Q',\mathbb{Z})$ is the zero map, equivalently $\iota^*[f_k] = 0$. Using the transfer homomorphism $\tau : H^2(Q',\mathbb{Z})\to H^2_{\phi_2}(Q,\mathbb{Z})$ and by the property $\tau\circ\iota^*([f]) = [Q:Q'][f] = 2[f]$ $(\forall\,[f]\in H^2_{\phi_2}(Q,\mathbb{Z}))$, we obtain $2[f] = 0$.

Let $[f_k]$ be a 2-cocycle of ${}_2\pi(k)$. Similarly as in the proof of Proposition 4.10 we obtain

$$[f_k] = k\cdot[f_1]. \tag{4.20}$$

As a consequence, $H^2_{\phi_2}(Q,\mathbb{Z})$ is isomorphic to $\mathbb{Z}_2$. □

The following is obvious using Proposition 4.11 and Proposition 4.12.

Corollary 4.13. *The group extension ${}_2\pi(k)$ is isomorphic to $\mathcal{B}_3$ or $\mathcal{B}_4$ in accordance with k is even or odd.*

Case 3: The group ${}_3\pi(k)$ has the following presentation for some $k\in\mathbb{Z}$;

$$\tilde{g}n\tilde{g}^{-1} = n^{-1},\ \tilde{h}n\tilde{h}^{-1} = n,\ \tilde{g}\tilde{h}\tilde{g}^{-1} = n^k\tilde{h}^{-1}. \tag{4.21}$$

Lemma 4.14. *The groups* ${}_3\pi(0)$, ${}_3\pi(k)$ *are isomorphic to* $\mathcal{G}_2$, $\Gamma(k)$ *respectively. (cf.* (4.7)*.)*

Proof. Let $\tilde{g}, \tilde{h}, n \in {}_3\pi(0)$ be as before. Put $\alpha = \tilde{g}$, $t_1 = \tilde{g}^2$, $t_2 = \tilde{h}$ and $t_3 = n$. Note that the group generated by α, t_1, t_2, t_3 coincides with ${}_3\pi(0)$. By using the relation (4.21), it is easy to check that:

$$\begin{aligned} &\alpha^2 = t_1, \\ &\alpha t_2 \alpha^{-1} = t_2^{-1}, \\ &\alpha t_3 \alpha^{-1} = t_3^{-1}. \end{aligned}$$

And so ${}_3\pi(0)$ is isomorphic to $\mathcal{G}_2$. (See [36].)

Suppose $\tilde{g}, \tilde{h}, n \in {}_3\pi(k)$ $(k \neq 0)$. By the relations (4.6) and (4.21), ${}_3\pi(k)$ is isomorphic to $\Gamma(k)$ (cf. (4.7)). □

Proposition 4.15. $H^2_{\phi_3}(G, \mathbb{Z})$ *is isomorphic to* $\mathbb{Z}$*.*

Proof. From Theorem 4.3 and Lemma 4.14, $\Gamma(k)$ represents the torsionfree element $[f_k]$ in $H^2_{\phi_3}(G, \mathbb{Z})$. Moreover as in the proof of Proposition 4.10, we can show that $[f_k] = k \cdot [f_1]$. Therefore $H^2_{\phi_3}(G, \mathbb{Z})$ is isomorphic to $\mathbb{Z}$. □

Case 4. The group ${}_4\pi(k)$ has the following presentation.

$$\tilde{g} n \tilde{g}^{-1} = n^{-1}, \tilde{h} n \tilde{h}^{-1} = n^{-1}, \tilde{g} \tilde{h} \tilde{g}^{-1} = n^k \tilde{h}^{-1}. \tag{4.22}$$

Put $\alpha = \tilde{g}\tilde{h}$. It is easy to check that

$$\alpha n \alpha^{-1} = n, \ \tilde{h} n \tilde{h}^{-1} = n^{-1}, \ \alpha \tilde{h} \alpha = n^k \tilde{h}^{-1}. \tag{4.23}$$

In view of (4.18), this implies that ${}_4\pi(k)$ is isomorphic to ${}_2\pi(k)$.

We have shown that any element of $H^2_{\phi_i}(Q, \mathbb{Z})$ can be realized an S^1-fibred nilBott manifold M_3, and obtain the following table:

		Case 1	Case 2 and 4	Case 3
	$H^2_\phi(Q, \mathbb{Z})$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
$\pi_1(M_3)$	$[f] = 0$	$\mathcal{B}_1$	$\mathcal{B}_3$	$\mathcal{G}_2$
	$[f] \neq 0$:torsion	$\mathcal{B}_2$	$\mathcal{B}_4$	-
	$[f]$:torsionfree	-	-	$\Gamma(k)$

4.3.4 Realization of S^1-fibration over T^2

Let $\mathbb{Z}^2$ be the fundamental group of a torus T^2 generated by α, β. Given a representation $\phi : \mathbb{Z}^2 \to \mathbb{Z} = \{\pm 1\}$, we shall show that any element of $H^2_\phi(\mathbb{Z}^2, \mathbb{Z})$ can be realized as an S^1-fibred nilBott manifold.

We must consider following cases of a representation ϕ:

Case 5. $\phi(\alpha) = 1,\ \phi(\beta) = 1,$
Case 6. $\phi(\alpha) = 1,\ \phi(\beta) = -1,$
Case 7. $\phi(\alpha) = -1,\ \phi(\beta) = -1.$

Suppose ϕ_i $(i = 5, 6, 7)$ is the representation ϕ for **Case i.** Any element of $H^2_{\phi_i}(\mathbb{Z}^2, \mathbb{Z})$ gives rise to a group extension

$$1 \to \mathbb{Z} \to \pi \xrightarrow{p} \mathbb{Z}^2 \to 1.$$

Then π is generated by $\tilde{\alpha}$, $\tilde{\beta}$, m such that $\langle m \rangle = \mathbb{Z}$ and $p(\tilde{\alpha}) = \alpha$, $p(\tilde{\beta}) = \beta$. There exists $k \in \mathbb{Z}$ which satisfies

$$\tilde{\alpha}\tilde{\beta}\tilde{\alpha}^{-1} = m^k \tilde{\beta} \tag{4.24}$$

Put $\pi = {}_i\pi(k)$ for each $k \in \mathbb{Z}$ and $[f_k]$ denotes the 2-cocycle of $H^2_{\phi_i}(\mathbb{Z}^2, \mathbb{Z})$ representing ${}_i\pi(k)$. Note that $[f_0] = 0$.

Case 5: The group ${}_5\pi(k)$ has the following presentation.

$$\tilde{\alpha} m \tilde{\alpha}^{-1} = m,\ \tilde{\beta} m \tilde{\beta}^{-1} = m,\ \tilde{\alpha}\tilde{\beta}\tilde{\alpha}^{-1} = m^k \tilde{\beta}, \tag{4.25}$$

for some $k \in \mathbb{Z}$. Compared these relations with (4.4),

Proposition 4.16. *The groups ${}_5\pi(0)$, ${}_5\pi(k)$ are isomorphic to $\pi_1(T^3)$, $\pi_1(\Delta(-k))$ respectively.*

Similarly as in the proof of Proposition 4.15, we obtain

Proposition 4.17. *$H^2_{\phi_5}(\mathbb{Z}^2, \mathbb{Z})$ is isomorphic to $\mathbb{Z}$.*

Case 6: The group ${}_6\pi(k)$ has the following presentation.

$$\tilde{\alpha} m \tilde{\alpha}^{-1} = m,\ \tilde{\beta} m \tilde{\beta}^{-1} = m^{-1},\ \tilde{\alpha}\tilde{\beta}\tilde{\alpha}^{-1} = m^k \tilde{\beta}, \tag{4.26}$$

for some $k \in \mathbb{Z}$.

Proposition 4.18. *The groups ${}_6\pi(0)$, ${}_6\pi(1)$ are isomorphic to $\mathcal{B}_1$, $\mathcal{B}_2$ respectively.*

Proof. First let $k = 0$. Put $m = \tilde{h}$, $\tilde{\alpha} = n$, $\tilde{\beta} = \tilde{g}$, then we can check easily that ${}_6\pi(0)$ is isomorphic to ${}_1\pi(0)$. So ${}_6\pi(0)$ is isomorphic to $\mathcal{B}_1$.

Suppose $k = 1$. Put $m = n$, $\tilde{\alpha} = \tilde{g}$, $m^{-1}\tilde{\beta} = \tilde{h}$, then it is easy to check that ${}_6\pi(1)$ is isomorphic to $\mathcal{B}_2$. □

Moreover similarly as in the proof of Proposition 4.12, we obtain

Proposition 4.19. *$H^2_{\phi_6}(\mathbb{Z}^2, \mathbb{Z})$ is isomorphic to $\mathbb{Z}_2$.*

Case 7: The group ${}_7\pi(k)$ has the following presentation.

$$\tilde{\alpha} m^{-1} \tilde{\alpha}^{-1} = m, \ \tilde{\beta} m \tilde{\beta}^{-1} = m^{-1}, \ \tilde{\alpha}\tilde{\beta}\tilde{\alpha}^{-1} = m^k \tilde{\beta}, \tag{4.27}$$

for some $k \in \mathbb{Z}$. Then it is easy to check that ${}_7\pi(k)$ is isomorphic to ${}_6\pi(k)$ if we put $g = \tilde{\alpha}\tilde{\beta}$.

We have shown that any element of $H^2_\phi(\mathbb{Z}^2, \mathbb{Z})$ can be realized an S^1-fibred nilBott manifold M_3, and we obtain the following table:

		Case 5	Case 6 and 7
	$H^2_\phi(\mathbb{Z}^2, \mathbb{Z})$	$\mathbb{Z}$	$\mathbb{Z}_2$
$\pi_1(M_3)$	$[f] = 0$	$\mathcal{G}_1$	$\mathcal{B}_1$
	$[f] \neq 0$:torsion	-	$\mathcal{B}_2$
	$[f]$:torsionfree	$\Delta(k)$	-

Chapter 5

Holomorphic torus Bott tower

5.1 Holomorphic torus-Bott tower

Suppose that there is a tower of fiber bundles:

$$M = M_n \to M_{n-1} \to \ldots \to M_1 \to \{\mathrm{pt}\}. \tag{5.1}$$

Each (M_m, J_m) is a complex manifold such that

$$T^1_{\mathbb{C}} \longrightarrow M_m \longrightarrow M_{m-1} \tag{5.2}$$

is a holomorphic fiber bundle $(m = 1, \ldots, n)$ which induces a group extension:

$$1 \to \mathbb{Z}^2 \to \pi_m \longrightarrow \pi_{m-1} \to 1. \tag{5.3}$$

For $m = 1$, $M_1 = T^1_{\mathbb{C}}$ with $\pi_1 = \mathbb{Z}^2$. Let (X_m, J_m) be the universal covering space of M_m $(m = 1, \ldots, n)$ such that $X_1 = \mathbb{C}$.

Definition 5.1. The *holomorphic torus-Bott tower* is a tower of (5.1) which satisfies the following condition:

(1) There is an equivariant holomorphic principal bundle:

$$(\mathbb{Z}^2, \mathbb{C}) \to (\pi_m, X_m, J_m) \xrightarrow{p_m} (\pi_{m-1}, X_{m-1}, J_{m-1})$$

associated with the group extension (5.3).

(2) Each π_m normalizes the holomorphic action of $\mathbb{C}$.

We call the top space M $(= M_n)$ a *holomorphic torus-Bott manifold* (*of depth* n).

There are several remarks. The condition (2) for m is equivalent to say that $T^1_{\mathbb{C}}{\rightarrow}M_m{\longrightarrow}M_{m-1}$ is a Seifert fiber space in the smooth case. It is not necessarily true that the universal covering X_m is biholomorphic to the product $\mathbb{C}\times X_{m-1}$. So contrary to the smooth case, holomorphic Seifert actions are not described explicitly on the product $\mathbb{C}\times X_{m-1}$ in general. However, our holomorphic Seifert actions on the universal covering of a holomorphic torus-Bott manifold can be described. In fact, let $(X,J)\,(=(X_n,J_n))$ be the universal covering of a holomorphic torus-Bott manifold $M=M_n$. Put $(X_{n-1},J_{n-1})=(\hat{X},\hat{J})$.

Proposition 5.2. *(X,J) is biholomorphic as a holomorphic principal bundle to the product $(\mathbb{C}\times\hat{X},J_0\times\hat{J})$.*

Proof. By Definition 5.1, $X_1=\mathbb{C}$. We assume inductively that $\hat{X}=X_{n-1}$ is biholomorphic to $\mathbb{C}^{n-1}$. By (2) of Definition 5.1, $\mathbb{C}{\rightarrow}X{\longrightarrow}\hat{X}$ is a holomorphic principal bundle. When A_h is the sheaf of germs of (local) holomorphic functions on $\hat{X}$, Oka's principle says that $H^1(\hat{X},A_h)=0$. See [16, p.167-8]. Thus (X,J) is holomorphically *bundle isomorphic* to the product $(\mathbb{C}\times\hat{X},J_0\times\hat{J})$. □

5.1.1 Holomorphic Seifert action

As a consequence of Proposition 5.2, the holomorphic action of π on (X,J) is a holomorphic action of π on $(\mathbb{C}\times\hat{X},J_0\times\hat{J})$. Assume that $(\hat{\pi},\hat{X},\hat{J})$ is a holomorphic action. Let $(\mathbb{Z}^2,\mathbb{C}){\rightarrow}(\pi,\mathbb{C}\times\hat{X},J)\xrightarrow{p}(\hat{\pi},\hat{X},\hat{J})$ be an equivariant holomorphic principal bundle as in (1) of Definition 5.1.

- The group extension $1{\rightarrow}\mathbb{Z}^2{\rightarrow}\pi{\longrightarrow}\hat{\pi}{\rightarrow}1$ represents a cocycle $f:\hat{\pi}\times\hat{\pi}{\rightarrow}\mathbb{Z}^2$ such that each element $\gamma\in\pi$ is viewed as $(n,\alpha)\in\mathbb{Z}^2\times\hat{\pi}$ with group law:

  $$(n,\alpha)(m,\beta)=(n+\phi(\alpha)(m)+f(\alpha,\beta),\alpha\beta).$$

  Here $\phi:\hat{\pi}{\rightarrow}\mathrm{Aut}(\mathbb{Z}^2)$ is the homomorphism induced by conjugation of π.

Since π normalizes the left translations $\mathbb{C}$ on $\mathbb{C}\times\hat{X}$ by (2) of Definition 5.1, we can describe the action of π explicitly;

- There is a holomorphic map $\chi(\alpha):(\hat{X},\hat{J}){\rightarrow}(\mathbb{C},J_0)$ for each $\alpha\in\hat{\pi}$ such that the action $(\pi,\mathbb{C}\times\hat{X})$ is described as

  $$(n,\alpha)(x,w)=(n+\bar{\phi}(\alpha)(x)+\chi(\alpha)(\alpha w),\alpha w) \tag{5.4}$$

  $({}^\forall\,(n,\alpha)\in\pi,{}^\forall\,(x,w)\in\mathbb{C}\times\hat{X}.)$ Here $\bar{\phi}:\hat{\pi}{\rightarrow}\mathrm{Aut}(\mathbb{C})$ is a unique extension of ϕ.

By the definition, (π,X) is a holomorphic Seifert action (cf. [9],[23],[5]).

5.1.2 Topology of holomorphic torus-Bott manifold

From (5.3) there is a homomorphism induced by conjugation: $\phi : \pi_{m-1} \to \mathrm{Aut}(\mathbb{Z}^2)$. Since each element of π_m is almost complex and normalizes $\mathbb{C}$, there exists a matrix $P \in \mathrm{GL}(2, \mathbb{R})$ such that

$$P^{-1} \cdot \phi(\pi_{m-1}) \cdot P \leq \mathrm{U}(1).$$

If we let $P^{-1} \cdot \phi(\alpha) \cdot P = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ for $\alpha \in \pi_{m-1}$, then the trace condition shows that $\cos\theta = 0, \pm\frac{1}{2}, \pm 1$. It follows respectively

$$\phi(\alpha) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{\pm 1}, \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^{\pm 1}, \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}. \tag{5.5}$$

So ϕ extends uniquely to an automorphism: $\bar{\phi} : \pi_{m-1} \to \mathrm{Aut}_J(\mathbb{C}) = \mathbb{C}^*$ such as

$$\bar{\phi}(\alpha) = \pm \mathbf{i},\ e^{\pm \mathbf{i}\pi/3} \text{ or } \pm 1 \ (\forall\, \alpha \in \pi_{m-1}) \text{ respectively.} \tag{5.6}$$

In particular $\bar{\phi}(\pi_{m-1})$ is a cyclic group of order $1, 2, 4$ or 6.

Lemma 5.3. *Each π_m is virtually nilpotent.*

Proof. As $\mathbb{Z}^2 = \pi_1$, we suppose inductively that π_{m-1} is virtually nilpotent. Since $\phi(\pi_{m-1}) \leq \mathrm{Aut}(\mathbb{Z}^2)$ is a finite cyclic group, we choose a finite index normal nilpotent subgroup Δ_{m-1} of π_{m-1} such that $\phi(\Delta_{m-1}) = \{1\}$. Then the group extension of (5.3) induces a central extension:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & \pi_m & \longrightarrow & \pi_{m-1} & \longrightarrow & 1 \\ & & \| & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & \Delta_m & \longrightarrow & \Delta_{m-1} & \longrightarrow & 1. \end{array} \tag{5.7}$$

And hence Δ_m is nilpotent which proves the induction step. □

Given a holomorphic torus-Bott manifold M, there is a holomorphic fiber bundle : $T^1_{\mathbb{C}} \to M \longrightarrow M_{n-1}$. As the fundamental group π of M is virtually nilpotent, there exist a simply connected nilpotent Lie group N and a discrete faithful homomorphism $\rho : \pi \to \Gamma \leq \mathrm{E}(N)$ such that the quotient N/Γ is an infranilmanifold. (Compare [1] for instance.) It follows from *Seifert rigidity* for nil-fiber ([19], see also [18],[23]) that

Proposition 5.4. *Any holomorphic torus-Bott manifold M is diffeomorphic to an infranilmanifold N/Γ.*

Moreover, the diffeomorphism h between them preserves the fiber, i.e. there is the commutative diagram of equivariant diffeomorphisms:

$$\begin{array}{ccccc} (\mathbb{Z}^2, \mathbb{C}) & \longrightarrow & (\pi, X) & \xrightarrow{p} & (\hat{\pi}, \hat{X}) \\ {\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle \tilde{h}}\downarrow & & {\scriptstyle \hat{h}}\downarrow \\ (\mathbb{Z}^2, \mathbb{C}) & \longrightarrow & (\Gamma, N) & \xrightarrow{p} & (\hat{\Gamma}, \hat{N}) \end{array} \tag{5.8}$$

5.2 Invariant metric on a nilpotent Lie group

5.2.1 Holomorphic infranilmanifolds

Let N be a simply connected nilpotent Lie group with left invariant complex structure J. Denote by $\mathrm{Aut}_J(N)$ the group of automorphisms of N which preserve J, i.e. $\alpha_* \circ J = J \circ \alpha_*$ on $T_1 N$. Choose a maximal compact subgroup K from $\mathrm{Aut}_J(N)$ and put $\mathrm{E}_J(N) = N \rtimes K$. Each element $h = (a, \alpha) \in \mathrm{E}_J(N)$ acts on N as $h(x) = a \cdot \alpha(x)$ ($^\forall x \in N$). Then $\mathrm{E}_J(N) = N \rtimes K$ acts holomorphically on N. If Γ is a discrete (torsionfree) uniform subgroup of $\mathrm{E}_J(N)$, a quotient N/Γ is said to be a *holomorphic infranilorbifold* (*infranilmanifold*). It is well known that a finite cover of N/Γ is a nilmanifold.

5.2.2 Construction of $\mathrm{E}(N)$-invariant complex structure

Let N be a simply connected nilpotent Lie group which has a central group extension: $1 \to \mathbb{C} \to N \xrightarrow{\pi} \hat{N} \to 1$. Let $\mathrm{E}(N) = N \rtimes K$ be the semidirect product. As $\mathbb{C}$ is normal in $\mathrm{E}(N)$, π induces an equivariant (continuous) homomorphism

$$\pi : (\mathrm{E}(N), N) \to (\mathrm{E}(\hat{N}), \hat{N}). \tag{5.1}$$

As $K \leq \mathrm{Aut}(N)$ normalizes $\mathbb{C}$, there is a homomorphism $\rho : K \to \mathrm{GL}(2, \mathbb{R})$. In order to be holomorphic on $\mathbb{C}$, we require that $\rho(K) \leq \mathrm{U}(1) \leq \mathrm{GL}(1, \mathbb{C}) = \mathrm{Aut}(\mathbb{C})$. Equivalently, for $^\forall k \in K$,

$$k_* \circ J_0 = J_0 \circ k_* \text{ on } T\mathbb{C}. \tag{5.2}$$

Suppose that $\hat{J}$ is a left invariant complex structure on the $2n-2$-dimensional nilpotent Lie group $\hat{N}$. Similarly as before, $\mathrm{E}_{\hat{J}}(\hat{N})$ denotes the holomorphic semidirect product $\hat{N} \rtimes \hat{K}$ of $\hat{N}$ with a compact group $\hat{K} \leq \mathrm{Aut}_{\hat{J}}(\hat{N})$.

Proposition 5.5. *There exists a* $\mathrm{E}(N)$*-invariant complex structure on* N *under the requirement* (5.2)*. Moreover,*

$$(\mathbb{C}, J_0) \to (N, J) \xrightarrow{\pi} (\hat{N}, \hat{J})$$

is a principal holomorphic bundle.

Proof. Choose an N-invariant Riemannian metric on N and average it by the compact group K. Since K normalizes N, this gives a $\mathrm{E}(N)$-invariant Riemannian metric g on N. Let $T\mathbb{C}^\perp = \{X \in TN \mid g(X, A) = 0, {}^\forall A \in T\mathbb{C}\}$. As g is $\mathrm{E}(N)$-invariant and $\mathbb{C}$ is normal in $\mathrm{E}(N)$, it is easy to see that $T\mathbb{C}^\perp$ is $\mathrm{E}(N)$-invariant. Then the projection $\pi : N \to \hat{N}$ induces an isomorphism $\pi_* : T\mathbb{C}^\perp \to T\hat{N}$ at each point of N. Define an almost complex structure J on $T\mathbb{C}^\perp$ by the following correspondence at each point of N:

$$\pi_* J X = \hat{J} \pi_* X. \tag{5.3}$$

Let J_0 be the standard complex structure on $\mathbb{C}^k$ ($k \geq 1$). If we note that $TN = T\mathbb{C} \oplus T\mathbb{C}^{\perp}$, then we define

$$J(A+X) = J_0 A + JX \quad (A \in T\mathbb{C}, X \in T\mathbb{C}^{\perp}). \tag{5.4}$$

It follows that J is an *almost complex* structure on N. Since $\mathrm{E}(N)$ leaves invariant $T\mathbb{C}^{\perp}$ and normalizes $\mathbb{C}$, the decomposition is preserved by any element $h \in \mathrm{E}(N)$; $h_* A + h_* X \in T\mathbb{C} \oplus T\mathbb{C}^{\perp}$. Using (5.1), the hypothesis that $\hat{J}$ is $\mathrm{E}(\hat{N})$-invariant shows that

$$\begin{aligned}\pi_*(h_* JX) &= \pi(h)_* \pi_*(JX) = \pi(h)_* \hat{J} \pi_*(X) \\ &= \hat{J} \pi(h)_* \pi_*(X) = \hat{J} \pi_*(h_* X) \\ &= \pi_*(J h_* X),\end{aligned}$$

and so $h_* JX = J h_* X$ ($\forall\, X \in T\mathbb{C}^{\perp}$). As $\mathbb{C}$ is the center of N, $x_* J_0 = J_0 x_*$ on $T\mathbb{C}$ ($\forall\, x \in N$). Each $\alpha \in K$ satisfies that $\alpha_* J_0 = J_0 \alpha_*$ on $T\mathbb{C}$ by our requirement (5.2). In particular, if $h = (x, \alpha) \in \mathrm{E}(N)$, then $h_* J_0 = J_0 h_*$ on $T\mathbb{C}$. Taking into account these equalities,

$$\begin{aligned}J h_*(A+X) &= J_0 h_* A + J h_* X \\ &= h_* J_0 A + h_* JX = h_* J(A+X),\end{aligned}$$

and hence J is $\mathrm{E}(N)$-invariant. Obviously $(\mathbb{C}, J_0) \to (N, J) \xrightarrow{\pi} (\hat{N}, \hat{J})$ is an almost complex principal fiber bundle with respect to J. Let $\varphi : (\pi^{-1}(U), J) \to (U \times \mathbb{C}, J_0 \times \hat{J})$ be a local trivialization isomorphism for this bundle. As $\hat{J}$ is a complex structure by the hypothesis, so is J on N. $\square$

5.2.3 Trivialization

Let $(\mathbb{C}, J_0) \to (N, J) \xrightarrow{\pi} (\hat{N}, \hat{J})$ be a principal holomorphic bundle from Proposition 5.5. We assume that $(\hat{N}, \hat{J})$ is biholomorphic to $(\mathbb{C}^{n-1}, J_0)$. By Proposition 5.2 we have

Corollary 5.6. (N, J) *is biholomorphic as a holomorphic principal bundle to the product* $(\mathbb{C} \times \hat{N}, J_0 \times \hat{J})$.

Let $\mathrm{E}_J(N) = N \rtimes K$ be the holomorphic semidirect product. Choose a torsionfree discrete cocompact subgroup Γ from $\mathrm{E}_J(N)$ so that N/Γ is a holomorphic infranilmanifold.

5.3 Holomorphic infranil action

We observe that a holomorphic infranilmanifold N/Γ will be a holomorphic Seifert manifold.

The central group extension $1 \to \mathbb{C} \to N \xrightarrow{\pi} \hat{N} \to 1$ is viewed as a holomorphic principal bundle by Proposition 5.5. Under the hypothesis in Subsection 5.2.3, Corollary 5.6 shows that $N = \mathbb{C} \times \hat{N}$ biholomorphically with group law

$$(x, z) \cdot (y, w) = (x + y + \mathsf{f}(z, w), z \cdot w). \tag{5.1}$$

Here $\mathsf{f} : \hat{N} \times \hat{N} \to \mathbb{C}$ is a 2-cocycle. Put $\mathrm{E}(N) = \mathrm{E}_J(N)$ for brevity. Since $\mathrm{E}(N)$ normalizes $\mathbb{C}$, there is a commutative diagram of the exact sequences:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{C} & \longrightarrow & N & \xrightarrow{\pi} & \hat{N} & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbb{C} & \longrightarrow & \mathrm{E}(N) & \xrightarrow{\pi} & \hat{N} \circ K & \longrightarrow & 1 \end{array} \tag{5.2}$$

where we put $\mathrm{E}(N)/\mathbb{C} = \hat{N} \circ K$. As $\mathrm{E}(N)$ is the semidirect product $N \rtimes K$, $\hat{N} \circ K$ has the group law; for $\alpha = a \circ k, \beta = b \circ h \in \hat{N} \circ K$,

$$\alpha \cdot \beta = ak(b) \circ kh.$$

As $K \leq \mathrm{Aut}(N)$, there is a homomorphism $\hat{\rho} : K \to \mathrm{Aut}(\hat{N})$. If we recall that $\hat{K}$ is the maximal compact subgroup of $\mathrm{Aut}(\hat{N})$, $\hat{\rho}(K) \leq \hat{K}$ up to conjugate. It follows that

$$\hat{N} \circ K = \hat{N} \rtimes \hat{\rho}(K) \leq \mathrm{E}(\hat{N}). \tag{5.3}$$

Let $\phi : \hat{N} \circ K \to \mathrm{Aut}(\mathbb{C})$ be a homomorphism induced by the conjugation from (5.2). Then $\mathrm{E}(N)$ is viewed as the set $\mathbb{C} \times (\hat{N} \circ K)$ with group law

$$(x, \alpha) \cdot (y, \beta) = (x + \phi(\alpha)(y) + \bar{\mathsf{f}}(\alpha, \beta), \alpha \cdot \beta) \tag{5.4}$$

in which $\bar{\mathsf{f}} : \hat{N} \circ K \times \hat{N} \circ K \to \mathbb{C}$ is a 2-cocycle extending f on $\hat{N} \times \hat{N}$ of (5.1). The action of $\mathrm{E}(N)$ on N is interpreted in terms of group law (5.4): $\mathrm{E}(N) \times \mathrm{E}(N) \to \mathrm{E}(N) \longrightarrow N$; let $\alpha = a \circ k \in \hat{N} \circ K$ with $(x, \alpha) \in \mathrm{E}(N)$ and $b \in \hat{N}$ with $(y, b) \in N$. Then

$$\begin{aligned} (x, \alpha) \cdot (y, b) &= (x + \phi(\alpha)(y) + \bar{\mathsf{f}}(\alpha, b), ak(b) \circ k) \\ &\mapsto (x + \phi(\alpha)(y) + \bar{\mathsf{f}}(\alpha, b), ak(b)) \in N. \end{aligned} \tag{5.5}$$

As in Section 5.2.1, $\mathrm{E}(N)$ normalizes $\mathbb{C}$ so the holomorphic action of $\mathrm{E}(N)$ on N induces a holomorphic action of $\hat{N} \circ K$ on $\hat{N}$ by $\alpha b = ak(b)$ $({}^\forall \alpha = a \circ k \in \hat{N} \circ K, {}^\forall b \in \hat{N})$. Associated with the group extensions of (5.2), we obtain a *holomorphic Seifert fibration* by the definition of Section 5.1.1:

$$(\mathbb{C}, \mathbb{C}) \longrightarrow (\mathrm{E}(N), N) \xrightarrow{\pi} (\hat{N} \circ K, \hat{N})$$

where $N = \mathbb{C} \times \hat{N}$ biholomorphically. Let ${}^\forall (y, w) \in N$. If $h = (x, \alpha) \in \mathrm{E}(N)$ with $\alpha\,(= a \cdot k) \in \hat{N} \circ K$, then as in (5.4) the *holomorphic Seifert action* implies that there is a holomorphic map $\mu(\alpha) : (\hat{N}, \hat{J}) \to (\mathbb{C}, J_0)$ such that

$$h(y, w) = (x + \phi(\alpha)(y) + \mu(\alpha)(\alpha w), \alpha w). \tag{5.6}$$

Using μ (cf. [23]), $\bar{\mathsf{f}} : \hat{N} \circ K \times \hat{N} \circ K \to \mathbb{C}$ is described as $\bar{\mathsf{f}}(\alpha, \beta) = \delta^1 \mu(\alpha, \beta)(w)$ $({}^\forall w \in \hat{N})$, i.e.

$$\begin{gathered} \bar{\mathsf{f}}(\alpha, \beta) = \phi(\alpha)(\mu(\beta)(\alpha^{-1} \cdot w)) + \mu(\alpha)(w) - \mu(\alpha\beta)(w) \\ ({}^\forall \alpha, \beta \in \hat{N} \circ K, {}^\forall w \in \hat{N}). \end{gathered} \tag{5.7}$$

Here the set $hol(\hat{N}, \mathbb{C})$ is a $\hat{N} \circ \hat{K}$-module defined by

$$(\alpha \cdot g)(w) = \phi(\alpha)(g(\alpha^{-1} \cdot w)) \;\; ({}^\forall g \in hol(\hat{N}, \mathbb{C}), {}^\forall \alpha \in \hat{N} \circ K). \tag{5.8}$$

5.3.1 Holomorphic Seifert manifold

Consider a torsionfree discrete uniform subgroup Γ lying in $\mathrm{E}(N) = \mathrm{E}_J(N)$:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & \Gamma & \stackrel{\pi}{\longrightarrow} & \hat{\Gamma} & \longrightarrow & 1 \\ & & \cap & & \cap & & \cap & & \\ 1 & \longrightarrow & \mathbb{C} & \longrightarrow & \mathrm{E}(N) & \stackrel{\pi}{\longrightarrow} & \hat{N} \circ K & \longrightarrow & 1. \end{array} \tag{5.9}$$

Here $\mathbb{Z}^2 = \mathbb{C} \cap \Gamma$ and $\hat{\Gamma} = \pi(\Gamma)$. Then the group extension of Γ is represented by a 2-cocycle $[f] \in H^2_\phi(\hat{\Gamma}; \mathbb{Z}^2)$ where $\phi = \phi_{|\hat{\Gamma}} : \hat{\Gamma} \to \mathrm{Aut}(\mathbb{Z}^2)$ is a homomorphism restricted to $\hat{\Gamma}$. Note that $\mathbb{Z}^2$ is a $\hat{\Gamma}$-module through ϕ. In view of (5.6), we have shown that

Proposition 5.7. *Given a holomorphic infranil-action of* $\hat{\Gamma}$ *(i.e.* $\hat{\Gamma} \leq \mathrm{E}_{\hat{J}}(\hat{N})$*), a holomorphic infranil-action of* Γ *on* (N, J) *is a holomorphic Seifert action of* Γ *on* $(\mathbb{C} \times \hat{N}, J_0 \times \hat{J})$ *which can be determined by a holomorphic map* $\mu(\alpha) : \hat{N} \to \mathbb{C}$ *for each* $\alpha \in \hat{\Gamma}$ *such as*

$$\begin{aligned} (n, \alpha)(x, w) = (n + \phi(\alpha)(x) + \mu(\alpha)(\alpha w), \alpha w) \\ ({}^\forall\, (n, \alpha) \in \Gamma, {}^\forall\, (x, w) \in N). \end{aligned} \tag{5.10}$$

Moreover, the cocycle f *representing the group extension of* Γ *in* (5.9) *satisfies* $\delta^1 \mu = f$ *as in* (5.7).

Comparing (5.5) with (5.10), it follows that

$$\bar{\mathsf{f}}(\alpha, w) = \mu(\alpha)(\alpha w) \ \ ({}^\forall\, \alpha \in \hat{\Gamma}, {}^\forall\, w \in \hat{N}). \tag{5.11}$$

5.4 Deformation of nilpotent Lie groups

Let $\mathrm{hol}(\hat{N}, \mathbb{C})$ be the set of all holomorphic maps from $(\hat{N}, \hat{J})$ to $\mathbb{C}$. It is endowed with a $\hat{\Gamma}$-module as in (5.8), similarly for $\mathrm{hol}(\hat{N}, T^1_{\mathbb{C}})$ and $\mathbb{Z}^2$ (cf. Section 5.3.1).

Recall that a short exact sequence $1 \to \mathbb{Z}^2 \stackrel{i}{\to} \mathbb{C} \stackrel{j}{\longrightarrow} T^1_{\mathbb{C}} \to 1$ induces a long cohomology exact sequence (cf. [23], [5]):

$$\begin{aligned} \to H^1_\phi(\hat{\Gamma}; \mathbb{Z}^2) \stackrel{i}{\longrightarrow} H^1_\phi(\hat{\Gamma}; \mathrm{hol}(\hat{N}, \mathbb{C})) \stackrel{j}{\longrightarrow} H^1_\phi(\hat{\Gamma}; \mathrm{hol}(\hat{N}, T^1_{\mathbb{C}})) \\ \stackrel{\delta}{\longrightarrow} H^2_\phi(\hat{\Gamma}; \mathbb{Z}^2) \to \cdots \end{aligned} \tag{5.1}$$

Put $\hat{\mu} = j \circ \mu : \hat{N} \to T^1_{\mathbb{C}}$ for a holomorphic function μ of Proposition 5.7. Then (5.10) implies that $\delta[\hat{\mu}] = [f]$ by the definition. For any element $[\nu] \in H^1(\hat{\Gamma}; \mathrm{hol}(\hat{N}, \mathbb{C}))$, we have an element $j[\nu] \cdot [\hat{\mu}]$ such that $\delta(j[\nu] \cdot [\hat{\mu}]) = [f]$. Note that j maps $\mu + \nu$ to $j\nu \cdot \hat{\mu}$. From Proposition 5.7, $\delta^1 \mu = f$ and so it follows $\delta^1(\mu + \nu) = f$ which still defines the same group extension: $1 \to \mathbb{Z}^2 \to \Gamma \longrightarrow \hat{\Gamma} \to 1$.

We study a holomorphic Seifert action of Γ by this replacement $\mu+\nu$ which is given by

$$(n,\alpha)(x,w)=(n+\phi(\alpha)(x)+\mu(\alpha)(\alpha w)+\nu(\alpha)(\alpha w),\alpha w)$$
$$(n\in\mathbb{Z}^2,\alpha\in\hat{\Gamma},(x,w)\in N). \tag{5.2}$$

Theorem 5.8. *There exists a nilpotent Lie group N' isomorphic to N such that the complex structure J is invariant under $\mathrm{E}(N')$. The above action (Γ,N) is equivariantly biholomorphic to an infranil-action of Γ' on N', (i.e. $\Gamma'\leq \mathrm{E}_J(N')$). Here Γ' is a discrete uniform subgroup isomorphic to Γ. Specifically the quotient N/Γ is biholomorphic to the holomorphic infranilmanifold N'/Γ'. (In particular $\Delta'=\Gamma'\cap N'$ is a finite index subgroup of Γ' such that N'/Δ' is a holomorphic nilmanifold.)*

Proof. First when we take a $\hat{\Gamma}$-module $\mathrm{Map}(\hat{N},\mathbb{C})$ consisting of smooth maps from $\hat{N}$ to $\mathbb{C}$ instead of $\mathrm{hol}(\hat{N},\mathbb{C})$, we note that

$$H^q_\phi(\hat{\Gamma};\mathrm{Map}(\hat{N},\mathbb{C}))=0\ \ (q\geq 1).$$

(See [7], [23].)
If $[\nu]\in H^1_\phi(\hat{\Gamma};\mathrm{hol}(\hat{N},\mathbb{C}))$ is relaxed to be in $H^1_\phi(\hat{\Gamma};\mathrm{Map}(\hat{N},\mathbb{C}))$, then there is an element $\lambda\in\mathrm{Map}(\hat{N},\mathbb{C})$ such that $\delta^1\lambda=\nu$, i.e. $\nu(\alpha)(\alpha w)=\delta^1\lambda(\alpha)(\alpha w)=\alpha\circ\lambda(\alpha w)-\lambda(\alpha w)$, it follows (cf. (5.8))

$$\nu(\alpha)(\alpha w)=\phi(\alpha)(\lambda(w))-\lambda(\alpha w)\ \ (^\forall\,\alpha\in\hat{\Gamma},{}^\forall\,w\in\hat{N}). \tag{5.3}$$

A function $\mathsf{f}':\hat{N}\times\hat{N}\to\mathbb{C}$ is defined to be

$$\mathsf{f}'(z,w)=\mathsf{f}(z,w)+\delta^1\lambda(z,w)\ \ (z,w\in\hat{N}). \tag{5.4}$$

As $1\to\mathbb{C}\to N\to\hat{N}\to 1$ is a central extension, $\delta^1\lambda(z,w)=z\cdot\lambda(w)-\lambda(z\cdot w)+\lambda(z)=\lambda(z)+\lambda(w)-\lambda(z\cdot w)$. It is easy to see that $\delta^1\mathsf{f}'=0$ so f' is a 2-cocycle in $H^2(\hat{N};\mathbb{C})$. Let $N'=\mathbb{C}\times\hat{N}$ be the product with group law:

$$(x,z)\circ(y,w)=(x+y+\mathsf{f}'(z,w),z\cdot w).$$

N' becomes a Lie group. Moreover, if $\varphi:N\to N'$ is a map defined by

$$\varphi(x,z)=(x-\lambda(z),z), \tag{5.5}$$

then

$$\begin{aligned}
&\varphi((x,z)\cdot(y,w))=\varphi(x+y+\mathsf{f}(z,w),z\cdot w)\\
&=(x+y+\mathsf{f}(z,w)-\lambda(z\cdot w),z\cdot w)\\
&=(x+y+\mathsf{f}(z,w)+\delta^1\lambda(z,w)-\lambda(z)-\lambda(w),z\cdot w)\\
&=(x+y+\mathsf{f}'(z,w)-\lambda(z)-\lambda(w),z\cdot w)\\
&=(x-\lambda(z),z)\circ(y-\lambda(w),w)=\varphi(x,z)\circ\varphi(y,w).
\end{aligned} \tag{5.6}$$

Thus $\varphi:N\to N'$ is a Lie group isomorphism.

Let $\lambda : \hat{N} \to \mathbb{C}$ be the map as above. We extend λ to $\hat{N} \circ K$. Let $\alpha = a \cdot k \in \hat{N} \circ K$. Since $K \leq \mathrm{Aut}(N)$, evaluated at $1 \in N$, we simply put

$$\bar{\lambda}(\alpha) = \lambda(a). \tag{5.7}$$

We can define a 2-cocycle $\bar{\mathsf{f}}' : (\hat{N} \circ K) \times (\hat{N} \circ K) \to \mathbb{C}$ to be

$$\bar{\mathsf{f}}'(\alpha, \beta) = \bar{\mathsf{f}}(\alpha, \beta) + \delta^1 \bar{\lambda}(\alpha, \beta) \ \ (\alpha, \beta \in \hat{N} \circ K), \tag{5.8}$$

where

$$\delta^1 \bar{\lambda}(\alpha, \beta) = \phi(\alpha)(\bar{\lambda}(\beta)) - \bar{\lambda}(\alpha\beta) + \bar{\lambda}(\alpha). \tag{5.9}$$

Then we have a group G as the set $\mathbb{C} \times (\hat{N} \circ K)$ with group law:

$$(x, \alpha) \circ (y, \beta) = (x + \phi(\alpha)(y) + \bar{\mathsf{f}}'(\alpha, \beta), \alpha\beta). \tag{5.10}$$

By construction, there is an exact sequence: $1 \to N' \to G \xrightarrow{\pi} K \to 1$. As N' is a simply connected nilpotent Lie group, it follows that $G = N' \rtimes K'$ for which π maps K' isomorphically onto K. In particular, $G = \mathrm{E}(N')$. As in (5.6), if we define $\bar{\varphi} : \mathrm{E}(N) \to \mathrm{E}(N') = G$ to be

$$\bar{\varphi}(x, \alpha) = (x - \bar{\lambda}(\alpha), \alpha), \tag{5.11}$$

then

$$\begin{aligned}
\bar{\varphi}((x, \alpha) \cdot (y, \beta)) &= (x + \phi(\alpha)(y) + \bar{\mathsf{f}}(\alpha, \beta) - \bar{\lambda}(\alpha\beta), \alpha\beta) \\
&= (x + \phi(\alpha)(y) + \bar{\mathsf{f}}(\alpha, \beta) + \delta^1 \bar{\lambda}(\alpha, \beta) \\
&\qquad\qquad - \phi(\alpha)(\bar{\lambda}(\beta)) - \bar{\lambda}(\alpha),\ \alpha\beta) \\
&= (x + \phi(\alpha)(y) + \bar{\mathsf{f}}'(\alpha, \beta) \\
&\qquad\qquad - \phi(\alpha)(\bar{\lambda}(\beta)) - \bar{\lambda}(\alpha),\ \alpha\beta) \\
&= (x - \bar{\lambda}(\alpha), \alpha) \circ (y - \bar{\lambda}(\beta), \beta) = \bar{\varphi}(x, \alpha) \circ \bar{\varphi}(y, \beta).
\end{aligned} \tag{5.12}$$

Hence $\bar{\varphi} : \mathrm{E}(N) \to \mathrm{E}(N')$ is an isomorphism. By the formula (5.11), $\bar{\varphi}_{|\mathbb{C}} = \mathrm{id}$ and the induced homomorphism $\hat{\varphi} : \hat{N} \to \hat{N}$ of $\bar{\varphi}$ is id on $\hat{N} \circ K$. There induces the following exact sequences:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{C} & \longrightarrow & \mathrm{E}(N) & \xrightarrow{\pi} & \hat{N} \circ K & \longrightarrow & 1 \\
 & & {\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle \bar{\varphi}}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow & & \\
1 & \longrightarrow & \mathbb{C} & \longrightarrow & \mathrm{E}(N') & \xrightarrow{\pi} & \hat{N} \circ K & \longrightarrow & 1.
\end{array} \tag{5.13}$$

We recall the infranil-action of $\mathrm{E}(N')$ on N'. As in (5.5), for $\alpha = a \circ k \in \hat{N} \circ K$ with $(x, \alpha) \in \mathrm{E}(N')$ and $w \in \hat{N}$ with $(y, w) \in N'$, it follows

$$\begin{aligned}
(x, \alpha) \circ (y, w) &= (x + \phi(\alpha)(y) + \bar{\mathsf{f}}'(\alpha, w), ak(w) \circ k) \\
&\mapsto (x + \phi(\alpha)(y) + \bar{\mathsf{f}}'(\alpha, w), \alpha w) \in N'
\end{aligned} \tag{5.14}$$

where $\alpha w = ak(w)$. So we put this infranil-action $(\mathrm{E}(N'), N')$ as

$$(x, \alpha) \circ' (y, w) = (x + \phi(\alpha)(y) + \bar{\mathsf{f}}'(\alpha, w), \alpha w) \tag{5.15}$$

Let $\Gamma \leq \mathrm{E}(N)$ be as above. As in (5.9), there is the commutative diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & \Gamma & \xrightarrow{\pi} & \hat{\Gamma} & \longrightarrow & 1 \\
 & & \mathrm{id}\downarrow & & \bar{\varphi}\downarrow & & \mathrm{id}\downarrow & & \\
1 & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & \bar{\varphi}(\Gamma) & \xrightarrow{\pi} & \hat{\Gamma} & \longrightarrow & 1 \\
 & & \cap & & \cap & & \cap & & \\
1 & \longrightarrow & \mathbb{C} & \longrightarrow & \mathrm{E}(N') & \xrightarrow{\pi} & \hat{N} \circ K & \longrightarrow & 1.
\end{array} \tag{5.16}$$

Using (5.11), (5.3) and (5.8), (5.9), the action (5.2) becomes

$$\begin{aligned}
(n,\alpha)(x,w) &= (n+\phi(\alpha)(x)+\mu(\alpha)(\alpha w)+\nu(\alpha)(\alpha w),\, \alpha w) \\
&= (n+\phi(\alpha)(x)+\bar{\mathsf{f}}(\alpha,w)+\phi(\alpha)(\bar{\lambda}(w))-\bar{\lambda}(\alpha w),\, \alpha w) \\
&= (n+\phi(\alpha)(x)+\bar{\mathsf{f}}(\alpha,w)+\delta^1\bar{\lambda}(\alpha,w)-\bar{\lambda}(\alpha),\, \alpha w) \\
&= (n+\phi(\alpha)(x)+\bar{\mathsf{f}}'(\alpha,w)-\bar{\lambda}(\alpha),\, \alpha w) \\
&= (n-\bar{\lambda}(\alpha),\alpha)\circ'(x,w) = \bar{\varphi}(n,\alpha)\circ'(x,w),
\end{aligned} \tag{5.17}$$

where $\circ'$ is defined in (5.14). Hence the action of (Γ, N) is equivalent with the infranil-action of $\bar{\varphi}(\Gamma)$ on N' defined in (5.15).

On the other hand, there is a $E(N')$-invariant complex structure J' on N' by Proposition 5.5 such that (N', J') is biholomorphic to $(\mathbb{C}\times\hat{N}, J_0\times\hat{J})$ by Corollary 5.6. By Proposition 5.7, there exists an element $\mu'(\alpha) \in \mathrm{hol}(\hat{N},\mathbb{C})$ $({}^{\forall}\,\alpha\in\hat{\Gamma})$ for which a holomorphic infranil-action of $\bar{\varphi}(\Gamma)$ on (N', J') is obtained as

$$\bar{\varphi}(n,\alpha)\circ'(x,w) = (n+\phi(\alpha)(x)+\mu'(\alpha)(\alpha w),\alpha w). \tag{5.18}$$

Compared this with (5.17), we obtain

$$\mu(\alpha)(\alpha w)+\nu(\alpha)(\alpha w) = \mu'(\alpha)(\alpha w). \tag{5.19}$$

For arbitrary $A \in T\mathbb{C}, V \in T\hat{N}$, calculate

$$\begin{aligned}
&(n,\alpha)_*J(A,V) = (n,\alpha)_*(J_0A,\hat{J}V) \\
&= (\phi(\alpha)(J_0A)+\mu(\alpha)_*(\alpha_*\hat{J}V)+\nu(\alpha)_*(\alpha_*\hat{J}V),\alpha_*\hat{J}V) \\
&= (J_0\phi(\alpha)(A)+J_0\mu(\alpha)_*(\alpha_*V)+J_0\nu(\alpha)_*(\alpha_*V),\hat{J}\alpha_*V) \\
&= (J_0\phi(\alpha)(A)+J_0\mu'(\alpha)_*(\alpha_*V),\hat{J}\alpha_*V) \\
&= J'(\phi(\alpha)(A)+\mu'(\alpha)_*(\alpha_*V),\alpha_*V) \\
&= J'\bar{\varphi}(n,\alpha)_*(A,V).
\end{aligned} \tag{5.20}$$

As $(n,\alpha)_*J = J(n,\alpha)_*$ on TN, it follows $J' = J$ on $\mathbb{C}\times\hat{N} = N = N'$. And hence the holomorphic action (Γ, N, J) is equivariantly biholomorphic to $(\varphi(\Gamma), N', J)$. Equivalently the quotient N/Γ is biholomorphic to the holomorphic infranilmanifold $N'/\bar{\varphi}(\Gamma)$. □

5.5 Holomorphic classification

Let M be a holomorphic torus-Bott manifold of dimension $2n$. By Definition 5.1, $X_1 = \mathbb{C}$. We assume inductively that X_{n-1} is biholomorphic to $\mathbb{C}^{n-1}$. By (2) of Definition 5.1, $\mathbb{C} \to X = X_n \longrightarrow \hat{X} = X_{n-1}$ is a holomorphic principal bundle. Thus by Corollary 5.6, (X, J) is biholomorphic to the product $(\mathbb{C} \times \hat{X}, J_0 \times \hat{J})$ as a holomorphic bundle. Hence the action on the universal covering (X, π, J) is identified with a holomorphic Seifert action $(\mathbb{C} \times \hat{X}, \pi, J_0 \times \hat{J})$ as in (5.4).

Consider the associated group extension $1 \to \mathbb{Z}^2 \to \pi \longrightarrow \hat{\pi} \to 1$ which represents a 2-cocycle $[f] \in H_\phi(\hat{\pi}; \mathbb{Z}^2)$. As $(\pi, \mathbb{C} \times \hat{X})$ is a holomorphic Seifert action, there is a holomorphic map $\chi(\alpha) : \hat{N} \to \mathbb{C}$ for each $\alpha \in \hat{\pi}$ such that

$$\begin{aligned} (n, \alpha)(x, w) = (n + \bar{\phi}(\alpha)(x) + \chi(\alpha)(\alpha w), \alpha w) \\ ({}^\forall\, (n, \alpha) \in \pi, {}^\forall\, (x, w) \in \mathbb{C} \times \hat{N}), \end{aligned} \tag{5.1}$$

which satisfies that

$$\delta[\hat{\chi}] = [f]. \tag{5.2}$$

By Corollary 5.4, X/π is diffeomorphic to an infranilmanifold N/π. Suppose that $(\hat{X}, \hat{\pi}, \hat{J})$ is equivariantly biholomorphic to $(\hat{N}, \hat{\pi}, \hat{J})$ for which $\pi \leq \mathrm{E}(N) = N \rtimes K$. Since $\phi : \hat{\pi} \to \mathrm{Aut}(\mathbb{Z}^2)$ satisfies that $\phi(\hat{\pi}) \leq \mathrm{U}(1)$ from (5.6), we may assume that K satisfies the requirement (5.2) of Proposition 5.5. (In fact, as N centralizes $\mathbb{C}$ and $N \rtimes K$ normalizes $\mathbb{C}$, the conjugation map $\rho : N \rtimes K \to \mathrm{GL}(2, \mathbb{R})$ satisfies that $\rho(N \rtimes K) = \rho(K) \leq \mathrm{O}(2)$ in general. Taking $\mathrm{U}(1) \leq \mathrm{O}(2)$, we choose $K_0 \leq K$ such that $\rho(K_0) \leq \mathrm{U}(1)$ instead of K. As $\rho(\pi) = \phi(\hat{\pi}) \leq \mathrm{U}(1)$, it follows $\pi \leq N \rtimes K_0$ which satisfies the requirement obviously.)

By Proposition 5.5, there exists a $\mathrm{E}(N)$-invariant complex structure J such that $\pi \leq \mathrm{E}_J(N)$, i.e. the action (N, π) is a holomorphic infranil-action. As (N, J) is biholomorphic to $(\mathbb{C} \times \hat{N}, J_0 \times \hat{J})$ by Corollary 5.6, Proposition 5.7 implies that there is a holomorphic map $\mu(\alpha) : \hat{N} \to \mathbb{C}$ such that

$$(n, \alpha)(x, w) = (n + \bar{\phi}(\alpha)(x) + \mu(\alpha)(\alpha w), \alpha w). \tag{5.3}$$

It follows also

$$\delta[\hat{\mu}] = [f]. \tag{5.4}$$

As both $[\hat{\chi}]$ and $[\hat{\mu}]$ belong to $H^1_\phi(\hat{\pi}, \mathrm{hol}(\hat{N}, \mathbb{C}))$, there exists an element $[\nu] \in H^1_\phi(\hat{\pi}, \mathrm{hol}(\hat{N}, \mathbb{C}))$ such that

$$[\hat{\mu}]^{-1}[\hat{\chi}] = [\hat{\nu}]. \tag{5.5}$$

This implies that $j(\chi(\alpha)(w)) = j(\mu(\alpha)(w) + \nu(\alpha)(w)) \in T^1_{\mathbb{C}}$ (${}^\forall\, w \in \hat{N}$). We may assume that (up to constant)

$$\chi = \mu + \nu : \hat{\pi} \longrightarrow \mathrm{hol}(\hat{N}, \mathbb{C}). \tag{5.6}$$

Theorem 5.9. *Let M be a holomorphic torus-Bott manifold of dimension $2n$ and (X, π, J) its universal covering. There exists a nilpotent Lie group N' with*

E(N')*-invariant complex structure* J *such that the action* (X, π, J) *is equivariantly biholomorphic to a holomorphic infranil-action* (N', π', J) $(\pi' \leq \mathrm{E}_J(N'))$. *Specifically, a* $2n$*-dimensional holomorphic torus-Bott manifold* M *is biholomorphic to a holomorphic infranilmanifold* N'/π'.

Proof. We suppose inductively that $(\hat{X}, \hat{\pi}, \hat{J})$ is equivariantly biholomorphic to $(\hat{N}, \hat{\pi}, \hat{J})$. Then the action (X, π) is equivariantly biholomorphic to a holomorphic action (N, π, J) such that

$$(n, \alpha)(x, w) = (n + x + \mu(\alpha)(\alpha w) + \nu(\alpha)(\alpha w), \alpha \cdot w).$$

Applying Theorem 5.8 to this action, there is a holomorphic infranil geometry $(\mathrm{E}_J(N'), N')$ such that the complex quotient N/π is biholomorphic to a holomorphic infranilmanifold N'/Γ' for a torsionfree discrete subgroup $\Gamma' \leq \mathrm{E}_J(N')$. □

5.6 Application

Let $M = M_n \to M_{n-1} \to \ldots \to M_1 \to \{\mathrm{pt}\}$ be a holomorphic torus-Bott tower as in (5.1). Each holomorphic fiber bundle induces a group extension

$$1 \to \mathbb{Z}^2 \to \pi_m \longrightarrow \pi_{m-1} \to 1$$

which represents a 2-cocycle in $H^2_\phi(\pi_{m-1}; \mathbb{Z}^2)$ $(m = 1, \ldots, n)$. See (5.3).

Definition 5.10. A holomorphic torus-Bott tower is of finite type if each 2-cocycle has finite order in $H^2_\phi(\pi_{m-1}; \mathbb{Z}^2)$. Otherwise (i.e. there exists a cocycle of infinite order), a holomorphic torus-Bott tower is said to be of infinite type.

5.6.1 Holomorphic torus-Bott manifold of finite type

Since $\mathrm{U}(n)$ is the maximal compact unitary subgroup in $\mathrm{GL}(n, \mathbb{C})$, the affine group $\mathrm{A}_\mathbb{C}(n) = \mathbb{C}^n \rtimes \mathrm{GL}(n, \mathbb{C})$ has the complex euclidean subgroup $\mathrm{E}_\mathbb{C}(n) = \mathbb{C}^n \rtimes \mathrm{U}(n)$. If Γ is a torsionfree discrete uniform subgroup in $\mathrm{E}_\mathbb{C}(n)$, then the quotient $\mathbb{C}^n/\Gamma$ is a compact complex euclidean space form. Γ is said to be a Bieberbach group.

Theorem 5.11. *If* M *is a* $2n$*-dimensional holomorphic torus-Bott manifold of finite type, then* M *is biholomorphic to a complex euclidean space form* $\mathbb{C}^n/\Gamma$ $(\Gamma \leq \mathrm{E}_\mathbb{C}(n))$. *Moreover the holonomy group* $L(\Gamma) \leq \mathrm{U}(n)$ *is isomorphic to the product* $\begin{pmatrix} H_1 & & & \\ & H_2 & & \\ & & \ddots & \\ & & & H_n \end{pmatrix}$ *where* H_i *is either one of* $\{1\}$, $\mathbb{Z}_2$, $\mathbb{Z}_4$ *or* $\mathbb{Z}_6$.

Proof. Put $(X,\pi)=(X_n,\pi_{n-1}),(\hat{\pi},\hat{X})=(\pi_{n-1},X_{n-1})$. Let

$$(\mathbb{Z}^2,\mathbb{C}){\to}(\pi,X)\stackrel{p}{\longrightarrow}(\hat{\pi},\hat{X}) \tag{5.1}$$

be an equivariant principal holomorphic bundle (cf. (5.1)). Inductively suppose that $\hat{X}/\hat{\pi}$ is biholomorphic to a complex euclidean space form $\mathbb{C}^{n-1}/\hat{\Gamma}$ ($\hat{\Gamma}\leq \mathrm{E}_{\mathbb{C}}(n-1)$). As $\hat{\pi}\cong\hat{\Gamma}$, $\hat{\pi}$ has a normal free abelian subgroup $\mathbb{Z}^{2(n-1)}$ of finite index. Consider the commutative diagram as in (5.9):

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & \pi & \longrightarrow & \hat{\pi} & \longrightarrow & 1\\
 & & || & & \cup & & \cup & & \\
1 & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & \Delta & \longrightarrow & \mathbb{Z}^{2(n-1)} & \longrightarrow & 1.
\end{array} \tag{5.2}$$

Note that $\phi(\hat{\pi})\leq \mathrm{Aut}(\mathbb{Z}^2)$ is a finite cyclic group. Taking a finite index subgroup if necessary, we may assume that the lower sequence is a central group extension. The cocycle of $H^2_{\phi}(\hat{\pi};\mathbb{Z}^2)$ restricts to an element of a free abelian group $H^2(\mathbb{Z}^{2(n-1)};\mathbb{Z}^2)$. Since the cocycle representing (5.2) is a torsion in $H^2_{\phi}(\pi_{m-1};\mathbb{Z}^2)$ by the hypothesis, it is zero in $H^2(\mathbb{Z}^{2(n-1)};\mathbb{Z}^2)$, i.e. the lower group extension splits; $\Delta\cong\mathbb{Z}^2\times\mathbb{Z}^{2(n-1)}=\mathbb{Z}^{2n}$.

On the other hand, M is biholomorphic to a holomorphic infranilmanifold N/Γ for $^{\exists}\,\Gamma\leq \mathrm{E}_J(\mathsf{N})$ by Theorem 5.9. In particular, Γ has a finite index subgroup Γ' isomorphic to $\mathbb{Z}^{2n}$. As Γ' is a discrete uniform subgroup of N, the Mal'cev uniqueness property implies that N is isomorphic to $\mathbb{C}^n$. (Note that N is isomorphic to a vector space $\mathbb{R}^{2n}$. The complex structure J on N is equivalent to the standard complex structure $J_0=J_0\times J_0$ on $\mathbb{C}^n=\mathbb{C}\times\mathbb{C}^{n-1}$ by Corollary 5.6. Thus (N,J) is holomorphically isomorphic to $\mathbb{C}^n$.) If we note that K belongs to $\mathrm{Aut}(\mathbb{C}^n)=\mathrm{GL}(n,\mathbb{C})$ in this case, it follows $K=\mathrm{U}(n)$ so that $\mathrm{E}_J(\mathsf{N})=\mathrm{E}_{\mathbb{C}}(n)$. Since $\Gamma\leq\mathrm{E}_{\mathbb{C}}(n)$, M is biholomorphic to a complex euclidean space form $\mathbb{C}^n/\Gamma$.

We may identify $M=\mathbb{C}^n/\Gamma$. Let $L:\mathrm{Aff}_{\mathbb{C}}(n)=\mathbb{C}^n\rtimes\mathrm{GL}(n,\mathbb{C}){\to}\mathrm{GL}(n,\mathbb{C})$ be the holonomy homomorphism. It remains to describe the structure of the holonomy group $L(\Gamma)$ of $\mathbb{C}^n/\Gamma$. First of all note that $L(\Gamma)\leq\mathrm{U}(n)$. The (Bieberbach) group Γ has an extension as in (5.2):

$$1{\to}\mathbb{Z}^2{\to}\Gamma\stackrel{p}{\longrightarrow}\hat{\Gamma}{\to}1 \tag{5.3}$$

where $\mathbb{C}^{n-1}/\hat{\Gamma}$ is a $2(n-1)$-dimensional complex euclidean space. As Γ normalizes $\mathbb{C}$ ($\geq\mathbb{Z}^2$), we have

$$L((n,\alpha))=\{\begin{pmatrix}\bar{\phi}(\alpha) & 0\\ 0 & B_\alpha\end{pmatrix}\}\leq\mathrm{U}(n)\ \ ({}^{\forall}\,(n,\alpha)\in\Gamma). \tag{5.4}$$

If we recall that $\hat{\Gamma}\leq\mathrm{E}_{\mathbb{C}}(n-1)=\mathbb{C}^n\rtimes\mathrm{U}(n-1)$, then the action of Γ_{n-1} on $\mathbb{C}^{n-1}$ is described as

$$\alpha(y)=(b_\alpha,B_\alpha)(y)=b_\alpha+B_\alpha(y)\ \ (\alpha\in\hat{\Gamma},y\in\mathbb{C}^{n-1}).$$

By the induction hypothesis we assume that $L(\hat{\Gamma}) = \{B_\alpha\} \leq \Pi_{i=2}^n H_i$ where each H_i is isomorphic to either one of $\{1\}$, $\mathbb{Z}_2$, $\mathbb{Z}_4$ or $\mathbb{Z}_6$.

Noting $H_1 = \phi(\{\alpha\}) = \{\pm 1\}$, $\{\pm \mathbf{i}\}$ or $\{e^{\pm \mathbf{i}\pi/3}\}$ from (5.6), it follows from (5.4) that $L(\Gamma) \leq \Pi_{i=1}^n H_i$. This proves the induction step. □

Remark 5.12. *By the hypothesis* $[f] \in H^2_\phi(\hat{\Gamma}; \mathbb{Z}^2)$ *has finite order, say* ℓ. *Let* $\ell \cdot f = \delta^1 \tilde{\lambda}$ *for some function* $\tilde{\lambda} : \hat{\Gamma} \to \mathbb{Z}^2$. *Putting* $\lambda = \ell / \tilde{\lambda} : \Gamma_{n-1} \to \mathbb{C}$, *it follows*

$$f = \delta^1 \lambda. \tag{5.5}$$

Associated with the extension (5.3), *we have another holomorphic Seifert action of* Γ *on* $\mathbb{C}^n$*:*

$$\begin{array}{r} (n, \alpha)(x, y) = (n + \phi(\alpha)(x) + \lambda(\alpha), \alpha y) \\ (^\forall (n, \alpha) \in \Gamma, ^\forall (x, y) \in \mathbb{C}^n). \end{array} \tag{5.6}$$

Then for $(n, \alpha) \in \Gamma$, *the Seifert action* (5.6) *of* Γ *on* $\mathbb{C} \times \mathbb{C}^{n-1} = \mathbb{C}^n$ *is identified with the euclidean action;*

$$(n, \alpha) \begin{bmatrix} x \\ y \end{bmatrix} = \left(\begin{bmatrix} n + \lambda(\alpha) \\ b_\alpha \end{bmatrix}, \begin{pmatrix} \phi(\alpha) & 0 \\ 0 & B_\alpha \end{pmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix}. \tag{5.7}$$

If we put

$$\rho((n, \alpha)) = \left(\begin{bmatrix} n + \lambda(\alpha) \\ b_\alpha \end{bmatrix}, \begin{pmatrix} \phi(\alpha) & 0 \\ 0 & B_\alpha \end{pmatrix} \right), \tag{5.8}$$

then this gives a faithful homomorphism $\rho : \Gamma \longrightarrow \mathrm{E}_\mathbb{C}(n)$. *We obtain a compact complex euclidean space form* $\mathbb{C}^n / \rho(\Gamma)$. *By the Bieberbach theorem,* Γ *is conjugate to* $\rho(\Gamma)$ *by some element* $f \in \mathrm{A}(2n) = \mathbb{R}^{2n} \rtimes \mathrm{GL}(2n, \mathbb{R})$. *Two complex euclidean space forms* $\mathbb{C}^n / \Gamma$ *and* $\mathbb{C}^n / \rho(\Gamma)$ *are affinely diffeomorphic. In general they are different holomorphic Bieberbach classes.*

Remark 5.13. *We have a similar result to an* S^1*-fibred nilBott manifold of finite type. In fact, it is diffeomorphic to an euclidean space form with holonomy group isomorphic to* $(\mathbb{Z}_2)^s$ $(0 \leq s \leq n)$. *(Compare [19].)*

5.6.2 Kähler Bott tower

An example of finite type is a Kähler torus-Bott manifold, i.e. a torus-Bott manifold which admits a Kähler structure. More precisely, let $T^1_\mathbb{C} \to M_m \xrightarrow{\mathsf{p}_m} M_{m-1}$ be a holomorphic torus-Bott tower as in (5.2). Suppose that

(1) Each M_m supports a Kähler form Ω_m.

(2) Let $\mathbb{C} \to X_m \xrightarrow{p_m} X_{m-1}$ be the equivariant principal holomorphic bundle in which p_m is a Kähler submersion.

(3) $\mathbb{C}$ leaves invariant Ω_m $(m = 1, \ldots, n)$.

Then (5.2) is said to be a *Kähler Bott tower*. The top space $M = M_n$ is said to be a Kähler Bott manifold.

The following theorem is inspired by the result of Carrell [5]. (See [19] also.)

Theorem 5.14. *Let (M, Ω) be a Kähler Bott manifold. Then M is biholomorphic to the complex euclidean space form $\mathbb{C}^n/\Gamma$ where $L(\Gamma) = \Pi_{i=1}^n H_i$.*

Proof. To apply Theorem 5.11, it suffices to show that each cocycle $[f]$ representing (5.3) is of finite order in $H^2_\phi(\pi_{m-1}; \mathbb{Z}^2)$. In fact, there is a central group extension $1 \to \mathbb{Z}^2 \to \Delta_m \xrightarrow{p_m} \Delta_{m-1} \to 1$ from (5.7). Put $T^1_{\mathbb{C}} = \mathbb{C}/\mathbb{Z}^2$, $Y_m = X_m/\Delta_m$, and $Y_{m-1} = X_{m-1}/\Delta_{m-1}$. Then M_m has a finite covering Y_m which admits a principal holomorphic fibration:

$$T^1_{\mathbb{C}} \to Y_m \xrightarrow{q_m} Y_{m-1}. \tag{5.9}$$

Then it is proved in [5], also [19] that the Kähler isometric action of $T^1_{\mathbb{C}}$ is *homologically injective*, i.e. the obit map $\mathrm{ev}(t) = ty$ at a point $y \in Y_m$ induces an injective homomorphism $\mathrm{ev}_* : H_1(T^1_{\mathbb{C}}; \mathbb{Z}) = \mathbb{Z}^2 \to H_1(Y_m; \mathbb{Z})$. This implies that Δ_m has a finite index normal splitting subgroup so the representative cocycle of π_m in $H^2_\phi(\pi_{m-1}; \mathbb{Z}^2)$ has finite order. (See [8] for details.) By Theorem 5.11, M is *biholomorphic* to a complex euclidean space form $\mathbb{C}^n/\Gamma$ with holonomy group $L(\Gamma) = \Pi_{i=1}^n H_i$. □

Remark 5.15. *It follows from the result of Hasegawa [14], Baues-Cortés [2] that a compact aspherical Kähler manifold with virtually solvable fundamental group is* biholomorphic *to a complex euclidean space form. As the fundamental group of a Kähler Bott manifold is virtually nilpotent by Lemma* 5.3*, the above theorem is obtained from this result except for the holonomy group characterization.*

5.7 Holomorphic torus-Bott tower of infinite type

We study a holomorphic torus-Bott tower of infinite type. It is hard to determine a *holomorphic classification* of holomorphic torus-Bott manifolds of *infinite type* in higher dimension. Recall the following facts about holomorphic torus-Bott manifolds of *infinite type.*

- The fundamental group is virtually nilpotent (but not abelian).
- A holomorphic torus-Bott manifold of infinite type is a non-Kähler manifold.

5.7.1 4-dimensional holomorphic torus-Bott manifolds

It follows from the classification of complex surfaces that a 4-dimensional holomorphic torus-Bott manifold is finitely covered by either $T^2_{\mathbb{C}}$ or $S^1 \times \mathcal{N}/\Delta$ where $\mathcal{N}$ is a 3-dimensional Heisenberg Lie group isomorphic to the 3×3-upper triangular unipotent matrices with lattice Δ.

Proposition 5.16. *A 4-dimensional holomorphic torus-Bott manifold is biholomorphic to either $T^2_{\mathbb{C}}/F$ or $S^1 \times \mathcal{N}^3/\Delta$ where F is a finite group of $\mathrm{U}(2)$ or Δ is a discrete uniform subgroup of $\mathcal{N} \rtimes \mathrm{U}(1)$.*

5.7.2 6-dimensional examples of infinite type

As a special case of 6-dimensional holomorphic torus-Bott manifolds of infinite type, there is a nontrivial holomorphic principal torus bundle over a complex 2-torus which is a holomorphic principal nilmanifold : $T^1_{\mathbb{C}} \to N_3/\Gamma \xrightarrow{q_3} T^2_{\mathbb{C}}$. N_3 is a 2-step nilpotent Lie group with a left invariant complex structure. There is a classification of six dimensional nilpotent Lie algebras with left invariant complex structure in [33], [35]. As b_1 is either 4 or 5 in this case except for $\mathbb{C}^3$, the classification gives

Proposition 5.17. *A 6-dimensional holomorphic torus-Bott manifold over a 4-dimensional complex euclidean space form is biholomorphic to the quotient of the following nilpotent Lie group by a cocompact subgroup acting properly discontinuously.*

- $\mathbb{C}^3$
- $\mathcal{N}^3 \times \mathcal{N}^3$ *(Lie algebra $\mathfrak{h}_2$).*
- $\mathbb{R}^+ \times \mathcal{N}^5$ *(Lie algebra $\mathfrak{h}_3$).*
- *The Iwasawa Lie group $\mathcal{L}_3$ (Lie algebra $\mathfrak{h}_5$).*
- *The Nilpotent Lie group corresponding to $\mathfrak{h}_4$.*
- *The Nilpotent Lie group corresponding to $\mathfrak{h}_6$.*
- $\mathbb{R}^3 \times \mathcal{N}^3$ *(Lie algebra $\mathfrak{h}_8$).*

Remark 5.18. *Here $\mathcal{N}_8 = \mathbb{R}^3 \times \mathcal{N}^3$ is viewed as $\mathbb{R} \times \mathbb{R} \to \mathcal{N}_8 \to \mathbb{R}^2 \times \mathbb{C}$. There is another exact sequence: $1 \to \mathbb{C} \to \mathcal{N}_8 \longrightarrow \mathbb{R}^+ \times \mathcal{N}^3 \to 1$ such that $[\mathcal{N}_8, \mathcal{N}_8] = \mathbb{R} \leq \mathbb{C}$. Note that this is a splitting exact sequence $\mathcal{N}_8 = \mathbb{C} \times (\mathbb{R}^+ \times \mathcal{N}^3)$ but the base space $\mathbb{R}^+ \times \mathcal{N}^3$ is not $\mathbb{C}^2$.*

It is interesting to find which non-Kähler geometric structure exists on a holomorphic torus-Bott manifold of infinite type. We have found two such classes in general dimension. The following result is obtained from [15] and [21].

Theorem 5.19.

(i) *A* $2n+2$*-dimensional compact infranilmanifold* M *admits a* locally conformal Kähler *structure if and only if* $M = \mathbb{R} \times \mathcal{N}/\Gamma$ *where* $\mathcal{N}$ *is the Heisenberg nilpotent Lie group and* $\Gamma \leq \mathbb{R} \times (\mathcal{N} \rtimes \mathrm{U}(n))$ *is a discrete cocompact subgroup. In this case* M *has the holomorphic torus fibration over the complex euclidean orbifold:*

$$T^1_{\mathbb{C}} \to M \longrightarrow \mathbb{C}^n/\Gamma.$$

Some finite cover M' *of* $\mathbb{R} \times \mathcal{N}/\Gamma$ *is a holomorphic torus-Bott manifold of infinite type:*

$$M' \xrightarrow{T^1_{\mathbb{C}}} T^n_{\mathbb{C}} \longrightarrow \cdots \longrightarrow \{pt\}.$$

(ii) *There exist a* $2(2n+1)$*-dimensional complex nilpotent Lie group* $\mathcal{L} = \mathcal{L}_{2n+1}$ *and a torsionfree discrete cocompact subgroup* Γ *of the semidirect product* $\mathcal{L} \rtimes (\mathrm{Sp}(n) \cdot S^1)$ *such that a* $2(2n+1)$*-dimensional complex infranilmanifold* $\mathcal{L}/\Gamma$ *admits a* complex contact *structure.* $\mathcal{L}/\Gamma$ *supports a holomorphic torus bundle over the quaternionic euclidean orbifold:*

$$T^1_{\mathbb{C}} \to \mathcal{L}/\Gamma \longrightarrow \mathbb{H}^n/\Delta.$$

Moreover, some finite cover M' *of* $\mathcal{L}/\Gamma$ *is a holomorphic torus-Bott manifold of infinite type:*

$$M' \xrightarrow{T^1_{\mathbb{C}}} T^{2n}_{\mathbb{C}} \longrightarrow \cdots \longrightarrow \{pt\}.$$

Here $\mathcal{L}_3$ is the Iwasawa complex nilpotent Lie group.

Bibliography

[1] O. Baues, *Infra-solvmanifolds and rigidity of subgroups in solvable linear algebraic groups*, Topology43 (4), (2004) 903-924.

[2] O. Baues and V. Cortés, *Aspherical Kähler manifolds with solvable fundamental group*, Geom. Dedicata 122 (2006) 215-229.

[3] K. Brown, *Cohomology of groups*, GTM, Springer-Verlag,1982.

[4] E. Calabi, *Closed locally euclidean* 4*-dimensional manifolds*, Bull. of AMS 63 (1957) p.135.

[5] J. Carrell, *Holomorphically injective complex toral actions*, Proc. Second conference on Compact transformation groups, Part II, Lecture notes in Math., 299 Springer, New York (1972) 205-236.

[6] S. Choi, M. Masuda and D. Y. Suh, *Topological classification of generalized Bott towers*, Trans. Amer. Math. Soc. 362 (2010) 1097-1112.

[7] P.E. Conner and F. Raymond, *Actions of compact Lie groups on aspherical manifolds*, Topology of Manifolds, Proceedings Inst. Univ. of Georgia, Athens, 1969 Markham (1970) 227-264.

[8] P.E. Conner and F. Raymond, *Injective operations of the toral groups*, Topology 10 (1971) 283-296.

[9] P.E. Conner and F. Raymond, *Holomorphic Seifert fiberings*, Proceedings of the second conference on compact transformation groups, Springer Lecture Notes in Math. 299 (1972) 124-204.

[10] K. Dekimpe, *Almost-Bieberbach groups:Affine and Polynomial structures*, Lecture Notes in Math. 1639 (1996), Springer-Verlag.

[11] D. Gromoll and J. Wolf, *Some relations between the metric structure and the algebraic structure of the fundamental group in manifolds of nonpositive curvature*, Bull. Amer. Math. Soc. 77 (1971).

[12] M. Grossberg and Y. Karshon, *Bott towers, complete integrability, and the extended character of representations*, Duke Math. J. 76 (1994) 23-58.

[13] S. Halperin, *Rational homotopy and torus actions*, *Aspects of Topology, London Math. Soc. Lecture Note* Ser. 93 (1985), 293-306.

[14] K. Hasegawa, *A note on compact solvmanifolds with Kähler structures*, *Osaka J. Math.* 43 (2006) 131-135.

[15] K. Hasegawa and Y. Kamishima, *Locally Conformally Kähler Structures on Homogeneous Spaces*, arXiv:1101.3693v11 [math.DG].

[16] F. Hirzebruch, *Topologicl methods in algebraic geometry*, New appendix and translation from the second German edition by R. L. E. Schwarzenberger, with an additional section by A. Borel. Die Grundlehren der Mathematischen Wissenschaften, Band 131 Springer-Verlag New York, Inc., New York 1966.

[17] Y. Kamishima, K.B. Lee and F. Raymond, *The Seifert construction and its applications to infranilmanifolds*, *Quart. J. Math.* 34 (1983) 433-452.

[18] Y. Kamishima, K.B. Lee and F. Raymond, *The Seifert construction and its applications to infranilmanifolds*, Quart. J. Math., Oxford (2), 34 (1983), 433-452.

[19] Y. Kamishima and M. Nakayama, *On the Holomorphic Torus-Bott Tower of Aspherical Manifolds*, Proceedings of the Steklov Institute of Mathematics 286 (2014) 250-264.

[20] Y. Kamishima and A. Nazra, *Seifert fibred structure and rigidity on real Bott towers*, *Contemp. Math.* 501 (2009), 103-122.

[21] Y. Kamishima and A. Tanaka, *On complex contact similarity manifolds*, *Journal of Mathematics Research.* 5 (4) (2013) 1-10.

[22] J.B. Lee and M. Masuda, *Topology of iterated S^1-bundles*, *Osaka J. Math.* 50 (2013) 847-869.

[23] K. B. Lee and F. Raymond, *Seifert fiberings*, Mathematical Surveys and Monographs, vol. 166 (2010).

[24] K. B. Lee and F. Raymond, *Topological, affine and isometric actions on flat Riemannian manifolds*, *J. Differential Geom.*16 (1981), 255-269.

[25] A. Nazra, *Diffeomorphism Classes of Real Bott Manifolds*, Tokyo J. Math. 34 (2011), 229-260.

[26] A. Nazra, *Geometry of Transformation Groups and Related Topics, Diffeomorphism Type of Real Bott Towers*, *RIMS Kokyuroku* 1612 (2008), 165-176.

[27] M. Nakayama, *On the S^1-fibred nil-Bott Tower*, *Osaka J. Math.* 51 (2014), 67-87.

[28] M. Masuda and T. Panov, *Semi-free circle actions, Bott towers, and quasitoric manifolds*, *Sb. Math.* 199 (2008), no. 7-8, 1201-1223

[29] J. Milnor, *On the 3-dimensional Brieskorn manifolds* $M(p,q,r)$, *Ann. of Math. Studies, Princeton Univ. Press* 84 (1975), 175–225.

[30] S. Murakami, *Sur certains espaces fibrés principaux holomorphes admettant des connexions holomorphes*, *Osaka Math. J.* 11 (1959) 43-62.

[31] V. Puppe, *Multiplicative aspects of the Halperin-Carlsson conjecture*, *Georgian Math. J.* 16 (2009), no. 2, 369-379.

[32] S. Rollenske, *Geometry of nilmanifolds with left-invariant complex structure and deformations in the large*, Proc. London Math. Soc. *99 (2)* (2009) 425-460.

[33] S. Salamon, *Complex structures on nilpoptent Lie algebras* J. Pure and *Appl. Algebra* 157 (2001) 311-333.

[34] M.S. Raghunathan, *Discrete subgroups of Lie groups*, *Ergebnisse Math. Grenzgebiete* vol. 68 Springer, Berlin, New York 1972.

[35] L. Ugarte, *Hermitian structures on six-dimensional nilmanifolds*, *Transformation Groups* 12 (1) (2007) 175-202.

[36] J. Wolf, *Spaces of constant curvature*, *McGraw-Hill, Inc.* (1967).