2-15-2011

Plane gravitational waves in real connection variables

F. Hinterleitner

Seth Major
Hamilton College, smajor@hamilton.edu

This document is the publisher's version of an article published in:
Physical Review D., vol. 83, no. 4 (2011): 044034. doi: 10.1103/PhysRevD.83.044034

PHYSICAL REVIEW D 83, 044034 (2011)

Plane gravitational waves in real connection variables

Franz Hinterleitner[1] and Seth Major[2]
[1]Department of Theoretical Physics and Astrophysics, Faculty of Science of the Masaryk University, Kotlářská 2, 611 37 Brno, Czech Republic
[2]Department of Physics, Hamilton College, Clinton, New York 13323 USA
(Received 24 June 2010; published 18 February 2011)

We investigate using plane-fronted gravitational wave space-times as model systems to study loop quantization techniques and dispersion relations. In this classical analysis we start with planar symmetric space-times in the real connection formulation. We reduce via Dirac constraint analysis to a final form with one canonical pair and one constraint, equivalent to the metric and Einstein equations of plane-fronted-with-parallel-rays waves. Because of the symmetries and use of special coordinates, general covariance is broken. However, this allows us to simply express the constraints of the consistent system. A recursive construction of Dirac brackets results in nonlocal brackets, analogous to those of self-dual fields, for the triad variables. Not surprisingly, this classical analysis produces no evidence for dispersion, i.e. a variable propagation speed of gravitational plane-fronted-with-parallel-rays waves.

DOI: 10.1103/PhysRevD.83.044034 PACS numbers: 04.60.Pp, 04.20.Fy, 04.30.Nk, 04.60.Ds

I. INTRODUCTION

The success in constraining modified dispersion relations [1–4] has renewed efforts to see whether, in the context of various approaches to quantum gravity, such modifications arise. This is interesting even in model systems where the quantization may be more unambiguously carried out and where it is possible to identify the origin of the modifications, should they appear. For instance, this has been explored in the context of polymer quantization of scalar fields in flat space-time [5,6]. In this case the origin of the modifications lies in the choice of classical polymer variables, in particular, the length scale required to express the exponentiated momentum variable, rather than in a granularity of spatial geometry. While for loop quantum gravity there are heuristic results suggesting that there might be modifications to dispersion relations [7–9], it would be interesting to investigate possible modifications in a model system in which both the origin of the modification is clear and in which the quantization may be completed. This paper explores whether the symmetry-reduced space-times of plane-fronted gravitational waves with parallel rays (pp) may be a suitable context in which to explore modifications to dispersion relations.

Classical plane gravitational waves are, like homogeneous and isotropic cosmological models, among the most simple exact solutions of general relativity (GR). Despite the nonlinearity of GR, due to the symmetry of the model, pulses of (pp)-waves travel without dispersion and leave space flat outside the pulse; they form a "wave sandwich" with the gravitational wave pulse between regions of flat space. In fact, the only gravitational waves with this flat space sandwich property are pp-waves [10]. Because of their simplicity, pp-waves promise to be good candidates to test quantization techniques for pure gravity. It is also intriguing that the Einstein equations for plane electromagnetic waves coupled to gravity take the same form as the pp-waves [11], suggesting that a quantization and study of dispersion relations of pp-waves could be extended to this Einstein-Maxwell theory model. So the quantization of pp-waves could yield answers to the questions on whether quantum effects of 3-dimensional geometry lead to dispersion of gravitational (or electromagnetic) waves and whether spatial granularity leads to an energy dependence of the speed of gravitation waves (or light).

In view of this eventual goal, we formulate polarized parallel plane waves in terms of the real connection variables, proceeding from rather general assumptions about homogeneity in two dimensions, to a form equivalent to a standard form in the literature, given below. Despite the simplicity of this well-known result, the canonical way to this goal is not trivial.

The metric of pp-waves propagating in z direction, in a "Rosen-type" chart, given by Misner, Thorne, and Wheeler [11] is

$$ds^2 = -dt^2 + L^2 e^{2\beta} dx^2 + L^2 e^{-2\beta} dy^2 + dz^2. \quad (1)$$

(See also Ehlers and Kundt [12].) This metric has a convenient interpretation for our purposes. The function L, called the "background factor," is determined by the free function β, called the "wave factor." Both L and β are functions of $v := t + z$ or $u := t - z$. L satisfies the Einstein equation

$$L'' + (\beta')^2 L = 0. \quad (2)$$

(In this equation the prime denotes a derivative with respect to u or v.) This single equation survives the reduction of GR. In it β' acts as a "time-dependent" angular frequency. In the light cone coordinates u—once $\beta(u)$, $L(0)$ and $L'(0)$ are specified—the function $L(u)$ is determined just as in a simple 1-dimensional mechanical system.

The above form of the metric as well as Eq. (2) are valid only for waves traveling in the positive *or* in the negative z direction. The combination of both, i.e. colliding waves, are a problem of a much higher degree of difficulty [13,14].

An important feature of the above coordinate system is that it does not globally cover the space-time inhabited by plane waves. One may see this even with a short pulse, with $\beta \neq 0$ only in an interval (u_-, u_+). For example choosing $L \equiv 1$ in the region of flat space in front of the pulse, at the location of the pulse, where $\beta \neq 0$, L'' becomes negative due to (2), and so L decreases inside the wave, as long as $L \geq 0$. (We assume the pulse to be short enough and not too strong so that $L > 0$ everywhere inside. For details of this approximation see [11].) In the flat space behind the pulse, when β is constant again, $L'' = 0$ and L is a linear function, which has to be matched smoothly to L at the end of the pulse. This leads necessarily to $L = 0$ at a certain value $u_2 > u_+$ somewhere behind the pulse, a coordinate singularity in flat space [11,15]. In this case the coordinate system is valid in the region $u < u_2$.

One can choose—and we shall do so—L to be a nonconstant linear function on both sides of the wave pulse, with one zero in front, at $u = u_1$, $u_1 < u_-$, and one behind it. In this case, the coordinates cover a slice of the gravitational wave sandwich. In detail, we have the three subintervals of the coordinate range (u_1, u_2):

(1) $u_1 < u \leq u_-$, $L' = \text{const} > 0$, $\beta = 0$, flat space in front of the pulse,
(2) $u_- < u < u_+$, $L'' < 0$, $\beta \neq 0$, the pulse, and
(3) $u_+ < u < u_2$, $L' = \text{const} < 0$, β constant, flat space behind the pulse.

The boundary conditions at $u = u_{1,2}$ are flat-space boundary conditions with constant β and $L = 0$ at the coordinate singularities. Despite the regions of flat space before and after the pulse, neighboring test particles in the xy plane accelerate and fall toward each other as the wave passes [11,15].

In the next section we review the canonical variables and the polarized Gowdy model of [16]. The consistent reduction to the pp-wave case is accomplished in Secs. II D and II E. The Dirac brackets are constructed in Sec. III. Time evolution in the preferred coordinates is discussed in Sec. III D. A note on the orthogonality of the connection and the Immirzi parameter is in Sec. IV. Finally, the results of the classical calculations are summarized in Sec. V.

II. SYMMETRY REDUCTION

A. The connection variables

We formulate the system, after a 3 + 1 split, with the usual densitized triads ${E^a}_i$ and connection components ${A_a}^i$ of the real connection formulation. (See Ref. [17] for a review.) We denote $a = x, y, z$ as a spatial and $i = 1, 2, 3$ as an su(2) Lie algebra index. By homogeneity, we further assume that on every spatial slice they are functions of z alone, with their time dependence to be determined by equations of motion. The system has a close formal analogy to the polarized Gowdy T^3 model analyzed by Banerjee and Date [16,18]. For this reason we have chosen essentially the same notation, so that at many points we can refer to these papers. Of course, the angular variable θ of [16] had to be changed to z and, as is clear from this notation, we are also not working with a compact spatial topology.

The symmetry reduction from full GR to the Gowdy model is carried out in a process outlined, for example, in the Appendix of [19]. In the present case the symmetry reduction is briefly the following: We start with a principal fiber bundle of $SU(2)$, the gauge group of loop quantum gravity, over a 3-dimensional space manifold. The spatial symmetry in the presence of pp-waves consists of translations in the x and y direction, the orbits of which are planes parallel to the xy-plane. The space manifold is decomposed into an orbit bundle with a one-dimensional basis manifold, the z-axis, called the reduced manifold. The "reduced bundle" is the trivial principal $SU(2)$ bundle over a single coordinate neighborhood $z_1 < z < z_2$ of the reduced manifold, where z_1 and z_2 correspond to the null coordinate boundaries u_1 and u_2 mentioned in the Introduction. The symmetry reduction proceeds with a decomposition of the bundle connection. The latter one is separated into the reduced connection, i.e. the restriction along z, ${A_z}^i(z)\tau_i$, and into scalar fields on the reduced manifold, ${A_x}^i(z)\tau_i$, ${A_y}^i(z)\tau_i$, where τ_i are the usual SU(2) generators. By a choice of gauge the reduced connection is assumed to lie in the subalgebra generated by τ_3, the scalar fields in the subspace spanned by τ_1 and τ_2, so that the matrices ${A_a}^i$ and the canonically conjugate densitized triad matrices become block-diagonal [20]:

$$\begin{aligned} &{E^z}_I = E^\rho_3 = 0, \\ &{A_z}^I = {A_\rho}^3 = 0, \quad \text{with} \quad \rho = x, y, \qquad I = 1, 2. \end{aligned} \tag{3}$$

As usual in loop quantum gravity, the connection ${A_a}^i$ is defined as the combination

$${A_a}^i = {\Gamma_a}^i - \gamma {K_a}^i \tag{4}$$

of the torsion-free spin connection ${\Gamma_a}^i$ and the extrinsic curvature ${K_a}^i$. The Barbero-Immirzi parameter is denoted with γ. These variables are subject to the usual constraints of canonical GR, the Gauß, the diffeomorphism, and the Hamiltonian constraint.

Because of the planar symmetry of the waves, the phase space variables are free of dependence on x or y. We restrict integration in the xy plane to a finite, fiducial patch with area A_o. Integrating the symplectic structure over this patch gives

$$\Omega = \frac{A_o}{\kappa' \gamma} \int dz (dA^3_z \wedge dE^z_3 + dA^I_\rho \wedge dE^\rho_I) \tag{5}$$

where $\kappa' = 4\pi G$. For the rest of this article we use $\kappa = \kappa'/A_o$. Following [16] we denote $E^z{}_3$ by $\mathcal{E}$ and $A_z{}^3$ by $\gamma\mathcal{A}$ and introduce polar coordinates in the "1-2" plane:

$$E^x{}_1 = E^x\cos\beta, \qquad E^x{}_2 = E^x\sin\beta,$$
$$E^y{}_1 = -E^y\sin\bar{\beta}, \qquad E^y{}_2 = E^y\cos\bar{\beta}, \tag{6}$$

$$A_x{}^1 = A_x\cos(\alpha+\beta), \qquad A_x{}^2 = A_x\sin(\alpha+\beta),$$
$$A_y{}^1 = -A_y\sin(\bar{\alpha}+\bar{\beta}), \qquad A_y{}^2 = A_y\cos(\bar{\alpha}+\bar{\beta}). \tag{7}$$

The canonically conjugate connection variables to the radial variables E^x and E^y, β and $\bar{\beta}$ are

$$K_x := \frac{1}{\gamma}A_x\cos(\alpha) \quad \text{and} \quad K_y := \frac{1}{\gamma}A_y\cos(\bar{\alpha}) \tag{8}$$

and

$$P^\beta := -E^xA_x\sin(\alpha) \quad \text{and} \quad P^{\bar{\beta}} := -E^yA_y\sin(\bar{\alpha}), \tag{9}$$

respectively.

B. The polarization condition

Following Banerjee and Date [16] we carry out a reduction to the polarized model by setting $\beta = \bar{\beta}$. This ensures that the Killing vectors ∂_x and ∂_y are orthogonal. This means that E^x and E^y become orthogonal in the sense that

$$E^x{}_iE^y{}_i = 0 \tag{10}$$

and that the spatial part of the metric (1) becomes diagonal. Denoting the spatial distance as ds^2

$$ds^2 = \mathcal{E}\frac{E^y}{E^x}dx^2 + \mathcal{E}\frac{E^x}{E^y}dy^2 + \frac{E^xE^y}{\mathcal{E}}dz^2. \tag{11}$$

At this point we do not yet specify the lapse function and the shift vector. After redefinition of the angular variables and their momenta

$$\xi := \beta - \bar{\beta}, \qquad P^\xi := \frac{P^\beta - P^{\bar{\beta}}}{2} \tag{12}$$

and

$$\eta := \beta + \bar{\beta}, \qquad P^\eta := \frac{P^\beta + P^{\bar{\beta}}}{2}, \tag{13}$$

the condition $\beta = \bar{\beta}$ may be imposed in the form of a second-class constraint, $\xi := \beta - \bar{\beta} = 0$, whose Poisson bracket with the Hamiltonian constraint does not vanish weakly. When $\{\xi, H\}$ is added to the constraints, then together with ξ it forms a pair of second-class constraints, weakly Poisson-commuting with all the other constraints. After introducing Dirac brackets, this pair of constraints can be imposed strongly, thus eliminating ξ and P^ξ [16].

The remaining phase space variables are $\mathcal{A}$, K_x, K_y, η and $\mathcal{E}$, E^x, E^y, P^η; their Dirac brackets are equal to the Poisson brackets, as they all have weakly vanishing Poisson brackets with the two strongly imposed constraints.

C. The gauge constraint

We have accomplished the symmetry reduction and imposed the polarization condition. In these variables the Gauß constraint G, the diffeomorphism constraint C and the Hamiltonian constraint H reduce to (see [16])

$$G = \frac{1}{\kappa\gamma}[\mathcal{E}' + 2P^\eta], \tag{14}$$

$$C = \frac{1}{\kappa}\left[K_x'E^x + K_y'E^y - \mathcal{E}'\mathcal{A} + \frac{1}{\gamma}\eta'P^\eta\right], \tag{15}$$

$$\begin{aligned} H = -\frac{1}{2\kappa\sqrt{E}}\Bigg[&\frac{\kappa^2}{2}G^2 + (K_xE^x + K_yE^y)\left(\frac{\eta'}{\gamma} + 2\mathcal{A}\right)\mathcal{E} \\ &+ 2K_xE^xK_yE^y + \frac{1}{2}\left\{(\mathcal{E}')^2 - \mathcal{E}^2\left(\frac{E^{y\prime}}{E^y} - \frac{E^{x\prime}}{E^x}\right)^2\right\} \\ &+ 2\left\{-\left(\frac{E^{x\prime}}{E^x} + \frac{E^{y\prime}}{E^y}\right)\mathcal{E}P^\eta + \mathcal{E}'P^\eta + 2\mathcal{E}P^{\eta\prime}\right\}\Bigg], \end{aligned} \tag{16}$$

where $E := \mathcal{E}E^xE^y$ is the determinant of the 3-metric. The prime means derivative with respect to z and κ is the gravitational constant. The total Hamiltonian is

$$H_{\text{tot}} = A_o\int dz(\lambda G + nC + NH). \tag{17}$$

The authors of [16] point out that, for the polarized Gowdy model, an orthogonality condition on the $A_a{}^i$ analogous to (10), namely,

$$A_x{}^iA_y{}^i = \gamma(K_xE^{x\prime} - K_yE^{y\prime}) = 0 \tag{18}$$

in the above variables, is not conserved under evolution. More precisely, the Poisson bracket of the condition with H does not vanish weakly; it would give rise to a further constraint, and so on, rendering the system inconsistent. Nevertheless, a careful analysis of the spin connection and extrinsic curvature derived from the metric (1) shows that pp-waves do satisfy this condition. We will come back to this issue in Sec. IV.

For the present we follow the simplifications in [16] one more step by strongly imposing the Gauß constraint together with the associated gauge fixing condition $\eta = 0$ and thus remove the variables η and P^η. Again the Dirac brackets of the remaining variables are equal to their Poisson brackets, defined by

$$\begin{aligned} \{F, G\} = \kappa\int dz\bigg(&\frac{\delta F}{\delta\mathcal{A}}\frac{\delta G}{\delta\mathcal{E}} - \frac{\delta F}{\delta\mathcal{E}}\frac{\delta G}{\delta\mathcal{A}} \\ &+ \frac{\delta F}{\delta K_\rho}\frac{\delta G}{\delta E^\rho} - \frac{\delta F}{\delta E^\rho}\frac{\delta G}{\delta K_\rho}\bigg). \end{aligned} \tag{19}$$

When we have carried this out, the diffeomorphism constraint (15) drops its last term and the Hamiltonian constraint becomes equal to

$$H = -\frac{1}{2\kappa}\frac{1}{\sqrt{E}}[2K_xE^xK_yE^y + 2(K_xE^x + K_yE^y)\mathcal{E}\mathcal{A}] + \frac{1}{4\kappa}\frac{1}{\sqrt{E}}\bigg[\mathcal{E}^2\bigg(\frac{E^{y\prime}}{E^y} - \frac{E^{x\prime}}{E^x}\bigg)^2 - 2\mathcal{E}\mathcal{E}'\bigg(\frac{E^{y\prime}}{E^y} + \frac{E^{x\prime}}{E^x}\bigg) + (\mathcal{E}')^2 + 4\mathcal{E}\mathcal{E}''\bigg]. \tag{20}$$

This last step reduced the system to three canonical pairs, related by two first-class constraints, so that the field theory has exactly one phase space degree of freedom per spatial point. As shown in Ref. [16], the algebra of the constraints is the correct one for canonical GR.

D. Reduction to pp-waves and the spatial Einstein equation

Now we reduce the theory to a model equivalent to the one formulated in metric variables by (1) and (2), again using Dirac's constraint analysis. The metric of (1) contains two functions, L and β, rather than the three functions $\mathcal{E}$, E^ρ in (11). The one Einstein equation (2) is not equivalent to the remaining constraints $C = 0$ or $H = 0$. So we need (at least) one more constraint. Comparing the spatial part of the metrics (1) with (11), we see that we need the primary constraint $g_{zz} = 1$, or

$$B := \mathcal{E} - E^xE^y = 0. \tag{21}$$

Of course there is no guarantee that the resulting system is consistent—after all, the polarized Gowdy is already reduced to two phase space degrees of freedom—but this system is simple enough so we can introduce the appropriate constraints in the special coordinate system.

For the reduced theory to be consistent, the new local constraint B must be preserved under evolution of the total Hamiltonian constraint. The Poisson bracket with the smeared-out diffeomorphism constraint

$$C[n] := A_o \int dz n(z)C(z) \tag{22}$$

is

$$\{B(z), C[n]\} = -n(z)B'(z) + 2n'(z)E^x(z)E^y(z). \tag{23}$$

Generally this is not weakly equal to zero, because the new constraint is not invariant under local diffeomorphisms due to the different nature of $\mathcal{E}$ and E^ρ. The variable $\mathcal{E}$ transforms as a scalar, whereas E^x and E^y transform as scalar densities, as can be seen from the Poisson brackets:

$$\{\mathcal{E}(z), C[n]\} = n(z)\mathcal{E}'(z),$$
$$\{E^\rho(z), C[n]\} = (n(z)E^\rho(z))'.$$

Only when the test function n is constant along z—meaning the shift vector depends only on t—is the combination B of $\mathcal{E}$ and E^ρ meaningful and conserved under the action of $C[n]$.

In the end the failure of B to be diffeomorphism invariant is not a surprise. Demanding $g_{zz} = 1$, we obviously restrict local diffeomorphism invariance. (In the special case (1), the shift vector is equal to zero.) The local constraint $C(z)$, in contrast to the global translation generator $C[n]$, becomes second-class after introducing the new constraint $B(z)$.

With the Hamiltonian constraint, smeared out with a (lapse) function N, the constraint B has the Poisson bracket

$$\{B(z), H[N]\} = N\bigg(\frac{(K_xE^x + K_yE^y)}{\sqrt{E}}B - 2\sqrt{E}\mathcal{A}\bigg). \tag{24}$$

B will thus be preserved under the evolution generated by H, only if we add a new constraint,

$$\mathcal{A} = 0. \tag{25}$$

The constraints $\mathcal{A}$ and B form a second-class conjugate pair:

$$\{\mathcal{A}(z), B(z')\} = \kappa\delta(z - z'). \tag{26}$$

The new constraint $\mathcal{A}$ must be preserved as well. It is diffeomorphism invariant in the full sense, and thus translation invariant, since

$$\{\mathcal{A}(z), C[n]\} = [n(z)\mathcal{A}(z)]' \tag{27}$$

is weakly equal to zero. The Poisson bracket of $\mathcal{A}$ with the Hamiltonian constraint is

$$\{\mathcal{A}, H[N]\} = \kappa N\bigg[\frac{\partial H}{\partial\mathcal{E}} - \bigg(\frac{\partial H}{\partial\mathcal{E}'}\bigg)' + \bigg(\frac{\partial H}{\partial\mathcal{E}''}\bigg)''\bigg] - \kappa N'\bigg[\frac{\partial H}{\partial\mathcal{E}'} - \bigg(\frac{\partial H}{\partial\mathcal{E}''}\bigg)'\bigg] + \kappa N''\frac{\partial H}{\partial\mathcal{E}''}. \tag{28}$$

The derivatives are

$$\frac{\partial H}{\partial\mathcal{E}} = -\frac{H}{2\mathcal{E}} - \frac{1}{\kappa\sqrt{E}}\bigg[(K_xE^x + K_yE^y)\mathcal{A} - \frac{1}{2}\bigg(\frac{E^xE^{y\prime} - E^{x\prime}E^y}{E^xE^y}\bigg)^2\mathcal{E} + \frac{1}{2}\frac{(E^xE^y)'}{E^xE^y}\mathcal{E}' - \mathcal{E}''\bigg],$$

$$\frac{\partial H}{\partial\mathcal{E}'} = \frac{1}{\kappa\sqrt{E}}\bigg[\mathcal{E}' - \frac{(E^xE^y)'}{E^xE^y}\mathcal{E}\bigg], \quad \text{and} \quad \frac{\partial H}{\partial\mathcal{E}''} = \frac{1}{\kappa\sqrt{E}}\mathcal{E}.$$

With the new constraints $\mathcal{A}$ and B taken into account, we have the following weak equivalences:

$$\frac{\partial H}{\partial\mathcal{E}} \approx \frac{E^xE^{y\prime\prime} + E^{x\prime\prime}E^y}{\kappa E^xE^y}, \qquad \frac{\partial H}{\partial\mathcal{E}'} \approx 0, \qquad \frac{\partial H}{\partial\mathcal{E}''} \approx \frac{1}{\kappa}. \tag{29}$$

Inserting this into (28) gives

$$\{\mathcal{A}, H[N]\} \approx -\frac{N}{E^xE^y}(E^{x\prime\prime}E^y + E^xE^{y\prime\prime}) - N''. \tag{30}$$

So we see that, as in the case of B with diffeomorphisms, $\mathcal{A}$ is not invariant under the local action of $H(z)$, so the full local Hamiltonian constraint becomes second-class, like the local diffeomorphism constraint $C(z)$. If we choose a lapse function N linear in z and introduce the further constraint

$$D := E^{x\prime\prime}E^y + E^xE^{y\prime\prime}, \tag{31}$$

then the constraint $\mathcal{A}$ is preserved under evolution.

For reasons that become more clear in later calculations, we make the more specialized choice $N = N(t)$. With the additional constraint $\partial_z N = 0$ on the Lagrange multiplier N, the system remains consistent, as can be checked using the constraint algebra. With this assumption, the Hamiltonian constraint is reduced to a global condition $H[N] = 0$; the associated symmetry transformation is an evolution in a global time. This choice is in accordance with the form of the metric (1), where the choice $N = 1$ is even more special [21].

The constraint D is interesting. Imposing $B = 0$, we may express

$$E^x = Le^{-\beta} \quad \text{and} \quad E^y = Le^{\beta} \tag{32}$$

and, after insertion into (31), the constraint equation $D = 0$ becomes $2L(\partial_z^2 L + (\partial_z\beta)^2 L) = 0$; this is the spatial part of the Einstein equation (2).

Now D Poisson-commutes trivially with $\mathcal{A}$ and B. Its Poisson bracket with $C[n]$, given $n' = 0$, is

$$\{D[f], C[n]\} = -nD[f'] \tag{33}$$

and vanishes weakly. However, the analysis is not complete since we have to be sure that $D = 0$ is preserved under the Hamiltonian constraint.

E. Consistency of the reduced system

Taking into account the constraints $\mathcal{A}$ and B, and under the condition $N = N(t)$, the Poisson bracket of D with H is, after integration by parts,

$$\{D[f], H[N]\} \approx \int dz f[N(K_xE^{x\prime\prime} + K_yE^{y\prime\prime} + K_x^{\prime\prime}E^x + K_y^{\prime\prime}E^y)]. \tag{34}$$

So far we have reduced a system on the 6-dimensional phase space with two first-class local constraints, $H(z)$ and $C(z)$, to one with five second-class local constraints $H(z)$, $C(z)$, $\mathcal{A}(z)$, $B(z)$ and $D(z)$ and two global evolution generators $H[N(t)]$ and $C[n(t)]$. Numerically, five constraints per space point would suffice to reduce six phase-space functions to one, corresponding to the free function β in (2), but consistency under time evolution requires more.

Even with the assumption that N is independent of z, D does not weakly Poisson-commute with $H[N]$. The bracket is equivalent to

$$\begin{aligned} &-N\int dz f(K_x^{\prime\prime}E^x + K_y^{\prime\prime}E^y + K_xE^{x\prime\prime} + K_yE^{y\prime\prime}) \\ &\quad =: -N\int dz f(z)J(z). \end{aligned} \tag{35}$$

The new constraint $J(z)$ can be expressed as a sum of similar terms including the derivative of D,

$$\begin{aligned} J = {} & E^x(K_x - E^{y\prime})'' + E^y(K_y - E^{x\prime})'' + E^{x\prime\prime}(K_x - E^{y\prime}) \\ & + E^{y\prime\prime}(K_y - E^{x\prime}) + D', \end{aligned} \tag{36}$$

or, alternatively,

$$\begin{aligned} J = {} & E^x(K_x + E^{y\prime})'' + E^y(K_y + E^{x\prime})'' + E^{x\prime\prime}(K_x + E^{y\prime}) \\ & + E^{y\prime\prime}(K_y + E^{x\prime}) - D'. \end{aligned} \tag{37}$$

The Poisson bracket of J with H is weakly equal to the second derivative of D (using $\mathcal{A}$ and B in the equivalence) plus additional terms

$$\begin{aligned} \{J[f], H[N]\} \approx N\int dz f\Big\{ & D'' - 2\Big(\frac{E^{x\prime}}{E^x} + \frac{E^{y\prime}}{E^y}\Big)(E^{x\prime}E^{y\prime} - K_xK_y)' \\ & + 2\Big[\Big(\frac{E^{x\prime}}{E^x}\Big)^2 + \Big(\frac{E^{y\prime}}{E^y}\Big)^2\Big](E^{x\prime}E^{y\prime} - K_xK_y) \\ & + 4(E^{x\prime\prime}E^{y\prime\prime} - K_x^{\prime}K_y^{\prime})\Big\}. \end{aligned} \tag{38}$$

We clearly need to check the Poisson bracket of $\{J, H\}$ with H. The constraints descended from J contain higher and higher derivatives, so this leads to an infinite tower of constraints; the system in this form is inconsistent. On the other hand, we know from the metric formulation of (1) and (2) that there is a consistent formulation for non-colliding waves with one configuration degree of freedom per point in light-cone coordinates. Obviously, there must be relations between the constraints to reduce the number of independent ones.

This observation suggests an obvious solution to the apparent inconsistency. We can restrict the phase space variables at the kinematical level so that they only support left- or right-moving waves [22]. The constraint J in the form (36) or (37) weakly vanishes when $K_x = \pm E^{y\prime}$ and $K_y = \pm E^{x\prime}$. Then J and $\{J, H\}$ are essentially D' and D'' and so also weakly vanish. Hence we impose either the "right-moving"

$$U_x := K_x - E^{y\prime} = 0 \quad \text{and} \quad U_y := K_y - E^{x\prime} = 0 \tag{39}$$

or the "left-moving"

$$V_x := K_x + E^{y\prime} = 0 \quad \text{and} \quad V_y := K_y + E^{x\prime} = 0 \tag{40}$$

as primary constraints. As shown in Appendix AA, these relations (together with the equation of motion, provide a consistent solution to the Einstein equations in terms of the triad and canonical momenta. The relations also anticipate that the canonical momentum of the metric variable E^ρ is equal to $\pm$ its spatial derivative for pp-waves.

In the following we work with the right-moving constraints U_ρ. The Poisson brackets of U_x and U_y are

$$\{U_x(z), U_y(z')\} = \{U_y(z), U_x(z')\} = 2\kappa\delta'(z-z') = 2\kappa\frac{\partial}{\partial z}\delta(z-z') = -\{U_y(z'), U_x(z)\}. \quad (41)$$

Note the antisymmetry in z and z' in spite of the symmetry under the exchange of U_x and U_y.

These right-moving constraints have nonvanishing Poisson brackets with B:

$$\{U_x(z), B(z')\} = \kappa E^y(z)\delta(z-z'), \quad \text{and} \quad \{U_y(z), B(z')\} = \kappa E^x(z)\delta(z-z'). \quad (42)$$

Introducing the multipliers u_ρ (not to be confused with the light-cone coordinate u in the Introduction) for the constraints U_ρ and h for D, we have the Poisson bracket

$$\{U_x[u_x], D[h]\} = \kappa\int dz\{[u_x(z)h(z)]'' + u_x(z)''h(z)\}E^y(z), \quad (43)$$

and a similar relation for $\{U_y[u_y], D[h]\}$. The Poisson brackets $\{U_\rho, H[N(t)]\}$ and $\{U_\rho, C[n(t)]\}$ vanish weakly. Thus, the constraints U_ρ are compatible with time evolution and their introduction solves the problem of the infinitely many constraints, thus making time evolution consistent. On the other hand, this introduction increases the number of second-class constraints to seven, which is definitely too many. What remains to be solved is this apparent overconstraining of the system.

Physically the reason for the constraints U or V lies in the fact that the full Hamiltonian constraint of plane gravitational waves applies to modes going into both the positive and the negative z direction and their mutual interaction. A superposition of left- and right-moving waves would introduce complicated interactions and spoil the simple form of the metric. In Sec. III D we will see that, under the conditions $U_\rho = 0$ or $V_\rho = 0$, the Hamiltonian constraint generates simple plane wave propagation.

III. DIRAC BRACKETS

In this section we construct the Dirac brackets of the local second-class constraints step by step (see below), according to algebraic relationships. The algebraic Poisson bracket structure associates the second-class constraints into two "pairs", $(\mathcal{A}, B)$ and (U_x, U_y), and three single constraints D, H, and C. In addition $\mathcal{A}$ and B do not contain derivatives and so are actually associated to each point z separately. In the course of the analysis the constraints C and H turn out to be dependent, more precisely, equivalent to D, so that the set of independent constraints reduces to the convenient number of five. An odd number of second-class constraints (per space point z) may appear incompatible with the standard construction of Dirac brackets [24], but not all of them are related exactly to one point; some of them contain derivatives.

For a mechanical system with second-class constraints C_i, $i = 1, 2, \ldots, 2n$, the Dirac bracket of two phase space functions F and G is defined as

$$\{F, G\}_d = \{F, G\} - \{F, C_i\}M_{ik}^{-1}\{C_k, G\} \quad (44)$$

in terms of Poisson brackets. The matrix M_{ik}^{-1} is the inverse of the matrix $M_{ik} = \{C_i, C_k\}$ of the Poisson brackets among the constraints. After the Dirac brackets are constructed, the constraints can be imposed strongly. This reduces the system to its physical degrees of freedom.

A helpful fact about Dirac brackets is that they can be constructed recursively, i.e. the construction of Eq. (44) can be carried out for any subset of second-class constraints, provided the matrix of their Poisson brackets is invertible [24]. After imposing these constraints strongly, the procedure can be repeated with the preliminary Dirac brackets replacing the Poisson brackets in Eq. (44). This possibility greatly facilitates the work with our constraints. In field theory, of course, the matrix multiplication in (44) implies integration.

A. Dirac brackets, version D1

Beginning with the pair $(\mathcal{A}, B)$, we have the Poisson brackets (26) and

$$M_{ik}^{-1}(z, z') = \frac{1}{\kappa}\delta(z-z')\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (45)$$

The ensuing Dirac brackets, version D1, are explicitly

$$\{F(z), G(z')\}_{\mathrm{D1}} = \{F(z), G(z')\} - \frac{1}{\kappa}\int dz''\{F(z), \mathcal{A}(z'')\}\{B(z''), G(z')\} + \frac{1}{\kappa}\int dz''\{F(z), B(z'')\}\{\mathcal{A}(z''), G(z')\}. \quad (46)$$

Because of the appearance of $\mathcal{A}$ in both the integrals on the right-hand side, neither the Dirac brackets of the variables E^ρ, K_ρ, nor those of the remaining constraints, differ from the corresponding Poisson brackets. We can simply impose $\mathcal{A}$ and B strongly. When this is done, U_x, U_y, and D are untouched, whereas C and H are simplified considerably: The diffeomorphism constraint drops its term $-\frac{1}{\kappa}\mathcal{E}'\mathcal{A}$ and becomes

$$C = \frac{1}{\kappa}(K_x'E^x + K_y'E^y), \quad (47)$$

whereas the Hamiltonian constraint boils down to

$$H = -\frac{1}{\kappa}[K_xK_y + E^{x\prime}E^{y\prime} - (E^xE^y)''], \quad (48)$$

with the second-derivative term not contributing to integrals with a z-independent test function. Without this term the last expression for H is similar to the Hamiltonian of two free Klein-Gordon fields, the nonlinearity of GR is now hidden in D, which is not conserved under the evolution generated by the Hamiltonian constraint. In the simplest case of constant lapse and shift, e.g. $N = n = 1$, D commutes weakly with C,

$$\{D, C[1]\} = E^x E^{y''''} + E^{x\prime} E^{y'''} + E^{x''} E^{y\prime} + E^{x'''} E^y = D', \tag{49}$$

but not with H,

$$\{D, H[1]\} = E^x K_x'' + E^{x''} K_x + E^y K_y'' + E^{y''} K_y, \tag{50}$$

as long as we do not introduce the constraints U_ρ.

B. Dirac brackets, version D2

The next pair of second-class constraints, (U_x, U_y), has the mutual Poisson brackets (41). To construct the inverse of the matrix of these brackets, we need the inverse of the derivative of a δ-function, denoted by $\delta^{(-1)}$, which satisfies the relation

$$\int dz'' \delta'(z - z'') \delta^{(-1)}(z'' - z') = \delta(z - z'). \tag{51}$$

Obviously $\delta^{(-1)}(z - z')$ is a step function plus an additive constant that is adjusted by demanding antisymmetry [25]:

$$\delta^{(-1)}(z - z') = \frac{1}{2}\,\mathrm{sign}(z - z'). \tag{52}$$

We construct the matrix N_{ik}^{-1} that plays an analogous role as M_{ik}^{-1} in (45),

$$N_{ik}^{-1}(z, z') = \frac{1}{4\kappa}\,\mathrm{sign}(z - z')\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{53}$$

With this matrix the next version of Dirac brackets $\{F, G\}_{\mathrm{D2}} = \{F, G\} - \{F, U_x\} N_{ik}^{-1} \{U_y, G\}$, becomes

$$\begin{aligned}\{F(z), G(z')\}_{\mathrm{D2}} = \{F(z), G(z')\} &- \frac{1}{4\kappa}\int_{z_-}^{z_+} dz'' dz''' \{F(z), U_x(z'')\} \\ &\times \mathrm{sign}(z'' - z''')\{U_y(z'''), G(z')\} \\ &- \frac{1}{4\kappa}\int_{z_-}^{z_+} dz'' dz''' \{F(z), U_y(z'')\} \\ &\times \mathrm{sign}(z'' - z''')\{U_x(z'''), G(z')\}.\end{aligned} \tag{54}$$

(The D1 brackets are the same as the Poisson brackets so the label is omitted.) In particular, the Dirac brackets of the remaining fundamental variables are the following:

$$\begin{aligned}
&\{K_x(z), K_x(z')\}_{\mathrm{D2}} = \{K_y(z), K_y(z')\}_{\mathrm{D2}} = 0,\\
&\{K_x(z), K_y(z')\}_{\mathrm{D2}} = -\frac{\kappa}{2}\delta'(z - z'),\\
&\{K_x(z), E^x(z')\}_{\mathrm{D2}} = \{K_y(z), E^y(z')\}_{\mathrm{D2}} = \frac{\kappa}{2}\delta(z - z'),\\
&\{K_x(z), E^y(z')\}_{\mathrm{D2}} = \{K_y(z), E^x(z')\}_{\mathrm{D2}} = 0,\\
&\{E^x(z), E^x(z')\}_{\mathrm{D2}} = \{E^y(z), E^y(z')\}_{\mathrm{D2}} = 0,\\
&\{E^x(z), E^y(z')\}_{\mathrm{D2}} = \frac{\kappa}{4}\mathrm{sign}(z - z').
\end{aligned} \tag{55}$$

The bracket relations between E^x and E^y may look awkward due to nonlocality. This is explained by the form of the constraints U_x and U_y. Integrating them yields the E's in form of an integral over K. The expression

$$\begin{aligned}E^y(z) &= \frac{1}{2}\left[\int_{z_-}^{z} K_x(z')dz' - \int_{z}^{z_+} K_x(z')dz'\right] \\ &= \frac{1}{2}\int_{z_-}^{z_+} \mathrm{sign}(z - z') K_x(z') dz'\end{aligned}$$

and its counterpart E^x from $U_y = 0$ make the nonlocality of their Dirac brackets plausible. These brackets are of the same form as those of the self-dual fields of [23]; see Sec. V.

To impose the U constraints strongly, we can express the K's in terms of the E's or vice versa, or choose one of the canonical pairs (K_x, E^x) and (K_y, E^y) as fundamental variables. To preserve the canonical structure, the latter choices would seem to be preferable, but in different calculations different choices may be suitable.

After the U's are imposed strongly, $C(z)$ and $H(z)$ become equivalent to $\frac{1}{\kappa}D(z)$, explicitly,

$$C = \frac{1}{\kappa}D + \frac{1}{\kappa}(U_x' E^x + U_y' E^y) \tag{56}$$

and

$$H = \frac{1}{\kappa}D - \frac{1}{\kappa}(U_x U_y + U_x E^{x\prime} + U_y E^{y\prime}), \tag{57}$$

which means that finally the number of independent local constraints is reduced to five and that $D(z)$ now implies also the global constraints $H[N]$ and $C[n]$. So the constraints U_ρ themselves solve the problem of overconstraining that arose after their introduction. Further, the fact that U_ρ (V_ρ) lead to $H[1] = \pm D[1]$ confirms that U/V single out left/right-moving wave modes. The integrated Hamiltonian constraint with $N \equiv 1$ becomes

$$H[1] = -\frac{2}{\kappa}\int dz E^{x\prime}(z) E^{y\prime}(z) = -\frac{2}{\kappa}\int dz K_x(z) K_y(z). \tag{58}$$

Finally, D commutes with the total Hamiltonian, which is now [for $N = 1$ and $n = 0$, according to the assumption in (1)] just $H[1]$,

$$\{D(z), H[1]\}_{\mathrm{D2}} = D'(z) \approx 0. \tag{59}$$

C. The final Dirac brackets

At this point we have one phase space degree of freedom, represented equivalently by one of the above-mentioned pairs of variables, one local constraint $D(z)$ per point z and one global one, $H[1]$, which is at the same time the generator of time evolution. The constraints $D(z)$ are second-class and their Dirac brackets, version D2, are rather complicated.

$$\{D(z), D(z')\}_{\mathrm{D2}} = \kappa[f(z,z')\delta'''(z-z') + g(z,z')\delta'(z-z') + h(z,z')\delta^{(-1)}(z-z')], \tag{60}$$

where $\delta^{(-1)}(z-z')$ was introduced in (52) and

$$f(z,z') = \frac{1}{2}[E^x(z)E^y(z') + E^x(z')E^y(z)],$$
$$g(z,z') = \frac{1}{2}[E^x(z)E^{y\prime\prime}(z') + E^{x\prime\prime}(z)E^y(z') + E^x(z')E^{y\prime\prime}(z) + E^{x\prime\prime}(z')E^y(z)],$$
$$h(z,z') = \frac{1}{2}[E^{x\prime\prime}(z)E^{y\prime\prime}(z') + E^{x\prime\prime}(z')E^{y\prime\prime}(z)]. \tag{61}$$

Let us denote by $\Delta(z,z')$ the inverse of $\{D(z), D(z')\}_{\mathrm{D2}}$, needed in the construction of the final Dirac brackets

$$\{F(z), G(z')\}_D = \{F(z), G(z')\}_{\mathrm{D2}} - (\kappa)^{-1} \times \int dz'' dz''' \{F(z), D(z'')\}_{\mathrm{D2}} \Delta(z'', z''') \times \{D(z'''), G(z')\}_{\mathrm{D2}}. \tag{62}$$

It is an antisymmetric function satisfying

$$\int dz'' \{D(z), D(z'')\}_{\mathrm{D2}} \Delta(z'', z') = \delta(z-z'). \tag{63}$$

We do not have the full solution to this equation. In Appendix B we calculate an approximation, demonstrating some qualitative features of the canonical structure rather than giving the exact Dirac brackets. In the following, Δ is understood as this approximation and $\{,\}_D$ as a representative part of the full Dirac bracket, constructed with Δ.

As already mentioned, when we apply five local constraints strongly, there remains one free variable. If we choose E^x, our fundamental Dirac brackets are those of E^x at different points, constructed according to (62). For this purpose we need the bracket

$$\begin{aligned}\{E^x(z), D(z'')\}_{\mathrm{D2}} &= E^x(z'')\{E^x(z), E^{y\prime\prime}(z'')\}_{\mathrm{D2}} + E^{x\prime\prime}(z'')\{E^x(z), E^y(z'')\}_{\mathrm{D2}} \\ &= \frac{\kappa}{2}E^x(z'')\delta'(z-z'') + \frac{\kappa}{4}E^{x\prime\prime}(z'')\mathrm{sign}(z-z'');\end{aligned} \tag{64}$$

$\{D(z'''), E^x(z')\}_{\mathrm{D2}}$ is calculated analogously. For our approximation of $\{E^x(z), E^x(z')\}_{\mathrm{D}}$ we take only the δ' parts of these brackets. In the following, Δ, calculated in (B11), is more conveniently expressed in terms of antiderivatives of the δ functions,

$$\delta^{(-3)}(z-z') = \frac{1}{4}|z-z'|(z-z'),$$
$$\delta^{(-5)}(z-z') = \frac{1}{48}|z-z'|(z-z')^3.$$

Putting these ingredients together, we have

$$\{E^x(z), E^x(z')\}_{\mathrm{D}} \approx \frac{\kappa}{4}\int dz'' dz''' E^x(z'')\delta'(z-z'') \times \left[\frac{\delta^{(-3)}(z''-z''')}{\Lambda(\frac{z''+z'''}{2})} + \frac{3}{4}\frac{(\Lambda'(\frac{z''+z'''}{2}))^2}{\Lambda^3(\frac{z''+z'''}{2})} \times \delta^{(-5)}(z''-z''')\right] E^x(z''')\delta'(z'''-z').$$

After integrating the δ' functions by parts we expand $E^x(z)$ and its first derivative around $\bar{z} = (z+z')/2$,

$$E^x(z) \approx E^x(\bar{z}) + E^{x\prime}(\bar{z})\frac{z-z'}{2} + \ldots$$

and $E^x(z')$, and make use of

$$\delta^{(-1)}(z-z')\cdot(z-z')^2 = \delta^{(-2)}(z-z')\cdot(z-z') = 2\delta^{(-3)}(z-z').$$

With all the variables evaluated at $\bar{z}$ (so that Λ corresponds to Λ_0 in Appendix B) and inserting finally $\Lambda = L^2$ and $E^x = Le^{-\beta}$, we find

$$\{E^x(z), E^x(z')\}_{\mathrm{D}} \approx \frac{\kappa}{8}e^{-2\beta}\mathrm{sign}(z-z') \times \left[1 - \left(2(\beta')^2 + \frac{5}{4}\beta''\right)(z-z')^2\right]. \tag{65}$$

In the flat space-time regions $z_1 < z < z_-$ and $z_+ < z < z_2$ of our coordinate domain, where $\beta = 0$ (and $L'' = 0$) the field E^x satisfies bracket relations analogous to the commutation relations of self-dual Klein-Gordon fields, considered in [23], which are constructed by restriction to waves going into one direction. The correction in the brackets for E^x is expressed purely in terms of the wave factor β. Although the approximation is rather qualitative, it is quite instructive for some insight into the influence of the gravitational Hamiltonian in the canonical structure of the self-dual fields [23].

For $\beta \neq 0$ the above expression can be interpreted as a low-order approximation of a gravitational correction. Were this quantized, this would appear as a variable Planck constant, as suggested by Hossenfelder [26], or a variable gravitational constant. Other corrections, however, do not fit into this scheme; they give rise to terms qualitatively different from (65).

D. Time evolution

The time evolution of a phase space function F is generated by its Dirac bracket with the total Hamiltonian. As already stated in the preceding section, by virtue of the U's, $H[N]$ becomes equivalent to $C[N]$ and so the total Hamiltonian becomes a generator of a rigid translation. (Had we chosen the V constraints, $H[N]$ would be equivalent to $-C[N]$.) This equivalence allows us to introduce $C[N]$ as a true total Hamiltonian, when we choose a lapse function. The most convenient choice $N \equiv 1$ means a constant unit of time.

The Hamiltonian constraint $H_{\rm tot} = H[1] = C[1]$ being first-class at every stage, its Dirac brackets with any phase space function are equal to the corresponding Poisson brackets, which are equal to the z-derivative, according to the nature of $C[1]$ as translation generator. Hence,

$$\dot{F}(x) = \{F(z), H[1]\}_D = \{F(z), C[1]\} = F'(z). \quad (66)$$

Equivalence of $H[N]$ with $\pm C[N]$ simply means that time evolution is a rigid space translation to the left or to the right, the same relation that characterizes self-dual fields [23].

This completes the Einstein equation by making all the variables depend on $t - z$ (or $t + z$, alternatively). So we have recovered the classical equation of motion (2) in a much reduced phase space. One can describe the system with a single function, e.g. $E^x(z)$, on (z_1, z_2) subject to the second-class constraint D.

IV. THE IMMIRZI PARAMETER AND THE POLARIZATION ANGLE

In this section we return to the orthogonality of the connection components (18) to show that this is satisfied by the reduced model. After strong imposition of all the constraints, the two-vectors $\vec{A}_x = (A_x{}^1, A_x{}^2)$ and $\vec{A}_y = (A_y{}^1, A_y{}^2)$ are orthogonal and there arises a simple relation between the angle α between $\vec{E}^x = (E^x_1, E^x_2)$ and $\vec{A}_x = (A^1_x, A^2_x)$ and the corresponding angle $\bar{\alpha}$.

The variables in polar coordinates from [16], (13)–(16), with $\beta = \bar{\beta} = 0$, corresponding to the gauge $\xi = \eta = 0$, are

$$\begin{aligned}
&E^x_1 = E^x, \quad E^x_2 = 0, \quad E^y_1 = 0, \quad E^y_2 = E^y;\\
&A^1_x = A_x \cos\alpha, \quad A^2_x = A_x \sin\alpha, \quad A^1_y = -A_y \sin\bar{\alpha},\\
&A^2_y = A_y \cos\bar{\alpha}. \quad (67)
\end{aligned}$$

From elementary calculations of the connection components Γ in terms of the E's we find in the gauge $\beta = \bar{\beta} = 0$

$$\Gamma^1_x = \Gamma^2_y = 0, \qquad \Gamma^2_x = -E^{y\prime}, \qquad \Gamma^1_y = -E^{x\prime}, \quad (68)$$

so that the diagonal components A^1_x and A^2_y contain only extrinsic curvature,

$$A^1_x = A_x \cos\alpha = \gamma K^1_x, \qquad A^2_y = A_y \cos\bar{\alpha} = \gamma K^2_y. \quad (69)$$

On the other hand, from the Gauß and the polarization constraint $\{\xi, H\} = 0$, we obtain [from Ref. [16]'s equation (A.14)],

$$A^2_x = A_x \sin\alpha = \Gamma_x, \qquad A^1_y = -A_y \sin\bar{\alpha} = -\Gamma_y; \quad (70)$$

thus the off-diagonal components are purely composed from Γ's. Now the vectors $\vec{A}$ have acquired the form

$$\vec{A}_x = (\gamma K_x, -E^{y\prime}), \qquad \vec{A}_y = (-E^{x\prime}, \gamma K_y), \quad (71)$$

where we have written $K_x = K^1_x$ and $K_y = K^2_y$. For the absolute squares of these vectors we get

$$(A_x)^2 = \gamma^2 (K_x)^2 + (E^{y\prime})^2 \quad \text{and} \quad (A_y)^2 = \gamma^2 (K_y)^2 + (E^{x\prime})^2. \quad (72)$$

With the constraints U_ρ (or V_ρ) this becomes

$$A_x = K_x \sqrt{1 + \gamma^2} \quad \text{and} \quad A_y = K_y \sqrt{1 + \gamma^2}. \quad (73)$$

Inserting into (67) and comparing with (69),

$$\begin{aligned}
A^1_x &= K_x \sqrt{1 + \gamma^2} \cos\alpha = \gamma K_x,\\
A^2_y &= K_y \sqrt{1 + \gamma^2} \cos\bar{\alpha} = \gamma K_y,
\end{aligned}$$

leads to a relation between the angles α and $\bar{\alpha}$ and the Immirzi parameter:

$$\alpha = \bar{\alpha} = \operatorname{arccot}\gamma, \quad (74)$$

so, in the end, after all gauge fixing, $\vec{A}_x$ and $\vec{A}_y$ are orthogonal, and orthogonality is compatible with the Hamiltonian constraint, when the latter reduces to a translation generator.

V. SUMMARY AND CONCLUSION

The principal aim of our considerations is the loop quantization of polarized gravitational plane waves to see whether dispersion relations would be modified, or if there are other effects from the granularity of the kinematic states such as a variable speed of gravitation or a variable Planck constant. One way to handle a quantum theory of plane waves is to quantize a more general system, such as a model with plane symmetry, analogous to the Gowdy model exploited here, and then to distinguish the subspace of left- or right-going modes of the full Hilbert space. But, as the formulation of basic operators in [18] shows, this turns out to be quite complicated.

In the present work we reduced the formalism of plane waves to the physical degree of freedom at the classical level. To derive a classical description of plane waves suitable for loop quantization, we started with the assumption of homogeneity of the spatial geometry in the transversal directions and a coordinate system extending in both directions beyond a gravitational pulse, so that the latter one is embedded between two slabs of flat space in these coordinates. The finite range of this coordinate system rendered the integrations over the remaining spatial coordinate finite. This setting is analogous to the Gowdy model described in [16], and we used the formalism developed therein. The reduction was completed with a Dirac constraint analysis.

This was done as follows. The description starts with a symmetry-reduced model with three configuration space degrees of freedom and the standard diffeomorphism and Hamiltonian constraints of GR, giving one field degree of freedom per spatial point. The first step of pp-wave reduction was carried out in Sec. II D. We restricted the metric in the triad variables to the simple diagonal form of (1) by introducing the constraint B. Preservation of this constraint led us to introduce the secondary constraints $\mathcal{A} = 0$ and $D = 0$, the latter one being the (spatial projection of) the classical Einstein equation [11]. Knowing that the symmetry reduced system cannot describe colliding waves and requiring consistency in the sense that further secondary constraints vanish, we introduced the right- or left-moving constraints U_ρ (or V_ρ) in Sec. II E. After restricting the fields to left- or right-moving modes by imposing constraints, we find a system analogous to the self-dual fields—also scalar fields propagating in one direction—described by Floreanini and Jackiw [23].

We impose these constraints in a special coordinate system, fixed by the constraint B, which breaks diffeomorphism invariance, but makes the form of the constraints U_ρ simple. Alternatively, the constraints distinguishing left-/right-going modes can be formulated in a fully diffeomorphism invariant manner. Work is underway on this approach [27].

In the second step of the reduction we used the "pairwise" structure of the Poisson algebra of the constraints to recursively construct Dirac brackets. In Sec. III, using $\mathcal{A}$ and B and then U_ρ, we constructed the first two versions of Dirac brackets. After imposing the constraints U_ρ, the constraint $D(z)$ commutes with the Hamiltonian, but becomes second class by itself. At this point the evolution has become trivial; all the complications are now in the D2-bracket relation of $D(z)$ and $D(z')$. To complete the canonical treatment of the problem, we constructed the final Dirac brackets. In the end the $D(z)$ are the remaining second-class constraints, leading to one variable (we have chosen E^x) with a very nontrivial Dirac bracket $\{E^x(z), E^x(z')\}_{\mathrm{D}}$, containing the step function $\mathrm{sign}(z - z')$, multiplied by an analytic function in $z - z'$. An approximation is given in Eq. (65). It is clear from this bracket that the canonical structure of the reduced system is obscure when using this variable.

This makes a full quantization elusive. We do not even know the exact closed form of the Dirac brackets. The lowest-order terms, however, turn out to be the analog of the commutation relation of a linear self-dual field [23] plus gravitational corrections.

The result of our preliminary classical considerations gives no suggestion of dispersion in these waves, which would provide an indication of an energy-dependent speed of gravitation. The reduction to left- or right-moving waves leads automatically to the equivalence of the Hamiltonian to the generator of spatial translations. This equivalence was not assumed from the beginning, as in other approaches, for example, in lightlike coordinates, but it appears as a result of the disentangling of otherwise colliding modes by the constraints U_ρ. In this way this complete reduction with the aid of Dirac brackets differs from other canonical approaches such as [28]. In a quantum theory, based on our classical analysis, where these generators are promoted to operators, an analogous result can be expected. The final Dirac bracket does hint at a modification of the quantum relations.

The nonlocal bracket of $E^x(z)$ (65) suggests a modification of the Planck constant (or the gravitational constant) in the first approximation, rather than a variable speed of light. In the framework of our approach (starting from unmodified GR), we expect that the space-time texture arising in a quantum theory of gravity would influence the fundamental structure of quantum theory, mainly the commutators and the uncertainty relations derived from them.

Finally, a remark on the approach to the reduction: After specializing to a z-independent shift vector and a lapse function of the same type in order to conserve B and the ensuing constraints, we could have abandoned the local constraints $C(z)$ and $H(z)$ and kept only the according global ones $C[1]$ and $H[1]$. This would mean to start with a theory different from GR, a theory without full diffeomorphism and time-reparametrization invariance. Nevertheless, we would have arrived at the same results, because on the constraint surface determined by $\mathcal{A}$, B, U_x, and U_y, the local constraints $H(z)$ and $C(z)$ are equivalent to $D(z)$. For this reason we did not have to make explicit use of the local diffeomorphism and Hamiltonian constraint in our work.

ACKNOWLEDGMENTS

The work was supported by the Ministry of Education of the Czech Republic, Contract No. MSM 0021622409. We thank Hamilton College, Masaryk University, and the Perimeter Institute, for hospitality and support. This research was supported in part by the Perimeter Institute for Theoretical Physics. We are grateful to Klaus Bering for pointing out Ref. [23] and for helpful discussions. We thank an anonymous referee for helping to clarify the canonical analysis.

APPENDIX A: THE RICCI TENSOR FOR THE METRIC diag$(-1, (E^y)^2, (E^x)^2, 1)$

To introduce canonical variables, we calculate from the Levi-Cività connection the extrinsic curvature components

$$K_x = \dot{E}^y, \qquad K_y = \dot{E}^x.$$

as canonical conjugate variables to the metric variables E^x and E^y. In terms of these canonical variables we get the five nonvanishing independent components of the Ricci tensor:

$$\begin{aligned} R_{00} &= -\left(\frac{\dot{K}_y}{E^x} + \frac{\dot{K}_x}{E^y}\right),\\ R_{03} &= -\left(\frac{K_y'}{E^x} + \frac{K_x'}{E^y}\right),\\ R_{33} &= -\left(\frac{E^{x\prime\prime}}{E^x} + \frac{E^{y\prime\prime}}{E^y}\right),\\ R_{11} &= E^y(\dot{K}_x - E^{y\prime\prime}) + \frac{E^y}{E^x}(K_xK_y - E^{x\prime}E^{y\prime}),\\ R_{22} &= E^x(\dot{K}_y - E^{x\prime\prime}) + \frac{E^x}{E^y}(K_xK_y - E^{x\prime}E^{y\prime}). \end{aligned}$$

The vacuum Einstein equations $R_{03} = 0$ and $R_{33} = 0$ are the constraints C and D, divided by $-E^xE^y$; the remaining ones contain time evolution. Here C and D appear as primary second-class constraints, unless we smear out C with a z-independent function. As in the main text, all the constraints are consistent when $U_\rho = 0$ or $V_\rho = 0$ and $\partial/\partial t = \pm\partial/\partial z$. For a quantization, this approach also requires Dirac brackets.

APPENDIX B: THE INVERSE OF $\{D(z), D(z')\}_{D2}$

In the defining Eq. (63) of $\Delta(z, z')$, with (64) inserted, the derivative of the δ function is shifted to its second argument: $\delta'(z - z'') = -\partial/\partial z''\delta(z - z'')$, and analogously the third derivative. Then, integrating by parts, we get

$$\begin{aligned} &\int dz''\left\{\frac{\partial^3}{\partial z''^3}[f(z, z'')\Delta(z'', z')] + \frac{\partial}{\partial z''}[g(z, z'')\Delta(z'', z')]\right\}\\ &\quad\times\delta(z - z'') + \int dz'' h(z, z'')\delta^{(-1)}(z - z'')\Delta(z'', z')\\ &= \delta(z - z'). \end{aligned} \tag{B1}$$

Consider the third derivative of the first square bracket,

$$\left(\frac{\partial^3}{\partial z''^3}f\right)\Delta + 3\left(\frac{\partial^2}{\partial z''^2}f\right)\frac{\partial}{\partial z''}\Delta + 3\left(\frac{\partial}{\partial z''}f\right)\frac{\partial^2}{\partial z''^2}\Delta + f\frac{\partial^3}{\partial z''^3}\Delta. \tag{B2}$$

This is multiplied by $\delta(z - z'')$ in (B1), so we need f and its derivatives at $z'' = z$.

$$\begin{aligned} f(z, z) &= E^x(z)E^y(z),\\ \frac{\partial f}{\partial z''}\bigg|_{z''=z} &= \frac{1}{2}(E^x(z)E^y(z))',\\ \frac{\partial^2 f}{\partial z''^2}\bigg|_{z''=z} &= \frac{1}{2}D(z) \approx 0,\\ \frac{\partial^3 f}{\partial z''^3}\bigg|_{z''=z} &= \frac{1}{4}(3D'(z) - (E^xE^y)'''). \end{aligned}$$

Similarly $g(z, z) = D(z)$ and $\partial/\partial z'' g(z, z'')|_{z''=z} = \frac{1}{2}D'(z)$, so the second square bracket does not contribute anything, when $D \approx 0$. Denoting $E^x(z)E^y(z) = L^2(z)$ by $\Lambda(z)$, we may write (B1) in the form

$$\begin{aligned} &\Lambda(z)\frac{\partial^3}{\partial z^3}\Delta(z, z') + \frac{3}{2}\Lambda'(z)\frac{\partial^2}{\partial z^2}\Delta(z, z') - \frac{1}{4}\Lambda'''(z)\Delta(z, z')\\ &\quad + \int dz'' h(z, z'')\delta^{(-1)}(z - z'')\Delta(z'', z') = \delta(z - z'). \end{aligned} \tag{B3}$$

In this equation Δ can be considered as a Green function of an integro-differential operator. Because of the required antisymmetry of the Dirac bracket, we are looking for an antisymmetric Green function.

The dominant coefficient function is Λ, the square of the background factor of the gravitational wave; its derivatives are smaller. To find an approximative part of the solution of this equation we first look for a solution of the differential part and leave the integral part for later iterative corrections. The leading term contains a third derivative, therefore the leading term in Δ is expected to contain the function

$$\begin{aligned} \delta^{(-3)}(z - z') &= \frac{1}{4}\mathrm{sign}(z - z')(z - z')^2\\ &= \frac{1}{4}|z - z'|(z - z'), \end{aligned} \tag{B4}$$

the third derivative of which is the delta function on the right-hand side. To find at least an approximation for Δ, we make an ansatz in the form of a product of this function by a symmetric function of z and z'. Further we assume this function to be analytic in some neighborhood of $z = z'$. Written in terms of $z + z'$ and $z - z'$, this function has only even powers in $z - z'$, so that

$$\Delta(z, z') = \frac{1}{4}\mathrm{sign}(z - z')[a_2(z - z')^2 + a_4(z - z')^4 + \ldots], \tag{B5}$$

with the coefficients a_i being (analytic) functions of $\bar{z} := \frac{z+z'}{2}$. For the derivatives with respect to z, we find

$$\Delta_{,z}(z, z') = \frac{\mathrm{sign}(z-z')}{4}\left[2a_2(z-z') + \frac{1}{2}a_2'(z-z')^2 + \ldots\right]$$

$$\Delta_{,zz}(z, z') = \frac{\mathrm{sign}(z-z')}{4}\left[2a_2 + 2a_2'(z-z') + \left(\frac{a_2''}{4} + 12a_4\right)(z-z')^2 + \ldots\right]$$

$$\Delta_{,zzz}(z, z') = a_2\delta(z-z') + \frac{\mathrm{sign}(z-z')}{4}\left[3a_2' + \left(\frac{3a_2''}{2} + 24a_4\right)(z-z') + \left(\frac{a_2'''}{8} + 18a_4'\right)(z-z')^2 + \ldots\right].$$

Inserting this into the first line of (B3) gives

$$\begin{aligned}&\Lambda a_2\delta(z-z') + \frac{3}{4}(\Lambda a_2' + \Lambda' a_2)\mathrm{sign}(z-z')\\ &\quad + \left(\frac{3}{8}\Lambda a_2'' + \frac{3}{4}\Lambda' a_2' + 6\Lambda a_4\right)|z-z'|\\ &\quad + \left(\frac{1}{32}\Lambda a_2''' + \frac{3}{32}\Lambda' a_2'' - \frac{1}{16}\Lambda''' a_2 + \frac{9}{2}\Lambda a_4' + \frac{9}{2}\Lambda' a_4\right)\\ &\quad \times |z-z'|(z-z') + \ldots . \end{aligned} \tag{B6}$$

Note that Λ and its derivatives are functions of z, whereas the a_i are functions of $\bar{z}$, and their derivatives refer to this argument.

To express everything in terms of $\bar{z}$ and $z - z'$, we expand $\Lambda(z) = \Lambda(\frac{z+z'}{2} + \frac{z-z'}{2})$ around $\bar{z}$:

$$\Lambda(z) = \Lambda_0 + \frac{1}{2}\Lambda_0'(z-z') + \frac{1}{8}\Lambda_0''(z-z')^2 + \ldots, \tag{B7}$$

where $\Lambda_0 = \Lambda(\bar{z})$. Inserting into (B6) and rearranging terms gives

$$\begin{aligned}&\Lambda_0 a_2\delta(z-z') + \frac{3}{4}(\Lambda_0 a_2' + \Lambda_0' a_2)\mathrm{sign}(z-z')\\ &\quad + \left(\frac{3}{8}\Lambda_0 a_2'' + \frac{9}{8}\Lambda_0' a_2' + \frac{3}{8}, \Lambda_0'' a_2 + 6\Lambda_0 a_4\right)|z-z'|\\ &\quad + \left(\frac{1}{32}\Lambda_0 a_2''' + \frac{9}{32}\Lambda_0' a_2'' + \frac{15}{32}\Lambda_0'' a_2' + \frac{1}{32}\Lambda_0''' a_2\right.\\ &\quad \left. + \frac{9}{2}\Lambda_0 a_4' + \frac{15}{2}\Lambda_0' a_4\right)|z-z'|(z-z') + \ldots .\end{aligned}$$

The coefficient of the δ function on the right-hand side of (B3) is one, therefore $\Lambda_0 a_2 = 1$. Now all primes denote derivatives with respect to $\bar{z}$, so we can split off the vanishing derivatives of $\Lambda_0 a_2$ and have

$$\begin{aligned}&\delta(z-z') + \left(\frac{3}{8}\Lambda_0' a_2' + 6\Lambda_0 a_4\right)|z-z'|\\ &\quad + \left(\frac{3}{8}\Lambda_0' a_2'' + \frac{3}{4}\Lambda_0'' a_2' + \frac{9}{2}\Lambda_0 a_4' + \frac{15}{2}\Lambda_0' a_4\right)\\ &\quad \times |z-z'|(z-z') + \ldots .\end{aligned} \tag{B8}$$

Inserting now $a_2 = \Lambda_0^{-1}$ and setting the coefficient of $|z - z'|$ equal to zero, we find

$$a_4 = \frac{1}{16}\frac{(\Lambda_0')^2}{\Lambda_0^3} \tag{B9}$$

or, in terms of L,

$$a_2 = \frac{1}{L^2(\bar{z})}, \qquad a_4 = \frac{1}{4}\frac{(L')^2(\bar{z})}{L^4(\bar{z})}. \tag{B10}$$

The coefficient of $|z - z'|(z - z')$ in the last term of (B8) cancels, so our ansatz (B5) leads to an antisymmetric approximation of the inversion of $\{D(z), D(z')\}_{\mathrm{D2}}$ in some neighborhood of $z = z'$,

$$\Delta(z, z') \approx \frac{|z-z'|(z-z')}{4L^2(\bar{z})}\left[1 + \frac{1}{4}\frac{(L')^2(\bar{z})}{L^2(\bar{z})}(z-z')^2\right]. \tag{B11}$$

This is an approximation to the differential part of the integro-differential Eq. (B3); iterating the integral part would not give contributions of this order, but only higher antiderivatives of δ.

On the other hand, had we begun with the integral part, we would have obtained δ' as first approximation, because the integral contains the first antiderivative. Beginning with this, we would get higher and higher derivatives of the δ function in the sequel. This does not show up in our Δ and so the ensuing expression for the Dirac brackets reveals only part, although an important part, of the consequences of imposing D strongly.

[1] T. Jacobson, S. Liberati, and D. Mattingly, arXiv:hep-ph/0110094; Phys. Rev. D **66**, 081302 (2002).
[2] T. Konopka and S. Major, New J. Phys. **4**, 57 (2002).
[3] S. Liberati and L. Maccione, Annu. Rev. Nucl. Part. Sci. **59**, 245 (2009).
[4] T. Jacobson, S. Liberati, and D. Mattingly, Ann. Phys. (N.Y.) **321**, 150 (2006).

[5] G. M. Hossain, V. Husain, and S. S. Seahra, Phys. Rev. D **80**, 044018 (2009).
[6] V. Husain and A. Kreienbuehl, Phys. Rev. D **81**, 084043 (2010).
[7] R. Gambini and J. Pullin, Phys. Rev. D **59**, 124021 (1999).
[8] J. Alfaro, H. A. Morales-Técotl, and L. F. Urrutia, Phys. Rev. Lett. **84**, 2318 (2000).
[9] J. Alfaro, H. A. Morales-Técotl, and L. F. Urrutia, Phys. Rev. D **65**, 103509 (2002).
[10] U. Yurtsever, Phys. Rev. D **37**, 2803 (1988).
[11] C. Misner, K. Thorne, and J. Wheeler, *Gravitation* (Freeman, New York, 1973), Sec. 35.9.
[12] J. Ehlers and W. Kundt, in *Gravitation: An Introduction to Current Research*, edited by L. Witten (Wiley, New York, 1962).
[13] J. B. Griffiths, *Colliding Waves in General Relativity*, (Oxford University, New York, 1991).
[14] J. B. Griffiths and J. Podolský, *Exact Space-Times in Einstein's General Relativity*, Monographs on Mathematical Physics (Cambridge University Press, Cambridge, England, 2009).
[15] H. Bondi, Nature (London) **179**, 1072 (1957).
[16] K. Banerjee and G. Date, Classical Quantum Gravity **25**, 105014 (2008).
[17] A. Ashtekar and J. Lewandowski, Classical Quantum Gravity **21**, R53 (2004).
[18] K. Banerjee and G. Date, Classical Quantum Gravity **25**, 145004 (2008).
[19] M. Bojowald, Living Rev. Relativity **8**, 11 (2005), http://www.livingreviews.org/lrr-2002-1.
[20] M. Bojowald, Classical Quantum Gravity **21**, 3733 (2004).
[21] The choice $N = N(t)$ is also essential for calculations of the algebra of constraints including Eq. (34) and $\{U_\rho, H[N]\} \approx 0$. The constraints U_ρ are defined in Eq. (39).
[22] There is precedent in the literature for this. Scalar fields in $1 + 1$ dimensions with the same property are called self-dual fields and are quantized, for example, by Floreanini and Jackiw [23]. The restriction of gravitational field variables to either left- or right-moving waves will lead to bracket relations analogous to those for the self-dual fields with corrections in the part of space-time where the wave factor β is nonconstant.
[23] R. Floreanini and R. Jackiw, Phys. Rev. Lett. **59**, 1873 (1987).
[24] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems* (Princeton University, Princeton, NJ, 1992).
[25] Up to boundary terms that do not apply when z is in the interval (z_1, z_2).
[26] S. Hossenfelder, Classical Quantum Gravity **23**, 1815 (2006); Phys. Rev. D **73**, 105013 (2006).
[27] F. Hinterleitner and S. Major (work in progress).
[28] P. C. Aichelburg and H. Balasin, Gen. Relativ. Gravit. **39**, 1075 (2007).