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Reaction thresholds in doubly special relativity

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Two theories of special relativity with an additional invariant scale, "doubly special relativity," are tested with calculations of particle process kinematics. Using the Judes-Visser modified conservation laws, thresholds are studied in both theories. In contrast with some linear approximations, which allow for particle processes forbidden in special relativity, both the Amelino-Camelia and Magueijo-Smolin frameworks allow no additional processes. To first order, the Amelino-Camelia framework thresholds are lowered and the Magueijo-Smolin framework thresholds may be raised or lowered.

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I. INTRODUCTION

Special relativity with an observer independent scale has been proposed as a modification to local Lorentz invariance [1–7]. The existence of an additional scale at high energy was motivated by a variety of studies including κ-deformed Poincaré algebras [2,8–12], heuristic semi-classical states in quantum gravity [13], and string theory [14]. The new scale may be an energy, momentum, or perhaps even a length. Despite our intuition from special relativity, the new relativity theories seem to demonstrate that it is *not* necessary to use a preferred reference frame when there is a distinguished scale [1]. Dubbed "doubly special relativity" (DSR) the theories maintain the relativity principle even with the inclusion of an invariant energy or momentum [1]. For the purposes of this paper, the distinguishing features of the new theories are the relativity principle and an invariant scale. To emphasize this we refer to them as "invariant scale relativity" (ISR). In ISR theories the speed of light may not be an observer invariant.[1] We study two example theories: the ISR of Amelino-Camelia and collaborators [1–3,5] and the ISR of Magueijo and Smolin [6,7]. Both proposals exploit a freedom to define non-linear transformations on momentum space, retaining the group properties of Lorentz transformations, and include an invariant scale.

Defined in momentum space the new ISR transformations raise many questions. For instance, is the relativity principle maintained? Indeed, what is the relativity principle in this new context? What is the corresponding spacetime associated with these theories?[2] How are composite particles described? Using particle process kinematics to test relativity in the ISR models, we focus on the first two questions and, to the extent possible, limit ourselves to the single particle sector.

Studies of process kinematics, together with current astrophysical observations, have been surprisingly successful in constraining specific proposals for modifications of special relativity requiring a preferred frame [16–19]. Thus far these studies have focused on modifications of dispersion relations with a term linear in the Planck scale. Further constraints may be imposed by ensuring consistency at lower energies via an effective field theory, as was done for dimension-5 operators by Myers and Pospelov [20]. Lehnert found constraints on dispersion relations arising from the additional considerations of coordinate invariance and non-dynamical tensor backgrounds which break Lorentz symmetry [21].

Kinematics is particularly well suited to non-linear realizations of the Lorentz group since both the spacetime picture and the effective dynamical framework of ISRs is not complete. To perform the analysis we need conservation laws. Judes and Visser derive modified conservation laws in Ref. [22] based on the observation that, since the physical energy-momenta in ISRs are non-linearly related to the formal energy-momenta, the ISR conservation laws may be found by appropriately applying the non-linear transformations to the usual additive conservation laws.

Given the success constraining modified dispersion relations in Refs. [16–19], we might expect that process kinematics could again be used to constrain the new invariant scale in ISRs. In fact, although this is the first general study, several such processes, including photo-production of pions occurring in high-energy-proton—cosmic-microwave-background-photon collisions [the Greisen-Zatsepin-Kuzmin (GZK) cutoff [24]], have been explored [5,7]. These calcu-

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[1]For instance, the modified dispersion relation $E^2=p^2+p^2E/E_p$ yields a velocity of [1]

$$v_\gamma(p):=\frac{dE}{dp}\approx 1+\frac{p}{E_p}$$

which depends on the reference frame for $p\neq E_p$.

[2]At the present, despite some progress [15], it is unclear precisely how this scale affects relativistic effects such as length contraction.

lations have been carried out in the leading order formalism. Here, making use of the Judes-Visser conservation laws, we present new, exact and first order calculations for the Magueijo-Smolin and Amelino-Camelia ISRs. Particle process kinematics does not limit parameters in the same manner as Refs. [16–19]. Instead, process kinematics shows how thresholds are modified and provides a perspective from which the notion of relativity may be sharpened. Indeed particle kinematics brings matters of principle to the fore in ISRs rather than numerical limits on parameters.

We present our results for Magueijo-Smolin ISR before turning to Amelino-Camelia ISR in Sec. III. We show that to first order Amelino-Camelia ISR lowers existing thresholds, whereas Magueijo-Smolin ISR may either lower or raise them. They allow no additional processes. We explore the issue of the uniqueness of particle process thresholds in Sec. IV and close with a brief discussion of the relativity principle in light of these results.

Throughout the article when we refer to the "Planck scale" we simply mean the invariant scale of the theory expected to correspond to $E_p = 1.3\times10^{19}$ GeV. The low-energy speed of light is set to unity. We generally calculate in $1+1$ for simplicity. However, in Sec. IV where the results depend on dimension, we work in $3+1$.

II. MAGUEIJO AND SMOLIN'S RELATIVITY WITH AN INVARIANT ENERGY

Fock [25] derives spacetime transformations for a system in which linear motion is covariant; if motion is rectilinear in one frame, then it is rectilinear in all inertial frames. He showed that the transformations from a frame x^μ to $x^{\mu\prime}$ must be of the form

$$x^{\mu\prime} = \frac{A^\mu + A^\mu_\nu x^\nu}{B + B_\alpha x^\alpha} \tag{1}$$

where A^μ, A^μ_ν, B, and B_μ are coordinate independent functions of velocity. Magueijo and Smolin found that these same transformations applied in momentum space introduce an invariant scale at high energy. They showed that the fractional linear transformations may be obtained by exponentiation of boost generators modified by a dilation $D = p_\nu \partial_p^\nu$ [6]:

$$K^i = L^i + \lambda p^i D \tag{2}$$

in which L^i is the unmodified Lorentz generator.

The resulting Magueijo-Smolin ISR may be defined by the physical energy-momenta for a single particle [6,22],

$$E = \frac{\epsilon}{1+\lambda\epsilon},$$

$$p = \frac{\pi}{1+\lambda\epsilon}, \tag{3}$$

and the modified dispersion relation

$$\frac{E^2 - p^2}{(1-\lambda E)^2} = \mu^2 \equiv \frac{m^2}{(1-\lambda m)^2}. \tag{4}$$

The quantities (ϵ,π), called "pseudo-energy-momenta," transform under the usual linear Lorentz transformations.

The presence of the pseudo-energy-momentum variables in the background does not necessarily mean that the ISR trivially reduces to SR. An "ISR physicist" would not measure—perhaps not even calculate—the pseudo-energy-momentum variables. We assume that the non-linearly realized variables are the physical ones. For notational convenience we use $E_p = 1/\lambda$ but this in no way is meant to suggest that there is an invariant length. Until the spacetime picture is complete we cannot be sure how the invariant scale relates to a possible length.

For many particle processes the total physical energy is given by the same expression although (ϵ,π) become the total pseudo-energy-momenta $(\epsilon_{tot},\pi_{tot})$.[3] Thus, Eqs. (3) also define modified energy-momentum conservation laws which, unlike the pseudo-energy-momenta, are not additive [22].

Before exploring process kinematics it is worth reviewing a couple of results on the invariant scale. As shown in Ref. [6], the theory has an invariant energy, E_p, such that if a particle has this energy in one frame, then it has the same energy in all frames (despite the change in momentum). The Magueijo-Smolin theory also has invariant "Planck scale null vectors" $(E_p, \pm E_p)$. Interpreting E_p as the invariant energy, we always take $\lambda > 0$. One might wonder whether the distinguished energy is included in the momentum space accessible to physical particles. Kinematic calculations suggest that it should not be included.

The root of the issue is the singularity in the pseudo-energy $\epsilon = E/(1-\lambda E)$ at $E = E_p$ where "anything can happen." By modified energy conservation, the total energy of N particles is

$$E_{\rm tot} = \frac{\displaystyle\sum_{i=1}^N \frac{E_i}{1-\lambda E_i}}{\displaystyle 1+\lambda\sum_{i=1}^N \frac{E_i}{1-\lambda E_i}} = \frac{1}{\lambda}\left[1 - \frac{1}{\displaystyle 1+\lambda\sum_{i=1}^N \frac{E_i}{1-\lambda E_i}}\right]. \tag{5}$$

This is always smaller than $E_p = 1/\lambda$, as long as all the E_i are smaller than the Planck scale energy. If one of the E_i is equal to E_p, then also the total energy is E_p, regardless of the number of particles and the values of the other, sub-Planckian energies.

Further curiosities appear for composite particles. Kinematically, a Planck-scale particle can decay to N particles (with N finite) as long as one of them has Planck-scale energy. One may similarly check that momentum is conserved.

[3] As is clear from the definition, we study the Magueijo-Smolin "classic" ISR of [6] rather than later variants which contain more than one scale [7].

Indeed the derivation holds for the Planck scale null vector as well. (See Refs. [3,7,23] for further complications in defining composite particles.) Thus, a Planck-scale particle is a source (or sink) for an arbitrary number of particles with energies less than or equal to the Planck scale. In addition, one may show that a finite number of sub-Planckian particles cannot interact to produce a Planck scale particle. Because of this closure property for $E<E_p$ particles under process kinematics and the pathologies of including these invariants in the physical energy-momentum space, we take Magueijo-Smolin ISR to be defined on the space of 4-momenta satisfying the modified conservation laws and $E<E_p$. (This is analogous to what is done in SR for infinite energies.)

Process kinematics is considerably simplified by the observation that conservation of the physical energy and momentum is equivalent to conservation of the pseudo-energy-momenta. To see this, consider an M to N particle process, with incoming pseudo-energy

$$\epsilon_o=\sum_{i=1}^{M}\epsilon_i=\sum_{i=1}^{M}E_i/(1-\lambda E_i)$$

and outgoing pseudo-energy $\epsilon_f=\Sigma_{j=1}^{N}\epsilon'_j$. Energy conservation $E_0=E_f$ then requires

$$\frac{\epsilon_o}{1+\lambda\epsilon_o}=\frac{\epsilon_f}{1+\lambda\epsilon_f}\tag{6}$$

which immediately implies that the total pseudo-energy is conserved. This in turn implies that the pseudo-momentum is conserved. However, note in particular that this result does not imply that the ISR results are identical to the results of SR kinematics. Further, the result is by no means generic to all ISRs but a simple consequence of the fractional modification. For instance, one might try a "time reversal" invariant theory with modifications of the form $(1+(\lambda\epsilon)^2)^{-1}$. The above argument obviously fails for such an ISR.

To compare process thresholds of the Magueijo-Smolin ISR with those of SR, we take the reaction of two incoming particles with masses m_1 and m_2, resulting in N outgoing particles with masses m_i, $i=3,\ldots,N+2$, in the center-of-mass (c.m.) system. Let $M:=\Sigma_{i=3}^{N+2}m_i$ and $M^{(2)}=\Sigma_{i=3}^{N+2}m_i^2$. Recall that the usual SR threshold in the c.m. system is given by

$$E^*_{\rm SR}=\frac{m_1^2-m_2^2+M^2}{2M}.\tag{7}$$

To find the ISR threshold the physical energies and masses in Eq. (7) are replaced by the corresponding pseudo-quantities,

$$\frac{E^*_{\rm ISR}}{1-\lambda E^*_{\rm ISR}}=\epsilon^*=\frac{\mu_1^2-\mu_2^2+\mu^2}{2\mu},\tag{8}$$

with $\mu:=\Sigma_{i=3}^{N+2}\mu_i$. From this we obtain $E^*_{\rm ISR}$ in terms of the ISR invariants $\mu_i=m_i/(1-\lambda m_i)$ and, after expansion with respect to λ, the first-order correction of the SR threshold energy:

$$E^*_{\rm ISR}\approx E^*_{\rm SR}\left[1-\lambda\left(E^*_{\rm SR}-\frac{4M(m_1^3-m_2^3)-2M^{(2)}(m_1^2-m_2^2)+2M^2M^{(2)}-M^4}{2M(m_1^2-m_2^2+M^2)}\right)\right].\tag{9}$$

In the case of equal ingoing masses, $m_1=m_2$, this simplifies to

$$E^*_{\rm ISR}\approx E^*_{\rm SR}+\lambda\frac{2M^{(2)}-M^2}{4}.\tag{10}$$

The sign of the correction is not generally definite; it depends on the values of the outgoing masses. In the case of two outgoing particles, nevertheless, the threshold is always raised, as Eq. (10) reduces to

$$E^*_{\rm ISR}\approx E^*_{\rm SR}+\frac{\lambda}{4}(m_3-m_4)^2.\tag{11}$$

This is not a generic result for the reaction of two different incoming particles, as we will see below.

An interesting example is the interaction of an ultra-high energetic proton from cosmic radiation with the cosmic microwave background, $p\gamma\rightarrow p\pi$, in which the proton loses energy to produce a pion. We assume in the following that, however physical momenta are defined for the composite proton and pion, the result is well approximated by the dispersion relation for an elementary particle. The SR threshold for this process leads to a cutoff in the cosmic particle spectrum, the GZK cutoff [24]. Recently, higher energy cosmic particles have been reported. To check whether the Magueijo-Smolin ISR could account for a raising of this threshold we specialize the above method. From Eq. (7) the special relativistic threshold is

$$E^*_{\rm SR}=\frac{(m_p+m_\pi)^2+m_p^2}{2(m_p+m_\pi)}.\tag{12}$$

In the Magueijo-Smolin ISR the corresponding relation is

$$\epsilon^*=\frac{(\mu_p+\mu_\pi)^2+\mu_p^2}{2(\mu_p+\mu_\pi)},\tag{13}$$

from which follows

$$E^{*}_{\rm ISR}=\frac{(\mu_p+\mu_\pi)^2+\mu_p^2}{2(\mu_p+\mu_\pi)+\lambda[(\mu_p+\mu_\pi)^2+\mu_p^2]}. \tag{14}$$

In first order in λ this is

$$E^{*}_{\rm ISR}\approx E^{*}_{\rm SR}-\lambda\frac{m_\pi^2(6m_p^2-m_\pi^2)}{4(m_p+m_\pi)^2}, \tag{15}$$

a *lowering* of the SR threshold energy in the c.m. system. To compare this with the GZK threshold in the cosmological frame, one performs a non-linear Lorentz transformation, which boosts E_γ to the energy of a far infrared background photon. This is done in Appendix A. However, like in ordinary Lorentz transformations, the boosted energy is a monotonic function of the original one and so Magueijo-Smolin ISR is not capable of raising the GZK threshold and explaining the apparent abundance of cosmic particles above the GZK cutoff [7].

We exhibit two exact kinematic calculations for the Magueijo-Smolin ISR in Appendix A. These are based on two processes of the basic QED vertex: vacuum Čerenkov radiation (VCR) $a\to a\gamma$ for a charged particle a and photon stability $\gamma\not\to e^+e^-$. These processes, both forbidden in SR, are of particular interest, because considerations of linear modifications of SR [16,17] indicate that they could be allowed in modified theories. From the exact calculations it follows that they are forbidden in the ISR as well.

It is no surprise that we obtain these results, for the Magueijo-Smolin theory does not admit additional kinematic solutions. The crux of the matter is the equivalence of the conservation of the physical energy-momenta and the pseudo-energy-momenta. Since the map between physical energy-momentum thresholds and pseudo-energy-momentum thresholds is one-to-one, the theory contains no additional solutions (see Sec. IV). If a process is forbidden in SR, it will remain forbidden in the Magueijo-Smolin ISR.

III. AMELINO-CAMELIA RELATIVITY WITH AN INVARIANT MOMENTUM

The next ISR we consider differs from the Magueijo-Smolin theory in a number of important ways. First, the Amelino-Camelia ISR does not simply contain a dilation in momentum space but represents a more drastic modification. This can be easily seen by comparing Eq. (2) to the first order form of the modified boost generators for Amelino-Camelia ISR [3]:

$$K^i=L^i+\lambda\left(\frac{1}{2}\eta^{\mu\nu}p_\mu p_\nu x^i+p^ip_jx^j\right). \tag{16}$$

The dilation is only on the 3-momenta and the non-linear action extends to the spacetime transformations. As a result of these non-linearities, it is often necessary to work with the physical energy momenta to obtain exact results for process kinematics. Second, the Amelino-Camelia ISR has a single invariant momentum $p_0=1/\lambda$ but the energy, as in SR, is unconstrained. The theory may again be defined by the relation to the pseudo-energy-momenta [22]:

$$E=\frac{1}{\lambda}\ln\left[1+\lambda\epsilon\sqrt{1+\frac{\lambda^2(\epsilon^2-\pi^2)}{4}}+\frac{\lambda^2(\epsilon^2-\pi^2)}{2}\right],$$

$$p=\pi e^{-\lambda E}\sqrt{1+\frac{\lambda^2(\epsilon^2-\pi^2)}{4}}. \tag{17}$$

The theory has a modified dispersion relation [22]

$$\cosh(\lambda E)=\cosh(\lambda m)+\frac{1}{2}\lambda^2p^2e^{\lambda E}. \tag{18}$$

This dispersion relation, to leading order [1], is identical to the modified dispersion relations studied in [16]. However, in the ISR context the energy-momentum conservation laws are modified as well [1,22].

As may be swiftly seen from the dispersion relations of Eq. (18), although there is an invariant momentum, no positive energy particle may obtain it. We consider only those particles with momentum less than the upper limit p_0. In the following we analyze the theory defined by Eqs. (17) and (18), the Judes-Visser conservation laws [22], and the restriction $p<1/\lambda$. For ease of reference we will refer to this theory as Amelino-Camelia ISR.

The calculation of leading order corrections to threshold energies in the c.m. frame begins with the observation that the invariant μ of the theory differs only in second order from the physical mass:

$$\mu=\frac{2}{\lambda}\sinh\frac{\lambda m}{2}\approx m+\lambda^2\frac{m^3}{24}. \tag{19}$$

From this it follows that the threshold pseudo-energy for a general $2\to N$ particle process, given by the right equality of Eq. (8), is

$$\epsilon^*=E^*_{\rm SR}+O(\lambda^2), \tag{20}$$

which greatly simplifies the calculation of the first order expression of the threshold energy $E^*_{\rm ISR}$ in Amelino-Camelia ISR. With the aid of Eq. (17),

$$E^*_{\rm ISR}\approx E^*_{\rm SR}-\frac{\lambda\pi_1^2}{2}. \tag{21}$$

Here π_1 is the pseudo-momentum of the ingoing particle, whose pseudo-energy is ϵ^*, given by

$$\pi_1^2=(\epsilon^*)^2-\mu_1^2=(E^*_{\rm SR})^2-m_1^2+O(\lambda^2). \tag{22}$$

From this we immediately find

$$E^*_{\rm ISR}\approx E^*_{\rm SR}-\frac{\lambda}{2}[(E^*_{\rm SR})^2-m_1^2], \tag{23}$$

which indicates a general lowering of threshold energies for $2\to N$ particle reactions. The modified GZK threshold is simply the above result with $m_1 = m_p$. Hence Amelino-Camelia ISR also lowers the threshold, so we cannot give an explanation of a possible raising of the GZK cutoff [5]. We note, however, that this result again depends on the assumption that the composite particle relations do not differ significantly from the SR relations.

We further illustrate the kinematics with the same processes studied before: VCR and photon stability. Both exact calculations are in Appendix B. As in SR, there is no VCR and the photon is stable in Amelino-Camelia ISR.

IV. UNIQUENESS OF PROCESS THRESHOLDS

The above results hold only if the map between the pseudo-variables and the physical variables is one-to-one. If this property holds, then there corresponds just one physical threshold for every threshold in special relativity. ISRs satisfy modified conservation laws in which the total energy-momenta

$$E_{tot} = F_\lambda(\epsilon_{tot}, \pi_{tot}),$$

$$p_{tot} = \pi_{tot} G_\lambda(\epsilon_{tot}, \pi_{tot}) \tag{24}$$

are conserved.

In this equation the total pseudo-energy-momenta $(\epsilon_{tot}, \pi_{tot})$ are functions of the physical energy-momenta. For a single particle,

$$\epsilon = f_\lambda^{-1}(E,p),$$

$$\pi = p g_\lambda^{-1}(E,p) \tag{25}$$

f_λ and g_λ may or may not be equivalent to F_λ and G_λ. For example, in Magueijo-Smolin ISR, $F_\lambda = \epsilon w_\lambda(\epsilon) = f_\lambda$ and $G_\lambda = w_\lambda(\epsilon) = g_\lambda$ with $w_\lambda(\epsilon) = 1/(1+\lambda\epsilon)$. So in Magueijo-Smolin ISR the "lowercase functions" are equivalent to "uppercase functions."

In the Amelino-Camelia ISR, however, the relevant equations are, for a single particle [22],

$$E = F_\lambda(\epsilon,\pi) = \frac{1}{\lambda}\ln[\lambda\epsilon\cosh(\lambda m/2) + \cosh(\lambda m)],$$

$$p = \pi G_\lambda(\epsilon,\pi) = \pi\cosh(\lambda m/2)e^{-\lambda E} \tag{26}$$

and

$$\epsilon = f_\lambda(E,p) = \frac{e^{\lambda E} - \cosh(\lambda m)}{\lambda\cosh(\lambda m/2)},$$

$$\pi = \pi g_\lambda(E,p) = \frac{p e^{\lambda E}}{\cosh(\lambda m/2)}, \tag{27}$$

which are simple inverses.

In contrast to the single particle case for which F_λ and G_λ may be written as functions only of ϵ and m, in the multiple particle case the total energy and momentum are given by

$$F_\lambda(\epsilon,\pi) = \frac{1}{\lambda}\ln\left[1 + \lambda\epsilon\sqrt{1 + \frac{\lambda^2}{4}(\epsilon^2 - \pi^2)} + \frac{\lambda^2}{2}(\epsilon^2 - \pi^2)\right],$$

$$G_\lambda(\epsilon,\pi) = \sqrt{1 + \frac{\lambda^2}{4}(\epsilon^2 - \pi^2)}, \tag{28}$$

in which ϵ and π are *sums* of the pseudo-energy-momentum variables for each particle. The functions are not identical; $F_\lambda \neq f_\lambda$ and $G_\lambda \neq g_\lambda$.[4]

Despite the apparent difference, the meaningful question is whether the mapping remains on-to-one. Suppose (E_0, p_0) is the total physical energy-momentum for the incoming particles obtained by summing the incoming particle pseudo-energy-momenta in Eqs. (24). These modified energy conservation laws are equations for surfaces in energy-momentum space. By the implicit function theorem, these surfaces determine solutions (generally, one-parameter families of solutions) only if the Jacobian of the functions is non-vanishing on their domain. More precisely, we require

$$\pi(\partial_\pi F_\lambda \partial_\epsilon G_\lambda - \partial_\pi G_\lambda \partial_\epsilon F_\lambda) - G_\lambda \partial_\epsilon F_\lambda \neq 0 \tag{29}$$

for $\epsilon \geq 0$ and $-\infty < \pi < \infty$. The derivatives are with respect to the pseudo-energy-momenta, e.g. $\partial_\pi = \partial/\partial\pi$. For Magueijo-Smolin ISR this reduces to

$$-1/(1+\lambda\epsilon)^3 \neq 0. \tag{30}$$

In the case of the Amelino-Camelia ISR, using Eqs. (28) for the four dimensional case it is

$$-\frac{e^{-3\lambda E(\epsilon,\vec{\pi})}}{1 + \lambda^2(\epsilon^2 - \vec{\pi}^2)/4} \tag{31}$$

which is negative-definite, as well.[5] Hence, both ISRs considered here have non-vanishing Jacobians and thus the mapping is bijective. The ISRs have no additional process thresholds.

V. DISCUSSION

Using exact and first order calculations of process kinematics we have tested Amelino-Camelia ISR and Magueijo-Smolin ISR in their "natural domain": momentum space. Unlike previous kinematic calculations, these results made

[4]These two expressions are equivalent for a single particle. In the multiple particle case the problem arises because there is no longer a mass which relates the two expressions. Nevertheless, it is easy to see that the expression $\epsilon^2 - \pi^2$ is always positive-semidefinite (zero in the case of a collection of photons). For example, in the case of two particles from $|\epsilon_1| \geq |\pi_1|$ and $|\epsilon_2| \geq |\pi_2|$ it follows that the absolute value of the sum $|\epsilon_1 + \epsilon_2|$ is also greater or equal than $|\pi_1 + \pi_2|$ and so $\epsilon_{tot}^2 - \pi_{tot}^2 \geq 0$. For more than two particles this can be generalized.

[5]In the 1+1 case we find the Jacobian to be

$$e^{-\lambda E}[\lambda^2(\epsilon^2 - \pi^2)/4]/[1 + \lambda^2(\epsilon^2 - \pi^2)/4].$$

use of the Judes-Visser conservation laws [22]. The first order calculations in the c.m. frame show that Amelino-Camelia ISR lowers threshold energies, whereas the Magueijo-Smolin ISR may raise or lower threshold energies, for all allowed processes in special relativity. The exact calculations exhibited in the Appendixes show that there is no vacuum Čerenkov radiation, forbidden in SR, and that photons are stable in these ISRs. Finally, by studying the map to pseudo-energy-momentum variables we demonstrated that no processes beyond those in SR are allowed.

These results show that, when using the Judes-Visser modified conservation laws, the GZK threshold is *lowered* in these ISRs. Although the "GZK paradox" created by the apparent over abundance of events above the GZK threshold is controversial [26,27], our analysis show that these ISRs do not provide a viable explanation of an apparent raising of the threshold. We note, however, that these results depend on both the form of the ISR energy-momentum conservation laws and the assumption on composite particles mentioned in Sec. II.

The kinematic results for the two example theories suggest two questions for any ISR: (i) Is the map between particle kinematic thresholds in the physical variables and the linear variables one-to-one? One source of trouble would be the existence of multiple threshold solutions which would require additional criteria to determine which solution is physical. (ii) Are there processes normally forbidden in special relativity? And at what energy and momentum do they occur?

In addition, in the ISR context we should expect covariance under the modified transformations without requiring the energy-momenta to take unphysical values. If agreement between observers requires an unphysical boundary point of the physical state space, then the theory is not relativistic.

These observations lead us to suggest sharpening the criteria of relativistic theories with an additional invariant scale. As in previous formulations of ISRs, (i) all modifications to special relativity must reduce to special relativity when the second invariant scale λ (E_p) vanishes (diverges). Physical solutions of the modified theories must reduce to the processes of special relativity in this limit. Any theories which have multiple threshold solutions which satisfy this criteria are unphysical. (ii) Processes normally forbidden in special relativity may only occur at the boundary (as determined by the additional scale) of the physical energy-momentum space. Therefore, ISRs can only shift processes (such as kinematic thresholds) or events but will not allow additional processes.

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APPENDIX A

1. Boost for the GZK threshold

To find the boost from the c.m. frame to the cosmological frame one can use the c.m. condition

$$\frac{P_p}{1-\lambda E_p} = -\frac{P_\gamma}{1-\lambda E_\gamma} \tag{A1}$$

to find E_γ, the energy of the photon in the c.m. frame. Boosting this energy to give ϵ, the energy of the far infrared photon in the cosmological frame gives γ:

$$\gamma = \frac{E_\gamma^2 + \epsilon^2 - 2E_\gamma^2\epsilon\lambda - 2\lambda E_\gamma \epsilon^2 + 2\lambda^2\epsilon^2 E_\gamma^2}{2E_\gamma \epsilon(1-\lambda E_\gamma)(1-\lambda\epsilon)}. \tag{A2}$$

With the modified dispersion relation, Eq. (12), and the equation for E_γ it is possible to use the above γ to boost the threshold back into the cosmological frame. The result, to leading order in λ (with $m \equiv m_p$), is

$$\begin{aligned} E^*_{\rm ISR} \approx{}& \frac{4\epsilon^2 m^2 + m_\pi^2(2m+m_\pi)^2}{4\epsilon m_\pi(2m+m_\pi)} - \lambda[m_\pi^4(m+m_\pi)(2m+m_\pi)^4 \\ &+16\epsilon^3 m^2\{\epsilon(m^3 - m^2 m_\pi - 3mm_\pi^2 - m_\pi^3) \\ &- m_\pi(6m^3 + 8m^2 m_\pi + 2mm_\pi^2 - m_\pi^3)\} \\ &-4\epsilon m_\pi^3(2m+m_\pi)^2(-2m^3 - 2m^2 m_\pi + m_\pi^3) \\ &+4\epsilon^2 m_\pi^2(2m+m_\pi)^2(2m^3 + 4m^2 m_\pi + 3mm_\pi^2 + m_\pi^3)]/ \\ &16\epsilon^2 m_\pi^2(m+m_\pi)(2m+m_\pi)^2. \end{aligned} \tag{A3}$$

Expanding this in leading terms assuming $m_\pi/m \ll 1$ and $\epsilon/m_\pi \ll 1$ one finds that

$$E^*_{\rm ISR} \approx \frac{mm_\pi}{2\epsilon} - \lambda\left(\frac{mm_\pi}{2\epsilon}\right)^2 = E^*_{\rm ISR} - \lambda(E^*_{\rm SR})^2, \tag{A4}$$

so, not surprisingly, the boost modifications swamp the mass modifications.

2. VCR

Vacuum Čerenkov radiation may occur in theories with modified dispersion relations, and indeed this process places strong limits on the extent of the modification [16]. Since ISRs apparently do not require a preferred frame, we can make use of the usual process kinematics techniques of SR. In the rest frame of the incoming charged particle let the energy-momentum be $(E_0, p_0) = (m_a, 0)$. We denote the product energy momenta as (E_a, p_a) and (E_γ, p_γ). The modified conservation of momentum immediately gives $\pi_a = -\pi_\gamma$. The modified conservation of energy is then

$$E_0 = E_{tot} = \frac{\epsilon_a + \epsilon_\gamma}{1+\lambda(\epsilon_a+\epsilon_\gamma)} = \frac{\epsilon_a - \pi_a}{1+\lambda(\epsilon_a - \pi_a)}. \tag{A5}$$

With the dispersion relation $(\epsilon_a-\pi_a)(\epsilon_a+\pi_a)=\mu_a^2$ one can re-express energy conservation as a simple polynomial in ϵ_a which has but one solution $(\epsilon_a,\pi_a)=(\mu_a,0)$. Therefore, since the photon physical momentum vanishes, VCR does not occur.

3. Photon stability

In the case of photon stability we use a different method that does not require a choice of reference frame. We denote the photon energy-momentum by (E_γ,p_γ) and the electron-positron pair energy-momenta by $(E_\pm,p_\pm)$. In Magueijo-Smolin ISR, the pseudo-momentum is conserved, so we have $\epsilon_{tot}=\epsilon_\gamma=\pi_\gamma$ with the last equality being true for massless particles. The relation gives the simple result

$$\frac{E_+}{1-\lambda E_+}-\frac{p_+}{1-\lambda E_+}=-\frac{E_-}{1-\lambda E_-}+\frac{p_-}{1-\lambda E_-}. \quad \text{(A6)}$$

With the energy and momentum of the outgoing particles separated we simply need to understand the behavior of one function. Using the dispersion relations of Eq. (4) we simply have

$$f(E_+)=-f(E_-) \quad \text{(A7)}$$

with

$$f(E)=\frac{E-\sqrt{E^2-m^2\left(\frac{1-\lambda E}{1-\lambda m}\right)^2}}{1-\lambda E}. \quad \text{(A8)}$$

The condition of Eq. (A7) is only satisfied at a root of $f(E)=0$. However, this only occurs when $E=E_p$. Since this point is excluded, the photon is stable.

APPENDIX B

1. VCR

The vacuum Čerenkov calculation proceeds as in Magueijo-Smolin ISR when one takes the rest frame of the incoming charged particle. In Amelino-Camelia ISR, however, the modified energy conservation becomes

$$m_a=\frac{1}{\lambda}\ln\left[1+\lambda\epsilon_{tot}\sqrt{1+\frac{\lambda^2\epsilon_{tot}^2}{4}}+\frac{\lambda^2}{2}\epsilon_{tot}^2\right] \quad \text{(B1)}$$

with

$$\epsilon_{tot}=\frac{e^{\lambda E_a}-\cosh(\lambda m_a)}{\lambda\cosh(\lambda m_a/2)}+\frac{e^{\lambda E_\gamma}-1}{\lambda}. \quad \text{(B2)}$$

The expression of Eq. (B1) simply gives, after a bit of algebra,

$$\epsilon_{tot}=\frac{2\sinh(\lambda m_a/2)}{\lambda}\equiv\mu_a. \quad \text{(B3)}$$

Since the pseudo-energy is equivalent to the pseudo-mass, it is not surprising that we find, from the definition of ϵ_{tot}, that $E_\gamma=0$ and $(E_a,p_a)=(m_a,0)$. As in SR, there is no VCR in Amelino-Camelia ISR.

2. Photon stability

In the Amelino-Camelia ISR, conservation of energy $E_\gamma=E_{tot}$ gives

$$\epsilon_\gamma=\epsilon_{tot}\sqrt{1+\frac{\lambda^2(\epsilon_{tot}^2-\pi_{tot}^2)}{4}}+\frac{\lambda(\epsilon_{tot}^2-\pi_{tot}^2)}{2}. \quad \text{(B4)}$$

But photons have the property that $\epsilon_\gamma^2=\pi_\gamma^2$. So we can use momentum conservation $p_\gamma=p_{tot}$ to simplify this. In fact,

$$\epsilon_\gamma^2=\pi_{tot}^2\left(1+\frac{\lambda^2}{4}(\epsilon_{tot}^2-\pi_{tot}^2)\right). \quad \text{(B5)}$$

Equating the two expressions for ϵ_γ^2 we have the result

$$\begin{aligned}0&=(\epsilon_{tot}^2-\pi_{tot}^2)\left[1+\frac{\lambda^2}{2}(\epsilon_{tot}^2-\pi_{tot}^2)\right.\\&\left.\quad+\lambda\epsilon_{tot}\sqrt{1+\frac{\lambda^2}{4}(\epsilon_{tot}^2-\pi_{tot}^2)}\right]\\&=(\epsilon_{tot}^2-\pi_{tot}^2)e^{\lambda E_{tot}}.\end{aligned} \quad \text{(B6)}$$

The first solution to Eq. (B6), when the first factor vanishes, gives $E=-m$. This is the result that one would obtain in SR by an analogous calculation. Since $E>0$, the "solution" is unphysical. For the same reason the second factor cannot vanish. Hence, there are no massive-particle solutions, so the photon is stable in the Amelino-Camelia framework as well.

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