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Gravity and BF theory defined in bounded regions

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Abstract

We study Einstein gravity in a finite spatial region. By requiring a well-defined variational principle, we identify all local boundary conditions, derive surface observables, and compute their algebra. The observables arise as induced surface terms, which contribute to a non-vanishing Hamiltonian. Unlike the asymptotically flat case, we find that there are an infinite number of surface observables. We give a similar analysis for $SU(2)$ BF theory. © 1997 Elsevier Science B.V.

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1. Introduction

Gravity has been studied mainly in the context of either closed or asymptotically flat space-times. The former applies to cosmology, whereas the latter applies to situations where the gravitating system is viewed from a flat environment at infinity. The asymptotically flat setting allows the identification of properties of the system, such as its energy and angular momentum [1–3]. In the case of asymptotically flat space-times, these conserved quantities, being integrals over a two-sphere, may be called "surface observables".

There exist, however, interesting solutions of the Einstein equations which do not fall into the closed or asymptotically flat category. These require a study of more general boundary conditions. Recently such boundary conditions have been studied. Brown and York [4] study the covariant gravity action for a spatially bounded region, and derive

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surface observables on a finite boundary (which are referred to as quasi-local quantities). Balachandran, Chandar and Momen [5] perform a similar analysis in the context of an inner boundary and an asymptotically flat outer boundary. Hawking and Horowitz [6] provide an analysis for asymptotic conditions other than the asymptotically flat one. The main approach underlying all these works is functional differentiability of the gravity action in the presence of boundaries, as in the initial work of Regge and Teitelboim [1].

The study of more general boundary conditions falls into two main categories: A gravitating system may be viewed as being enclosed in a finite spatial region, or, as in the asymptotically flat situation, in a two-sphere at infinity. In this paper we study the former case with an emphasis on presenting *all* possible boundary conditions, and finding observables of the theory.

Observables for gravity, or any other generally covariant field theory, may be defined as phase space functionals that commute weakly with the first class constraints of the theory. For a four-dimensional theory in a finite spatial region, observables may be classified into "bulk" and "surface" observables. The former are integrals over the spatial region, while the latter are integrals over the surface bounding the spatial region.

There are (at least) two reasons why it may be useful to find surface observables and their algebra for a gravitating system in a bounded spatial region. The first reason has to do with black hole entropy. Specifically the questions are: What are the microscopic degrees of freedom of a black hole? Where do these degrees of freedom reside?

Recently there has been a proposal, originating in string theory, for the statistical mechanical interpretation of black hole entropy [7]. In the weak coupling limit of string theory, there are bound states of D-branes labeled by charges *which are the same* as the charges on the extremal black holes. The degeneracy of these bound states is taken to represent the microscopic degrees of freedom of the black holes - which arise only in the strong coupling limit. It is remarkable that this degeneracy leads to the correct entropy formula for black holes. However, essential to the identification of these states as black hole microstates is the extrapolation of the degeneracy calculation from weak to strong coupling (known as the "non-renormalization theorem"). This extrapolation obscures the space-time origin of the microscopic degrees of freedom in the strong coupling limit (where there is a black hole), as well as the location of the degrees of freedom. Furthermore, as this idea applies only to extremal and near extremal black holes, it does not work for the Schwarzschild black hole. Therefore this string theory approach so far provides only an indirect answer to the two questions.

Another conjectured solution, investigated in detail by Carlip [8] for a black hole in $(2+1)$ dimensions [9], provides the following answer to these questions: The microscopic degrees of freedom of a black hole are those of a theory induced on the horizon. This horizon forms the (null) boundary of the system. "Surface observables" for the whole system are observables of the induced boundary theory. The answer arises by first noticing that $(2+1)$-gravity with a cosmological constant may be expressed as a Chern-Simons theory [10,11]. This theory, on a manifold with boundary, induces the two-dimensional WZNW theory on the boundary. Since $(2+1)$-gravity has a finite number of degrees of freedom, and the WZNW theory has an infinite number, this effect of inducing the WZNW theory on the boundary is referred to as the "bulk gauge degrees of freedom becoming dynamical on the boundary." The conserved currents of this theory form a Kac–Moody algebra, as do the surface observables. Quantization of the surface observable algebra gives a Hilbert space of states associated with the boundary, from which the entropy is determined. It is not clear whether this approach will work for $(3+1)$-gravity.

The second reason for a full investigation of surface observables is the "holographic hypothesis" [12,13]. This hypothesis rests on the assumption that the maximum allowed entropy in a region bounded by a spherical surface of area A is $A/4$, corresponding to a black hole that just fits in the surface. This finite entropy implies a phase space of finite volume, and hence a finite-dimensional Hilbert space for the system. 't Hooft [12] further argued that this leads to the striking conclusion that physical degrees of freedom must be associated with the boundary of the region: If the entropy of a bounded system not containing a black hole were proportional to its volume, then one could add matter until the system becomes a black hole and the entropy becomes proportional to the area. The entropy would decrease in such a process, and lead to an apparent violation of the second law of thermodynamics. One solution to this conundrum is to hypothesize that the entropy of a bounded system must always be proportional to the boundary area. This follows if the degrees of freedom are associated only with the boundary.

It may be possible to verify this hypothesis using the present work, if one can quantize the algebra of boundary observables such that the resulting representation space has finite dimension. If the observable algebra is infinite-dimensional this may not be possible unless only a finite, and somehow "representative" subset is quantized. In the context of canonical gravity and specific boundary conditions, a quantization of a set of surface observables, including an area observable, has been studied recently [14].

To investigate these issues, we provide an analysis for boundary conditions at a finite spatial boundary for Einstein gravity, and also for the topological BF theory in four dimensions. In two specific cases, we exhibit the surface observables of the system, and compute their algebra. There is earlier work on the finite boundary case for gravity in Refs. [4,15], and for Abelian BF theory in Ref. [16].

We begin in the next subsection by reviewing the derivation of the ADM surface observables for asymptotically flat general relativity. Following this is a brief discussion of the spatially closed case in the Ashtekar Hamiltonian formulation. This sets the stage for our discussion of Einstein gravity in a bounded spatial region. In Section 2 we give a general procedure for constructing surface observables, followed by a discussion of the possible local boundary conditions and corresponding surface observables for general relativity in a finite spatial region. Section 3 contains a similar analysis for the topological $SU(2)$ BF theory. The generalization of the results to gauge groups other than $SU(2)$ is immediate. The final section presents our conclusions and contains a comparison with other studies of gravity in a finite spatial region.

1.1. The asymptotically flat case: A brief review

The fundamental difference between the variational principles for the Einstein equations for spatially open versus closed space-times is that the former requires proper treatment of asymptotic boundary conditions; surface terms need to be added to the action [3] to make the variational principle well defined. One crafts these surface terms so that their variation cancels the surface terms arising from the variation of the "bulk" part of the action.

The standard asymptotically flat space-time is defined so that for large proper radial coordinate r (spatial infinity), the space-time metric behaves like the Schwarzschild metric

$$ds^2|_{r\to\infty} = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(\delta_{ab} + M\frac{x^a x^b}{r^3}\right) dx^a dx^b, \tag{1}$$

where δ_{ab} is the Euclidean three-metric and x^a are the asymptotic Cartesian coordinates. The definition of the phase space for asymptotically flat space-times includes specific fall-off conditions for the spatial metric $q_{ab}(x,t)$, its conjugate momentum $\pi^{ab}(x,t)$, and for the lapse and shift functions $N(x,t)$ and $N^a(x,t)$. These conditions completely determine the allowed gauge symmetries. In the asymptotically flat case, space-time diffeomorphisms are restricted to Poincaré transformations in the asymptotic (Minkowski) region. The lapse and shift functions have the asymptotic form

$$\begin{aligned} &N \to \alpha + \delta_{ab}\beta^b x^a + O(1/r), \\ &N^a \to \alpha^a + \epsilon^a{}_{bc}\phi^b x^c + O(1/r), \end{aligned} \tag{2}$$

where α and α^a are time and space translations, and β^a and ϕ^a are boost and spatial rotation parameters. The fall-off conditions on the metric and its conjugate momenta, guided by the Schwarzschild metric, are

$$\begin{aligned} &q_{ab} \to \delta_{ab} + \frac{f_{ab}(\theta,\phi)}{r} + O(1/r^2), \\ &\pi^{ab} \to \frac{p^{ab}(\theta,\phi)}{r^2} + O(1/r^3). \end{aligned} \tag{3}$$

The fall-off conditions, by themselves, are not sufficient to make the Hamilton variational principle well defined; one must add boundary terms to the action [1]. With these choices, together with parity conditions on the angle dependent tensors f_{ab} and p^{ab}, the surface terms that need to be added to the (3+1) action for the compact case are precisely the ADM four-momentum, angular momentum and boost charge [1].

Functional differentiability of the (3+1) action, or equivalently, the constraints, requires surface terms to be added to the action. The full Hamiltonian becomes a linear combination of constraints *plus* surface terms. Evaluated on a classical solution, the initial value constraints vanish leaving a non-vanishing "surface Hamiltonian". This

[3] Here, we take the view that the action *is* the action for the compact case.

Hamiltonian is the sum of conserved charges corresponding to the lapse and shift functions in the asymptotic region.

While the full Hamiltonian is functionally differentiable one still has to check that its Poisson algebra closes. The algebra does close and reduces to the Poincaré algebra when evaluated on a solution. Thus, the algebra of the full Hamiltonian with itself *necessarily* gives the surface observable algebra.

We can then ask if there are surface observables *other* than those which are already included in the full Hamiltonian. Are there other phase space functionals associated to the boundary which commute with the full Hamiltonian? An immediate attempt might be to see if more freedom can be introduced into the lapse and shift functions which parameterize the ADM surface observables. As an example consider the candidate observable defined using the diffeomorphism generator

$$\int_{\Sigma} d^3x\, \pi^{ab}\mathcal{L}_M g_{ab} \approx -2\int_{\partial\Sigma} d^2S_a\, M_b \pi^{ab},$$

where the vector field M^a is now arbitrary. The Poisson bracket of this functional with the full Hamiltonian gives a non-vanishing surface term *unless* $M^a \to \epsilon^a{}_{bc}\phi^b x^c$; that is, it is an observable only if it reduces to the familiar ADM angular momentum. One can check similarly that no new surface observables arise using the Hamiltonian constraint.[4]

Intuitively, one expects a connection between the freedom in the lapse and shift at the boundary, and the number of surface observables: A reduction in the number of gauge transformations at a boundary should correspond to an increase in the number of surface observables. As we will see, this expectation only partially true.

For gravity in a bounded region, to be discussed in Section 2, we follow a method similar to the one used above. While we do not work with an asymptotic region, with its corresponding forms for the lapse, shift and phase space variables, there are nonetheless significant restrictions on the boundary variables. These form the possible boundary conditions for gravity. Before proceeding to this, we first review the canonical theory on a compact manifold in the Ashtekar variables.

1.2. The compact case: Constraints and algebra

For space-time manifolds $\mathcal{M} = \Sigma \times R$, where Σ is closed, the (3+1) action for vacuum, Riemannian general relativity in the Ashtekar variables is

$$S[E^{ai}, A^i_a; \Lambda^i, N^a, N] = \frac{1}{\kappa}\int_{t_1}^{t_2} dt \int_{\Sigma} d^3x\, [E^{ai}\dot{A}^i_a - N\mathcal{H} - N^a\mathcal{D}_a - \Lambda^i G^i]. \tag{4}$$

[4] There is more freedom in the boundary observables than has been manifested so far [3]. The lapse and shift functions of Eq. (2) can have additional angle dependent functions. These are the so-called "super translations," which are transformations on the two-sphere at infinity; these are in addition to the translations, rotations and boosts already present in Eq. (2).

The conjugate phase space variables are an $SU(2)$ connection A_a^i and a densitized, inverse triad E^{ai}, which satisfy

$$\{A_a^i(x), E_j^b(y)\} = \kappa\,\delta_a^b \delta_j^i \delta(x,y), \tag{5}$$

with $\kappa = 16\pi G$. (We set $\kappa = 1$ in the following.) The Lagrange multipliers N, N^a, and Λ^i are, respectively, the lapse, shift, and $SU(2)$ gauge rotation parameters. Varying the action with respect to these functions gives the first class constraints

$$\mathcal{G}^i \equiv -D_a E^{ai} \approx 0; \tag{6}$$

$$\mathcal{D}_a \equiv -E^{bi}\partial_a A_b^i + \partial_b(E^{bi}A_a^i) \approx 0; \tag{7}$$

$$\mathcal{H} \equiv -\epsilon^{ijk} F_{ab}^k E^{ai} E^{bj} \approx 0, \tag{8}$$

where $D_a\lambda^i = \partial_a \lambda^i + \epsilon^{ijk} A_a^j \lambda^k$ and $F_{ab}^i = \partial_a A_b^i - \partial_b A_a^i + \epsilon^{ijk} A_a^j A_b^k$. These constraints generate gauge transformations via the Poisson bracket. The smeared diffeomorphism constraint[5] $D(N) = -\int d^3x\, N^a \mathcal{D}_a$ satisfies the expected relations

$$\{A_a^i, D(N)\} = \mathcal{L}_N A_a^i, \qquad \{E^{ai}, D(N)\} = \mathcal{L}_N E^{ai}. \tag{9}$$

As the manifold has no boundary, there is some freedom in writing the constraints. For instance, integrating the second term of the diffeomorphism constraint (Eq. (7)) by parts one finds

$$D(N) = \int_\Sigma d^3x\, E^{ai} \mathcal{L}_N A_a^i. \tag{10}$$

The Hamiltonian constraint has density weight +2 so the lapse function has density weight −1. The resulting constraint

$$H(N) = \int_\Sigma d^3x\, N \epsilon^{ijk} E^{ai} E^{bj} F_{ab}^k \tag{11}$$

generates time evolution via the Poisson bracket.

Classically, the constraints satisfy the following algebra [17]:

$$\{G(N), G(M)\} = -G([N,M]), \tag{12}$$

$$\{D(N), G(M)\} = -G(\mathcal{L}_N M), \tag{13}$$

$$\{D(N), D(M)\} = -D([N,M]), \tag{14}$$

$$\{G(N), H(M)\} = 0, \tag{15}$$

$$\{D(N), H(M)\} = -H(\mathcal{L}_N M), \tag{16}$$

$$\{H(N), H(M)\} = D(K) + G(A_a K^a), \tag{17}$$

[5] The theory originally found by a Legendre transform [17] has the vector constraint $V(N) = \int_\Sigma d^3x\, N^a E^{bi} F_{ab}^i$. However, the Gauss constraint may be combined with this to give the diffeomorphism constraint $D(M) = V(M) - G(A_a^i M^a) = \int d^3x\, M^a [E^{bi}\partial_a A_b^i - \partial_b(E^{bi}A_a^i)]$ used above.

where $K^a = E^{ai}E^{bi}\,(N\partial_b M - M\partial_b N)$.

A more complete discussion of the spatially closed case can be found in Ref. [17]. The asymptotically flat case is presented in Refs. [17,18]. In the next section we consider the possible boundary conditions for gravity in a finite spatial region.

2. Gravity in a bounded spatial region

We consider spatial slices Σ with boundary $\partial\Sigma$. The boundaries are taken to be "orthogonal" in the sense that the normal n_a to the spatial boundary is orthogonal to the timelike direction of the foliation. (This condition does not rule out asymptotic boundaries or bifurcate horizons.) Though our discussion focuses on a single boundary, the analysis can be extended easily to a boundary with disjoint regions. In this case one can choose separate boundary conditions and surface terms for each disjoint region of the boundary. Our analysis proceeds in the following steps:

(1) When a boundary is present the variations of the (3+1) action (4) with respect to the phase space variables E^{ai} and A_a^i are not defined. To define the theory, one must add appropriate surface terms to the action and impose boundary conditions. There are a number of ways to do this, and we list all the possible choices.

(2) We find the full Hamiltonian H_F (constraints plus surface terms), which is a function of all gauge parameters, and compute the algebra of H_F with itself. This Poisson bracket should close in the same way that the constraint algebra closes; if necessary, we impose additional boundary conditions to ensure that it does. This completes the definition of the theory, and also identifies the surface terms in H_F as (at least some) of the surface observables.

(3) Finally we ask if there are any other surface observables that commute with the full Hamiltonian H_F. Since the boundary conditions on the lapse and shift functions are not as stringent as in the asymptotically flat case, we check to see if additional surface observables may be found by introducing more freedom into the surface parts of H_F by replacing the lapse and shift by more general functions. Once all the surface observables have been determined in this way, we compute their algebra. This sets the stage for quantization.

At the end of this procedure we have a well-defined theory, its surface observables, and their algebra. The full Hamiltonian of the theory is functionally differentiable and satisfies a consistent algebra. On a solution, the full Hamiltonian may have non-vanishing terms which are integrals on the boundary $\partial\Sigma$. These are the surface observables.

Below we list the possible boundary conditions, and then follow the rest of the procedure for two choices of spatial boundary conditions. In the first case, all gauge parameters are set to zero on the spatial boundary, while in the second the triad (and therefore the metric) is fixed on the spatial boundary.

2.1. Boundary conditions

We consider only local boundary conditions. Instead of requiring that integrals on the boundary vanish, we list the stronger conditions that the integrand vanishes. In this sense, our list is only a complete list of *local* boundary conditions for gravity. For certain cases, such as where the boundary is a sphere, it is possible to introduce global boundary conditions. Then, as in the asymptotically flat case, one can impose parity conditions on the fields at the boundary to make undesirable surface integrals vanish.

For completeness we first mention the conditions on the timelike three-boundary $\partial\mathcal{M}$, before listing the conditions on the spatial two-boundary $\partial\Sigma$. The variation with respect to A_a^i of the first term in the $(3+1)$ action (4) gives the surface term

$$\int_\Sigma d^3x\, E^{ai}\delta A_a^i|_{t_1}^{t_2}.$$

This can be made to vanish by requiring A_a^i to be fixed on $\partial\mathcal{M}$, or by subtracting

$$\int_{t_1}^{t_2} dt \int_\Sigma d^3x \frac{d}{dt}(E^{ai}A_a^i)$$

from the action and requiring E^{ai} to be fixed on $\partial\mathcal{M}$.

For the remainder of the paper we focus on the spatial two-boundary $\partial\Sigma$. The variation of the action (4) contains the variations of each constraint with respect to the phase space variables. These can contribute a surface term to the full, finite boundary Hamiltonian. Of course, in order to obtain the correct initial value constraints for vacuum gravity, or BF theory, the gauge parameters (Lagrange multipliers) have to be fixed on $\partial\Sigma$. Precisely how these parameters are fixed may depend, as we discuss below, on what choices are made for the phase space variables on $\partial\Sigma$, and/or what gauge invariances on the boundary one would like.

The following is the list of possibilities that gives functional differentiability of the initial value constraints. Every mutually consistent choice from this list defines a possible finite boundary theory. We can of course change our starting point, and begin with a $(3+1)$ action that already has an arbitrary surface term, rather than the action (4). This obviously increases arbitrarily the possibilities for defining theories in finite spatial regions. For example, one could add a Chern–Simons term for the timelike three-boundary $R \times \partial\Sigma$ as has been done by Smolin [14]. This leads to source terms for the Gauss law constraint, and new possibilities for boundary conditions.

Gauss constraint

The variation of the Gauss constraint is [6]

[6] We have overall minus signs in all constraints because of the signs in the $(3+1)$ action (4).

$$\delta G(\Lambda) = -\int_{\Sigma} d^3x \left[\Lambda^i \epsilon^{ijk}(\delta A^j_a E^{ak} + A^j_a \delta E^{ak}) - (\partial_a \Lambda^i)\delta E^{ai} + \delta\Lambda^i D_a E^{ai}\right]$$
$$- \int_{\partial\Sigma} d^2x\, n_a \Lambda^i \delta E^{ai}, \tag{18}$$

where n_a is the normal to the boundary two-surface $\partial\Sigma$. The variation of the gauge parameter simply yields the constraint. Functional differentiability with respect to the phase space variables requires vanishing integrand in the surface term, which leads to at least one of the following conditions:

(i) Vanishing gauge transformations on the boundary

$$\Lambda^i|_{\partial\Sigma} = 0.$$

(ii) Boundary conditions involving the triad:

(a) Fixed boundary "area density"

$$n_a\, \delta E^{ai}|_{\partial\Sigma} = 0.$$

That this condition fixes the area density may be seen as follows. Let $a^i = n_a E^{ai}$. The surface area of the boundary is $\int_{\partial\Sigma} d^2x \sqrt{q} \equiv \int_{\partial\Sigma} d^2x \sqrt{a^i a_i}$. Fixing the area density means that $a^i \delta a_i|_{\partial\Sigma} = 0$ which is implied by the above condition.

(b) Fixed boundary triad $\delta E^{ai}|_{\partial\Sigma} = 0$, or

(c) $\Lambda^i \delta E^{ai}|_{\partial\Sigma} = 0$.

(iii) Addition of the surface term [7]

$$+\int_{\partial\Sigma} d^2x\, n_a \left(\Lambda^i E^{ai}\right).$$

These boundary conditions may be placed independently on different disjoint parts of the boundary. One could also take a combination of cases, such as (i) and (ii).

Diffeomorphism constraint

The variation of the diffeomorphism constraint gives

$$\delta D(N^a) = -\int_{\Sigma} d^3x \left[\delta E^{ai} \mathcal{L}_N A^i_a - \delta A^i_a \mathcal{L}_N E^{ai} + \delta N^a \mathcal{D}_a\right]$$
$$+ \int_{\partial\Sigma} d^2x\, n_b \left[N^a \delta(A^i_a E^{bi}) - N^b E^{ai} \delta A^i_a\right]. \tag{19}$$

[7] One is in fact free to add any multiple of this surface term. Functional differentiability then induces further conditions on fields and gauge parameters on the boundary. Work is in progress on such special cases.

There are five choices which guarantee functional differentiability:

(i) Vanishing diffeomorphisms on the boundary

$$N^a|_{\partial\Sigma} = 0. \tag{20}$$

This case is effectively the same as for manifolds without boundary. The spatial diffeomorphism constraint in this case may be rewritten as

$$D(N) = -\int_\Sigma d^3x\, A_a^i \mathcal{L}_N E^{ai}. \tag{21}$$

(ii) Addition of the boundary term

$$-\int_{\partial\Sigma} d^2x\, n_b (N^a A_a^i E^{bi})$$

to $D(N)$, *and* restriction of the normal component of the shift on the boundary

$$n_a N^a|_{\partial\Sigma} = 0.$$

That is, the shift function on the boundary two-surface $\partial\Sigma$ must be tangential to the boundary. For this choice, imposing functional differentiability on $D(N)$ does not require that diffeomorphisms vanish on the boundary.

(iii) Addition of the same boundary term as in (ii) *and* fixed connection on the boundary

$$\delta A_a^i|_{\partial\Sigma} = 0.$$

(iv) Fixed triad (and hence metric) on the boundary,

$$\delta E^{ai}|_{\partial\Sigma} = 0$$

and addition of the boundary term

$$-\int_{\partial\Sigma} d^2x\, n_b \left[N^a A_a^i E^{bi} - N^b E^{ai} A_a^i\right],$$

with the shift function free on the boundary.

(v) Fixed fields on the boundary

$$\delta E^{ai}|_{\partial\Sigma} = \delta A_a^i|_{\partial\Sigma} = 0.$$

Hamiltonian constraint

The variation of the Hamiltonian constraint (11) is

$$\delta H(N) = -\int_{\Sigma} d^3x\, 2\epsilon^{ijk}\Big[(NE^{bj}F^k_{ab})\delta E^{ai} + (\epsilon^{klm}NE^{ai}E^{bj}A^m_b)\delta A^l_a$$
$$-\partial_a(NE^{ai}E^{bj})\delta A^k_b + \frac{1}{2}E^{ai}E^{bj}F^k_{ab}\delta N\Big] - \int_{\partial\Sigma} d^2x\, n_a\, 2\epsilon^{ijk}NE^{ai}E^{bj}\delta A^k_b. \tag{22}$$

Functional differentiability requires at least one of the following:

(i) Vanishing lapse on the boundary

$$N|_{\partial\Sigma} = 0.$$

This eliminates the possibility of having a boundary Hamiltonian, and hence dynamics and quasi-local energy. However, it may be appropriate for space-times containing a bifurcate Killing horizon.

(ii) The triad satisfies

$$n_a E^{ai}|_{\partial\Sigma} = 0$$

which restricts the metric on the boundary to be tangential. This requires the spatial three-metric to be degenerate on the boundary.

(iii) Boundary conditions involving the connection:

(a) The variation of the tangential part of the connection vanishes on the boundary

$$n_{[a}\delta A^i_{b]}|_{\partial\Sigma} = 0,$$

(b) or, the connection's variation vanishes $\delta A^i_a|_{\partial\Sigma} = 0$.

(iv) Addition of the surface term

$$+\int_{\partial\Sigma} d^2x\, 2N\epsilon^{ijk}A^i_a E^{aj}\, n_b\, E^{bk}$$

and

$$\delta E^{ai}|_{\partial\Sigma} = 0.$$

This fixes the boundary two-metric. The surface term leads to the quasi-local energy [4] and becomes the usual ADM surface energy in the asymptotically flat case.

Adding a cosmological constant term to the Hamiltonian constraint does not contribute any new surface terms to the variation because it does not contain any derivatives, therefore the above choices of boundary conditions remain the same.

The asymptotically flat case in the Ashtekar variables has been worked out [18]. The fall-off conditions on the lapse and shift are the same as for the ADM variables (Section 1.1) while the fall-off conditions on the phase space variables are

$$A^i_a = \frac{a^i_a(\theta,\phi)}{r^2} + O(1/r^3), \qquad E^{ai} = e^{ai} + \frac{f^{ai}(\theta,\phi)}{r} + O(1/r^2), \tag{23}$$

where a_a^i and f^{ai} are functions on the sphere at infinity, and e^{ai} is a dreibein such that $e^{ai}e^{bi} = \delta^{ab}$.

We now consider two specific cases of functionally differentiable actions from the above list, and continue with steps (2) and (3) for each case. Any other case may be similarly treated.

2.2. The case $\Lambda^i|_{\partial\Sigma} = N^a|_{\partial\Sigma} = N|_{\partial\Sigma} = 0$

Perhaps the simplest choice of boundary conditions is the case for which all gauge parameters vanish on the boundary. This corresponds to case (i) for each of the constraints. The action is exactly the same as for the closed case (4), and therefore the constraint algebra is just as in Eqs. (12)-(17). The Hamiltonian remains a linear combination of constraints; all the surface integrals vanish identically. There are no surface observables which arise as surface terms in the action.

Turning to step (3) above, we ask if there are any surface observables. One might expect that the reduction in gauge freedom should give many surface observables: As the phase space variables on the boundary are completely unconstrained, all the gauge degrees of freedom in the interior become true degrees of freedom on the boundary. This does indeed occur, but the reduction of the gauge freedom does not correspond directly to new observables in each case. Rather, as we now see, there is an infinite number of observables, but not an infinite number for each gauge parameter.

To find the explicit form of the observables, consider the functionals

$$\mathcal{O}_G(\lambda) = \int_\Sigma d^3x\, E^{ai} D_a \lambda^i; \tag{24}$$

$$\mathcal{O}_D(M) = \int_\Sigma d^3x\, A_a^i \mathcal{L}_M E^{ai}; \tag{25}$$

$$\mathcal{O}_H(L) = \int_\Sigma d^3x\, \epsilon^{ijk} \left[-2A_b^k \partial_a \left(L E^{ai} E^{bj} \right) + L E^{ai} E^{bj} \epsilon^{klm} A_a^l A_b^m \right], \tag{26}$$

where λ^i, M^a, and L are (at this stage) arbitrary, and *unconnected* with the gauge parameters Λ^i, N^a, and N. These functionals are obtained by integrating the constraints by parts, discarding the surface terms, and replacing the gauge parameters with the functions λ M^a and L. This approach was followed by Balachandran, Chandar, and Momen in Ref. [5]. Since E^{ai} and A_a^i are free on the boundary, functional differentiability is guaranteed if we require $L|_{\partial\Sigma} = 0$ and $n_a M^a|_{\partial\Sigma} = 0$, leaving λ^i arbitrary. It is important to note that functional differentiability eliminates $\mathcal{O}_H$ as an observable. The remaining functionals are surface observables in that they are weakly equal to surface integrals

$$\mathcal{O}_G(\lambda) \approx -\int_{\partial\Sigma} d^2x\, n_a \lambda^i E^{ai}, \tag{27}$$

$$\mathcal{O}_D(M) \approx \int_{\partial\Sigma} d^2x\, n_b E^{bi} M^a A^i_a. \tag{28}$$

It is easy to see that the non-zero $\mathcal{O}_G$ and $\mathcal{O}_D$ have weakly vanishing Poisson brackets with the constraints; any possible surface terms in their Poisson brackets with the constraints vanish because the gauge parameters Λ^i, N^a, and N vanish on the boundary.

Given the definitions of the observables, the algebra [8] is the expected one

$$\{\mathcal{O}_G(\lambda), \mathcal{O}_G(\mu)\} = \mathcal{O}_G(\lambda \times \mu), \tag{29}$$

$$\{\mathcal{O}_D(M), \mathcal{O}_D(P)\} = \mathcal{O}_D([M,P]), \tag{30}$$

$$\{\mathcal{O}_G(\lambda), \mathcal{O}_D(M)\} = -\mathcal{O}_G(\mathcal{L}_M \lambda). \tag{31}$$

Thus, we see that restricting the gauge freedom on the boundary generates surface observables. However, as the case of $\mathcal{O}_H$ above shows, there need not be any direct correspondence between reducing gauge degrees of freedom on the boundary and increasing the number of boundary observables. The connection is more subtle; the new degrees of freedom give more observables for the kinematic constraints, but not for the Hamiltonian constraint.

2.3. *Fixed boundary metric*

The case of fixed triad on the spatial boundary $\delta E^{ai}|_{\partial\Sigma} = 0$, and hence fixed boundary metric, is a more interesting case. It has been studied before, although not entirely along the lines we follow. Brown and York studied this case starting from the standard metric action [4], and gave definitions for quasi-local quantities associated with the finite boundary. Lau performed an analysis similar for the fixed metric case in the new variables [15]. These works begin with the covariant action rather than the (3+1) action for the spatially closed case, and do not exhibit an algebra of surface observables.

Fixed boundary metric means case (ii) for the Gauss law and case (iv) for the Hamiltonian constraint, but more than one possibility for the diffeomorphism constraint. The possible diffeomorphism cases are (i), (ii) and (iv). Among these, we consider (ii) because it gives the minimal restriction on the shift function, as well as a well-defined algebra. This gives the (3+1) action

$$\begin{aligned} S[E^{ai}, A^i_a; \Lambda^i, N^a, N] = &\int dt \int d^3x [E^{ai}\dot{A}^i_a - N\mathcal{H} - N^a\mathcal{D}_a - \Lambda^i G^i] \\ &+ \int dt \int_{\partial\Sigma} d^2x (2Nn_b \epsilon_{ijk} A^i_a E^{aj} E^{bk}) \\ &- \int dt \int_{\partial\Sigma} d^2x\, (n_b N^a A^i_a E^{bi}), \end{aligned} \tag{32}$$

[8] It is possible to show in general that the Poisson brackets of two functionals is functionally differentiable [19].

where N^a must be tangential to the boundary at the boundary. Although we started with only the condition of fixed metric on the boundary, the additional condition $n_a N^a|_{\partial\Sigma} = 0$ was induced (by choice (ii) for diffeomorphisms). In general additional conditions on the boundary may be induced by the Hamiltonian algebra and by requiring the boundary conditions to be preserved in time.

Although this action represents a well-defined variational principle, we are still free to add to it a surface term which is a function of the fixed boundary data. This is an ambiguity in any variational principle. For gravity, this freedom has been utilized [4,6] to normalize the values of the various surface observables relative to a reference solution. This is done by subtracting the action of the reference solution from the action of the solution of interest. Such normalizations may be necessary in order to avoid divergences of the action, as in the asymptotically flat case, where integrations are over all space. Here we consider finite spatial regions so the action (32) is well defined and divergence-free as it stands.

The full Hamiltonian H_F is a linear combination of constraints plus surface terms, and is identified from Eq. (32),

$$H_F[E^{ai}, A_a^i; \Lambda^i, N^a, N] = \int_\Sigma d^3x \, \left[N\mathcal{H} + N^a \mathcal{D}_a + \Lambda^i G^i \right] + \int_{\partial\Sigma} d^2x \, n_b \left[2N\epsilon_{ijk} A_a^i E^{aj} E^{bk} - N^a A_a^i E^{bi} \right]. \tag{33}$$

Denoting the Hamiltonian constraint plus its corresponding surface term by H', and the diffeomorphism constraint plus its surface term by C, the algebra of the full Hamiltonian contains

$$\{G(\Lambda), G(\Omega)\} = G(\Lambda \times \Omega) + \int_{\partial\Sigma} d^2x \, n_c E^{ci} (\Lambda \times \Omega)^i, \tag{34}$$

$$\{H'(M), H'(N)\} = -4C(K) + G(A_a K^a) - \int_{\partial\Sigma} d^2x (n_a E^{ai})(A_b^i K^b), \tag{35}$$

where $K^a := E^{ai} E^{bi} (M \partial_b N - N \partial_b M)$. Similar surface terms also arise in the Poisson brackets $\{G(\Lambda), H'(N)\}$ and $\{G(\Lambda), C(M)\}$. All such surface terms ought to vanish in order to have an anomaly-free algebra. This may be accomplished by requiring the lapse functions to be constant on the boundary and the Gauss parameters to vanish on the boundary. No additional constraint on the shift function is required (other than the already imposed $n_a N^a|_{\partial\Sigma} = 0$).

For consistency it is also necessary that our boundary conditions be preserved under evolution. This leads to further conditions. The piece of H_F which generates non-trivial evolution of E^{ai} is H' (the Hamiltonian constraint plus its surface term). This leads to the condition

$$\dot{E}^{ai}|_{\partial\Sigma} = 2N\epsilon^{ijk} D_b (E^{aj} E^{bk})|_{\partial\Sigma} = 0. \tag{36}$$

The simplest solution of this is to require that the lapse N vanish on the boundary. This is rather limiting, however, because it means that the quasi-local energy observable vanishes. The only other possibility is that the fixed boundary dreibein satisfy Eq. (36). We choose the latter possibility – the boundary metric is required to be static. We note that no further conditions are necessary, in particular, the connection A_a^i is free to vary on the boundary with no consequences for functional differentiability. In summary, the theory is defined with the following conditions:

$$\delta E^{ai}|_{\partial\Sigma} = 0, \qquad \dot{E}^{ai}|_{\partial\Sigma} = 0, \tag{37}$$

$$n_a N^a|_{\partial\Sigma} = 0, \qquad \Lambda|_{\partial\Sigma} = 0, \qquad \partial_a N|_{\partial\Sigma} = 0. \tag{38}$$

The surface observables are just the surface terms in the action in Eq. (32), with the lapse N fixed to be constant. We note that while there is only one quasi-local energy observable [9]

$$\begin{aligned} \mathcal{O}_H(N) &= \int_\Sigma d^3x\, \epsilon^{ijk} \left[2\partial_b \left(N E^{ai} E^{bj}\right) A_a^k - \epsilon^{klm} E^{ai} E^{bj} A_a^l A_b^m\right] \\ &\approx 2N \int_{\partial\Sigma} d^2x\, (n_b \epsilon_{ijk} A_a^i E^{aj} E^{bk}), \end{aligned} \tag{39}$$

there are an infinite number of "momentum" observables

$$\mathcal{O}_D(N^a) = \int_\Sigma d^3x\, E^{ai} \mathcal{L}_N A_a^i \approx \int_{\partial\Sigma} d^2x (n_b N^a A_a^i E^{bi}) \tag{40}$$

parameterized by vector fields N^a subject to $n_a N^a|_{\partial\Sigma} = 0$. These are the generalization of the ADM momentum and angular momentum for finite boundary.

As in the last section, we can ask if there are any other surface observables defined like those of Eq. (24)–(26). One might think that there should be an infinite number of Gauss observables as in Eq. (24) because the gauge parameters Λ^i vanish on the boundary here (just as in the last subsection). However, the algebra $\{H_F, \mathcal{O}_G\}$ contains the piece $\{C(N), \mathcal{O}_G(\lambda)\}$ which weakly equals a surface term unless $\lambda^i|_{\partial\Sigma} = 0$. Thus, there are no surface observables other than the two above.

The algebra of surface observables is *necessarily* the same as the algebra of H_F with itself. Indeed, the addition of boundary terms may be viewed as accomplishing nothing but the functional differentiability of the constraints,

$$\{\mathcal{O}_D(N), \mathcal{O}_D(M)\} = -\mathcal{O}_D\left(\mathcal{L}_N M\right),$$

$$\{\mathcal{O}_H(N), \mathcal{O}_H(M)\} = \mathcal{O}_D(K), \tag{41}$$

$$\{\mathcal{O}_H(N), \mathcal{O}_D(M)\} = \mathcal{O}_H\left(\mathcal{L}_N M\right), \tag{42}$$

[9] While this is another derivation, the surface term (39) *is* the same as the quasi-local energy in Ref. [4], where it arises by varying the action with respect to the boundary lapse function. Setting the (constant) lapse here to one ensures that the normalizations are the same.

where K is given after Eq. (17).

A comparison of the results of this and the last subsection shows that all surface observables are contained in the full Hamiltonian, except for the case (Section 2.2)

$$\Lambda|_{\partial\Sigma} = N^a|_{\partial\Sigma} = \partial_a N|_{\partial\Sigma} = 0.$$

3. BF theory in a bounded region

We now turn to BF theory and apply the same procedure. The topological BF theory in four dimensions has action

$$S = \int_M \mathrm{Tr}\left[B \wedge F + \frac{\alpha}{2} B \wedge B\right], \tag{43}$$

where $F(A)$ is the curvature of a Yang-Mills gauge field and B is a Lie algebra valued two-form. We consider the case of gauge group SU(2), and four-manifold $M = \Sigma \times R$, in which space Σ has a boundary. The (3+1)-decomposition of this action leads to the phase space variables A_a^i, $E^{ai} = \epsilon^{abc} B_{bc}^i$, and the first class constraints

$$G^i \equiv D_a E^{ai} = 0, \tag{44}$$

$$f^{ai} \equiv \epsilon^{abc} F_{bc}^i + \alpha E^{ai} = 0. \tag{45}$$

On a three-manifold without boundary with $\alpha = 0$, the theory has two sets of Dirac observables [21]. One set depends on loops, and the other on loops and closed two-surfaces in Σ. The first is the trace of the holonomy of A_a^i based on loops γ, $T^0[A](\gamma) = \mathrm{Tr}\, U_\gamma[A]$, and the second set is

$$T^1[A, E](\gamma, S) = \int_S d^2\sigma\, n_a \mathrm{Tr}[E^a(\sigma) U_\gamma(\sigma, \sigma)], \tag{46}$$

where n_a is the unit normal to the surface S, and σ is the base point of the loops γ. These are obviously invariant under the Gauss constraint and a calculation shows that they are also invariant under the second constraint. On the constraint surface, these observables capture information about non-contractible loops and closed two-surfaces in Σ. For example, for $\Sigma = S^1 \times S^2$, there is one observable of each type on the constraint surface.

We would like to find what additional observables, other than the above bulk ones, arise when Σ has boundary. [10]

We therefore follow the procedure for gravity outlined in the previous section. Since the second term in the action does not contain spatial derivatives, the following applies to both zero and non-zero cosmological constant.

[10] Boundary observables and their quantization for Abelian BF theory have been extensively discussed in a series of papers by Balachandran et al. [16].

3.1. Boundary conditions

The functional differentiability conditions for the Gauss law are as already outlined above in Section 2.1. The constraint (45) with gauge parameter V_a^i is

$$F(V) = -\int d^3x\, V_c^i \left(\epsilon^{abc}F_{ab}^i + \alpha E^{ai}\right). \tag{47}$$

Its variation is

$$\delta F(V) = 2\int_{\partial\Sigma} d^2x\, \epsilon^{abc} n_a \delta A_b^i V_c^i - \int_{\Sigma} d^3x\, \left[2\epsilon^{abc}\left(\partial_a V_c^i \delta A_b^i - \epsilon^{ijk} V_c^i A_a^j \delta A_b^k\right) + \alpha V_a^i \delta E^{ai}\right]. \tag{48}$$

Functional differentiability leads to the following choices:

(i) $\epsilon^{abc} n_b V_c^i|_{\partial\Sigma} = 0$;

(ii) $V_a^i|_{\partial\Sigma} = 0$;

(iii) Addition of the boundary term

$$+2\int_{\partial\Sigma} d^2x\, \epsilon^{abc} n_a A_b^i V_c^i.$$

(iv) Conditions involving the connection

(a) $\delta A_a^i|_{\partial\Sigma} = 0$,

or

(b) $n_{[b}\delta A_{a]}^i|_{\partial\Sigma} = 0$,

Perhaps the most interesting case is to keep the gauge transformations unrestricted on the boundary. We therefore consider cases (iii) for both the Gauss constraint and the BF theory constraint (47). The (3+1) action for this case is

$$S[E^{ai}, A_a^i; \Lambda^i, V_a^i] = \int dt \int_{\Sigma} d^3x [E^{ai}\dot{A}_a^i - V_a^i f^{ai} - \Lambda^i G^i] + \int dt \int_{\partial\Sigma} d^2x n_a \left[2\epsilon^{abc} A_b^i V_c^i + E^{ai}\Lambda^i\right]. \tag{49}$$

From this we identify the Hamiltonian

$$\begin{aligned} H_F[E^{ai}, A_a^i; \Lambda^i, V_a^i] &= \int_{\Sigma} d^3x\, [V_a^i f^{ai} - \Lambda^i G^i] + \int_{\partial\Sigma} d^2x\, n_a \left[2\epsilon^{abc} A_b^i V_c^i + E^{ai}\Lambda^i\right] \\ &= \int_{\Sigma} d^3x\, [2\epsilon^{abc} D_b V_a^i A_c^i - \alpha V_a^i E^{ai} + E^{ai} D_a \Lambda^i]. \end{aligned} \tag{50}$$

As before, for any specific choice of boundary conditions, we must calculate the algebra of the full Hamiltonian with itself. If the algebra does not close then we need further conditions on the gauge parameters Λ^i and V_a^i. Denoting $H_F = f'(V) + G'(\Lambda)$ where f' and G' are the constraints plus their corresponding surface terms, we find that the algebra $\{H_F(V,\Lambda), H_F(W,\mu)\}$ contains

$$\{G'(\Lambda), G'(\mu)\} = G'(\Lambda \times \mu), \tag{51}$$

$$\{G'(\Lambda), f'(V)\} = 2f'(\Lambda \times V) + 2\int_{\partial\Sigma} d^2x\, \epsilon^{abc} n_a \Lambda^i \partial_b V_c^i, \tag{52}$$

$$\{f'(V), f'(W)\} = \alpha \int_{\partial\Sigma} d^2x\, n_a \epsilon^{abc} W_b^i V_c^i \tag{53}$$

where $\Lambda \times V_c = \epsilon^{ijk} \Lambda^j V_c^k$. We do not want V_a^i to vanish on the boundary because this would give a vanishing surface observable. An alternative choice is to require V_a^i to be curl-free on the boundary so that the surface term in Eq. (52) vanishes. Finally, for $\alpha \neq 0$, we require the surface term in Eq. (53) to vanish. The least restrictive way to ensure this is to require all the field V_c^i, etc., to tend to a fixed value on the boundary. This completes the list of conditions.

As for gravity, on the constraint surface, the bulk parts of the full Hamiltonian vanish leaving the surface terms as the surface observables

$$\mathcal{O}_G(\Lambda) \approx -\int_{\partial\Sigma} d^2x\, n_a \Lambda^i E^{ai}, \tag{54}$$

$$\mathcal{O}_F(V) \approx -2\int_{\partial\Sigma} d^2x\, \epsilon^{abc} n_a A_b^i V_c^i. \tag{55}$$

These observables are parameterized by Λ^i and V_a^i, and therefore are infinite in number. Continuing to step (3), we find that no more observables are obtained by generalizing the parameterizing functions λ^i and V_c^i as in Section 2.2.

The observable algebra is

$$\{\mathcal{O}_G(\Lambda), \mathcal{O}_G(\Omega)\} = \mathcal{O}_G(\Lambda \times \Omega),$$

$$\{\mathcal{O}_F(V), \mathcal{O}_F(W)\} = 0, \tag{56}$$

$$\{\mathcal{O}_F(V), \mathcal{O}_G(\lambda)\} = \mathcal{O}_F(\Lambda \times V). \tag{57}$$

Finally, since the bulk observables for vanishing cosmological constant ($\alpha = 0$) based on loops and surfaces commute with the full Hamiltonian, the bulk observables commute with the surface observables.

3.2. *Quantization*

In the connection representation, BF theory with non-zero cosmological constant, α, has a unique solution [20]. The quantum constraints are

$$D_a \frac{\delta}{\delta A^i_a} \psi[A] = 0, \tag{58}$$

$$\left(\epsilon^{abc} F^i_{bc} + \alpha \frac{\delta}{\delta A^i_a} \right) \psi[A] = 0. \tag{59}$$

A regularization is not necessary here because the equations are linear in the momenta. The unique solution, for Σ without boundary, is the Chern–Simons state

$$\psi[A] = \exp \left[-\frac{1}{\alpha} \int_\Sigma \mathrm{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \right]. \tag{60}$$

For spaces with boundary, the variation of the Chern–Simons state is well defined only if the connection is fixed on the boundary. With this additional condition, although there are an infinite number of boundary observables, the Hilbert space has only one state. Its eigenvalue depends on the surface terms in the Hamiltonian. This example shows that an infinite number of boundary observables does not preclude a finite-dimensional Hilbert space for the bounded system. In fact, if we take the case (iv) for the BF theory constraint we have only the Gauss observable and its algebra. Since this is just the angular momentum algebra, quantization can give a finite-dimensional state space. Thus this theory, though meager in content, is not inconsistent with the holographic hypothesis.

4. Discussion

Beginning with the Hamiltonian action, we studied Einstein gravity and BF theory in finite spatial regions. The action had to be augmented by surface terms in order to generate the correct equations of motion. After giving a complete list of local boundary conditions, we identified surface observables and computed their algebra. These observables naturally arose from the surface terms added to the action. We noted that additional surface observables may be generated in some cases (as in Section 2.2) by replacing the gauge parameters with more general functions.

The procedure given here is similar in spirit to that of Regge and Teitelboim [1]. Imposing functional differentiability on the (3+1)-action results, in most cases, in the addition of surface terms to the action. So when a boundary is present, there is non-vanishing full Hamiltonian which is a linear combination of constraints plus surface terms. Evaluated on a solution, this Hamiltonian gives the non-vanishing surface observables. All conditions on the functions parameterizing the observables are derived by requiring that the algebra of the full Hamiltonian remain anomaly-free. In the interior, this Poisson bracket gives the algebra of constraints. When restricted to the boundary, it gives the algebra of surface observables. Also, except for the case studied in Section 2.2, there are no surface observables other than those which already make up the full Hamiltonian – as we saw any attempt to "generalize" the parameters in the surface terms leads to undesirable surface terms in the algebra.

Although our discussion was restricted to the case of a single finite boundary, other cases are easy to incorporate. For example, for a static black hole, one would have both an inner boundary at the horizon and an asymptotic boundary. One could choose the boundary conditions given in Section 2.2 on the horizon and the standard conditions in the asymptotic region [18]. For a black hole in a bounded region, one would need to augment the procedure to include an additional finite inner boundary. On this boundary one could use the conditions given in Section 2.3.

More generally, this procedure could be used for relating an observed space-time to an observer space-time. This "relative state formalism" is an extension of the study of asymptotic space-times and closely related to methods of topological field theory [22]. The system may be expressed as a known classical solution matched with a gravitational system of interest. By cutting the (compact) space Σ in two pieces, say σ_1 and σ_2, we could express the space as

$$\Sigma = \sigma_1 \cup \sigma_2 .$$

The full Hamiltonian of the theory would split into full Hamiltonians on each subspace

$$H_\Sigma = H_{\sigma_1} + H_{\sigma_2}$$

with, typically, the same bulk pieces and surface terms of opposite sign in the individual Hamiltonians H_{σ_i}. States in one region are then expressed relative to the states in the other region. Such a formalism might provide a tractable approach to quantization.

Although our primary focus is classical, we comment briefly on quantization in the connection representation. For general relativity with a cosmological constant α, it is known that the exponential of the Chern–Simons integral (60) is a formal solution to all the quantum constraints [23]. In the fixed metric case, the action of the full Hamiltonian on this state gives a non-zero answer determined by the action of the surface terms. This state is formally an eigenstate of the full Hamiltonian with eigenvalue

$$\frac{1}{\alpha^2} \int_{\partial\Sigma} d^2x\, 2n_b \epsilon_{ijk} A_a^i B^{aj} B^{bk} - \frac{1}{\alpha} \int_{\partial\Sigma} d^2x\, n_b N^a A_a^i B^{bi}, \tag{61}$$

where $B^{ai} = \epsilon^{abc} F_{bc^i}$ is the restriction of the magnetic field to the boundary. This state is the *only* solution of the constraints of the topological BF theory. Thus this solution, though an eigenstate of the Hamiltonian, corresponds to a topological sector of general relativity.

It is interesting to see how our results can be made a part of the ongoing developments in non-perturbative quantum gravity, quite apart from extending the classical case to null inner boundaries. For example, given a system such as a black hole, one would select appropriate boundary conditions, and find the action and the surface observables. From a suitable Poisson algebra of observables and one state, the GNS construction provides a way of finding an inner product, Hilbert space, and representation of the observable algebra. A not unrelated direction is to fully explore the relative state formalism by introducing a dynamical boundary, in which the theory is essentially the bulk theory, find observables, and again use the GNS construction to built a quantum theory. In this manner, these suggestions provide a basis for exploring the role of surface theories in the context of gravitational entropy and the holographic hypothesis.

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References

[1] T. Regge and C. Teitelboim, Ann. Phys. 88 (1974) 286.
[2] R. Beig and N. Murchadha, Ann. Phys. 174 (1978) 463.
[3] A. Ashtekar and R. Hansen, J. Math. Phys. 19 (1978) 1542.
[4] J.D. Brown and J.W. York, Phys. Rev. D 47 (1993) 1407; 47 (1993) 1420.
[5] A.P. Balachandran, L. Chandar and A. Momen, Nucl. Phys. B 461 (1996) 581; Edge states in canonical gravity, gr-qc/9506006; Int. J. Mod. Phys. A 12 (1997) 625, and references therein; see also A. Momen, Edge Dynamics for BF theories and gravity, hep-th/9609226.
[6] S. Hawking and G. Horowitz, Class. Quant. Grav. 13 (1996) 1487; a generalization to non-orthogonal boundaries is presented in S. Hawking and C. Hunter, Class. Quant. Grav. 13 (1996) 2735.
[7] A. Strominger and C. Vafa, Phys. Lett. B 379 (1996) 99; see also J. Maldacena, hep-th/9607235, and references therein.
[8] S. Carlip, Phys. Rev. D 51 (1995) 632; The statistical mechanics of the three-dimensional euclidean black hole, UCD-96-13, gr-qc/9606043.
[9] M. Bañados, C. Teitelboim and J. Zanelli, Phys. Rev. Lett. 69 (1992) 1849; M. Bañados, M. Henneaux, C. Teitelboim and J. Zanelli, Phys. Rev. D 48 (1993) 1506.
[10] A. Achucarro and P.K. Townsend, Phys. Lett B 180 (1986) 89.
[11] E. Witten, Nucl. Phys. B 311 (1988/89) 46.
[12] G. 't Hooft, On the quantization of space and time, *in* Quantum Gravity: Proceedings, ed. M.A. Markov, V.A. Brezin, V.P. Frolov (World Scientific, Singapore, 1988).
[13] L. Susskind, J. Math. Phys. 36 (1995) 6377.
[14] Lee Smolin, J. Math. Phys. 36 (1995) 6417; Linking topological field theory and non-perturbative quantum gravity, Penn State preprint CGPG-95/4-5 (1995).
[15] S. Lau, Class. Quant. Grav. 13 (1996) 1509.
[16] A.P. Balachandran and P. Teotonio-Sobrinho, Helv. Phys. Acta 65 (1992) 723; A.P. Balachandran, G. Bimonte and P. Teotonio-Sobrinho, Mod. Phys. Lett. A 8 (1993) 1305; A.P. Balachandran and P. Teotonio-Sobrinho, Int. J. Mod. Phys. A 9 (1994) 1569, and references therein.
[17] A. Ashtekar, New perspectives in canonical gravity (Bibliopolis, Naples, 1988); Lectures on non-perturbative canonical gravity (World Scientific, Singapore, 1991).
[18] T. Thiemann, Class. Quant. Grav. 12 (1995) 181.
[19] J.D. Brown and M. Henneaux, J. Math. Phys. 27 (1986) 489.
[20] G.T. Horowitz, Commun. Math. Phys. 125 (1989) 417.
[21] V. Husain, Phys. Rev. D 43 (1991) 1803; loop variables for BF theory are also discussed in Y. Bi and J. Gegenberg, Class. Quant. Grav. 11 (1994) 883.
[22] J. Baez, BF theory and quantum gravity in four dimensions, in preparation.
[23] H. Kodama, Phys. Rev. D 42 (1990) 2548.