The biharmonic mean

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# What do Biharmonic mean?

Marco Abrate, Stefano Barbero, Umberto Cerruti, Nadir Murru
University of Turin, Department of Mathematics
Via Carlo Alberto 10, Turin
Italy
marco.abrate@unito.it, stefano.barbero@unito.it
umberto.cerruti@unito.it, nadir.murru@unito.it

**Abstract**

In this paper we describe some well–known means and their properties, focusing on the relationship with integer sequences. In particular, the harmonic numbers, deriving from the harmonic mean, motivate the definition of a new kind of mean that we call *biharmonic mean*. Biharmonic mean allows us to introduce the biharmonic numbers and to provide a new characterization of primes. Biharmonic numbers appear to be very interesting for some properties of divisibility which have been studied in our paper by means of linear recurrent sequences and diophantine equations.

## 1 Introduction

The need to explore Nature and establish from direct observation its rules, encouraged ancient thinkers in finding appropriate mathematical tools, able to extrapolate numerical data. One of the oldest method used to combine observations in order to give an unique approximate value is the *arithmetic mean*. It has been probably used for the first time in the third century B.C., by ancient Babylonian astronomers, to determine the positions of celestial bodies. The mathematical concept of arithmetic mean was first enhanced by the greek astronomer Hipparchus (190–120 B.C.) and some other greek mathematicians, following the Pythagoric ideals, have also introduced or rigorously defined further kinds of means. For example Archytas (428–360 B.C.) named the *harmonic mean* and used it in his theory of music and in an algorithm for doubling the cube. His disciple Eudoxus (408–355 B.C.) introduced the *contraharmonic mean* while he was developing the studies on proportions. So in parallel with the practical use of many numerical means in various sciences, a deep exploration of their arithmetic and geometric properties took place during the centuries. The book of Bullen [2] is a classical reference for a good survey about means and their history.

In this paper we especially focus our attention to some interesting arithmetic aspects related to the most used means in many fields of Mathematics. We also define a new kind of mean, showing how it also allows us to give a new characterization of prime numbers.

We start recalling the classical definitions and properties of the involved means.

**Definition 1.** *Let $a_1, a_2, \dots, a_t$ be $t$ positive real numbers, then we define their*

- arithmetic mean

$$\mathcal{A}(a_1, a_2, \dots, a_t) = \frac{a_1 + a_2 + \dots + a_t}{t}; \tag{1}$$

- geometric mean

$$\mathcal{G}(a_1, a_2, \dots, a_t) = \sqrt[t]{a_1 a_2 \cdots a_t}; \tag{2}$$

- harmonic mean

$$\mathcal{H}(a_1, a_2, \dots, a_t) = \left(\frac{1}{t}\left(\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_t}\right)\right)^{-1}; \tag{3}$$

- contraharmonic mean

$$\mathcal{C}(a_1, a_2, \dots, a_t) = \frac{a_1^2 + a_2^2 + \dots + a_t^2}{a_1 + a_2 + \dots + a_t}. \tag{4}$$

We have the well–known inequalities

$$\mathcal{H}(a_1, a_2, \dots, a_t) \le \mathcal{G}(a_1, a_2, \dots, a_t) \le \mathcal{A}(a_1, a_2, \dots, a_t) \le \mathcal{C}(a_1, a_2, \dots, a_t)$$

and only if we consider two positive real numbers $a$ and $b$, we always have the following beautiful relations:

$$\mathcal{A}(a,b) = \mathcal{A}(\mathcal{H}(a,b), \mathcal{C}(a,b)), \tag{5}$$

$$\mathcal{G}(a,b) = \mathcal{G}(\mathcal{H}(a,b), \mathcal{A}(a,b)). \tag{6}$$

A very interesting question is to determine whether at least one of these equalities holds for some choice of more than two positive real numbers. In 1948 Oystein Ore, ([4], [5]), introduced the idea of *harmonic number* finding some related properties and a quite surprising answer to this question. He applied the four means to all the divisors of a positive integer $n$. If we indicate the set of divisors of $n$ as

$$D(n) = \{d_1, d_2, \ldots, d_t\}$$

and for the sake of simplicity we pose

$$\begin{aligned} A(n) &= \mathcal{A}(d_1, d_2, \ldots, d_t), \\ G(n) &= \mathcal{G}(d_1, d_2, \ldots, d_t), \\ H(n) &= \mathcal{H}(d_1, d_2, \ldots, d_t), \\ C(n) &= \mathcal{C}(d_1, d_2, \ldots, d_t), \end{aligned}$$

recalling that the divisor function is

$$\sigma_x(n) = \sum_{i=1}^{t} d_i^x, \quad x \in \mathbb{N} \tag{7}$$

the first immediate result of Ore can be summarized in the following theorem.

**Theorem 1.**

$$A(n) = \frac{\sigma_1(n)}{\sigma_0(n)}, \tag{8}$$

$$G(n) = \sqrt{n}, \tag{9}$$

$$H(n) = \frac{n\sigma_0(n)}{\sigma_1(n)}, \tag{10}$$

$$C(n) = \frac{\sigma_2(n)}{\sigma_1(n)}. \tag{11}$$

*Proof.* We give a proof only for equalities (10) and (9) since (8) and (11) clearly arise from (1) and (4) respectively. From equation (3) clearly

$$H(n) = \left(\frac{1}{t}\left(\frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_t}\right)\right)^{-1}$$

and the sum $\frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_t}$, when reduced to the least common denominator $n$, gives as numerator $\sigma_1(n)$ (the sum of all the divisors of $n$). Moreover, since $t = \sigma_0(n)$ (the number of all the divisors of $n$), relation (10) easily follows. Now, to prove equation (9), we start from (2)

$$G(n) = \sqrt[t]{d_1 d_2 \cdots d_t} = \sqrt[\sigma_0(n)]{\prod_{i=1}^{\sigma_0(n)} d_i}$$

distinguishing the two cases $n = m^2$ or $n \neq m^2$. When $n \neq m^2$, $n$ has an even number of divisors, so we multiply $d_i$ with $\frac{n}{d_i}$ for all $i = 1, \ldots, \frac{\sigma_0(n)}{2}$, finding $\prod_{i=1}^{\sigma_0(n)} d_i = n^{\frac{\sigma_0(n)}{2}}$ and we are done. On the other hand if $n = m^2$, then $t = \sigma_0(n)$ is odd. As before, we multiply $d_i$ with $\frac{n}{d_i}$, but in this case we can do this only for every $d_i \neq m$, finding

$$\prod_{i=1}^{\sigma_0(n)} d_i = n^{\frac{\sigma_0(n)-1}{2}} m = (m^2)^{\frac{\sigma_0(n)-1}{2}} m = m^{\sigma_0(n)}.$$

Thus

$$G(n) = \sqrt[\sigma_0(n)]{\prod_{i=1}^{\sigma_0(n)} d_i} = \sqrt[\sigma_0(n)]{m^{\sigma_0(n)}} = m = \sqrt{n}.$$

□

A straightforward consequence of these results, observed by Ore in [4], shows how an equality similar to (6) holds if we take into account the elements of $D(n)$. We have

**Corollary 1.** *For any positive integer $n$*

$$G(n) = \mathcal{G}(H(n), A(n)). \tag{12}$$

*Proof.* From what we stated in the previous theorem, we clearly obtain

$$\mathcal{G}(H(n), A(n)) = \sqrt{H(n) \cdot A(n)} = \sqrt{\frac{n\sigma_0(n)}{\sigma_1(n)} \frac{\sigma_1(n)}{\sigma_0(n)}} = \sqrt{n} = G(n).$$

□

Before considering another fashinating question, we want to point out that equality (12) can be interpreted also as a formal identity. For example when $n = p^2 q$, where $p$ and $q$ are two primes, $D(n) = \{1, p, q, pq, p^2, p^2 q\}$ and we have from (12)

$$\mathcal{G}(1, p, q, pq, p^2, p^2 q) = \mathcal{G}(\mathcal{H}(1, p, q, pq, p^2, p^2 q), \mathcal{A}(1, p, q, pq, p^2, p^2 q)). \quad (13)$$

The astonishing fact is that this equality also holds if we substitute $p$ and $q$ with any other couple of positive *real* numbers. For example if we use in (13) $\sqrt{2}$ and $\pi$ instead of $p$ and $q$ respectively, we obtain the identity

$$\mathcal{G}(1, \sqrt{2}, \pi, \sqrt{2}\,\pi, 2, 2\pi) = \mathcal{G}(\mathcal{H}(1, \sqrt{2}, \pi, \sqrt{2}\,\pi, 2, 2\pi), \mathcal{A}(1, \sqrt{2}, \pi, \sqrt{2}\,\pi, 2, 2\pi)).$$

which is not so immediate. In general, as we have previously observed, for a set of $t$ positive distinct real numbers $a_1, \ldots, a_t$ randomly choosed, the equality

$$\mathcal{G}(a_1, a_2, \ldots, a_t) = \mathcal{G}(\mathcal{H}(a_1, a_2, \ldots, a_t), \mathcal{A}(a_1, a_2, \ldots, a_t)),$$

is false.

The second question is pretty natural: when do the means applied to the divisors of an integer $n$ also give a result which is an integer? The case of $G(n)$ is not so interesting because $G(n)$ is an integer if and only if $n$ is a square.

All the integers $n$ such that $A(n)$ is also an integer are the terms of the sequence A003601 in OEIS [6]:

$$1, 3, 5, 6, 7, 11, 13, 14, 15, 17, 19, 20, 21, 22, 23, 27, 29, 30, 31, 33, 35, 37, \ldots \quad (14)$$

These integers are the so–called *arithmetic numbers*.
Moreover, every integer $n$ giving an integer value for $C(n)$ belongs to the sequence A020487:

$$1, 4, 9, 16, 20, 25, 36, 49, 50, 64, 81, 100, 117, 121, 144, 169, 180, 196, 200, 225, \ldots.$$

Maybe the most important case is related to the harmonic mean. Ore provided the following definition.

**Definition 2.** *A positive integer $n$ is an* harmonic (divisor) number *(or* Ore number*) if $H(n)$ is an integer.*

The first harmonic numbers are

$$1, 6, 28, 140, 270, 496, 672, 1638, 2970, 6200, 8128, 8190, 18600, 18620, 27846, 30240, \ldots$$

and they form the sequence $A001599$. The corresponding values for $H(n)$ are listed in $A001600$

$$1, 2, 3, 5, 6, 5, 8, 9, 11, 10, 7, 15, 15, 14, 17, 24, \ldots.$$

Ore also proved that all perfect numbers are harmonic numbers.

In the next section, moving from the beautiful properties shown before, we define a new kind of mean that we will call *biharmonic mean*. Starting from the biharmonic mean we will define the biharmonic numbers, which appear to be very interesting since they allow us to provide a new characterization for prime numbers. Moreover, the investigation of composite biharmonic numbers will lead to the study of interesting divisibility properties, involving consecutive terms of certain linear recurrent sequences.

## 2 Biharmonic mean and biharmonic numbers

Here we introduce a new mean: the *biharmonic mean*.

**Definition 3.** *We call the* biharmonic mean *of the $t$ positive real numbers $a_1, a_2, \ldots, a_t$, the positive real number*

$$\mathcal{B}(a_1, a_2, \ldots, a_t) = \mathcal{A}(\mathcal{H}(a_1, \ldots, a_t), \mathcal{C}(a_1, \ldots, a_t)) = \frac{\mathcal{H}(a_1, a_2, \ldots, a_t) + \mathcal{C}(a_1, a_2, \ldots, a_t)}{2}$$

*corresponding to the arithmetic mean of the harmonic and contraharmonic means evaluated for all the $a_i$.*

From Eq. (5) we know that the biharmonic mean is clearly equal to the arithmetic mean when $t = 2$. But we also know that this fact is generally false when $t > 2$. Following Ore's idea we define the biharmonic numbers by means of an analogous function to $H(n)$.

**Definition 4.** *Let us consider a positive integer $n$ with $D(n) = \{d_1, d_2, \ldots, d_t\}$. We define $B(n)$ as the biharmonic mean of the divisors of $n$*

$$B(n) = \mathcal{B}(d_1, d_2, \ldots, d_t).$$

*We call an integer $n$ biharmonic number if $B(n)$ is an integer.*

From this definition we clearly have

$$B(n) = \mathcal{B}(d_1, d_2, \ldots, d_t) = \frac{\mathcal{H}(d_1, d_2, \ldots, d_t) + \mathcal{C}(d_1, d_2, \ldots, d_t)}{2} = \frac{H(n) + C(n)}{2},$$

and, thanks to Theorem 1, we can easily find a closed form for $B(n)$

$$B(n) = \frac{H(n) + C(n)}{2} = \frac{\frac{n\sigma_0(n)}{\sigma_1(n)} + \frac{\sigma_2(n)}{\sigma_1(n)}}{2} = \frac{n\sigma_0(n) + \sigma_2(n)}{2\sigma_1(n)}. \tag{15}$$

Investigating the occurrence of biharmonic numbers among positive integers, we initially find the sequence

$$1, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 35, 37, 41, 43, 47, 53, 59, 61, \ldots$$

which is , except from 1 and 35, very similar to the sequence of prime numbers. This is not so strange, because if $n$ is prime $B(n) = A(n)$.

**Theorem 2.** *Every odd prime number $p$ is a biharmonic number and $B(p) = \dfrac{p+1}{2}$*

*Proof.* If $p$ is an odd prime we have $\sigma_0(p) = 2, \sigma_1(p) = 1 + p, \sigma_2(p) = 1 + p^2$, thus

$$B(p) = \frac{2p + 1 + p^2}{2(1 + p)} = \frac{p + 1}{2}$$

□

But the very interesting fact is that we can *characterize* odd prime numbers using $B(n)$, because we now will prove that the converse of the previous theorem is also true.

**Theorem 3.** *An odd integer $n \neq 1$ such that $B(n) = \frac{n+1}{2}$ is a prime.*

*Proof.* First of all we observe that the equality $B(n) = \frac{n+1}{2}$ corresponds to

$$(n+1)\sigma_1(n) - (\sigma_2(n) + n\sigma_0(n)) = 0. \tag{16}$$

Let us consider two possible cases: $n = k^2$ or $n \neq k^2$. When $n = k^2$ with $k \neq 1$, we have $\sigma_0(n) = 2m + 1$ for some $m > 0$ and

$$D(n) = \{d_1, d_2, ..., d_m, d_{m+1}, d_{m+2}, ..., d_{2m}, k\},$$

where we pose $d_i = \frac{n}{d_{m+i}}$ for $i = 1, ..., m$.
Clearly by definition

$$\sigma_1(n) = \sum_{i=1}^{2m} d_i + k \tag{17}$$

and

$$\sigma_2(n) = \sum_{i=1}^{2m} d_i^2 + k^2. \tag{18}$$

We find from (17) that

$$(n+1)\sigma_1(n) = \sum_{i=1}^{2m} nd_i + nk + \sum_{i=1}^{2m} d_i + k = \sum_{i=1}^{m} d_i^2 d_{m+i} + \sum_{i=1}^{m} d_i d_{m+i}^2 + nk + \sum_{i=1}^{2m} d_i + k$$

Rearranging all the terms in a suitable way and remembering that $n = k^2$, we obtain

$$(n+1)\sigma_1(n) = \sum_{i=1}^{m} (d_i + d_{m+i})(d_i d_{m+i} + 1) + k^3 + k. \tag{19}$$

On the other hand from (18) the following relation holds

$$\begin{aligned} \sigma_2(n) + n\sigma_0(n) &= \sum_{i=1}^{2m} d_i^2 + k^2 + (2m+1)n = \sum_{i=1}^{2m} d_i^2 + \sum_{i=1}^{2m} d_i \frac{n}{d_i} + 2k^2 = \\ &= \sum_{i=1}^{2m} d_i^2 + 2\sum_{i=1}^{m} d_i d_{m+i} + 2k^2 = \sum_{i=1}^{m} (d_i + d_{m+i})^2 + 2k^2. \end{aligned} \tag{20}$$

Now using (19) and (20) we have

$$(n+1)\sigma_1(n) - (\sigma_2(n) + n\sigma_0(n)) = \sum_{i=1}^{m} (d_i + d_{m+i})(d_i d_{m+i} + 1) + k^3 + k - 2k^2 - \sum_{i=1}^{m} (d_i + d_{m+i})^2,$$

and we finally get

$$(n+1)\sigma_1(n)-(\sigma_2(n)+n\sigma_0(n))=\sum_{i=1}^{m}(d_i+d_{m+i})(d_i-1)(d_{m+i}-1)+k(k-1)^2. \tag{21}$$

This equality tell us that in this case the first member of (16) will never be equal to 0 if $n \neq 1$.
If $n \neq k^2$, $\sigma_0(n) = 2m$ for some $m \geq 1$. With calculations similar to those we have done before, we easily find that

$$(n+1)\sigma_1(n)-(\sigma_2(n)+n\sigma_0(n))=\sum_{i=1}^{m}(d_i+d_{m+i})(d_i-1)(d_{m+i}-1). \tag{22}$$

When $m > 1$ the only summand equal to 0 corresponds to the couple of trivial divisors 1 and $n$, thus equality (16) occurs only when $\sigma_0(n) = 2$, or, in other words, if $n$ is prime. □

Clearly, it is interesting to study properties of composite biharmonic numbers. The non–prime biharmonic numbers most similar to prime numbers are the semiprime biharmonic numbers, i.e., numbers $n$ such that $n = pq$, for $p, q$ primes, and $B(n) \in \mathbb{N}$. In this case, we have

$$B(n) = B(pq) = \frac{(p+q)^2+(pq+1)^2}{2(p+1)(q+1)}.$$

Semiprime biharmonic numbers belong to a more wide set of integers that we will call *crystals* for their beautiful properties. Let us consider the following function of integers:

$$\mathbf{B}(a,b) = \frac{(a+b)^2+(ab+1)^2}{2(a+1)(b+1)}. \tag{23}$$

**Definition 5.** *An odd number $n$ is a* crystal *if and only if $n = ab$, with $a, b > 1$ and $\mathbf{B}(a,b) \in \mathbb{N}$.*

In the following section, we determine all the crystals by means of a particular linear recurrent sequence.

## 3 Divisibility properties

In this section we will characterize all the pairs of odd integers $a, b$ such that $\mathbf{B}(a, b) \in \mathbb{N}$ by using recurrent sequences and integer points over certain conics.
First of all, we highlight that $\mathbf{B}(a, b) \in \mathbb{N}$ is equivalent to different divisibility properties involving the numbers $a, b$.

**Proposition 1.** *Given two integer odd numbers $a, b$, the following statements are equivalent*

1. $\mathbf{B}(a, b) \in \mathbb{N}$
2. $\mathbf{F}(a, b) = \dfrac{(ab+1)^2}{(a+1)(b+1)} \in \mathbb{N}$
3. $\mathbf{P}(a, b) = \dfrac{(a+b)(ab+1)}{(a+1)(b+1)} \in \mathbb{N}$
4. $\mathbf{Q}(a, b) = \dfrac{(a+b)^2}{(a+1)(b+1)} \in \mathbb{N}$

*Proof.*
Since

$$\mathbf{B}(a, b)+\mathbf{P}(a, b) = \frac{(a+1)(b+1)}{2}, \quad \mathbf{P}(a, b)+\mathbf{Q}(a, b) = a+b, \quad \mathbf{F}(a, b)+\mathbf{P}(a, b) = ab+1,$$

we clearly have

$$\mathbf{B}(a, b) \in \mathbb{N} \Leftrightarrow \mathbf{P}(a, b) \in \mathbb{N}, \quad \mathbf{P}(a, b) \in \mathbb{N} \Leftrightarrow \mathbf{Q}(a, b) \in \mathbb{N}, \quad \mathbf{F}(a, b) \in \mathbb{N} \Leftrightarrow \mathbf{P}(a, b) \in \mathbb{N}.$$

□

In order to characterize the crystals we need some preliminar result about the diophantine equation

$$(x + y - 1)^2 - wxy = 0$$

with $x, y$ unknown and $w \in \mathbb{N}$ a given parameter. This equation has been solved in positive integers by the authors in [1], using a particular recurrent sequence. The authors proved the following

**Theorem 4.** *The pair $(x, y)$ is a positive integer solution of the diophantine equation*

$$(x+y-1)^2 - wxy = 0,$$

*with $w \in \mathbb{N}$, if and only if $(x, y) = (u_n(w), u_{n-1}(w))$ for a given index $n \geq 1$, where $(u_n(w))_{n=0}^{+\infty}$ is the sequence defined by*

$$\begin{cases} u_0(w) = 0, \quad u_1(w) = 1 \\ u_{n+1}(w) = (w-2)u_n(w) - u_{n-1}(w) + 2, \quad \forall n \geq 2 \end{cases} .$$

When there will be no possibility of confusion, we will omit in the next the dependence from $w$. The sequence $(u_n)_{n=0}^{+\infty}$ can be written as a linear recurrent sequence of order 3:

$$\begin{cases} u_0 = 0, \quad u_1 = 1, \quad u_2 = w, \\ u_{n+2} = (w-1)u_{n+1} - (w-1)u_n + u_{n-1}, \quad \forall n > 2 \end{cases} . \tag{24}$$

Indeed, in general if $(p_n)_{n=0}^{+\infty}$ is a linear recurrent sequence of order $m$ with characteristic polynomial $f(t) = t^m - \sum_{h=1}^{m} f_h t^{m-h}$ and initial conditions $p_0, ..., p_{m-1}$, then the sequence $(q_n)_{n=0}^{+\infty}$ satisfying the following recurrence

$$q_n = \sum_{h=1}^{m} f_h q_{n-h} + k$$

and initial conditions $p_0, ..., p_{m-1}$ is a linear recurrent sequence of degree $m+1$ with characteristic polynomial $(x-1)f(x)$ and initial conditions $p_0, ..., p_{m-1}, p_m + k$ (see, e.g., [3]). Thus, from

$$x^3 - (w-1)x^2 + (w-1)x - 1 = (x-1)(x^2 - (w-2)x + 1)$$

we have that the sequence $(u_n)_{n=0}^{+\infty}$ satisfies the recurrence (24). This sequence is related to the linear recurrent sequence $(a_n(w))_{n=0}^{+\infty}$ defined by

$$\begin{cases} a_0(w) = 0, \quad a_1(w) = 1, \\ a_n(w) = \sqrt{w}a_{n-1}(w) - a_{n-2}(w), \quad \forall n \geq 2 \end{cases} . \tag{25}$$

The relation between sequences $(u_n)_{n=0}^{+\infty}$ and $(a_n)_{n=0}^{+\infty}$ is determined by the following function

$$\theta(x) = 2x^2 - 1 \quad \forall x \in \mathbb{R},$$

and stated in the following proposition.

**Proposition 2.** *For any index $n$ we have*

$$\theta(a_n) = 2u_n - 1.$$

*Proof.* The sequence $(a_n)_{n=0}^{+\infty}$ recurs with characteristic polynomial $x^2 - \sqrt{w}x + 1$ whose companion matrix is

$$F = \begin{pmatrix} 0 & 1 \\ -1 & \sqrt{w} \end{pmatrix}.$$

By definition of $\theta$ we have $\theta(a_n) = 2a_n^2 - 1$, $\forall n \geq 0$ and the sequence $(\theta(a_n))_{n=0}^{+\infty}$ is a linear recurrent sequence whose characteristic polynomial is the characteristic polynomial of the matrix $F \otimes F$, where $\otimes$ is the Kronecker product (see [3]). In this case, we have

$$F \otimes F = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & \sqrt{w} \\ 0 & -1 & 0 & \sqrt{w} \\ 1 & -\sqrt{w} & -\sqrt{w} & w \end{pmatrix}$$

whose characteristic polynomial is

$$(x-1)^2(x^2 - (w-2)x + 1).$$

Thus the minimal polynomial whereby $(\theta(a_n))_{n=0}^{+\infty}$ recurs is the same characteristic polynomial of the sequence $(u_n)_{n=0}^{+\infty}$. Finally, observing that

$$\theta(a_0) = 2a_0^2 - 1 = 2u_0 - 1 = 0, \quad \theta(a_1) = 2a_1^2 - 1 = 2u_1 - 1 = 1,$$

$$\theta(a_2) = 2a_2^2 - 1 = 2u_2 - 1 = 2w - 1,$$

we have the thesis. ∎

Moreover, two consecutive elements of the sequence $(a_n)_{n=0}^{+\infty}$ corresponds to a point belonging to the conic

$$C(w) = \{(x, y) \in \mathbb{R} : x^2 + y^2 - \sqrt{w}xy = 1\}$$

with $w \in \mathbb{N}$.

**Proposition 3.** *For any integer $n > 0$, we have*

$$(a_n, a_{n-1}) \in C(w)$$

*Proof.* In the proof of the previous proposition, we have observed that $F$ is the companion matrix of the characteristic polynomial of the sequence $(a_n)_{n=0}^{+\infty}$. Thus, we have

$$F^n = \begin{pmatrix} -a_{n-1} & a_n \\ -a_n & a_{n+1} \end{pmatrix}.$$

Since $\det(F) = 1$, we have $\det(F^n) = 1$, i.e.,

$$a_n^2 - a_{n-1}a_{n+1} = 1$$

and by Eqs. (25)

$$a_n^2 - a_{n-1}(\sqrt{w}a_n - a_{n-1}) = a_n^2 + a_{n-1}^2 - \sqrt{w}a_n a_{n-1} = 1.$$

$\square$

In the following proposition, we highlight the relation between points over the conic $C(w)$ and points over the conic

$$C_2(w) = \{(x, y) \in \mathbb{R} : (x + y - 1)^2 = wxy\}.$$

**Proposition 4.** *Let $C(w), C_2(w), C_3(w)$ be the following conics*

$$C(w) = \{(x, y) \in \mathbb{R} : x^2+y^2-\sqrt{w}xy = 1\}, \quad C_2(w) = \{(x, y) \in \mathbb{R} : (x+y-1)^2 = wxy\},$$

$$C_3(w) = \{(x, y) \in \mathbb{R} : (x + y)^2 = w(x + 1)(y + 1)\},$$

*with $w \in \mathbb{N}$. For any pair of positive real numbers $x, y \in \mathbb{R}^+$ we have*

$$(x, y) \in C(w) \Leftrightarrow (\theta(x), \theta(y)) \in C_3(w)$$

$$(x, y) \in C(w) \Leftrightarrow (x^2, y^2) \in C_2(w)$$

$$(x, y) \in C_2(w) \Leftrightarrow (2x - 1, 2y - 1) \in C_3(w).$$

*Proof.* Remembering that

$$\mathbf{Q}(a, b) = \frac{(a + b)^2}{(a + 1)(b + 1)},$$

we obtain

$$\mathbf{Q}(\theta(x), \theta(y)) = \frac{(x^2 + y^2 - 1)^2}{x^2y^2}$$

and

$$\mathbf{Q}(\theta(x), \theta(y)) = w \Leftrightarrow \frac{(x^2+y^2-1)^2}{x^2y^2} = w.$$

Now, we get

$$\frac{(x^2+y^2-1)^2}{x^2y^2} = w \Leftrightarrow \frac{x^2+y^2-1}{xy} = \sqrt{w}$$

and finally

$$\mathbf{Q}(\theta(x), \theta(y)) = w \Leftrightarrow x^2+y^2-\sqrt{w}xy = 1.$$

Moreover, we have

$$(x, y) \in C(w) \Leftrightarrow x^2+y^2-1 = \sqrt{w}xy$$

and squaring both members

$$(x, y) \in C(w) \Leftrightarrow (x^2+y^2-1)^2 = wx^2y^2 \Leftrightarrow (x^2, y^2) \in C_2(w).$$

Finally, $(x, y) \in C_2(w) \Leftrightarrow (2x-1, 2y-1) \in C_3(w)$ since

$$\mathbf{Q}(2x-1, 2y-1) = \frac{(x+y-1)^2}{xy}.$$

□

Now, we are ready to classify all the crystals in the following theorem.

**Theorem 5.** *An odd number $N = ab$ is a crystal if and only if there exist $w, n \in \mathbb{N}$ such that*

$$a = \theta(a_n), \quad b = \theta(a_{n-1}).$$

*Proof.* 1. "$\Leftarrow$"

If $a = \theta(a_n)$ and $b = \theta(a_{n-1})$ (i.e., $\theta(a_n)$ and $\theta(a_{n-1})$ are odd positive integers), then by Proposition 3 $(a, b) \in C(w)$ and by previous proposition we have $\mathbf{Q}(a, b) = w$, so $N = ab$ is a crystal.

2. "$\Rightarrow$"

Let $N = ab$ be a crystal. By Propostion 1 there exists a positive integer $w$ such that $\mathbf{Q}(a, b) = w$ and by Proposition 4 we know that

$$\left(\frac{a+1}{2}, \frac{b+1}{2}\right) \in C_2(w).$$

Thus, by Theorem 4 there exists an index $n$ such that

$$\left(\frac{a+1}{2}, \frac{b+1}{2}\right) = (u_n, u_{n-1})$$

and finally by Proposition 2 we obtain

$$a = \theta(a_n), \quad b = \theta(a_{n-1}).$$

$\square$

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