Random Matrix Products in Wireless Multiantenna Sytems

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Introduction

Modeling a multiantenna wireless channel via a product of independent random matrices captures the main geometrical and electromagnetic features of the communication link. Upon a proper tuning of the various parameters (e.g. marginal distribution of each matrix entry, size of each matrix factor, etc.) the product model, early introduced by Müller [1], is suitable to model different scenarios, across several generations of wireless systems. Among the various applications of random matrix theory in the performance analysis of wireless systems represented by product models, we focus hereinafter on a finite-blocklength setting. Specifically, we evaluate the so-called channel *dispersion*, a metric useful to determine the impact of channel dynamics and antenna selection rules on the communication rate, for an isotropic (i.e. unitarily invariant in law) channel. Then, we provide the statistics of the mutual information corresponding to non-isotropic product channels, paving the way to the characterization of the dispersion in more realistic scenarios.

Finite blocklength analysis

On a noisy communication channel, the maximal cardinality of a codebook of blocklength n which can be decoded with block error probability no greater than ϵ is denoted as $M^*(n,\epsilon)$; while its exact computation is mostly out of reach, an effective approximation is provided by

$$M^*(n,\epsilon) = nC - \sqrt{nV}Q^{-1}(\epsilon) + O(\log n), \tag{1}$$

with C the capacity of the noisy channel at hand and V its dispersion, while $Q^{-1}(\cdot)$ denotes the inverse of the Gaussian Q-function and $O(\cdot)$ is intended according to the usual Landau notation. For a multiantenna block-fading channel with n_t transmit and n_r receive antennas, with coherence time given by T, the input-output relation at block k (spanning time instants from $(k-1)T$ to kT), is given by

$$\mathbf{Y}_k = \mathbf{H}_k\mathbf{X}_k + \mathbf{Z}_k \quad k = 1,\ldots,n, \tag{2}$$

with $\mathbf{Y}$ the $n_r \times T$-valued channel output, $\mathbf{X}$ the input matrix of size $n_t \times T$ with iid zero-mean complex Gaussian entries with variance $\gamma = P/n_t$ [2, Thm.6], $\mathbf{Z}$ an $n_r \times T$-matrix of additive white Gaussian noise.

Assuming that a power constraint $\sum_{k=1}^n ||\mathbf{X}_k||_F^2 \leq nTP$ is enforced, and that the receiver is equipped with full information about the channel state, the mutual information per channel use is a random function of the input power level P, i.e. $\mathcal{I}(P) = \log\det\left(I_{n_r} + \frac{P}{n_t}\mathbf{H}\mathbf{H}^\dagger\right)$, and the channel capacity and dispersion appearing in (1) can be written, respectively, as [3, 2]

$$C(P) = \mathbb{E}\left[\mathcal{I}(P)\right] \tag{3}$$

$$\frac{V}{\log^2(e)n_{\min}} = \frac{Tn_{\min}}{\log^2(e)}\mathrm{var}\left(\mathcal{I}(P)\right) + \mathbb{E}\left[1 - \frac{1}{(1+\gamma\lambda)^2}\right] + \gamma^2\mathbb{E}[\alpha^2] - \frac{\gamma^2 n_{\min}}{n_t}\mathbb{E}^2[\alpha], \tag{4}$$

where $\alpha = \frac{\lambda}{1+\gamma\lambda}$, λ being a single, unordered eigenvalue of $\mathbf{H}\mathbf{H}^\dagger$, whose rank is denoted by $n_{\min}$. Notice that the expression of V in (3) holds whenever the joint eigenvalue distribution of $\mathbf{H}\mathbf{H}^\dagger$ is a symmetric function. Since most of the (even non-isotropic) fading laws commonly adopted in multi-antenna settings exhibit such a symmetry feature, we stick to such a simplified version of [2, Eq.(12)].

Channel Model

The process $\mathbf{H}_k$ is iid, with generic (matrix-valued) sample given by

$$\mathbf{H}_k = \left(\mathbf{\Sigma}_{rx}^{1/2}\mathbf{H}_M\mathbf{H}_{M-1}\cdot\ldots\cdot\mathbf{H}_1\mathbf{\Sigma}_{tx}^{1/2}\right)_k. \tag{7}$$

In (7), $\mathbf{H}_\ell \in \mathbb{C}^{N_\ell\times N_{\ell-1}}$, $\ell = 1,\ldots,M$ contain iid random entries distributed as $\mathcal{CN}(0,1)$, and the square matrices $\mathbf{\Sigma}_{tx}^{1/2}$ and $\mathbf{\Sigma}_{rx}^{1/2}$ of proper size, represent, the spatial correlation among transmit (resp. receive) antennas. Their eigenvalues, assumed distinct for sake of simplicity, are denoted by $\{\sigma_1^{\mathrm{tx}},\ldots,\sigma_{n_t}^{\mathrm{tx}}\}$ and $\{\sigma_1^{\mathrm{rx}},\ldots,\sigma_{n_r}^{\mathrm{rx}}\}$, respectively. We also assume, wlog, $n_t = n_0 = n_{\min}$, and define the auxiliary coefficients $\nu_\ell = N_\ell - n_{\min}$ $\ell = 1,\ldots,M$.

In absence of spatial correlation at both ends of the link, $\mathbf{\Sigma}_{rx}^{1/2} = \mathbf{I}_{n_r}$ and $\mathbf{\Sigma}_{tx}^{1/2} = \mathbf{I}_{n_t}$, respectively. In this case, a closed-form expression for C has been provided in [4]. As to the dispersion V, it can be obtained upon replacement in (3) of the following quantities:

$$\mathbb{E}[\alpha] = \frac{\kappa}{\gamma}\sum_{i,j=1}^{n_{\min}}\frac{\mathcal{D}_{i,j}}{\gamma^j}G_{1,M+1}^{M+1,1}\left(\begin{matrix}-j\\-j,\nu_M,\ldots,\nu_2,\nu_1+i-1\end{matrix}\middle|\frac{1}{\gamma}\right)$$

$$\mathbb{E}[\alpha^2] = \frac{\kappa}{\gamma^2}\sum_{i,j=1}^{n_{\min}}\frac{\mathcal{D}_{i,j}}{\gamma^j}G_{1,M+1}^{M+1,1}\left(\begin{matrix}-j-1\\-j,\nu_M,\ldots,\nu_2,\nu_1+i-1\end{matrix}\middle|\frac{1}{\gamma}\right)$$

$$\mathbb{E}\left[1-\frac{1}{(1+\gamma\lambda)^2}\right] = 1-\kappa\sum_{i,j=1}^{n_{\min}}\frac{\mathcal{D}_{i,j}}{\gamma^j}G_{1,M+1}^{M+1,1}\left(\begin{matrix}1-j\\2-j,\nu_M,\ldots,\nu_2,\nu_1+i-1\end{matrix}\middle|\frac{1}{\gamma}\right),$$

with κ the density-normalizing coefficient of the joint density of the eigenvalues of $\mathbf{H}\mathbf{H}^\dagger$ (see [4, Formula (21)]), and $\mathcal{D}_{i,j}$ denoting the $(i-j)$-th cofactor of the square matrix of size $n_{\min}$, with generic entry $\Gamma(\nu_1+i+j-1)\prod_{\ell=2}^M\Gamma(\nu_\ell+j)$. Finally, $\mathrm{var}\left(\mathcal{I}(P)\right) = \mathbb{E}_{\lambda_i,\lambda_j}\left[\log(1+\gamma\lambda_1)\log(1+\gamma\lambda_2)\right] - C^2$.

Non-isotropic channels

In the presence of spatial correlation at either ends of the link, the marginal law of the matrix-valued channel process is no longer isotropic, and the explicit dependence on the eigenvalues of the spatial correlation matrices appear. In presence of spatial correlation at a single end of the link, the one equipped with fewer antennas, [5] provided an expression for C; the corresponding expression for fully-correlated channel appeared in [7]. As to the dispersion V, its characterization is by far more challenging than that of C, since it involves some spectral statistics of the channel matrix process which have not been explicitly characterized yet.

Information Density

According to [2], the operational expression of the channel dispersion is obtained by evaluating the quantity

$$\frac{1}{T}\mathrm{var}\left[i(\mathbf{X};\mathbf{Y},\mathbf{H})|\mathbf{X}\right],$$

where the single-letter information density is defined as

$$i(x;y,h) \triangleq \log\frac{dP_{\mathbf{Y},\mathbf{H}|\mathbf{X}=x}}{dP^*_{\mathbf{Y},\mathbf{H}}}(y,h), \tag{5}$$

where P^* denotes a capacity-achieving output distribution and can be alternatively cast as [2]

$$i(x;y,h) = \frac{T}{2}\mathcal{I}(P) + \sum_{\ell=1}^{n_{\min}}f(\rho_\ell), \tag{6}$$

with $f(\cdot)$ an algebraic function of the non-zero singular values of the matrix h. The expression (6) leads to the second line in (3), otherwise stated, it reduces the evaluation of V to an (although possibly involved) expectation w.r.t. the law of a single and a randomly chosen pair of eigenvalues of $\mathbf{H}\mathbf{H}^\dagger$. Unfortunately, for several fading types (e.g. correlated Ricean, product channel with double correlation, just to mention two representative cases), explicit expressions for the eigenvalue densities are not available, despite a remarkable progress in the study of finite-sized random matrices spectra during the last decade [8, and refs therein].

Ongoing work: closed-form output densities

A characterization of the coherent dispersion, overcoming the limitation of dealing with an isotropic fading process, can be obtained by working on its rigorous definition (5), once the conditional output density $P_{\mathbf{Y},\mathbf{H}|\mathbf{X}=x}$ of the block-fading channel (2) is available in closed-form. Such a task is doable for many fading processes, beyond the isotropic case, as shown in [9, 10]. In particular, the output density is expected to have expression

$$P_{\mathbf{Y},\mathbf{H}|\mathbf{X}=x} = \kappa\det\left(g_i(y_j^2)\right)_{i,j=1,\ldots,n_{\min}}$$

with κ a density-normalizing constant and $g(\cdot)$ a function depending on the fading law and y_j^2 the j-th squared non-zero singular value of $\mathbf{Y}$. Notice that, as remarked in [2], $P^*_{\mathbf{Y},\mathbf{H}} = P_{\mathbf{H}}P^*_{\mathbf{Y}|\mathbf{H}}$, the last factor being under our assumptions a complex Gaussian law with zero mean and covariance matrix given by $\mathbf{I}_{n_r} + \frac{P}{n_t}\mathbf{H}\mathbf{H}^\dagger$.

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