The cohomology of the Grassmannian is a gl_n-module

The Cohomology of the Grassmannian is a $\mathfrak{gl}_n$-module

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Abstract

The integral singular cohomology ring of the Grassmann variety parametrizing r-dimensional subspaces in the n-dimensional complex vector space is naturally an irreducible representation of the Lie algebra $\mathfrak{gl}_n(\mathbb{Z})$ of all the $n \times n$ matrices with integral entries. The simplest case, $r = 1$, recovers the well known fact that any vector space is a module over the Lie algebra of its own endomorphisms. The other extremal case, $r = \infty$, corresponds to the bosonic vertex representation of the Lie algebra $\mathfrak{gl}_\infty(\mathbb{Z})$ on the polynomial ring in infinitely many indeterminates, due to Date, Jimbo, Kashiwara and Miwa.

In the present article we provide the structure of this irreducible representation explicitly, by meaans of a distinguished Hasse-Schmidt derivation on an exterior algebra, borrowed from Schubert Calculus

1 Introduction

It is well known from the undergraduate linear algebra courses, that any vector space is a module over the Lie algebra of its own endomorphisms. Less popular, but classical and well established in the literature, is the fact that a polynomial ring in infinitely many indeterminates is a module over the Lie algebra $\mathfrak{gl}_\infty$ of all the matrices $(a_{ij})_{i,j\in\mathbb{Z}}$, whose entries are all zero but finitely many. The explicit form of this representation, over any field containing the rationals, is due to Date, Jimbo, Kashiwara and Miwa, as reported in the milestone article [4] (see also [14]).

It turns out that these two seemingly different situations are bridged up by the general observation that the singular cohomology of the complex Grassmannian $G(r,n)$ is a module over the Lie algebra of $n \times n$ matrices. In this paper we present the explicit description of such a module structure by means of the Schubert derivation originally introduced in [5], see also [8, 11]. Its extension studied in [10] also enables us to deal with the $r = \infty$ case, so offering an alternative deduction of the classical expression of the DJKM bosonic vertex representation. Our description also links vertex operators in the boson-fermion correspondence to the Schubert Calculus as phrased in [5] (see also [7]).

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1.1 Cohomology rings of Complex Grassmannians. Let $r, n \in \mathbb{N}\cup\{\infty\}$ such that $0 \leqslant r \leqslant n$. The singular cohomology ring of the complex Grassmann variety $G(r,n)$ will be denoted by $B_{r,n}$, to be understood in the following extended sense. If r, n are both finite, then $B_{r,n} = H^*(G(r,n),\mathbb{Z})$ is the singular cohomology ring of the usual finite–dimensional Grassmann variety parametrizing r-dimensional subspaces of $\mathbb{C}^n$. If $r < \infty$ and $n = \infty$, then $B_r := B_{r,\infty} = H^*(G(r,\infty))$, where $G(r,\infty)$ is the ind-variety [13] corresponding to the chain of inclusions

$$\cdots \hookrightarrow G(r,n-1) \hookrightarrow G(r,n) \hookrightarrow G(r,n+1) \hookrightarrow \cdots.$$

In this case $B_r = \mathbb{Z}[e_1,\ldots,e_r]$, a polynomial ring in r indeterminates (see e.g. [1]), which is graded by giving degree i to each indeterminate e_i.

If both $r, n = \infty$, instead, $Gr(\infty) := G(r,\infty)$ is the Universal Grassmann Manifold (UGM) introduced by Sato (see the survey [19]), and which is the same as the ind-Grassmannian constructed in [3, 13]. In this case the ring $B := B_\infty$ is the projective limit of B_r in the category of graded modules and, concretely, $B = \mathbb{Z}[e_1, e_2, \ldots]$, a polynomial ring in infinitely many indeterminates.

Let now $\bigwedge M_n = \bigoplus_{r\geqslant 0}\bigwedge^r M_n$ be the exterior algebra of the free abelian group $M_n := \bigoplus_{0\leqslant i<n}\mathbb{Z}b_i$, with basis $\mathbf{b} := (b_i)_{0\leqslant i<n}$. Consider the Lie algebra

$$\mathfrak{gl}_n(\mathbb{Z}) := \{A \in \mathrm{End}_{\mathbb{Z}}(M_n)\,|\,Ab_j = 0 \text{ for all but finitely many } j \in \mathbb{N}\}, \tag{1}$$

with respect to the usual commutator.

For $r < \infty$, let $\mathcal{P}_{r,n}$ be the set of all the partitions whose Young diagram is contained in a $r \times (n-r)$-rectangle. Then there is a natural $\mathbb{Z}$-module isomorphism

$$\phi_{r,n} : B_{r,n} \to \bigwedge^r M_n,$$

which maps the basis $\Delta_{\boldsymbol{\lambda}}(H_r)$ (see Section 1.4) of Schur polynomials of $B_{r,n}$ to a natural basis $[\mathbf{b}]^r_{\boldsymbol{\lambda}}$ of $\bigwedge^r M_n$, both labeled by $\mathcal{P}_{r,n}$ (see e.g. [9, Formula (25)]). It turns out that $\bigwedge M_n$ is a natural representation of $\mathfrak{gl}_n(\mathbb{Z})$ and that the exterior powers $\bigwedge^r M_n$ are precisely its irreducible sub-representations.

The case $r = n = \infty$ needs a few adjustment, because the ring B is not isomorphic to any finite exterior power. It must be replaced by the fermionic Fock space (FFS), a suitable irreducible representation of a canonical Clifford algebra supported on the direct sum of $\mathfrak{M} := \bigoplus_{j\in\mathbb{Z}}\mathbb{Z}b_j$ with its restricted dual. The FFS is naturally a $\mathfrak{gl}_\infty(\mathbb{Z})$-module as well ([16, Section 4.3]). So, in general,

$$\boxed{B_{r,n} \text{ is a module over the Lie algebra } \mathfrak{gl}_n(\mathbb{Z}), \text{ for all } r \leqslant n \in \mathbb{N}\cup\{\infty\}.}$$

If $r = \infty$, the $\mathfrak{gl}_\infty(\mathbb{Q})$ structure of $B \otimes_{\mathbb{Z}} \mathbb{Q}$ is described by the bosonic vertex representation due to Date, Jimbo, Kashiwara and Miwa (DJKM) [4, 14]. It amounts to determine the shape of the generating function $\mathcal{E}(z,w) = \sum_{i,j\in\mathbb{Z}}\mathcal{E}_{ij}z^i w^{-j}$ of all the elementary endomorphisms $\mathcal{E}_{ij} \in \mathrm{End}_{\mathbb{Q}}(\mathfrak{M}\otimes\mathbb{Q})$, defined by $\mathcal{E}_{ij}b_k = b_i\delta_{jk}$ acting on B. In [10] the notion of Schubert derivation, a priori only defined on an exterior algebra, is extended to the FFS. As a byproduct, we offered an alternative deduction of the DJKM generating function $\mathcal{E}(z,w)$.

The present paper, instead, is concerned with the description of the generating function $\mathcal{E}(z,w)_n := \sum_{0\leqslant i,j<n}\mathcal{E}_{ij}\cdot z^i w^{-j}$ as acting on $B_{r,n}$ for $r < \infty$ and any $n \in \mathbb{N}\cup\{\infty\}$, see Sections 4 formula (25) and 5, formula (32). The formulas obtained in this case are new and their deductions are cheap.

1.2 Plan of the paper. We first consider the case $n = \infty$, and find two equivalent, although looking different, expressions for the action of $\mathcal{E}(z, w) := \sum_{i,j\geqslant 0} \mathcal{E}_{ij} z^i w^{-j}$ on B_r, see section 4 and 5. They have interesting complementarty features. The former is useful for explicit computations, and can be implemented as well when $r = \infty$ (Cf. [10, Section 7]). The latter visibly shows its close relationship with the DJKM representation, also because the shape of the approximated vertex operators in terms of Schubert derivations is exactly the same as that occurring in non-approximated DJKM ones, as shown in [10].

Section 6 is eventually devoted to make explicit the $gl_n(\mathbb{Z})$ structure of $B_{r,n}$, for finite n. It is obtained from the $gl_\infty(\mathbb{Z})$–structure of B_r, by projecting it through the canonical epimorphism $B_r \to B_{r,n}$. We provide a few examples to show how our formulas work to write explicit expressions for the product $\mathcal{E}(z, w)_n$ with elements of $B_{r,n}$. It is something that can be done automatically on a computer.

1.3 Statement of the results. For a more precise description of the outputs of this paper, let us introduce some further piece of notation. The canonical $gl_n(\mathbb{Z})$-module structure on $\bigwedge M_n$, where $n \in \mathbb{N} \cup \{\infty\}$, is defined by mapping each $A \in gl_n(\mathbb{Z})$ to $\delta(A) \in \mathrm{End}_Z(\bigwedge M_n)$ such that:

$$\begin{cases} \delta(A)\mathbf{u} & = & A \cdot \mathbf{u}, & \forall \mathbf{u} \in M_n = \bigwedge^1 M_n, \\ \delta(A)(\mathbf{v} \wedge \mathbf{w}) & = & \delta(A)\mathbf{v} \wedge \mathbf{w} + \mathbf{v} \wedge \delta(A)\mathbf{w}, & \forall \mathbf{v}, \mathbf{w} \in \bigwedge M_n \end{cases}. \tag{2}$$

Since every $\mathbf{u} \in \bigwedge M_n$ is a finite linear combination of monomials of some given degree, the initial condition and the Leibniz rule determine the map $\delta(A)$ over all $\bigwedge M_n$. An easy check shows that the commutator $[\delta(A), \delta(B)] \in \mathrm{End}_{\mathbb{Z}}(\bigwedge M_n)$ is equal to $\delta([A, B])$, where $[A, B]$ is the commutator in $gl_n(\mathbb{Z})$. The composition of δ with the restriction map to the r-th degree of the exterior algebra, $A \mapsto \delta(A)_{|\bigwedge^r M_n}$, turns $\bigwedge^r M_n$ into a representation of $gl_n(\mathbb{Z})$ for any r. This is easily seen to be irreducible, because any basis element of $\bigwedge^r M_n$ can be transported to any other via a suitable element of $gl_n(\mathbb{Z})$.

1.4 Let us set $M := M_\infty$. In the ring $B_r := \mathbb{Z}[e_1, \ldots, e_r]$ consider the generic polynomial of degree r:

$$E_r(z) := 1 - e_1 z + \cdots + (-1)^r e_r z^r \in B_r[z]$$

and the sequence $H_r := (h_j)_{j\in\mathbb{Z}}$ implicitly defined by

$$\sum_{i\geqslant 0} h_i z^i := \frac{1}{E_r(z)} \in B_r[[z]].$$

It is well known that $B_r = \bigoplus_{\boldsymbol{\lambda}\in\mathcal{P}_r} \mathbb{Z}\Delta_{\boldsymbol{\lambda}}(H_r)$ (see [18, Proposition (3.2)] or [8, Corollary 5.8.3] for an alternative deduction), where $\mathcal{P}_r$ denotes the set of all partitions of length at most r and $\Delta_{\boldsymbol{\lambda}}(H_r)$ is the Schur determinant $\det(h_{\lambda_j - j+i})_{1\leqslant i,j<n}$. We have a $\mathbb{Z}$-module isomorphism $\phi_r : B_r \mapsto \bigwedge^r M$ given by $\Delta_{\boldsymbol{\lambda}}(H_r) \mapsto [\mathbf{b}]^r_{\boldsymbol{\lambda}}$, where

$$[\mathbf{b}]^r_{\boldsymbol{\lambda}} := b_{r-1+\lambda_1} \wedge \cdots \wedge b_{\lambda_r} \in \bigwedge^r M.$$

Consider now the following two sequences of elements of the polynomial ring $B_r[z^{-1}]$:

$$\overline{\sigma}_-(z)H_r := (\overline{\sigma}_-(z)h_j)_{j\in\mathbb{Z}} \qquad \text{and} \qquad \sigma_-(z)H_r := (\sigma_-(z)h_j)_{j\in\mathbb{Z}},$$

where

$$\overline{\sigma}_-(z)h_j = h_j - \frac{h_{j-1}}{z} \qquad \text{and} \qquad \sigma_-(z)h_j = \sum_{i\geqslant 0} \frac{h_{j-i}}{z^i}.$$

Extending the action of $\sigma_-(z)$ and $\overline{\sigma}_-(z)$ to all basis element of B_r, through the rule

$$\sigma_-(z)\Delta_{\boldsymbol{\lambda}}(H_r) = \Delta_{\boldsymbol{\lambda}}(\sigma_-(z)H_r) \qquad \text{and} \qquad \overline{\sigma}_-(z)\Delta_{\boldsymbol{\lambda}}(H_r) = \Delta_{\boldsymbol{\lambda}}(\overline{\sigma}_-(z)H_r), \tag{3}$$

provides two well defined $\mathbb{Z}$-linear maps $\overline{\sigma}_-(z), \sigma_-(z) : B_r \to B_r[z^{-1}]$. Set now

$$\delta(z, w) := \sum_{i,j\geqslant 0} \delta(\mathcal{E}_{ij}) z^i w^{-j},$$

and define $\mathcal{E}(z, w) := \sum_{i,j\geqslant 0} \mathcal{E}_{ij} \cdot z^i w^{-j} : B_r \to B_r[[z, w^{-1}]$ via the equality

$$\phi_r(\mathcal{E}(z, w)P(e_1, \ldots, e_r)) = \delta(z, w)\phi_r(P(e_1, \ldots, e_r)),$$

where $P(e_1, \ldots, e_r)$ is an arbitrary $\mathbb{Z}$-polynomial in $e_1, \ldots, e_r$.

Let now set, as a notation:

$$\Delta_{\boldsymbol{\lambda}}(w^{-\boldsymbol{\lambda}}, \overline{\sigma}_-(z)H_r) := \begin{vmatrix} w^{-\lambda_1} & w^{1-\lambda_2} & \cdots & w^{r-1-\lambda_r} \\ h_{\lambda_1+1} - \dfrac{h_{\lambda_1}}{z} & h_{\lambda_2} - \dfrac{h_{\lambda_2-1}}{z} & \cdots & h_{\lambda_r+r-2} - \dfrac{h_{\lambda_r+r-3}}{z} \\ \vdots & \vdots & \ddots & \vdots \\ h_{\lambda_1+r-1} - \dfrac{h_{\lambda_1+r-2}}{z} & h_{\lambda_2+r-2} - \dfrac{h_{\lambda_2+r-3}}{z} & \cdots & h_{\lambda_r} - \dfrac{h_{\lambda_r-1}}{z}. \end{vmatrix}, \tag{4}$$

with which we are now in position to state the first main result of this paper.

Theorem 4.3. *The $\mathcal{E}(z, w)$-image in $B_r[z, w^{-1}]$ of a basis element $\Delta_{\boldsymbol{\lambda}}(H_r) \in B_r$, is:*

$$\mathcal{E}(z, w)\Delta_{\boldsymbol{\lambda}}(H_r) = \frac{z^{r-1}}{w^{r-1}} \cdot \frac{1}{E_r(z)} \Delta_{\boldsymbol{\lambda}}(w^{-\boldsymbol{\lambda}}, \overline{\sigma}_-(z)H_r). \tag{5}$$

In other words, the product $\mathcal{E}_{ij} \cdot \Delta_{\boldsymbol{\lambda}}(H_r)$ is the coefficient of $z^i w^{-j}$ in the expansion of the right–hand side of (5). A second equivalent expression for the action of $\mathcal{E}(z, w)$ is provided by:

Theorem 5.7. *The following formula holds*

$$\mathcal{E}(z, w) = \left(1 - \frac{z}{w}\right)^{-1} \left(1 - \frac{z^r}{w^r} \Gamma_r(z, w)\right), \tag{6}$$

where

$$\Gamma_r(z, w) := \frac{E_r(w)}{E_r(z)} \overline{\sigma}_-(z)\sigma_-(w).$$

Equation (6) recalls the shape of the bosonic representation of the Lie algebra $\mathcal{A}_\infty$ of all the matrices with finitely many non-zero diagonals: see [4, 14, 16] and, from now on, also [10, Section 9].

Suppose now that $n < \infty$. In this case

$$B_{r,n} = \frac{B_r}{(h_{n-r+1}, \ldots, h_n)}.$$

Denote by $\pi_{r,n} : B_r \to B_{r,n}$ the canonical epimorphism. Then

$$B_{r,n} = \bigoplus_{\boldsymbol{\lambda} \in \mathcal{P}_r} \mathbb{Z} \cdot \pi_{r,n} \Delta_{\boldsymbol{\lambda}}(H_r) = \bigoplus_{\boldsymbol{\lambda} \in \mathcal{P}_r} \mathbb{Z} \cdot \Delta_{\boldsymbol{\lambda}}(H_{r,n}),$$

where $H_{r,n} = \pi_{r,n} H_r = (\pi_{r,n}(h_j))_{j \in \mathbb{Z}} = (1 = h_0, h_1, h_2, \ldots, h_{n-r})$. As remakerd, $B_{r,n}$ is a $\mathfrak{gl}_n(\mathbb{Z})$-module. Denote $\mathcal{E}(z, w)_n := \sum_{0 \leqslant i,j < n} \mathcal{E}_{ij} z^i w^{-j}$.

Theorem 6.4. *The following equality holds in* $B_{r,n}[z, w^{-1}]$*:*

$$\mathcal{E}(z, w)_n \Delta_{\boldsymbol{\lambda}}(H_{r,n}) = \pi_{r,n}(\mathcal{E}(z, w) \Delta_{\boldsymbol{\lambda}}(H_r)). \tag{7}$$

Equality (7) means that the $\mathfrak{gl}_n(\mathbb{Z})$-action on an element of $B_{r,n}$ is obtained putting $h_{n-r+1+j} = 0$ for all $j > 0$ in the expression (5). For example:

$$\begin{aligned}
\mathcal{E}(z, w)_4 \Delta_{(2,2)}(H_{2,4}) &= \frac{1}{w^2}\left(-h_2 - h_1 h_2 z + h_2^2 z^2\right) + \frac{1}{w^3}\left(-h_1 - (h_1^2 - h_2) z + h_2^2 z^3\right) \\
&= \frac{1}{w^2}[(e_2 - e_1^2) + (e_1 e_2 - e_1^3) z + (e_1^4 - 2e_1^2 e_2 + e_2^2) z^2] \\
&- \frac{1}{w^3}[e_1 + e_2 z - (e_1^4 - 2e_1^2 e_2 + e_2^2) z^3] \in B_{2,4}[[z, w^{-1}].
\end{aligned}$$

2 Preliminaries and Notation

2.1 A partition is a monotonic non increasing sequence of non-negative integers $\lambda_1 \geqslant \lambda_2 \geqslant \ldots$ all zero but finitely many, said to be *parts*. The length $\ell(\boldsymbol{\lambda})$ of a partition $\boldsymbol{\lambda}$ is the number of non zero parts. We denote by $\mathcal{P}$ the set of all partitions, by $\mathcal{P}_r := \{\boldsymbol{\lambda} \in \mathcal{P}_r \,|\, \ell(\boldsymbol{\lambda}) \leqslant r\}$ and by $\mathcal{P}_{r,n}$ the set of all partitions of length at most r whose Young diagram is contained in a $r \times (n - r)$ rectangle. The partitions form an additive semigroup: if $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{P}$, then $\boldsymbol{\lambda} + \boldsymbol{\mu} \in \mathcal{P}$ [18, Chap. I.1]. If $\boldsymbol{\lambda} := (\lambda_1, \lambda_2, \ldots)$, we denote by $\boldsymbol{\lambda}^{(i)}$ the partition obtained by removing the i-th part:

$$\boldsymbol{\lambda}^{(i)} := (\lambda_1 \geqslant \lambda_{i-1} \geqslant \widehat{\lambda_i} \geqslant \lambda_{i+1} \geqslant \ldots),$$

where $\widehat{\ }$ means removed. By (1^j) we mean the partition with j parts equal to 1.

2.2 In the following M will denote the free abelian group $\bigoplus_{i \geqslant 0} \mathbb{Z} \cdot b_i$ with basis $\mathbf{b} := (b_i)_{i \geqslant 0}$. For $\boldsymbol{\lambda} \in \mathcal{P}_r$, let

$$[\mathbf{b}]^r_{\boldsymbol{\lambda}} := b_{r-1+\lambda_1} \wedge \cdots \wedge b_{\lambda_r}. \tag{8}$$

Clearly $\bigwedge^r M := \bigoplus_{\boldsymbol{\lambda} \in \mathcal{P}_r} \mathbb{Z}[\mathbf{b}]^r_{\boldsymbol{\lambda}}$. The *restricted dual* of M is $M^* := \bigoplus_{i \geqslant 0} \mathbb{Z}\beta_i$, where $\beta_i \in \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ is such that $\beta_i(b_j) = \delta_{ij}$. There is a natural well known identification between $(\bigwedge M_n)^*$ and $\bigwedge M_n^*$, see e.g. [9, Section 2.6].

2.3 Hasse-Schmidt derivations on $\bigwedge M$. Let z denote an arbitrary formal variable. A *Hasse-Schmidt derivation* (HS) [8] on $\bigwedge M$ is a homomorphism of abelian groups $\mathcal{D}(z) : \bigwedge M \to \bigwedge M[[z]]$ such that

$$\mathcal{D}(z)(\mathbf{u} \wedge \mathbf{v}) = \mathcal{D}(z)\mathbf{u} \wedge \mathcal{D}(z)\mathbf{v}, \tag{9}$$

for all $\mathbf{u}, \mathbf{v} \in \bigwedge M$. Writing $\mathcal{D}(z)$ as $\sum_{j\geqslant 0} D_j z^j$, equation (9) is equivalent to

$$D_j(\mathbf{u} \wedge \mathbf{v}) = \sum_{i=0}^{j} D_i\mathbf{u} \wedge D_{j-i}\mathbf{v}. \qquad \forall j \geqslant 0.$$

2.4 If $\mathcal{D}(z)$ is a HS–derivation on $\bigwedge M$ and D_0 is invertible, there exists $\overline{\mathcal{D}}(z) := \sum_{i\geqslant 0}(-1)^i\overline{\mathcal{D}}_i z^i \in \mathrm{End}_{\mathbb{Z}}(\bigwedge M)[[z]]$ such that $\overline{\mathcal{D}}(z)\mathcal{D}(z) = \mathcal{D}(z)\overline{\mathcal{D}}(z) = 1$. The map $\overline{\mathcal{D}}(z)$ is a HS-derivations, said to be the *inverse* of $\mathcal{D}(z)$. Thus the two integration by parts formulas hold:

$$\mathcal{D}(z)\mathbf{u} \wedge \mathbf{v} = \mathcal{D}(z)(\mathbf{u} \wedge \overline{\mathcal{D}}(z)\mathbf{v}) \qquad \text{and} \qquad \mathbf{u} \wedge \overline{\mathcal{D}}(z)\mathbf{v} = \overline{\mathcal{D}}(z)(\mathcal{D}(z)\mathbf{u} \wedge \mathbf{v}). \tag{10}$$

As remarked in [12], the second of (10) is the generalization (holding also for free A-module of infinite rank) of the Cayley-Hamilton theorem, which in [6] is also extended in the “tropical” context of Grassmann semi-algebras.

2.5 The transposed HS–derivation. For all $\eta \in \bigwedge M^*$, let $\mathcal{D}^T(z)\eta$ be the unique element of $\bigwedge M^*$ such that

$$\mathcal{D}^T(z)(\eta)(\mathbf{u}) = \eta(\mathcal{D}(z)(\mathbf{u})),$$

for all $\mathbf{u} \in \bigwedge M$. By [9, Proposition 3.8], $\mathcal{D}^T(z)$ is a HS derivation on $\bigwedge M^*$ said to be *the transposed* of $\mathcal{D}(z)$. Integration by parts (10) implies the following equality for transposed HS–derivations.

2.6 Proposition. *For all* $\eta \in M^*$ *and each* $\mathbf{u} \in \bigwedge^r M$

$$\mathcal{D}^T(z)\eta \lrcorner \mathbf{u} = \overline{\mathcal{D}}(z)\,(\eta \lrcorner \mathcal{D}(z)\mathbf{u})) \tag{11}$$

Proof. By definition of contraction of an exterior vector against a linear form (see e.g. [2, Ann. 15.3], for all $\zeta \in \bigwedge^{r-1} M^*$:

$$\zeta(\mathcal{D}(z)^T\eta \lrcorner \mathbf{u}) = (\mathcal{D}(z)^T\eta \wedge \zeta)(\mathbf{u}).$$

Now we apply the first of integration by parts (10):

$$(\mathcal{D}(z)^T\eta \wedge \zeta)(\mathbf{u}) = \mathcal{D}(z)^T(\eta \wedge \overline{\mathcal{D}}(z)^T\zeta)(\mathbf{u})$$

from which, by definition of transposistion,

$$(\eta \wedge \overline{\mathcal{D}}(z)^T\zeta)\mathcal{D}(z)\mathbf{u} = \overline{\mathcal{D}}(z)^T\zeta(\eta \lrcorner \mathcal{D}(z)\mathbf{u}) = \zeta\left[\overline{\mathcal{D}}(z)(\eta \lrcorner \mathcal{D}(z)\mathbf{u})\right]$$

which proves (11). ∎

3 Schubert derivations on $\bigwedge M$

It is easy to check that a HS-derivation on $\bigwedge M$ is uniquely determined by its restriction to the first degree $M = \bigwedge^1 M$ of the exterior algebra (Cf. [8, Ch. 4]). Let $\sigma_+(z) := \sum_{i\geqslant 0}\sigma_i z^i : \bigwedge M_n \to \bigwedge M_n[[z]]$ and $\sigma_-(z) := \sum \sigma_{-i}z^{-i} : \bigwedge M \to \bigwedge M[[z^{-1}]]$ be the unique HS-derivations such that for all $i \in \mathbb{Z}$, $\sigma_i b_j = b_{i+j}$ if $i+j \geqslant 0$ and 0 otherwise. Let $\overline{\sigma}_+(z)$ and $\overline{\sigma}_-(z)$ be, respectively, the inverse HS-derivations of $\sigma_+(z)$ and $\sigma_-(z)$ in the algebra $\mathrm{End}_{\mathbb{Z}}(\bigwedge M_n)[[z^{\pm 1}]]$.

3.1 Definition. *The HS-derivations* $\sigma_\pm(z)$ *and* $\overline{\sigma}_\pm(z)$ *are called* Schubert derivations.

Let $\Delta_{\boldsymbol{\lambda}}(\sigma_+) = \det(\sigma_{\lambda_j-j+i})_{1\leqslant i,j\leqslant r}$. Giambelli's formula for Schubert derivations [8, Corollary 5.8.2] or [9, Formula (3.2)] says that $[\mathbf{b}]^r_{\boldsymbol{\lambda}} = \Delta_{\boldsymbol{\lambda}}(\sigma_+)[\mathbf{b}]^r_0$. It enables us to equip $\bigwedge^r M$ with a structure of B_r-module by declaring that

$$h_i[\mathbf{b}]^r_{\boldsymbol{\lambda}} := \sigma_i[\mathbf{b}]^r_{\boldsymbol{\lambda}}. \tag{12}$$

Thus $\bigwedge^r M$ can be thought of as a free B_r-module of rank 1 generated by $[\mathbf{b}]^r_0$, such that $[\mathbf{b}]^r_{\boldsymbol{\lambda}} = \Delta_{\boldsymbol{\lambda}}(H_r)[\mathbf{b}]^r_0$. In particular

$$\sigma_+(z)[\mathbf{b}]^r_{\boldsymbol{\lambda}} = \frac{1}{E_r(z)}[\mathbf{b}]^r_{\boldsymbol{\lambda}}. \tag{13}$$

3.2 Using the B_r-module structure of $\bigwedge^r M$ one can define $\overline{\sigma}_-(z)$ and $\sigma_-(z)$ as maps $B_r \to B_r[z^{-1}]$, by setting

$$(\sigma_-(z)\Delta_{\boldsymbol{\lambda}}(H_r))[\mathbf{b}]^r_0 = \sigma_-(z)[\mathbf{b}]^r_{\boldsymbol{\lambda}} \qquad \text{and} \qquad (\overline{\sigma}_-(z)\Delta_{\boldsymbol{\lambda}}(H_r))[\mathbf{b}]^r_0 = \overline{\sigma}_-(z)[\mathbf{b}]^r_{\boldsymbol{\lambda}}.$$

A simple application of the definition shows, as in [9, Proposition 5.3], that the equalities

$$\sigma_-(z)h_j = \sum_{i=0}^{j}\frac{h_{j-i}}{z^i} \qquad \text{and} \qquad \overline{\sigma}_-(z)h_j = h_j - \frac{h_{j-1}}{z}, \tag{14}$$

hold in B_r for all $r \geqslant 1$.

3.3 Proposition. *Let* $\sigma_-(z)H_r$ *(resp.* $\overline{\sigma}(z)H_r$*) stands for the sequence* $(\sigma_-(z)h_j)_{j\in\mathbb{Z}}$ *(resp.* $(\overline{\sigma}_-(z)h_j)_{j\in\mathbb{Z}}$*). If* $\ell(\boldsymbol{\lambda}) \leqslant r$ *then*

$$\sigma_-(z)\Delta_{\boldsymbol{\lambda}}(H_r) = \Delta_{\boldsymbol{\lambda}}(\sigma_-(z)H_r) \qquad \text{and} \qquad \overline{\sigma}_-(z)\Delta_{\boldsymbol{\lambda}}(H_r) = \Delta_{\boldsymbol{\lambda}}(\overline{\sigma}_-(z)H_r).$$

Proof. According to [9, Theorem 5.7], by using a general determinantal formula due to Laksov and Thorup as in [17, Main Theorem]. ∎

3.4 Lemma. *For all* $\boldsymbol{\lambda} \in \mathcal{P}_r$ *the following equalities hold:*

$$b_0 \wedge \overline{\sigma}_+(z)[\mathbf{b}]^r_{\boldsymbol{\lambda}} = z^r\overline{\sigma}_-(z)\left([\mathbf{b}]^r_{\boldsymbol{\lambda}+(1^r)} \wedge b_0\right) = z^r\overline{\sigma}_-(z)[\mathbf{b}]^{r+1}_{\boldsymbol{\lambda}}. \tag{15}$$

Proof. One argues by induction on $r \geqslant 1$. For $r = 1$ one has:

$$\begin{aligned} b_0 \wedge \overline{\sigma}_+(z)b_\lambda &= b_0 \wedge (b_\lambda - b_{\lambda+1}z) \\ &= -b_0 \wedge z(b_{\lambda+1} - b_\lambda z^{-1}) \\ &= z(b_{\lambda+1} - b_\lambda z^{-1}) \wedge b_0 = z\overline{\sigma}_-(z)(b_{\lambda+1}\wedge b_0) = z\overline{\sigma}_-(z)[\mathbf{b}]^2_{(\lambda)}. \end{aligned}$$

Assume (15) holds for all $1 \leqslant s \leqslant r-1$. Then

$$
\begin{aligned}
b_0\wedge\overline{\sigma}_+(z)[\mathbf{b}]^r_{\boldsymbol{\lambda}} &= b_0\wedge\overline{\sigma}_+(z)\left(b_{r-1+\lambda_1}\wedge[\mathbf{b}]^{r-1}_{\boldsymbol{\lambda}^{(1)}}\right) && \text{(decomposition of } [\mathbf{b}]^r_{\boldsymbol{\lambda}})\\
&= b_0\wedge\overline{\sigma}_+(z)b_{r-1+\lambda_1}\wedge\overline{\sigma}_+(z)[\mathbf{b}]^{r-1}_{\boldsymbol{\lambda}^{(1)}} && (\overline{\sigma}_+(z)\text{ is a HS-derivation})\\
&= z\overline{\sigma}_-(z)b_{r-1+\lambda_1+1}\wedge(b_0\wedge\overline{\sigma}_+(z)[\mathbf{b}]^{r-1}_{\boldsymbol{\lambda}^{(1)}}) && (\text{case } r=1)\\
&= z\overline{\sigma}_-(z)b_{r-1+\lambda_1+1}\wedge z^{r-1}\overline{\sigma}_-(z)([\mathbf{b}]^{r-1}_{\boldsymbol{\lambda}^{(1)}+(1^{r-1})}\wedge b_0) && \text{(inductive hypothesis)}\\
&= z^r\overline{\sigma}_-(z)([\mathbf{b}]^r_{\boldsymbol{\lambda}+(1^r)}\wedge b_0). && (\overline{\sigma}_-(z)b_0=b_0 \text{ and } \overline{\sigma}_-(z)\text{ is a HS-derivation})\\
&= z^r\overline{\sigma}_(z)[\mathbf{b}]^{r+1}_{\boldsymbol{\lambda}}. && \text{(definition of } [\mathbf{b}]^{r+1}_{\boldsymbol{\lambda}})
\end{aligned}
$$

■

3.5 It is convenient to introduce one more formal variable w. Define

$$
\mathbf{b}(z) := \sum_{j\geqslant 0} b_j z^j \qquad \text{and} \qquad \boldsymbol{\beta}(w) = \sum_{j\geqslant 0}\beta_j w^{-j-1}.
$$

Then $\mathbf{b}(z) = \sigma_+(z)b_0$. Moreover $\boldsymbol{\beta}(w) = w^{-1}\sigma_-^T(w)\beta_0$. Indeed $(\sigma_{-i}^T\beta_j)(b_k) = \beta_j(\sigma_{-i}b_k) = \beta_j(b_{k-i}) = \delta_{i+j,k} = \beta_{j+i}(b_k)$.

Let $\Gamma_r(z) : B_r \to B_{r+1}[[z]]$ and $\Gamma^*_w(z) : B_r \to B_{r-1}[w^{-1}]$ be the operators implicitly defined by:

$$
(\Gamma_r(z)\Delta_{\boldsymbol{\lambda}}(H_r))[\mathbf{b}]^{r+1}_0 = z^{-r}\mathbf{b}(z)\wedge[\mathbf{b}]^r_{\boldsymbol{\lambda}} \tag{16}
$$

$$
(\Gamma^*_r(w)\Delta_{\boldsymbol{\lambda}}(H_r))[\mathbf{b}]^{r-1}_0 = w^r\boldsymbol{\beta}(w)\lrcorner[\mathbf{b}]^r_{\boldsymbol{\lambda}}. \tag{17}
$$

Clearly $\Gamma_r(z), \Gamma^*_r(z)$ are the finite r case of the bosonic vertex operators as in [16]. We now use the following notation:

$$
\Delta_{\boldsymbol{\lambda}}(w^{-\boldsymbol{\lambda}}, H_{r-1}) := \begin{vmatrix} w^{-\lambda_1} & w^{-\lambda_2+1} & \cdots & w^{-\lambda_r+r-1}\\ h_{\lambda_1+1} & h_{\lambda_2} & \cdots & h_{\lambda_r+r-2}\\ \vdots & \vdots & \ddots & \vdots\\ h_{\lambda_1+r-1} & h_{\lambda_2+r-2} & \cdots & h_{\lambda_r}\end{vmatrix} \tag{18}
$$

that keeps track of the fact that all the h_j occurring in the determinant (18) live in the ring B_{r-1}.

3.6 Proposition.

$$
\Gamma_r(z)\Delta_{\boldsymbol{\lambda}}(H_r) = \frac{1}{E_{r+1}(z)}\Delta_{\boldsymbol{\lambda}}(\overline{\sigma}_-(z)H_{r+1}) \tag{19}
$$

and

$$
\Gamma^*_r(w)\Delta_{\boldsymbol{\lambda}}(H_r) = \Delta_{\boldsymbol{\lambda}}(w^{-\boldsymbol{\lambda}}, H_{r-1}), \tag{20}
$$

Proof. Let us prove (19) first. One has

$$\begin{aligned}
\mathbf{b}(z)\wedge[\mathbf{b}]^r_{\boldsymbol{\lambda}} &= \sigma_+(z)b_0\wedge[\mathbf{b}]^r_{\boldsymbol{\lambda}} && \text{(definition of } \sigma_+(z))\\
&= \sigma_+(z)(b_0\wedge\overline{\sigma}_+(z)[\mathbf{b}]^r_{\boldsymbol{\lambda}}) && \text{(integration by parts)}\\
&= \sigma_+(z)(z^r\overline{\sigma}_-(z)[\mathbf{b}]^r_{\boldsymbol{\lambda}+(1^r)}\wedge b_0) && \text{(Formula 15)}\\
&= z^r\sigma_+(z)\overline{\sigma}_-(z)([\mathbf{b}]^r_{\boldsymbol{\lambda}+(1^r)}\wedge b_0) && (\overline{\sigma}_-(z)b_0=b_0 \text{ and}\\
& && \overline{\sigma}_-(z) \text{ is a HS derivation})\\
&= \frac{z^r}{E_{r+1}(z)}\left(\sigma_-(z)[\mathbf{b}]^{r+1}_{\boldsymbol{\lambda}}\right) && \text{(by the } B_r\text{-module structure}\\
& && \text{(12) of } {\textstyle\bigwedge^r} M)\\
&= \frac{z^r}{E_{r+1}(z)}\left(\overline{\sigma}_-(z)\Delta_{\boldsymbol{\lambda}}(H_{r+1})\right)[\mathbf{b}]^{r+1}_0 && \text{(definition of } \overline{\sigma}_-(z)\\
& && \text{as a map } B_r\to B_r[z^{-1}])\\
&= \frac{z^r}{E_{r+1}(z)}\Delta_{\boldsymbol{\lambda}}(\overline{\sigma}_-(z)H_{r+1})[\mathbf{b}]^{r+1}_0. && \text{(Proposition 3.3)}.
\end{aligned}$$

To prove (20), instead, the best is acting by direct computation:

$$\begin{aligned}
&\boldsymbol{\beta}(w)\lrcorner(b_{r-1+\lambda_1}\wedge\cdots\wedge b_{\lambda_r})\\
&= w^{-r-\lambda_1}[\mathbf{b}]^{r-1}_{\boldsymbol{\lambda}^{(1)}} - w^{-r+1-\lambda_2}[\mathbf{b}]^{r-1}_{\boldsymbol{\lambda}^{(2)}+(1)} + \cdots + (-1)^{r-1}w^{-\lambda_r-1}[\mathbf{b}]^{r-1}_{\lambda^{(r)}+(1^{r-1})}\\
&= w^{-r}\left(w^{-\lambda_1}\Delta_{\boldsymbol{\lambda}^{(1)}}(H_{r-1}) - w^{1-\lambda_2}\Delta_{\boldsymbol{\lambda}^{(1)}}(H_{r-1}) + \cdots + (-1)^{r-1}w^{r-1-\lambda_r}\Delta_{\boldsymbol{\lambda}^{(r)}+(1^{r-1})}\right)[\mathbf{b}]^{r-1}_0\\
&= w^{-r}\begin{vmatrix} w^{-\lambda_1} & w^{-\lambda_2+1} & \cdots & w^{-\lambda_r+r-1}\\ h_{\lambda_1+1} & h_{\lambda_2} & \cdots & h_{\lambda_r+r-2}\\ \vdots & \vdots & \ddots & \vdots\\ h_{\lambda_1+r-1} & h_{\lambda_2+r-2} & \cdots & h_{\lambda_r}\end{vmatrix}[\mathbf{b}]^{r-1}_0 = w^{-r}\Delta_{\boldsymbol{\lambda}}(w^{-\boldsymbol{\lambda}},H_{r-1})[\mathbf{b}]^{r-1}_0,
\end{aligned}$$

from which the desired expression of $\Gamma^*_r(w)\Delta_{\boldsymbol{\lambda}}(H_r)$. ■

4 The $gl_\infty(\mathbb{Z})$ structure of B_r. First description

Let $\mathcal{E}_{ij} := b_i\otimes\beta_j \in gl_\infty(\mathbb{Z})$.

4.1 Proposition.

$$\delta(\mathcal{E}_{ij})[\mathbf{b}]^r_{\boldsymbol{\lambda}} = b_i\wedge(\beta_j\lrcorner[\mathbf{b}]^r_{\boldsymbol{\lambda}}).$$

Proof. It is an easy check, provided one invokes the very definition of the contraction operator as a derivation of degree -1 of the exterior algebra. ■

4.2 Let $\delta(z,w)=\sum_{i,j\geqslant 0}\delta(\mathcal{E}_{ij})z^iw^{-j}$. Then $\delta(z,w)[\mathbf{b}]^r_{\boldsymbol{\lambda}}\in\bigwedge^r M[[z,w^{-1}]$. Define $\mathcal{E}(z,w):B_r\to B_r[[z,w^{-1}]$ through the equality:

$$(\mathcal{E}(z,w)\Delta_{\boldsymbol{\lambda}}(H_r))[\mathbf{b}]^r_0=\delta(z,w)[\mathbf{b}]^r_{\boldsymbol{\lambda}}.$$

4.3 Theorem.

$$\mathcal{E}(z,w)\Delta_{\boldsymbol{\lambda}}(H_r)=\frac{z^{r-1}}{w^{r-1}}\frac{1}{E_r(z)}\overline{\sigma}_-(z)\Delta_{\boldsymbol{\lambda}}(w^{-\boldsymbol{\lambda}},H_r)=\frac{z^{r-1}}{w^{r-1}}\frac{1}{E_r(z)}\Delta_{\boldsymbol{\lambda}}(w^{-\boldsymbol{\lambda}},\overline{\sigma}_-(z)H_r).\tag{21}$$

Proof. Since $\bigwedge^r M$ is a free B_r-module generated by $[\mathbf{b}]^r_0$, it suffices to expresses $\delta(z,w)[\mathbf{b}]^r_{\boldsymbol{\lambda}}$ as a $B_r[[z,w^{-1}]$-multiple of $[\mathbf{b}]^r_0$. One has:

$$\begin{aligned}\delta(z,w)[\mathbf{b}]^r_0 &= \mathbf{b}(z)\wedge(w\boldsymbol{\beta}(w)\lrcorner[\mathbf{b}]^r_{\boldsymbol{\lambda}})\\ &= \sigma_+(z)b_0\wedge(w\cdot w^{-r}\Gamma^*_r(w)\Delta_{\boldsymbol{\lambda}}(H_r)[\mathbf{b}]^{r-1}_0)\\ &= w^{-r+1}\sigma_+(z)(b_0\wedge\overline{\sigma}_+(z)\big(\Gamma^*_r(w)\Delta_{\boldsymbol{\lambda}}(H_r)\big)[\mathbf{b}]^{r-1}_0).\end{aligned}\tag{22}$$

Since $(\Gamma^*_r(w)\Delta_{\boldsymbol{\lambda}}(H_r))[\mathbf{b}]^{r-1}_0$ is a finite linear combination $\sum_{\boldsymbol{\mu}\in\mathcal{P}_{r-1}}a_{\boldsymbol{\mu}}(w)[\mathbf{b}]^{r-1}_{\mu}$:

$$\begin{aligned}b_0\wedge\overline{\sigma}_+(z)\sum_{\boldsymbol{\mu}}a_{\boldsymbol{\mu}}(w)[\mathbf{b}]^{r-1}_{\boldsymbol{\lambda}} &= \sum_{\boldsymbol{\mu}}a_{\boldsymbol{\mu}}(w)(b_0\wedge\overline{\sigma}_+(z)[\mathbf{b}]^{r-1}_{\boldsymbol{\lambda}})\\ &= \sum_{\boldsymbol{\mu}}a_{\boldsymbol{\mu}}(w)\left(z^{r-1}\overline{\sigma}_-(z)[\mathbf{b}]^{r-1}_{\boldsymbol{\mu}+(1^{r-1})}\wedge b_0\right)\\ &= z^{r-1}\overline{\sigma}_-(z)\sum a_{\boldsymbol{\mu}}(w)[\mathbf{b}]^r_{\boldsymbol{\mu}}\\ &= z^{r-1}\overline{\sigma}_-(z)\Delta_{\boldsymbol{\lambda}}(w^{-\boldsymbol{\lambda}},H_r)[\mathbf{b}]^r_0.\end{aligned}\tag{23}$$

Plugging (23) into (22) one finally obtains the equality

$$\delta(z,w)[\mathbf{b}]^r_{\boldsymbol{\lambda}}=\frac{z^{r-1}}{w^{r-1}}\sigma_+(z)(\overline{\sigma}_-(z)\Delta_{\boldsymbol{\lambda}}(w^{-\boldsymbol{\lambda}},H_r)[\mathbf{b}]^r_0.\tag{24}$$

Using the B_r-module structure of $\bigwedge^r M$ over B_r, one may replace $\sigma_+(z)$ by $1/E_r(z)$ in (24), getting:

$$(\mathcal{E}(z,w)\Delta_{\boldsymbol{\lambda}}(H_r))[\mathbf{b}]^r_0\delta(z,w)[\mathbf{b}]^r_{\boldsymbol{\lambda}}=\frac{z^{r-1}}{w^{r-1}}\frac{1}{E_r(z)}(\overline{\sigma}_-(z)\Delta_{\boldsymbol{\lambda}}(w^{-\boldsymbol{\lambda}},H_r)[\mathbf{b}]^r_0,\tag{25}$$

from which, by comparing the coefficients of $[\mathbf{b}]^r_0$ on either side of (25), and using Proposition 3.3, precisely (21). ∎

4.4 Example. Let us compute $\mathcal{E}(z,w)e_2$ in $B_2[[z,w^{-1}]$. Remind that

$$e_2=\Delta_{(1,1)}(H_2):=\begin{vmatrix}h_1 & 1\\ h_2 & h_1\end{vmatrix}$$

and corresponds to the basis element $[\mathbf{b}]^2_{(1,1)}:=b_2\wedge b_1\in\bigwedge^2 M$, In particular we expect that $\mathcal{E}_{ij}e_2=0$ for all $j\notin\{1,2\}$. By applying the recipe:

$$\mathcal{E}(z,w)e_2 \quad = \quad \frac{z}{w}(1+h_1z+h_2z^2+\cdots)\begin{vmatrix}w^{-1} & 1\\ h_2-\dfrac{h_1}{z} & h_1-\dfrac{1}{z}\end{vmatrix}$$

$$
\begin{aligned}
&= \frac{1}{w}\frac{h_1-h_2z}{E_2(z)}+\frac{1}{w^2}\frac{h_1z-1}{E_2(z)}\\
&= \left[\frac{1}{w}(h_1-h_2z)+\frac{1}{w^2}(h_1z-1)\right](1+h_1z+h_2z^2+h_3z^3+\cdots)
\end{aligned}
\tag{26}
$$

So, for instance

$$
\mathcal{E}_{4,2}e_2 = \text{coefficient of } z^4w^{-2} \text{ of } (26)= h_1h_3-h_4=\Delta_{(3,1)}(H_2)=e_1^2e_2-e_2^2.
$$

5 The $gl_\infty(\mathbb{Z})$ structure of B_r. Second description

We now compute an equivalent expression of the generating function $\mathcal{E}(z,w)$ which recalls the shape of the bosonic vertex representation of the Lie algebra $\mathcal{A}_\infty$ of the matrices of infinite size with only finitely many non-zero diagonals [16, Section 5.4] or [14, pp. 946–947].

5.1 Recall that $\mathbf{b}$ and $\boldsymbol{\beta}$ satisfy the Clifford algebras relations:

$$
b_i\wedge(\beta_j\lrcorner)+\beta_j\lrcorner(b_i\wedge)=\delta_{ij}.
$$

Thus

$$
w\left(\mathbf{b}(z)\wedge\boldsymbol{\beta}(w)\lrcorner[\mathbf{b}]^r_{\boldsymbol{\lambda}}+\boldsymbol{\beta}(w)\lrcorner(\mathbf{b}(z)\wedge[\mathbf{b}]^r_{\boldsymbol{\lambda}})\right)=\sum_{i\geqslant 0}\frac{z^i}{w^i}[\mathbf{b}]^r_{\boldsymbol{\lambda}}=i_{w,z}\frac{w}{w-z}[\mathbf{b}]^r_{\boldsymbol{\lambda}},
\tag{27}
$$

where, following [15, p. 18], the $i_{w,z}$ means that we are considering the expansion of $w/(w-z)$ in power series of z/w. We can then compute the $gl_\infty(\mathbb{Z})$-action of B_r by first computing $w\boldsymbol{\beta}(w)\lrcorner(\mathbf{b}(z)\wedge[\mathbf{b}]^r_{\boldsymbol{\lambda}})$ and subtracting it from the right-hand side of (27).

5.2 Lemma. *For all* $i\geqslant 1$*:*

$$
\sigma_-(w)b_{n+i}=\sigma_i\sigma_-(w)b_n+\frac{1}{w^{n+1}}\sigma_-(w)b_{i-1}.
\tag{28}
$$

Proof. In fact

$$
\begin{aligned}
\sigma_-(w)b_{n+i} &= b_{n+i}+\frac{b_{n+i-1}}{w}+\cdots+\frac{b_i}{w^n}+\frac{1}{w^{n+1}}\sigma_-(w)b_{i-1}\\
&= \sigma_i\sigma_-(w)b_n+\frac{1}{w^{n+1}}\sigma_-(w)b_{i-1},
\end{aligned}
$$

as desired. ■

5.3 Lemma. *The following commutation rule holds:*

$$
\sigma_-(w)\sigma_+(z)b_0=i_{w,z}\frac{w}{w-z}\sigma_+(z)\overline{\sigma}_-(w)b_0.
$$

Proof. Indeed

$$
\sigma_-(w)\sigma_+(z)b_0=\sigma_-(w)(b_0+b_1z+b_2z^2+b_3z^3+\cdots)
$$

$$
=b_0+\left(\frac{b_0}{w}+b_1\right)z+\left(\frac{b_0}{w^2}+\frac{b_1}{w}+b_2\right)z^2+\left(\frac{b_0}{w^3}+\frac{b_1}{w^2}+\frac{b_2}{w}+b_3\right)z^3+\cdots
$$

$$= \left(1 + \frac{z}{w} + \frac{z^2}{w^2} + \cdots\right) b_0 + \left(1 + \frac{z}{w} + \frac{z^2}{w^2} + \cdots\right) b_1 z + \left(1 + \frac{z}{w} + \frac{z^2}{w^2} + \cdots\right) b_2 z^2 + \cdots$$

$$= \left(1 + \frac{z}{w} + \frac{z^2}{w^2} + \cdots\right) \sigma_+(z) b_0 = i_{w,z} \frac{w}{w-z} \sigma_+(z)\sigma_-(w) b_0. \qquad \blacksquare$$

5.4 Lemma. *For all* $n \geqslant 0$*:*

$$\sigma_-(w)\sigma_+(z) b_n = \sigma_+(z)\sigma_-(w) b_n + \frac{1}{w^n} i_{w,z} \frac{z}{w-z} \sigma_+(z)\sigma_-(w) b_0. \tag{29}$$

Proof. First we use formula (28):

$$\begin{aligned}
\sigma_-(w)\sigma_+(z) b_n &= \sigma_-(w) b_n + \textstyle\sum_{i\geqslant 1} \sigma_-(w) b_{n+i} z^i && \text{(definition of } \sigma_+(z)\text{)} \\
&= \sum_{i\geqslant 0} \sigma_i \sigma_-(w) b_n z^i + \sum_{i\geqslant 1} \frac{1}{w^{n+1}} \sigma_-(w) b_{i-1} z^i && \text{(by eq. (28))} \\
&= \sigma_+(z)\sigma_-(w) b_n + \frac{z}{w^{n+1}} \sigma_-(w)\sigma_+(z) b_0 && \text{(again definition of } \sigma_+(z)\text{)} \\
&= \sigma_+(z)\sigma_-(w) b_n + \frac{z}{w^{n+1}} i_{w,z} \frac{w}{w-z} \sigma_+(z)\sigma_-(w) b_0 && \text{(Lemma 5.3)} \\
&= \sigma_+(z)\sigma_-(w) b_n + \frac{1}{w^n} i_{w,z} \frac{z}{w-z} \sigma_+(z)\sigma_-(w) b_0. && \text{(simplification)}
\end{aligned}$$

$\blacksquare$

In particular, for $n = 0$:

$$\begin{aligned}
\sigma_-(w)\sigma_+(z) b_0 &= \sigma_+(z)\sigma_-(w) b_0 + i_{w,z} \frac{z}{w-z} \sigma_+(z)\sigma_-(w) b_0 \\
&= i_{w,z} \frac{w}{w-z} \sigma_+(z)\sigma_-(w) b_0.
\end{aligned} \tag{30}$$

5.5 Proposition. *Let* $\boldsymbol{\lambda} \in \mathcal{P}_r$*. Then:*

$$\sigma_-(w)\sigma_+(z) [\mathbf{b}]^{r+1}_{\boldsymbol{\lambda}} = i_{w,z} \frac{w}{w-z} \sigma_+(z)\sigma_-(w) [\mathbf{b}]^{r+1}_{\boldsymbol{\lambda}}.$$

Proof. By using (29) and (30):

$$\begin{aligned}
&\sigma_-(w)\sigma_+(z) [\mathbf{b}]^{r+1}_{\boldsymbol{\lambda}} \\
= \;&\sigma_-(w)\sigma_+(z) \left(b_{r+\lambda_1} \wedge \cdots \wedge b_{\lambda_r} \wedge b_0\right) \\
= \;&\sigma_-(w)\sigma_+(z) b_{r+\lambda_1} \wedge \cdots \wedge \sigma_-(w)\sigma_+(z) b_{1+\lambda_r} \wedge \sigma_-(w)\sigma_+(z) b_0
\end{aligned}$$

$$= \bigwedge_{i=1}^{r} \left(\sigma_+(z)\sigma_-(w)b_{r-i+1+\lambda_i} + \frac{z}{w^{r+1+\lambda_1}} i_{w,z}\frac{w}{w-z}\sigma_+(z)\sigma_-(w)b_0 \right) \wedge i_{w,z}\frac{w}{w-z}\sigma_+(z)\sigma_-(w)b_0$$

$$= \quad i_{w,z}\frac{w}{w-z}\sigma_+(z)\sigma_-(w)[\mathbf{b}]^{r+1}_{\boldsymbol{\lambda}}. \qquad \blacksquare$$

5.6 Lemma. For all $\boldsymbol{\lambda} \in \mathcal{P}_r$:

$$\overline{\sigma}_-(w)(\beta_0 \lrcorner [\mathbf{b}]^{r+1}_{\boldsymbol{\lambda}}) = w^{-r}\overline{\sigma}_+(w)[\mathbf{b}]^{r}_{\boldsymbol{\lambda}}.$$

Proof. If $r = 1$, then

$$\overline{\sigma}_-(w)(\beta_0 \lrcorner b_{1+\lambda} \wedge b_0) = w^{-1}(b_\lambda - b_{\lambda+1}w) = w^{-1}\overline{\sigma}_+(w)b_\lambda$$

and the property holds for $r = 1$. For $r \geqslant 1$ it follows by using the fact that $\overline{\sigma}_-(w)$ is a HS derivation. In fact

$$\overline{\sigma}_-(w)(\beta_0 \lrcorner [\mathbf{b}]^{r+1}_{\boldsymbol{\lambda}}) = \overline{\sigma}_-(w)[\mathbf{b}]^{r}_{\boldsymbol{\lambda}} = [\overline{\sigma}_-(w)\mathbf{b}]^{r}_{\boldsymbol{\lambda}} = w^{-r}\overline{\sigma}_+(z)[\mathbf{b}]^{r}_{\boldsymbol{\lambda}}$$

$\blacksquare$

5.7 Theorem. *Let*

$$\Gamma_r(z,w) := \frac{E_r(w)}{E_r(z)}\sigma_-(w)\overline{\sigma}_-(z). \tag{31}$$

Then

$$\mathcal{E}(z,w) = i_{w,z}\frac{w}{z-w}\left(\frac{z^r}{w^r}\Gamma_r(z,w) - 1\right). \tag{32}$$

Proof. By properly expanding the expression $w\boldsymbol{\beta}(w)\lrcorner(\mathbf{b}(z) \wedge [\mathbf{b}]^{r}_{\boldsymbol{\lambda}})$ and using 3.5 one obtains:

$$\begin{aligned}
w\boldsymbol{\beta}(w)\lrcorner(\mathbf{b}(z) \wedge [\mathbf{b}]^{r}_{\boldsymbol{\lambda}}) &= z^r\sigma_-^T(w)\beta_0\lrcorner\left(\sigma_+(z)\overline{\sigma}_-(z)[\mathbf{b}]^{r+1}_{\boldsymbol{\lambda}}\right) && \text{(expression of } \mathbf{b}(z) \text{ and } \boldsymbol{\beta}(w) \text{ through Schubert derivations)}\\
&= z^r\overline{\sigma}_-(w)\left(\beta_0\lrcorner\sigma_-(w)\sigma_+(z)\overline{\sigma}_-(z)[\mathbf{b}]^{r+1}_{\boldsymbol{\lambda}}\right) && \text{(integration by parts (11))}\\
&= \frac{z^r}{w^r}\overline{\sigma}_+(w)\sigma_-(w)\sigma_+(z)\overline{\sigma}_-(z)[\mathbf{b}]^{r}_{\boldsymbol{\lambda}} && \text{(by Lemma 5.6)}\\
&= \frac{z^r}{w^r}i_{w,z}\frac{w}{w-z}\overline{\sigma}_+(w)\sigma_+(z)\left(\sigma_-(w)\overline{\sigma}_-(z)[\mathbf{b}]^{r}_{\boldsymbol{\lambda}}\right) && \text{(Proposition 5.5)}\\
&= \frac{z^r}{w^r}i_{w,z}\frac{w}{w-z}\frac{E_r(w)}{E_r(z)}\sigma_-(w)\overline{\sigma}_-(z)[\mathbf{b}]^{r}_{\boldsymbol{\lambda}}. && \text{(invoking the } B_r\text{-module structure of } \textstyle\bigwedge^r M)
\end{aligned}$$

Therefore

$$\mathcal{E}(z,w) = i_{w,z}\frac{w}{w-z} - \frac{z^r}{w^r}i_{w,z}\frac{w}{w-z}\frac{E_r(w)}{E_r(z)}\sigma_-(w)\overline{\sigma}_-(z) = i_{w,z}\frac{w}{z-w}\cdot\left(\frac{z^r}{w^r}\Gamma_r(z,w) - 1\right),$$

as desired. $\blacksquare$

5.8 In the ring $B_r \otimes_{\mathbb{Z}} \mathbb{Q}$ one can define the sequence $(x_1, x_2, \ldots)$ related to $(e_1, \ldots, e_r)$ by the relation

$$\exp\left(\sum_{i\geqslant 1} x_i z^i\right) E_r(z) = 1$$

Reference [9, Theorem 7.1] shows that, for $r = \infty$:

$$\overline{\sigma}_-(z) = \exp\left(-\sum_{i\geqslant 1} \frac{1}{iz^i}\frac{\partial}{\partial x_i}\right) \qquad \text{and} \qquad \sigma_-(w) = \exp\left(-\sum_{i\geqslant 1} \frac{1}{iw^i}\frac{\partial}{\partial x_i}\right).$$

Thus:

$$\Gamma_\infty(z,w) = \frac{E_\infty(w)}{E_\infty(z)}\sigma_-(w)\overline{\sigma}_-(z) = \exp\left(\sum_{i\geqslant 1} x_i(z^i - w^i)\right)\exp\left(-\sum_{i\geqslant 1}\frac{1}{i}\left(\frac{1}{z^i} - \frac{1}{w^i}\right)\frac{\partial}{\partial x_i}\right).$$

the classical expression of the vertex operator involved in the DJKM bosonic representation of $\mathfrak{gl}_\infty(\mathbb{Q})$.

6 The $\mathfrak{gl}_n(\mathbb{Z})$ structure of $B_{r,n}$.

6.1 Recall that $M_n := \bigoplus_{0\leqslant j<n} \mathbb{Z}\cdot b_j$. The abelian group $\sigma_n M := \bigoplus_{j\geqslant n} \mathbb{Z}\cdot b_j$ is a sub-module of M and sits into the split exact sequence:

$$0 \to \sigma_n M \to M \to M_n \to 0,$$

so that M_n can be identified with the quotient $M/\sigma_n M$. Similarly, the module $\bigwedge^r M_n$ sits into the bottom exact sequence of the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (h_{n-r+1}, \ldots, h_n) & \hookrightarrow & B_r & \xrightarrow{\pi_{r,n}} & B_{r,n} & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & \bigwedge^{r-1} M \wedge \sigma_n M & \hookrightarrow & \bigwedge^r M & \xrightarrow{\pi_{r,n}} & \bigwedge^r M_n & \longrightarrow & 0 \end{array}$$

whose vertical arrows are multiplication by $[\mathbf{b}]_0^r$ and where abusing notation we have denoted by $\pi_{r,n}$ both the canonical projection $B_r \to B_{r,n}$ and $\bigwedge^r M \to \bigwedge^r M_n$.

6.2 Let $I_{r,n}$ the ideal $(h_{n-r+1}, \ldots, h_n)$. Under the B_r-module structure (12) of $\bigwedge^r M$

$$I_{r,n}[\mathbf{b}]_0^r = \sigma_n M \wedge \bigwedge^{r-1} M. \tag{33}$$

Proof. Indeed, $I_{r,n}[\mathbf{b}]_0^r \subseteq \sigma_n M \wedge \bigwedge^{r-1} M$, because

$$h_{n-r+1+j}[\mathbf{b}]_0^r = b_{n+j} \wedge b_{r-2} \wedge \cdots \wedge b_0 \in \sigma_n M \wedge \bigwedge^{r-1} M,$$

for all $j \geqslant 0$. Conversely, if $[\mathbf{b}]_{\boldsymbol{\lambda}}^r \in \sigma_n M \wedge \bigwedge^{r-1} M$, then $\lambda_1 \geqslant n-r+1$. Since for all $\boldsymbol{\lambda} \in \mathcal{P}_r$ the Schur polynomial $\Delta_{\boldsymbol{\lambda}}(H_r)$ belongs to the ideal generated by the its first column $(h_{\lambda_1}, h_{\lambda_1+1} \ldots, h_{\lambda_1+r_1})$, it follows that, being $\lambda_1 \geqslant n - r + 1$, the inclusion $(h_{\lambda_1}, h_{\lambda_1+1}, \ldots, h_{\lambda_1+r-1}) \subseteq I_{r,n}$ holds, i.e. $[\mathbf{b}]_{\boldsymbol{\lambda}}^r \in I_{r,n}[\mathbf{b}]_0^r$. ∎

6.3 Let

$$\delta(z,w)_n := \sum_{0\leqslant i,j<n} \delta(b_i\otimes\beta_j)z^i w^{-j},$$

and define

$$\mathcal{E}(z,w)_n = \sum_{0\leqslant i,j<n} b_i\otimes\beta_j\cdot z^i w^{-j}.$$

as a map $B_{r,n}\to B_{r,n}[z,w^{-1}]$ through the equality:

$$(\mathcal{E}(z,w)_n\Delta_{\boldsymbol{\lambda}}(H_{r,n}))\left([\mathbf{b}]^r_0+\sigma_n M\wedge\bigwedge^{r-1}M\right)=\delta(z,w)_n\left([\mathbf{b}]^r_{\boldsymbol{\lambda}}+\sigma_n M\wedge\bigwedge^{r-1}M\right). \tag{34}$$

6.4 Theorem. *The* $\mathfrak{gl}_n(\mathbb{Z})$*-module structure of* $B_{r,n}$ *is described by:*

$$\mathcal{E}(z,w)_n\Delta_{\boldsymbol{\lambda}}(H_{r,n})=\frac{z^{r-1}}{w^{r-1}}\pi_{r,n}\left(\frac{1}{E_r(z)}\right)\pi_{r,n}\overline{\sigma}_-(z)\Delta_{\boldsymbol{\lambda}}(w^{-\boldsymbol{\lambda}},H_r), \tag{35}$$

or, more explicitly:

$$\mathcal{E}(z,w)_n\Delta_{\boldsymbol{\lambda}}(H_{r,n})=\frac{z^{r-1}}{w^{r-1}}(1+h_1z+\ldots+h_{n-r}z^{n-r})\pi_{r,n}\overline{\sigma}_-(z)\Delta_{\boldsymbol{\lambda}}(w^{-\boldsymbol{\lambda}},H_r). \tag{36}$$

Proof. Since

$$\delta(z,w)_n\left(\sigma_n M\wedge\bigwedge^{r-1}M\right)\subseteq\sigma_n M\wedge\bigwedge^{r-1}M,$$

as a simple exercise shows, it follows that $\mathcal{E}(z,w)_n I_{r,n}\subseteq I_{r,n}$. Therefore:

$$\begin{aligned}\mathcal{E}(z,w)_n\Delta_{\boldsymbol{\lambda}}(H_{r,n})\left([\mathbf{b}]^r_0+\bigwedge^{r-1}M\wedge\sigma_n M\right) &= \delta(z,w)_n\left([\mathbf{b}]^r_{\boldsymbol{\lambda}}+\sigma_n M\wedge\bigwedge^{r-1}M\right)\\ &= \delta(z,w)[\mathbf{b}]^r_{\boldsymbol{\lambda}}+\bigwedge^{r-1}M\wedge\sigma_n M\\ &= (\mathcal{E}(z,w)\Delta_{\boldsymbol{\lambda}}(H_r))[\mathbf{b}]^r_0+\sigma_n M\wedge\bigwedge^{r-1}M,\end{aligned} \tag{37}$$

i.e., in other words, $\mathcal{E}(z,w)_n\Delta_{\boldsymbol{\lambda}}(H_{r,n})=\pi_{r,n}\left(\mathcal{E}(z,w)\Delta_{\boldsymbol{\lambda}}(H_r)\right)$. Using Theorem 4.3 and the fact that $\pi_{r,n}$ is a epimorphism, one finally obtains (35). Equality (36) follows from noticing that

$$\pi_{r,n}\left(\frac{1}{E_r(z)}\right)=\pi_{r,n}\left(\sum_{i\geqslant 0}h_iz^i\right)=1+h_1z+\cdots+h_{n-r}z^{n-r}.$$

∎

6.5 Remark. It is important to notice that $\pi_{r,n}\circ\overline{\sigma}_-(z)\neq\overline{\sigma}_-(z)\circ\pi_{r,n}$ and that

$$\pi_{r,n}\overline{\sigma}_-(z)\Delta_{\boldsymbol{\lambda}}(w^{-\boldsymbol{\lambda}},H_r)=\overline{\sigma}_-(z)\Delta_{\boldsymbol{\lambda}}(w^{-\boldsymbol{\lambda}},\pi_{r,n}H_r)=\overline{\sigma}_-(z)\Delta_{\boldsymbol{\lambda}}(w^{-\boldsymbol{\lambda}},H_{r,n})$$

only if $n>r-1+\lambda_1$. Thus formula (35) is already in its best possible shape.

6.6 Example. Let us evaluate $\mathcal{E}(z,w)_4\Delta_{(2,2)}(H_{2,4}) \in B_{2,4}[z,w^{-1}]$, where $B_{2,4}$ is th cohomology (or Chow) ring of the Grassmannian $G(2,4)$. Recall that in this case $h_i = c_i(\mathcal{Q}_2)$, the i-th Chern class of the universal quotient bundle over it. According to the recipe, we first compute

$$\sigma_-(z)\Delta_{(2,2)}(w^{-(2,2)}, H_2) = \begin{vmatrix} w^{-2} & w^{-1} \\ h_3 - \dfrac{h_2}{z} & h_2 - \dfrac{h_1}{z} \end{vmatrix}.$$

Projecting ont $B_{2,4}$ via $\pi_{2,4}$ amounts to set h_3 to 0. Then we muliply by $\pi_{2,4}(1/E_2(z)) = 1 + h_1 z + h_2 z^2$ and by z/w. Eventually one obtains:

$$\begin{aligned} \mathcal{E}(z,w)_4\Delta_{(2,2)}(H_{2,4}) &= \frac{z}{w}(1+h_1z+h_2z^2)\begin{vmatrix} w^{-2} & w^{-1} \\ -\dfrac{h_2}{z} & h_2 - \dfrac{h_1}{z} \end{vmatrix} = \\ &= h_2\frac{1}{w^2} + h_1h_2\frac{z}{w^2} + h_2^2\frac{z^2}{w^2} - h_1\frac{1}{w^3} - (h_1^2 - h_2)\frac{z}{w^3} + h_2^2\frac{z^2}{w^3}. \end{aligned} \tag{38}$$

So, for instance,

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Delta_{(2,2)}(H_{2,4}) = \mathcal{E}_{1,2}\Delta_{(2,2)}(H_{2,4}) = h_1h_2$$

6.7 Example. In 4.4 we have computed

$$\mathcal{E}_{4,2}e_2 = h_1h_3 - h_4.$$

This is zero in $B_{2,4} := B_2/(h_3, h_4)$. Indeed

$$\begin{aligned} \mathcal{E}(z,w)_2e_2 &= \left[\frac{1}{w}(h_1 - h_2z) + \frac{1}{w^2}(h_1z - 1)\right](1 + h_1z + h_2z^2) \\ &= [e_1 + e_2z + (2e_1^2e_2 - e_1^4 - e_2^2z^3]w^{-1} + [-1 + e_2z^2 + (e_1^3 - e_1e_2)z^3]w^{-2}. \end{aligned}$$

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