On the extendability of parallel sections of linear connections

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Abstract

Let $\pi : E \to M$ be a vector bundle over a simply connected manifold, ∇ a linear connection in π, and $\sigma : U \to E$ a ∇-parallel section of π defined on a connected open subset U of M. We give sufficient conditions on U in order to extend σ to the whole of M. We mainly concentrate on the case when M is a 2-dimensional simply connected manifold.

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1 Introduction

Many interesting problems in Differential Geometry can be formulated in terms of the existence of *parallel sections* of suitably defined linear connections in certain vector bundles. For example, if (M, g) is a pseudo-Riemannian manifold, the existence of a Killing vector field turns out to be equivalent to the existence of a parallel section of the Kostant connection $\widetilde{\nabla}$ in the vector bundle $TM \oplus \mathfrak{so}(TM)$ over M (which is canonically isomorphic to $TM \oplus \Lambda^2 M$ via the metric) defined as follows (see also [4, 11, 17]):

$$\widetilde{\nabla}_X(Y, \mathcal{A}) := \Big(\nabla_X Y - \mathcal{A}(X),\, \nabla_X \mathcal{A} - \mathrm{R}(X, Y)\Big), \tag{1}$$

where X is a vector field on M, ∇_X is the covariant derivative associated with the Levi–Civita connection of (M, g), and R the curvature.

Similarly, a projective vector field, i.e., a vector field which preserves the geodesics, understood as unparametrized curves, is a parallel section of a suitable linear connection in the adjoint *tractor bundle* associated with the so-called projective structure [7] (the general theory behind is developed in [8], where the projective case is briefly discussed). More interesting examples can be found in the context of tractor calculus. For instance, the property of (M, g) of being conformally flat is equivalent to the flatness of the tractor connection discussed in [3]. In a broader perspective, the existence of an Einstein metric in the conformal class $[g]$ amounts to the possibility of finding a parallel section of the tractor connection [15].

The examples above represent the main contexts where the existence of a parallel section plays a prominent role. Such an existence can be sometimes established only on an open and dense subset $U \subset M$, so that it is natural to ask under which conditions the parallel section can be made global. Standard examples (see Section 2) show

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that, in general, these conditions are non-trivial, and suitable assumptions on U and M must be made. Even more common are the circumstances where the existence of a parallel section can be proved in the neighborhood of almost every point of M, naturally leading to the problem of gluing such sections into a global one. Such kind of situation must be faced, for instance, in the study of the singular (local) action of the Lie algebra of projective vector fields on a 2-dimensional pseudo-Riemannian manifold (M, g). Indeed, in [5] it has been proved that the set of points on which such an action is locally regular (i.e., the points possessing a neighborhood foliated by orbits of constant dimension) forms an open dense subset of M, and that a Killing vector field (which is a parallel section of the connection (1)) exists in a neighborhood of any regular point.

The following problem sums up the above questions.

Problem 1. *Let $\pi : E \to M$ be a vector bundle over a simply connected manifold M and ∇ a linear connection in π. Let $\sigma : U \to E$ be a non-zero ∇-parallel section defined on a connected, open, and dense subset U of M. Does σ extend to a parallel section to the whole of M?*

There are two circumstances in which Problem 1 admits a (not so hard) positive solution. One of them occurs when π is a rank–one vector bundle. In such a case, the existence of a non-zero parallel section on a dense subset implies that the curvature tensor R^∇ vanishes identically, so that the connection is flat and σ can be extended to the whole of M. The other case occurs when π has rank two and it is endowed with a metric compatible with ∇. Now, the existence of a parallel section σ on a dense subset implies that R^∇ vanishes identically and, again, σ can be extended to the whole of M.

In Section 2 we give examples showing that neither the hypothesis of connectedness of U nor the hypothesis of simply-connectedness of M can be discarded. However, if the complement $F := M \setminus U$ has higher codimension we can drop the hypothesis of simply-connectedness. Namely, we will prove Theorem 1 below, which is the first result of this paper.

Theorem 1. *Let $\pi : E \to M$ be a vector bundle over a connected manifold M and ∇ a linear connection in π. Let $\sigma : U \to E$ be a non-zero ∇-parallel section defined on an open subset U of M whose complement $F := M \setminus U$ is contained in a smooth submanifold of codimension greater or equal to 2. Then σ can be extended to the whole of M as a parallel section.*

In [6] R. Bryant discussed the existence of a Killing vector field defined on a compact Riemann surface M minus a finite number of points: thanks to Theorem 1, such a problem can now be easily solved (see Remark 4).

In Section 3 we introduce two conditions, herewith denoted by $\mathcal{R}$ and $\mathcal{R}^+$, which can be fulfilled by an open subset $U \subset \mathbb{R}^2$. We prove that, under the condition $\mathcal{R}$ (resp., $\mathcal{R}^+$), a ∇-parallel section $\sigma : U \to E$ of a vector bundle $E \to \mathbb{R}^2$, where ∇ is a metric (resp., general) connection, can be extended to the whole of $\mathbb{R}^2$. More precisely, we will prove Theorem 2 below, which is the second result of this paper.

Theorem 2. *Let $\pi : E \to \mathbb{R}^2$ be a vector bundle endowed with a linear connection ∇. Let $\sigma : U \to E$ be a ∇-parallel section defined on the open and dense subset $U \subset \mathbb{R}^2$. Then σ can be extended to the whole of $\mathbb{R}^2$ as a ∇-parallel section if at least one of the following conditions holds:*

(i) there exists a ∇-parallel metric g on E and the domain U of σ satisfies the condition $\mathcal{R}$;

(ii) the domain U of σ satisfies the condition $\mathcal{R}^+$.

In Section 3.2 we show that an open and connected subset U of $\mathbb{R}^2$ satisfies the condition $\mathcal{R}$ if its complement $\mathbb{R}^2 \setminus U$ is a compact set of (Lebesgue) measure zero.

As an interesting application of Theorem 2, item (*ii*), we prove that a Killing vector field defined on a Riemann surface minus a segment[1] can be always extended to the whole of the Riemann surface. Such extension can be also obtained by using a radial extension (see Remark 5).

[1]Here by a segment we mean a segment in some coordinate system.

Remark 1. *By the uniformization theorem, a 2-dimensional simply connected manifold is either the plane $\mathbb{R}^2$ or the sphere S^2. In order to adapt our extension theorem (Theorem 2) to the case of a vector bundle $E \to \mathbb{S}^2$, it suffices to remove a point from U and restrict the vector bundle to $\mathbb{R}^2$.*

Even though Theorem 1 and Theorem 2 can be used for extending projective (in particular Killing, affine, homothetic) vector fields, they cannot be applied to the class of conformal vector fields (Section 6). The point here is that such vector fields, in the 2-dimensional case, are not parallel sections of a vector bundle endowed with a linear connection.

Finally, in Section 7 we study the extendibility of Killing vector fields by using the Kostant connection.

It is worth mentioning that the main results of this paper have been used in [16] in the context of tractor connections.

2 Why the hypotheses of connectedness of Problem 1 are essential

The examples below show that both the hypothesis of connectedness of U and that of simply-connectedness of M in Problem 1 are essential. To begin with, we show that it is not possible to drop the hypothesis on the connectedness of U.

Example 1. *Let $\mathbb{R}^2 = \{(x, y)\}$ be the standard Euclidean plane and X the Killing vector field (which, we recall, is a particular parallel section of the connection $\widetilde{\nabla}$, see (1)) defined on $U = \mathbb{R}^2 \setminus \{(x, y) \,|\, y = 0\}$ as follows:*

$$X = \begin{cases} (1,0) & \text{if} \quad y > 0 \\ (0,1) & \text{if} \quad y < 0 \end{cases}$$

It is obvious that X cannot be extended to the whole of $\mathbb{R}^2$.

Now we list several examples regarding the hypothesis that M is simply connected.

Example 2. *Let $E := \mathbb{S}^1 \times \mathbb{R}$ be the trivial vector bundle over the circle $\mathbb{S}^1$ and $\mathbf{e} : p \in \mathbb{S}^1 \to (p, 1)$ a section of E. The non-exact 1-form $\mathrm{d}\theta$ on $\mathbb{S}^1$, where θ is the angle coordinate, gives rise to a connection ∇ in E, i.e., the derivative of the section $\mathbf{e}$ is given by*

$$\nabla \mathbf{e} := \mathrm{d}\theta \otimes \mathbf{e}\,.$$

Let $p \in \mathbb{S}^1$ and $U := \mathbb{S}^1 \setminus \{p\}$. Since U is an interval, there exists, by parallel transporting $\mathbf{e}_p$ along U, a non-zero ∇-parallel section $\sigma : U \to E$. Since the equation $\frac{df}{d\theta} + f = 0$ has no non-zero periodic solutions, σ cannot be extended to $\mathbb{S}^1$.

Example 3. *Let $\Sigma \subset \mathbb{R}^3$ be a Möbius strip in the Euclidean space, which is the standard example of a non-orientable surface. Denote by $\nu(\Sigma)$ the normal bundle of Σ endowed with the normal connection $\nabla^\perp$. It is well-known that, by removing the central circle γ of Σ, one obtains a cylinder $U := \Sigma \setminus \gamma$, which is a connected and orientable open subset. So, the restriction to U of the normal bundle $\nu(\Sigma)$ admits a $\nabla^\perp$-parallel section σ. Since the Möbius strip is not orientable, the section σ cannot be extended to the whole of Σ.*

Another way of constructing examples is to use a flat connection in a vector bundle over a compact Riemann surface Σ of genus ≥ 1. Indeed, it is well-known (see e.g. [1, page 559]) that an irreducible representation ρ of $\pi_1(\Sigma)$ gives rise to a flat connection γ whose holonomy group is $\rho(\pi_1(\Sigma))$. Thus, such a connection admits no global parallel sections. On the other hand, a compact Riemann surface is obtained from a polygon by gluing together pairs of edges. The interior U of such a polygon is simply connected so that the restriction of γ to U gives rise to a flat bundle over U, implying the existence of a globally defined ∇-parallel section σ on U. As explained above, the section σ cannot be extended to the whole of Σ.

3 The conditions $\mathcal{R}$ and $\mathcal{R}^+$

Let $U \subseteq \mathbb{R}^2$ be an open subset. Definitions 1 and 2 below clarify, respectively, the meaning of the conditions $\mathcal{R}$ and $\mathcal{R}^+$ on U.

Definition 1. *An open subset $U \subset \mathbb{R}^2$ satisfies the condition $\mathcal{R}$ if there exists a point $p_0 \in U$ and a dense subset $V \subset U$ such that for any $p \in V$ there is a compact subset K_p containing the segment $\overline{p_0 p}$ such that for any $\epsilon > 0$ there exist disjoint subsegments $I_i \subset \overline{p_0 p}$ and piecewise smooth curves $\gamma_i \subset U$, $i = 1, \ldots, n$ with the following properties: $\overline{p_0 p} \setminus \bigcup_i I_i \subset U$ and the concatenation $I_i \sharp \gamma_i$ of I_i and γ_i forms, for each $i \in \{1, \ldots, n\}$, a continuous piecewise smooth Jordan curve bounding a region S_i contained in the compact subset K_p such that*

$$\sum_{i=1}^{n} \mu(S_i) \leq \epsilon$$

where μ is the Lebesgue measure.

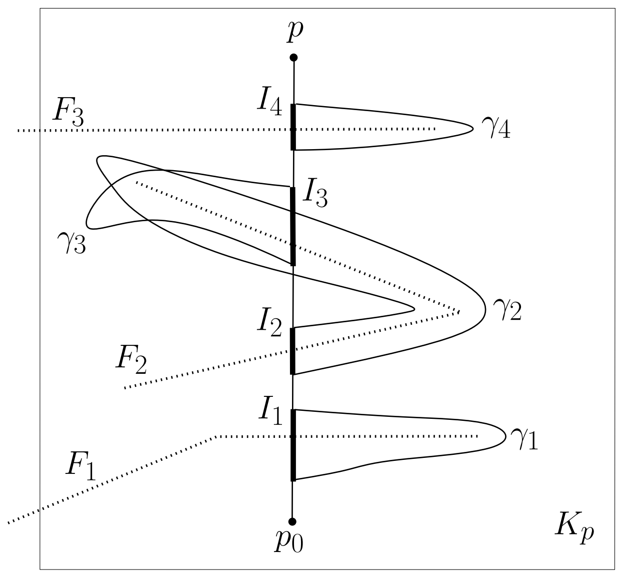

Figure 1: Example of condition $\mathcal{R}$ for the complement U of the closed set $F = F_1 \cup F_2 \cup F_3$.

Definition 2. *An open subset $U \subset \mathbb{R}^2$ satisfies the condition $\mathcal{R}^+$ if there exists a point $p_0 \in U$ and a dense subset $V \subset U$ such that for any $p \in V$ there is a compact subset K_p containing the segment $\overline{p_0 p}$ such that for any $\epsilon, G > 0$ there exist disjoint subsegments $J_i \subset \overline{p_0 p}$ and piecewise smooth curves $\gamma_i \subset U$, $i = 1, \ldots, n$ with the following properties: $\overline{p_0 p} \setminus \bigcup_i J_i \subset U$ and the concatenation $J_i \sharp \gamma_i$ of J_i and γ_i forms, for each $i \in \{1, \ldots, n\}$, a continuous piecewise smooth Jordan curve bounding a region S_i contained in the compact subset K_p such that*

$$e^{\frac{\mathrm{G}}{2}\mathrm{L}_\gamma} \sum_{i=0}^{n-1} e^{\mathrm{G}\mathrm{L}_i} \mu(S_i) < \epsilon$$

where μ is the Lebesgue measure, L_γ is the sum of the lengths of the curves γ_i, and

$$\mathrm{L}_i = \max_{s \in [0,1]} \{\mathrm{length}(h_{is})\}, \quad h_{is}(t) := h_i(t, s),$$

where h_i is a homotopy of the region S_i relative to the endpoints of J_i deforming γ_i to the segment J_i.

3.1 Example: the complement of a segment satisfies the condition $\mathcal{R}^+$

For any segment $I \subset \mathbb{R}^2$, we show that the complement $U := \mathbb{R}^2 \setminus I$ is an open subset fulfilling the condition $\mathcal{R}^+$. Without loss of generality we can assume that $I = \{(x, 0) : 0 \leq x \leq 1\}$. Let $p_0 = (\frac{1}{2}, \frac{1}{2}) \in U$. Set $V = U$ and let $p = (p_x, p_y) \in V = U$. Notice that if the segment $\overline{p_0 p}$ is disjoint from I then the property $\mathcal{R}^+$ holds trivially. So, assume that $\overline{p_0 p} \cap I$ is not empty. Let $K_p \subset \mathbb{R}^2$ be a closed ball centered at $(0, 0)$ whose interior contains both the interval I and the point p. Observe that p_0 is an internal point of K_p. Let G and ϵ be as Definition 2. Let

$$T_\delta = \{q \in \mathbb{R}^2 : \mathrm{dist}(q, I) \leq \delta\}$$

be the set of points whose distance from I is less or equal to δ. Fix $\delta < \min\{\frac{1}{2}, p_y\}$ small enough such that T_δ is contained in the interior of K_p. Let p_1, p_2 be the two points of the intersection $\overline{p_0 p} \cap \partial T_\delta$, and assume that p_1 is the closest one top_0. Set $p_3 := p$ and $\gamma_0 = \overline{p_0 p_1}$. The curve γ_1 is one of the two connected components of $\partial T_\delta \setminus \{p_1, p_2\}$, and $\gamma_2 := \overline{p_2 p_3}$. Now $\mu(S_0) = \mu(S_2) = 0$ and

$$\mu(S_1) \leq \mu(T_\delta) = 2\delta + \pi\delta^2\,;$$

$$L_\gamma \leq \mathrm{dist}(p, p_0) + \mathrm{perimeter}(T_\delta) = \mathrm{dist}(p, p_0) + 2 + 2\pi\delta.$$

Set $h(x, s) := x(sp_2 + (1 - s)p_1) + (1 - x)\gamma_1(s)$, where $\gamma_1(s)$ is a parametrization of γ_1 from p_1 to p_2 and $(x, s) \in [0, 1] \times [0, 1]$. Then h is the homotopy required in Definition 2; since S_1 is convex, we see that the curves $h(x, \cdot)$ are always contained in S_1. The length of the curves $h(x, \cdot)$ are always bounded by the length of γ_1. Then

$$e^{\frac{G}{2} L_\gamma} e^{G L_1} \mu(S_1) \leq e^{\frac{G}{2}(\mathrm{dist}(p, p_0) + 2 + 2\pi\delta)} e^{G(2+2\pi\delta)} (2\delta + \pi\delta^2)\,.$$

Now it is clear that for $\epsilon > 0$ there exists a number $\delta > 0$ such that

$$e^{\frac{G}{2} L_\gamma} e^{G L_1} \mu(S_1) \leq e^{\frac{G}{2}(\mathrm{dist}(p, p_0) + 2 + 2\pi\delta)} e^{G(2+2\pi\delta)} (2\delta + \pi\delta^2) < \epsilon\,.$$

This shows that the complement of I satisfies the property $\mathcal{R}^+$.

3.2 The case when $\mathbb{R}^2 \setminus U$ is a compact set of Lebesgue measure zero

Here we prove the following proposition.

Proposition 1. *Let $U \subset \mathbb{R}^2$ be an open and connected subset whose complement $F = \mathbb{R}^2 \setminus U$ is a compact set of measure zero. Then U satisfies the property $\mathcal{R}$.*

Proof. Let us fix a point $p_0 \in U$. Let $p \in U$ be any other point. Since F is compact there exists a disc K_p containing F and the segment $\overline{p_0 p}$ in its interior. Let ϵ be small enough such that F can be covered with an union $\mathbf{D}$ of discs contained in K_p and whose measure $\mu(\mathbf{D})$ is smaller than ϵ. In view of the compactness of F, we can assume that $\mathbf{D}$ is a finite union of discs. The set $\mathbf{D}$ has a finite number of connected components. Assume now that the connected components of $\mathbf{D}$ intersecting the segment $\overline{p_0 p}$ are also simply connected. By starting at p_0 and moving along the segment $\overline{p_0 p}$ we will meet a first point a_1 belonging to the boundary of one of the components of $\mathbf{D}$. In the case when a_1 belongs to several connected components we just select one of them and call it $\mathbf{D}_1$. Since $\mathbf{D}_1$ is simply connected, by following its boundary, we will meet the segment $\overline{p_0 p}$ at a point $b_1 \in \overline{p_0 p}$ such that both the segment $I_1 = \overline{a_1 b_1}$ and the boundary curve γ_1 from a_1 to b_1 are as in the condition $\mathcal{R}$, i.e., their concatenation $I_1 \sharp \gamma_1$ is a continuous piecewise smooth Jordan curve bounding a region S_1 contained in $\mathbf{D}_1$. Now, by starting at the point b_1 and moving towards p along $\overline{p_0 p}$, we will meet another point a_2 of the boundary of one of the connected components of $\mathbf{D}$. Now we can deal with a_2 much as we did with a_1. Namely, by following the boundary of the respective connected component, we will get another point $b_2 \in \overline{p_0 p}$ such that both the segment $I_2 = \overline{a_2 b_2}$ and the boundary curve γ_2 from a_2 to b_2 are as in the condition $\mathcal{R}$, i.e., their concatenation $I_2 \sharp \gamma_2$ is a continuous piecewise smooth Jordan curve bounding a region S_2 contained in the respective connected component. Then, by starting at b_2, we can repeat the above reasoning to construct a finite sequence of segments I_i and boundary curves γ_i, $i = 1, \cdots, N$ as in the condition $\mathcal{R}$. Since all the regions S_i are included in $\mathbf{D}$, we get that

$$\sum_{i=1}^{N} \mu(S_i) \leq \mu(\mathbf{D}) \leq \epsilon\,,$$

showing that, under the hypothesis that all the connected components of $\mathbf{D}$ intersecting the segment $\overline{p_0 p}$ are simply connected, the condition $\mathcal{R}$ holds.

Now assume that a connected component $\mathbf{A}$ of $\mathbf{D}$ is not simply connected.

Claim: There are finitely many connected and simply connected subsets $\mathbf{A}_1, \cdots, \mathbf{A}_k \subset \mathbf{A}$ such that

$$F \cap \mathbf{A} \subset \bigcup_{j=1}^{k} \mathbf{A}_j\,,$$

where $\mathbf{A}_j$, $j = 1, \cdots, k$, are described as follows. For each $j \in \{1, \cdots, k\}$ there exists $n_j \in \mathbb{N}$ such that $\mathbf{A}_j = \bigcup_{i=1}^{n_j} \mathbf{W}_{ij}$, where each $\mathbf{W}_{ij}$ is either a disc or a *broken disc*. Here by a broken disc we mean one of the two connected components of $\mathbf{B} \setminus \gamma$ where $\mathbf{B}$ is a disc and γ is a simple polygonal curve, made up of a finite number of segments, starting at $x \in \partial\mathbf{B}$ and ending at $y \in \partial\mathbf{B}$, $x \neq y$.

To prove the above claim we show that the complement $\mathbf{A}' = \mathbb{R}^2 \setminus \mathbf{A}$ has a finite number of connected components. Indeed, since $\mathbf{A}$ is a finite union of discs, the common boundary $\partial\mathbf{A} = \partial\mathbf{A}'$ is the union of a finite number of arcs of disc. Let us denote by $\mathcal{A}$ the finite set of such arcs, i.e., $\partial\mathbf{A} = \cup_{\beta\in\mathcal{A}}\beta$ and let $2^{\mathcal{A}}$ be its power set.

Observe that, since $\mathbf{A}$ is bounded, there is just one unbounded connected component $\mathbf{T}$ of $\mathbf{A}'$. Let M and N be two different bounded connected components of $\mathbf{A}'$. Both the boundaries ∂M and ∂N are made up of a finite number of arcs of $\mathcal{A}$. Moreover, if an arc of $\mathcal{A}$ belongs to ∂M, then it cannot belong to ∂N. For each connected component M of $\mathbf{A}'$, let $\mathcal{A}M$ be the set of arcs $\beta \in \mathcal{A}$ belonging to ∂M. Let $\mathcal{C}$ be the set of connected components of $\mathbf{A}'$. Then the map $f : \mathcal{C} \to 2^{\mathcal{A}}$ defined by $f(M) := \mathcal{A}M$ is injective. This shows that $\mathbf{A}'$ has a finite number of connected components.

Let $\mathbf{T}, M_1, \cdots, M_c$, $c \in \mathbb{N}$, be the (disjoint) connected components of $\mathbf{A}'$. Let $t \in \mathrm{int}(\mathbf{T})$ and $x \in \mathrm{int}(M_1)$. Observe that $t, x \in U$. Since U is open and connected, there exists a simple polygonal curve $\gamma \subset U$, made up of a finite number of segments, starting at x and ending at t. Set $\mathbf{A}^1 := \mathbf{A} \setminus \gamma$. Observe that $\mathbf{A}^1$ is the union of a finite number of discs or broken discs. Notice also that

$$F \cap \mathbf{A} \subset F \cap \mathbf{A}^1\,.$$

Moreover, the number of bounded connected components of $(\mathbf{A}^1)' := \mathbb{R}^2 \setminus \mathbf{A}^1$, which we denote by c_1, is less or equal to $c - 1$. So, if $c_1 = 0$, then $\mathbf{A}^1$ is simply connected with a finite number of connected components $\mathbf{A}_1, \cdots \mathbf{A}_k$ as we claimed. If $c_1 > 0$, then we can repeat the same reasoning we used to construct $\mathbf{A}^1$ from $\mathbf{A}$. Thus, we get a sequence of subsets $\mathbf{A} \supset \mathbf{A}^1 \supset \mathbf{A}^2 \supset \cdots$ such that the number of connected components of $(\mathbf{A}^i)' := \mathbb{R}^2 \setminus \mathbf{A}^i$ is strictly decreasing, where $\mathbf{A}^i$ is a finite union either of discs or broken discs and

$$F \cap \mathbf{A} \subset F \cap \mathbf{A}^i$$

for each i. Thus, there exists a number i_0 such that $\mathbb{R}^2 \setminus \mathbf{A}^{i_0}$ has just one connected component, so that any connected component of $\mathbf{A}^{i_0}$ is simply connected. This proves the claim above.

This concludes the proof of Proposition 1 as we can assume that any connected component $\mathbf{A}$ of $\mathbf{D}$ is simply connected and argue as before. $\square$

Remark 2. *As a by-product of the proof we have the following result. Let $F \subset \mathbb{R}^2$ be a compact subset of Lebesgue measure zero μ and $\epsilon > 0$ a positive real number. Then there exists a finite number of disjoint connected and simply connected subsets $\mathbf{A}_i$, $i = 1, \cdots, k$, such that $F \subset \bigcup_{i=1}^{k} \mathbf{A}_i$ and $\sum_{i=1}^{k} \mu(\mathbf{A}_i) < \epsilon$.*

Remark 3. *Notice that the hypothesis of compactness on F is just used to find a simply connected compact set K_p containing all the connected components of the intersection of F and the segment $\overline{p_0 p}$ in its interior. So the above proof applies also to non-compact subsets F whose all connected components are compact.*

4 Proof of Theorem 1

Here it follows the proof of Theorem 1.

Proof. Since the domain of σ is assumed to be dense, it is enough to show that σ can be extended around any point of F. So, let $p \in F$ be a point where σ is not defined. Let $S \subset M$ be the submanifold of codimension ≥ 2 which contains F. Then there exists a local coordinate system $(x_1, \cdots, x_m)$ of M centered at p such that S is locally described by the system $\{x_1 = x_2 = \cdots = x_{m-s} = 0\}$, where $s = \dim(S)$. Let ϵ be small enough such that an open ball B_q centered at $q = (\epsilon, 0, 0, \cdots, 0) \in M$ is contained in the domain of the coordinate system $(x_1, \cdots, x_m)$ and $p \in B_q$. Observe that q belongs to the open subset U. Consider the smooth section $\widetilde{\sigma}$ defined on B_q by parallel transporting $\widetilde{\sigma}(q) := \sigma(q)$ along the radial lines through the point q. If $\mathcal{L}$ is a radial line through the point q which does not intersect S, then

$$\widetilde{\sigma}|_{\mathcal{L}} = \sigma|_{\mathcal{L}} .$$

Observe that the subset $G \subset B_q$ made of the points $x \in B_q$ such that the radial line $\overline{xq}$ does not intersect S is dense in B_q. Indeed, the radial lines through q which intersect S are contained in the intersection of B_q with the hyperplane $x_2 = 0$. Then,

$$\widetilde{\sigma}|_{B_q \bigcap U} = \sigma|_{B_q \bigcap U} .$$

Since $B_q \bigcap U$ is dense in B_q, $\widetilde{\sigma}$ is a ∇-parallel section on B_q. Now it is clear that σ has been extended as a parallel section to $U \bigcup B_q$. $\square$

Remark 4. *The above result can be used to give a different solution to the problem of the extension of a Killing vector field defined on a Riemannian surface minus a finite number of points, discussed by R. Bryant in [6].*

5 Proof of Theorem 2

The idea behind the proof of Theorem 2 consists in using a suitable estimate, involving the curvature of the connection, to control the parallel transport along curves. To this aim, we need the following lemma.

Lemma 1. *Let $a \in \mathbb{R}$. Let $f(t)$ and $g(t)$ be two continuous functions for $t \geq a$. Let $u(t)$ be a C^1 function for $t \geq a$. If*

$$\begin{cases} u'(t) \leq f(t)u(t) + g(t) , & t \geq a \\ u(a) = u_0 \end{cases} \tag{2}$$

then

$$u(t) \leq u_0 e^{\int_a^t f(x)dx} + \int_a^t g(s) e^{\int_s^t f(x)dx} ds. \tag{3}$$

Observe that the right-hand side term of (3) *is the solution to the Cauchy problem obtained by imposing the equality in the system* (2).

Proof. A direct computation shows that (2) can be written as

$$\frac{d}{ds}\left(u(s)e^{\int_s^t f(x)dx}\right) \leq g(s)e^{\int_s^t f(x)dx}\,, \quad s \geq a\,, \quad u(a) = u_0$$

and integrating over s from a to t we obtain (3). $\square$

Proposition 2. *Let $\pi : E \to \mathbb{R}^2$ be a vector bundle endowed with a linear connection ∇ and g a metric on π (not necessarily compatible with the connection ∇). Let γ_0 and γ_1 be two curves starting at $p \in \mathbb{R}^2$ and ending at $q \in \mathbb{R}^2$. Let $\gamma : [0,1] \times [0,1] \to \mathbb{R}^2$ be a smooth homotopy between γ_0 and γ_1 relative to the endpoints p, q which is* 1-1 *when restricted to* $(0,1) \times [0,1]$. *Let $S := \gamma([0,1] \times [0,1])$. Then*

$$\|\tau_{\gamma_0}(\xi_p) - \tau_{\gamma_1}(\xi_p)\|_g \leq \|\xi_p\|_g \,\mathrm{R}\, e^{\mathrm{GL}} \mu(S) \tag{4}$$

where $\tau_{\gamma_i}(\xi_p)$ is the parallel transport from p to q of $\xi_p \in \pi^{-1}(p)$ along γ_i, $\mu(S)$ is the area of S w.r.t. the Lebesgue measure μ of $\mathbb{R}^2$, R is a constant depending only on the metric g and on the curvature tensor R^∇ of ∇ on S, $\mathrm{L} = \max_{s\in[0,1]}\{\mathrm{length}(\gamma_s)\}$, $\gamma_s(t) := \gamma(t,s)$, and G is a constant controlling the norm of the tensor ∇g on S, i.e. the constants G and R depend only on the image S of the homotopy and not on the homotopy itself.

Proof. We denote by $\|\xi\|_g^2 := g(\xi,\xi)$ the square of the norm of the vector ξ_p of the fiber $E_p := \pi^{-1}(p)$. If v is a tangent vector of $\mathbb{R}^2$, its norm $\|v\|$ is taken w.r.t. the flat standard Riemannian metric.
Regarding the curvature tensor R^∇ as a map $\mathrm{R}^\nabla : \Lambda^2 T_{(x,y)}\mathbb{R}^2 \to \mathrm{End}(E_{(x,y)})$, we have

$$g(\mathrm{R}^\nabla(v \wedge w)\eta, \xi) = g(\mathrm{R}^\nabla(v, w)\eta, \xi)$$

with $v, w \in T_{(x,y)}\mathbb{R}^2$ and $\eta, \xi \in E_{(x,y)}$. Since S is compact, there exists a constant R such that

$$g(\mathrm{R}^\nabla(v \wedge w)\eta, \xi) \leq \mathrm{R}\|v \wedge w\|\, \|\xi\|_g \,\|\eta\|_g$$

for all tangent vectors v, w of S and $\eta, \xi \in \pi^{-1}(S)$, where $\|v \wedge w\|$ is the area of the parallelogram spanned by v, w.
We denote by ∂_t and ∂_s, respectively, the vector fields $\frac{\partial\gamma}{\partial t}$ and $\frac{\partial\gamma}{\partial s}$, both tangent to S at the point $\gamma(t,s)$. Let us define $X(t,s)$ as the parallel transport of $\xi_p \in \pi^{-1}(p)$ along γ_s at the instant t (see Figure 2). We have

$$\|\tau_{\gamma_0}(\xi_p) - \tau_{\gamma_1}(\xi_p)\|_g = \|X(1,1) - X(1,0)\|_g \leq \int_0^1 \left\|\frac{D}{ds}X(1,s)\right\|_g ds. \tag{5}$$

The symbol $\frac{D}{ds}X(t,s)$ stands for the covariant derivative along the curve $s \to \gamma_t(s) := \gamma(t,s)$ (i.e. t is fixed) associated with ∇. Thus, for $(t,s) \in (0,1) \times (0,1)$, $\frac{D}{ds}X(t,s) = \nabla_{\partial_s}X(t,s)$ and $\frac{D}{ds}X(1,s) = \frac{\partial X(1,s)}{\partial s}$ is the derivative in the vector space E_q of the curve $X(1,s) \in E_q$, see Chapter 2 of [13] for more details. So, the above estimate is obtained by applying the fundamental theorem of the integral calculus.
The tensor $(\nabla_v g)(\xi,\eta) := v\big(g(\xi,\eta)\big) - g(\nabla_v\xi,\eta) - g(\xi,\nabla_v\eta)$ is continuous so that, by the compactness of S, there exists a real constant G such that

$$(\nabla_v g)(\xi,\eta) \leq \mathrm{G}\|v\|\,\|\xi\|_g\,\|\eta\|_g$$

where $\|v\|$ is the norm of the tangent vector v of S and $\eta, \xi \in \pi^{-1}(S)$. Then

$$\partial_t\|X\|_g^2 = \nabla_{\partial_t} g(X,X) \leq \mathrm{G}\|\partial_t\|\,\|X\|_g^2\,,$$

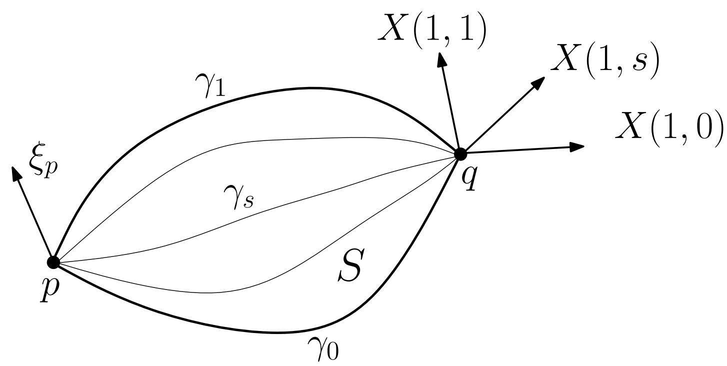

Figure 2: $X(t,s)$ is constructed by parallel transporting ξ_p along γ_s.

so that, in view of Lemma 1, we obtain

$$\|X(t,s)\|_g \leq \|\xi_p\|_g \left(e^{\int_0^t \mathrm{G}\|\partial_t\|_{(t',s)}dt'}\right)^{\frac{1}{2}} \leq \|\xi_p\|_g e^{\frac{\mathrm{G\,L}}{2}} \tag{6}$$

where $\mathrm{L} = \max\limits_{s\in[0,1]}\{\text{length}(\gamma_s)\}$.

Now we estimate $\left\|\frac{DX}{ds}\right\|_g$. From the equation

$$\partial_t \left\|\frac{DX}{ds}\right\|^2 = 2g\left(\frac{D}{dt}\frac{DX}{ds}\,,\,\frac{DX}{ds}\right) + \nabla_{\partial_t} g\left(\frac{DX}{ds}\,,\,\frac{DX}{ds}\right) = 2g\left(\mathrm{R}^{\nabla}(\partial_t,\partial_s)X\,,\,\frac{DX}{ds}\right) + \nabla_{\partial_t} g\left(\frac{DX}{ds}\,,\,\frac{DX}{ds}\right)$$

and the above inequalities we get

$$\partial_t \left\|\frac{DX}{ds}\right\|_g^2 \leq 2\mathrm{R}\,\|\partial_t\wedge\partial_s\|\;\|\xi_p\|_g e^{\frac{\mathrm{G\,L}}{2}}\;\left\|\frac{DX}{ds}\right\|_g + \mathrm{G}\,\|\partial_t\|\;\left\|\frac{DX}{ds}\right\|_g^2,$$

which implies

$$\partial_t \left\|\frac{DX}{ds}\right\|_g \leq \mathrm{R}\,\|\partial_t\wedge\partial_s\|\;\|\xi_p\|_g e^{\frac{\mathrm{G\,L}}{2}} + \frac{\mathrm{G}\,\|\partial_t\|}{2}\left\|\frac{DX}{ds}\right\|_g.$$

By Lemma 1 we obtain

$$\left\|\frac{DX(t,s)}{ds}\right\|_g \leq \int_0^t \mathrm{R}\,\|\partial_t\wedge\partial_s\|_{(t',s)}\;\|\xi_p\|_g e^{\frac{\mathrm{G\,L}}{2}} e^{\left(\int_{t'}^t \frac{\mathrm{G}\,\|\partial_t\|_{(t'',s)}}{2}dt''\right)} dt'$$

and, consequently,

$$\left\|\frac{DX(t,s)}{ds}\right\|_g \leq \|\xi_p\|_g\,\mathrm{R}\,e^{\mathrm{GL}}\int_0^t \|\partial_t\wedge\partial_s\|_{(t',s)}\;dt'.$$

Finally, from equation (5), we have

$$\|\tau_{\gamma_0}(\xi_p) - \tau_{\gamma_1}(\xi_p)\|_g \leq \int_0^1 \left\|\frac{D}{ds}X(1,s)\right\|_g ds \leq \|\xi_p\|_g\,\mathrm{R}\,e^{\mathrm{GL}}\int_0^1\int_0^1 ||\partial_t\wedge\partial_s||\;dtds,$$

which is that we wanted to show. $\square$

Proof of Theorem 2. Assume that condition (i) of Theorem 2 holds. Let $p_0 \in U$ be the point given by the condition $\mathcal{R}$ (see Definition 1) and ξ the smooth section defined on the whole of $\mathbb{R}^2$ obtained by parallel transporting $\xi(p_0) := \sigma(p_0)$ along the radial straight lines starting at p_0. Observe that $\sigma \equiv \xi$ near p_0. We first prove that $\sigma(p) = \xi(p)\ \forall\, p \in V$, where V is a dense subset of the domain U of σ. Fix $p \in V$ and a compact K_p containing the segment $\overline{p_0 p}$ as in Definition 1. We relabel the segments I_i (and the corresponding curves γ_i and regions S_i) of Definition 1 in order to obtain a sequence of subsegments on the oriented segment $\overrightarrow{p_0 p}$. Let a_i and b_i be the endpoints of I_i. The strategy is to apply the estimate of Proposition 2 to each region S_i. Since g is compatible with ∇, the constant G which appears in Proposition 2 is equal to zero. Since the regions S_i are inside the compact set K_p, we have an uniform bound R for the norm of the curvature tensor R^∇ on K_p. We have that

$$\begin{aligned}
\|\xi(p) - \sigma(p)\|_g = \|\xi(b_n) - \sigma(b_n)\|_g &= \|\tau_{I_n}\xi(a_n) - \tau_{\gamma_n}\sigma(a_n)\|_g = \\
&= \|\tau_{I_n}\xi(a_n) - \tau_{I_n}\sigma(a_n) + \tau_{I_n}\sigma(a_n) - \tau_{\gamma_n}\sigma(a_n)\|_g \\
&\le \|\tau_{I_n}\xi(a_n) - \tau_{I_n}\sigma(a_n)\|_g + \|\tau_{I_n}\sigma(a_n) - \tau_{\gamma_n}\sigma(a_n)\|_g \\
&\le \|\xi(a_n) - \sigma(a_n)\|_g + \|\tau_{I_n}\sigma(a_n) - \tau_{\gamma_n}\sigma(a_n)\|_g \\
&\le \|\xi(b_{n-1}) - \sigma(b_{n-1})\|_g + \|\tau_{I_n}\sigma(a_n) - \tau_{\gamma_n}\sigma(a_n)\|_g
\end{aligned}$$

Since the region S_n, whose boundary is formed of the segment I_n and the curve γ_n, is simply connected (by Definition 1), we can use the Riemann mapping theorem to map S_n in a 1-1 way onto the unit disc $\Delta \subset \mathbb{R}^2$. Under such a mapping, the segment I_n and the curve γ_n are mapped, respectively, into two complementary arcs δ_1 and δ_2 of the unit circle. Hence we can construct a smooth 1-1 homotopy $\gamma : [0,1] \times [0,1] \to \Delta$ between δ_1 and δ_2 relative to their endpoints. The pullback of γ by the aforementioned Riemann mapping is a homotopy between I_n and γ_n. Then, by applying Proposition 2 to the region S_n, we obtain

$$\|\xi(b_n) - \sigma(b_n)\|_g \le \|\xi(b_{n-1}) - \sigma(b_{n-1})\|_g + \mathrm{R}\|\sigma(a_n)\|_g \mu(S_n) = \|\xi(b_{n-1}) - \sigma(b_{n-1})\|_g + \mathrm{R}\|\sigma(p_0)\|_g \mu(S_n)$$

By repeating the above procedure for $j = n-1, \cdots, 1$ we get

$$\|\xi(b_j) - \sigma(b_j)\|_g \le \|\xi(b_{j-1}) - \sigma(b_{j-1})\|_g + \mathrm{R}\|\sigma(p_0)\|_g \mu(S_j)$$

and, consequently,

$$\|\xi(p) - \sigma(p)\|_g = \|\xi(p_n) - \sigma(p_n)\|_g \le \mathrm{R}\|\sigma(p_0)\|_g \sum_{i=1}^{n} \mu(S_i) \le \mathrm{R}\|\sigma(p_0)\|_g\, \epsilon\,.$$

In view of the arbitrariness of ϵ, $\xi = \sigma$ on $V \subset U$. Thus, $\xi \equiv \sigma$ on U as V is dense in U. Finally, ξ is parallel on the whole of $\mathbb{R}^2$ due to the fact that U is a dense subset of $\mathbb{R}^2$. This proves the theorem under the hypothesis of item (i).

Now, assume that condition (ii) of Theorem 2 holds. Let $p_0 \in U$ be the point given by the condition $\mathcal{R}^+$ (see Definition 2). Let g be a metric on the vector bundle $\pi : E \to \mathbb{R}^2$ such that $\|\sigma(p_0))\|_g = 1$. As in the previous case, let ξ be the smooth section defined on the whole of $\mathbb{R}^2$ obtained by parallel transporting $\xi(p_0) := \sigma(p_0)$ along the radial straight lines starting at p_0. We shall prove that $\sigma(p) = \xi(p)\ \forall p \in V$, where V is a dense subset of the domain U of σ. Fix $p \in V$ and K_p containing the segment $\overline{p_0 p}$ as in Definition 2. Let G be a bound for the norm of tensor ∇g on the compact subset K_p. By using the same notations we introduced in the previous case, we have that

$$\begin{aligned}\|\xi(p)-\sigma(p)\|_g = \|\xi(p_n)-\sigma(p_n)\|_g &= \|\tau_{I_{n-1}}\xi(p_{n-1})-\tau_{\gamma_{n-1}}\sigma(p_{n-1})\|_g = \\ &= \|\tau_{I_{n-1}}\xi(p_{n-1})-\tau_{I_{n-1}}\sigma(p_{n-1})+\tau_{I_{n-1}}\sigma(p_{n-1})-\tau_{\gamma_{n-1}}\sigma(p_{n-1})\|_g \\ &\leq \|\tau_{I_{n-1}}\xi(p_{n-1})-\tau_{I_{n-1}}\sigma(p_{n-1})\|_g+\|\tau_{I_{n-1}}\sigma(p_{n-1})-\tau_{\gamma_{n-1}}\sigma(p_{n-1})\|_g \\ &\leq \|\xi(p_{n-1})-\sigma(p_{n-1})\|_g e^{\frac{\mathrm{G}}{2}\|p_n-p_{n-1}\|}+\|\tau_{I_{n-1}}\sigma(p_{n-1})-\tau_{\gamma_{n-1}}\sigma(p_{n-1})\|_g \\ &\leq \|\xi(p_{n-1})-\sigma(p_{n-1})\|_g e^{\frac{\mathrm{G}}{2}\|p_n-p_{n-1}\|}+\mathrm{R}\|\sigma(p_{n-1})\|_g e^{\mathrm{GL}_{n-1}}\mu(S_{n-1}) \\ &\leq \|\xi(p_{n-1})-\sigma(p_{n-1})\|_g e^{\frac{\mathrm{G}}{2}\|p_n-p_{n-1}\|}+\mathrm{R}e^{\frac{\mathrm{GL}_\gamma}{2}}e^{\mathrm{GL}_{n-1}}\mu(S_{n-1})\end{aligned}$$

where the last two inequalities are obtained in view of Proposition 2 and inequality (6).
By repeating the above procedure for $j = n-1, \cdots, 1$ we get

$$\|\xi(p_j)-\sigma(p_j)\|_g \leq \|\xi(p_{j-1})-\sigma(p_{j-1})\|_g e^{\frac{\mathrm{G}}{2}\|p_j-p_{j-1}\|}+\mathrm{R}e^{\frac{\mathrm{GL}_\gamma}{2}}e^{\mathrm{GL}_{j-1}}\mu(S_{j-1})$$

and, consequently,

$$\|\xi(p)-\sigma(p)\|_g = \|\xi(p_n)-\sigma(p_n)\|_g \leq \mathrm{R}e^{\frac{\mathrm{G}}{2}\mathrm{L}_\gamma}e^{\frac{\mathrm{G}}{2}\|p-p_0\|}\sum_{i=0}^{n-1}e^{\mathrm{GL}_i}\mu(S_i) \leq \mathrm{R}e^{\frac{\mathrm{G}}{2}\|p-p_0\|}\,\epsilon\,.$$

In view of the arbitrariness of ϵ, $\xi = \sigma$ on $V \subset U$. Thus, $\xi \equiv \sigma$ on U as V is dense in U. Finally, ξ is parallel on the whole of $\mathbb{R}^2$ due to the fact that U is a dense subset of $\mathbb{R}^2$. This proves the theorem under the hypothesis of item (ii). □

6 Extendability of projective vector fields and non-extendability of conformal ones

We recall, from the introduction, that projective vector fields can be regarded as parallel sections of a suitably constructed linear connection. Thus, a projective vector field defined on a open set $U \subset \mathbb{R}^2$ satisfying the condition $\mathcal{R}^+$ of Definition 2 can be extended to the whole of $\mathbb{R}^2$. We underline that such result applies also to Killing, affine, and homothetic vector fields as they are special projective vector fields.
One can ask if this result holds for some more general class of vector fields, for instance for that of conformal ones. Below we see that, in dimension 2, this is not the case. Indeed, in dimension 2, conformal Killing vector fields cannot be regarded as parallel sections of a linear connection in a vector bundle, whereas, for dimension greater than 2, they can be seen as parallel sections of the so called *Geroch* connection [14, 19].
It is well-known that a conformal Killing vector field X of the Euclidean plane $\mathbb{R}^2$ is given by a holomorphic function f. In fact, if

$$X(x,y) = u(x,y)\frac{\partial}{\partial x}+v(x,y)\frac{\partial}{\partial y}$$

is a conformal Killing vector field defined on U, then $f(z) = u(z)+\mathrm{i}v(z)$ belongs to the set of holomorphic functions $\mathcal{O}(U)$ on U. Indeed the flow F_t^X consists of holomorphic maps, i.e., $F_t^X \in \mathcal{O}(U)$ for small values of t. Since the operators $\frac{\mathrm{d}}{\mathrm{d}t}$ and $\overline{\partial}$ commute, we have that

$$\left(\overline{\partial}\circ\frac{\mathrm{d}}{\mathrm{d}t}\right)F_t^X = \left(\frac{\mathrm{d}}{\mathrm{d}t}\circ\overline{\partial}\right)F_t^X = 0$$

which shows that $f(z) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}F_t^X(z)$ is holomorphic.

The function $1/z$ defines a conformal Killing vector field X on the open and connected subset $U = \mathbb{C} \setminus \{0\}$. Since X is unbounded near 0 (i.e. the Euclidean length of the vector field X goes to infinity when approaching the origin), it cannot be extended to the plane $\mathbb{C}$. Observe that if a bounded conformal Killing vector field defined on an open set U minus a discrete subset, then by Riemann's extension theorem it can be extended to the whole of U (see [18] for a general discussion regarding arbitrary 2-dimensional pseudo-Riemannian metrics).

Now we give an example of a *bounded* conformal Killing vector field X defined on $\mathbb{R}^2$ minus a segment that cannot be extended to the whole of $\mathbb{R}^2$. As explained in [20, page 5], in order to construct the Riemann surface of $w^2 = (z-r)\cdot(z-s)$, $r \neq s \in \mathbb{C}$, we cut $\mathbb{C} = \mathbb{R}^2$ along a segment I connecting the branching points r, s, thus obtaining two single-valued branches, i.e. two holomorphic functions $w_1(z)\,, w_2(z) : \mathbb{C} \setminus I \to \mathbb{C}$. Observe that both the functions w_1, w_2 are bounded. Therefore, by considering $f(z) = w_1(z)$, we get a bounded conformal Killing vector field which cannot be extended to the whole of $\mathbb{R}^2$.

Remark 5. *The above example shows that Bryant's argument to solve the problem described in [6] (see also Remark 4) can not be used to extend a Killing vector field X defined on a Riemann surface minus a segment on the whole of the Riemann surface. However, by using the Kostant connection and considering a radial extension of the parallel section associated with X, we see that X can be extended also in this case.*

7 Extension of Killing vector fields of $(\mathbb{R}^2, g)$

In this section we prove the following theorem.

Theorem 3. *Let $\mathbb{R}^2$ be the plane endowed with a Riemannian metric g. Let κ be the Gaussian curvature of g. Assume that the differential of κ is nowhere vanishing. Let $U \subset \mathbb{R}^2$ be a connected, open and dense subset. If (U, g) admits a Killing vector field X, then it extends to a Killing vector field of $(\mathbb{R}^2, g)$.*

For the proof of this theorem we will use the Kostant connection.

7.1 Local description of the Kostant connection

Let (x, y) be local isothermal coordinates about a point of $(\mathbb{R}^2, g)$, i.e. the metric g is given by

$$ds^2 = \lambda(dx^2 + dy^2)\,.$$

Let J be the complex structure given by $J(\partial_x) = \partial_y$ and $J(\partial_y) = -\partial_x$. Recall that J is parallel w.r.t. the Levi-Civita connection of ds^2.

Consider the bundle $T\mathbb{R}^2 \oplus \mathfrak{so}(T\mathbb{R}^2)$ endowed with the Kostant connection $\widetilde{\nabla}$ given by equation (1). The sections $\xi_1 := (\partial_x, 0)$, $\xi_2 := (\partial_y, 0)$ and $\xi_3 := (0, J)$ are linearly independent, so that they form a frame of $T\mathbb{R}^2 \oplus \mathfrak{so}(T\mathbb{R}^2)$. In order to prove Theorem 3, we need the following technical lemma.

Lemma 2. *Let $\mathrm{R}^{\widetilde{\nabla}}$ be the curvature tensor of $\widetilde{\nabla}$ and κ the Gaussian curvature of g. The matrices of the operators $\mathrm{R}^{\widetilde{\nabla}}_{\partial_x\partial_y}$, $(\widetilde{\nabla}_{\partial_x}\mathrm{R}^{\widetilde{\nabla}})_{\partial_x\partial_y}$ and $(\widetilde{\nabla}_{\partial_y}\mathrm{R}^{\widetilde{\nabla}})_{\partial_x\partial_y}$ w.r.t. the frame $\{\xi_1, \xi_2, \xi_3\}$ are*

$$\mathrm{R}^{\widetilde{\nabla}}_{\partial_x\partial_y} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\kappa_x\lambda & -\kappa_y\lambda & 0 \end{pmatrix}$$

$$(\widetilde{\nabla}_{\partial_x}\mathrm{R}^{\widetilde{\nabla}})_{\partial_x\partial_y} = \begin{pmatrix} 0 & 0 & 0 \\ \kappa_x\lambda & \kappa_y\lambda & 0 \\ * & * & -\kappa_y\lambda \end{pmatrix}$$

$$(\widetilde{\nabla}_{\partial_y}\mathrm{R}^{\widetilde{\nabla}})_{\partial_x\partial_y} = \begin{pmatrix} 0 & 0 & 0 \\ \kappa_x\lambda & \kappa_y\lambda & 0 \\ * & * & -\kappa_x\lambda \end{pmatrix}$$

Proof. The proof of the above lemma is based on straightforward computations. Let $\xi = (Z, 0)$ be a section of the Kostant bundle. Then

$$\widetilde{\nabla}_{\partial_x}\xi = (\nabla_{\partial_x}Z, -\mathrm{R}(\partial_x, Z))$$

where $\mathrm{R}(X,Y)Z = \kappa(X \wedge Y)Z$. Then

$$\widetilde{\nabla}_{\partial_y}\widetilde{\nabla}_{\partial_x}\xi = (\nabla_{\partial_y}\nabla_{\partial_x}Z + \mathrm{R}(\partial_x, Z)(\partial_y), -\nabla_y\mathrm{R}(\partial_x, Z) - \mathrm{R}(\partial_y, \nabla_{\partial_x}Z))$$

So

$$\begin{aligned} \mathrm{R}^{\widetilde{\nabla}}_{\partial_x\partial_y}\xi &= (\nabla_{\partial_x}\nabla_{\partial_y}Z + \mathrm{R}(\partial_y, Z)(\partial_x), -\nabla_x\mathrm{R}(\partial_y, Z) - \mathrm{R}(\partial_x, \nabla_{\partial_y}Z)) \\ &\quad -(\nabla_{\partial_y}\nabla_{\partial_x}Z + \mathrm{R}(\partial_x, Z)(\partial_y), -\nabla_y\mathrm{R}(\partial_x, Z) - \mathrm{R}(\partial_y, \nabla_{\partial_x}Z)) \\ &= (0, \nabla_y\mathrm{R}(\partial_x, Z) + \mathrm{R}(\partial_y, \nabla_{\partial_x}Z) - \nabla_x\mathrm{R}(\partial_y, Z) - \mathrm{R}(\partial_x, \nabla_{\partial_y}Z)) \\ &(0, (-\kappa\langle Z, \partial_y\rangle)_yJ + \kappa\langle\nabla_{\partial_x}Z, \partial_x\rangle J - (\kappa\langle Z, \partial_x\rangle)_xJ + \kappa\langle\nabla_{\partial_y}Z, \partial_y\rangle J) \end{aligned}$$

Then

$$\mathrm{R}^{\widetilde{\nabla}}_{\partial_x\partial_y}\xi_1 = -\kappa_x\lambda\ \xi_3\,, \quad \mathrm{R}^{\widetilde{\nabla}}_{\partial_x\partial_y}\xi_2 = -\kappa_y\lambda\ \xi_3\,, \quad \mathrm{R}^{\widetilde{\nabla}}_{\partial_x\partial_y}\xi_3 = 0$$

Now we compute the covariant derivatives of the sections ξ_1, ξ_2, ξ_3.

$$\begin{aligned} \widetilde{\nabla}_{\partial_y}\xi_1 &= (\nabla_{\partial_y}\partial_x\,, -\mathrm{R}(\partial_y, \partial_x)) = (\nabla_{\partial_y}\partial_x\,, \kappa\partial_x \wedge \partial_y) = (\nabla_{\partial_y}\partial_x\,, -\kappa\lambda J) = \\ &= \left(\frac{\lambda_y}{2\lambda}\partial_x + \frac{\lambda_x}{2\lambda}\partial_y\,, -\kappa\lambda J\right) = \frac{\lambda_y}{2\lambda}\xi_1 + \frac{\lambda_x}{2\lambda}\xi_2 - \kappa\lambda\xi_3 \end{aligned}$$

$$\widetilde{\nabla}_{\partial_x}\xi_1 = (\nabla_{\partial_x}\partial_x\,, 0) = \left(\frac{\lambda_x}{2\lambda}\partial_x - \frac{\lambda_y}{2\lambda}\partial_y, 0\right) = \frac{\lambda_x}{2\lambda}\xi_1 - \frac{\lambda_y}{2\lambda}\xi_2$$

$$\widetilde{\nabla}_{\partial_y}\xi_2 = (\nabla_{\partial_y}\partial_y\,, 0) = \left(-\frac{\lambda_x}{2\lambda}\partial_x + \frac{\lambda_y}{2\lambda}\partial_y, 0\right) = -\frac{\lambda_x}{2\lambda}\xi_1 + \frac{\lambda_y}{2\lambda}\xi_2$$

$$\widetilde{\nabla}_{\partial_x}\xi_2 = (\nabla_{\partial_x}\partial_y\,, -\mathrm{R}(\partial_x, \partial_y)) = \frac{\lambda_y}{2\lambda}\xi_1 + \frac{\lambda_x}{2\lambda}\xi_2 + \kappa\lambda\xi_3$$

$$\begin{aligned} \widetilde{\nabla}_{\partial_y}\xi_3 &= (-J(\partial_y), 0) = (\partial_x, 0) = \xi_1 \\ \widetilde{\nabla}_{\partial_x}\xi_3 &= (-J(\partial_x), 0) = (-\partial_y, 0) = -\xi_2 \end{aligned}$$

Now we are in position to compute the covariant derivatives of $\mathrm{R}^{\widetilde{\nabla}}$.

$$
\begin{aligned}
(\widetilde{\nabla}_{\partial_x}\mathrm{R}^{\widetilde{\nabla}})_{\partial_x\partial_y}\xi_1 &= \widetilde{\nabla}_{\partial_x}(\mathrm{R}^{\widetilde{\widetilde{\nabla}}}_{\partial_x\partial_y}\xi_1) - \mathrm{R}^{\widetilde{\widetilde{\nabla}}}_{\nabla_{\partial_x}\partial_x\,\partial_y}\xi_1 - \mathrm{R}^{\widetilde{\widetilde{\nabla}}}_{\partial_x\,\nabla_{\partial_x}\partial_y}\xi_1 - \mathrm{R}^{\widetilde{\widetilde{\nabla}}}_{\partial_x\partial_y}\widetilde{\nabla}_{\partial_x}\xi_1 \\
&= \widetilde{\nabla}_{\partial_x}(\mathrm{R}^{\widetilde{\widetilde{\nabla}}}_{\partial_x\partial_y}\xi_1) - \mathrm{R}^{\widetilde{\widetilde{\nabla}}}_{\frac{\lambda_x}{2\lambda}\partial_x\,\partial_y}\xi_1 - \mathrm{R}^{\widetilde{\widetilde{\nabla}}}_{\partial_x\,\frac{\lambda_x}{2\lambda}\partial_y}\xi_1 - \mathrm{R}^{\widetilde{\widetilde{\nabla}}}_{\partial_x\partial_y}\widetilde{\nabla}_{\partial_x}\xi_1 \\
&= \widetilde{\nabla}_{\partial_x}(\mathrm{R}^{\widetilde{\widetilde{\nabla}}}_{\partial_x\partial_y}\xi_1) - \frac{\lambda_x}{\lambda}\mathrm{R}^{\widetilde{\widetilde{\nabla}}}_{\partial_x\,\partial_y}\xi_1 - \mathrm{R}^{\widetilde{\widetilde{\nabla}}}_{\partial_x\partial_y}\widetilde{\nabla}_{\partial_x}\xi_1 \\
&= \widetilde{\nabla}_{\partial_x}(-\kappa_x\lambda\xi_3) - \frac{\lambda_x}{\lambda}\mathrm{R}^{\widetilde{\widetilde{\nabla}}}_{\partial_x\,\partial_y}\xi_1 - \mathrm{R}^{\widetilde{\widetilde{\nabla}}}_{\partial_x\partial_y}\widetilde{\nabla}_{\partial_x}\xi_1 \\
&= (-\kappa_x\lambda)_x\xi_3 - \kappa_x\lambda\widetilde{\nabla}_{\partial_x}\xi_3 - \frac{\lambda_x}{\lambda}\mathrm{R}^{\widetilde{\widetilde{\nabla}}}_{\partial_x\,\partial_y}\xi_1 - \mathrm{R}^{\widetilde{\widetilde{\nabla}}}_{\partial_x\partial_y}\widetilde{\nabla}_{\partial_x}\xi_1 \\
&= (-\kappa_x\lambda)_x\xi_3 - \kappa_x\lambda\widetilde{\nabla}_{\partial_x}\xi_3 + \lambda_x\kappa_x\xi_3 - \mathrm{R}^{\widetilde{\widetilde{\nabla}}}_{\partial_x\partial_y}\widetilde{\nabla}_{\partial_x}\xi_1 \\
&= (-\kappa_x\lambda)_x\xi_3 + \kappa_x\lambda\xi_2 + \lambda_x\kappa_x\xi_3 - \mathrm{R}^{\widetilde{\widetilde{\nabla}}}_{\partial_x\partial_y}\widetilde{\nabla}_{\partial_x}\xi_1 \\
&= \kappa_x\lambda\xi_2 - \kappa_{xx}\lambda\xi_3 - \mathrm{R}^{\widetilde{\widetilde{\nabla}}}_{\partial_x\partial_y}\widetilde{\nabla}_{\partial_x}\xi_1 \\
&= \kappa_x\lambda\xi_2 - \kappa_{xx}\lambda\xi_3 - \mathrm{R}^{\widetilde{\widetilde{\nabla}}}_{\partial_x\partial_y}\left(\frac{\lambda_x}{2\lambda}\xi_1 - \frac{\lambda_y}{2\lambda}\xi_2\right) \\
&= \kappa_x\lambda\xi_2 - \kappa_{xx}\lambda\xi_3 + \frac{\lambda_x}{2\lambda}\kappa_x\lambda\xi_3 - \kappa_y\lambda\frac{\lambda_y}{2\lambda}\xi_3 \\
&= \kappa_x\lambda\xi_2 + \left(\frac{\lambda_x\kappa_x - \kappa_y\lambda_y}{2} - k_{xx}\lambda\right)\xi_3 \\
(\widetilde{\nabla}_{\partial_x}\mathrm{R}^{\widetilde{\nabla}})_{\partial_x\partial_y}\xi_2 &= \widetilde{\nabla}_{\partial_x}(\mathrm{R}^{\widetilde{\widetilde{\nabla}}}_{\partial_x\partial_y}\xi_2) - \mathrm{R}^{\widetilde{\widetilde{\nabla}}}_{\nabla_{\partial_x}\partial_x\,\partial_y}\xi_2 - \mathrm{R}^{\widetilde{\widetilde{\nabla}}}_{\partial_x\,\nabla_{\partial_x}\partial_y}\xi_2 - \mathrm{R}^{\widetilde{\widetilde{\nabla}}}_{\partial_x\partial_y}\widetilde{\nabla}_{\partial_x}\xi_2 \\
&= \widetilde{\nabla}_{\partial_x}(\mathrm{R}^{\widetilde{\widetilde{\nabla}}}_{\partial_x\partial_y}\xi_2) - \mathrm{R}^{\widetilde{\widetilde{\nabla}}}_{\frac{\lambda_x}{2\lambda}\partial_x\,\partial_y}\xi_2 - \mathrm{R}^{\widetilde{\widetilde{\nabla}}}_{\partial_x\,\frac{\lambda_x}{2\lambda}\partial_y}\xi_2 - \mathrm{R}^{\widetilde{\widetilde{\nabla}}}_{\partial_x\partial_y}\widetilde{\nabla}_{\partial_x}\xi_2 \\
&= \widetilde{\nabla}_{\partial_x}(\mathrm{R}^{\widetilde{\widetilde{\nabla}}}_{\partial_x\partial_y}\xi_2) - \frac{\lambda_x}{\lambda}\mathrm{R}^{\widetilde{\widetilde{\nabla}}}_{\partial_x\,\partial_y}\xi_2 - \mathrm{R}^{\widetilde{\widetilde{\nabla}}}_{\partial_x\partial_y}\widetilde{\nabla}_{\partial_x}\xi_2 \\
&= \widetilde{\nabla}_{\partial_x}(-\kappa_y\lambda\xi_3) - \frac{\lambda_x}{\lambda}\mathrm{R}^{\widetilde{\widetilde{\nabla}}}_{\partial_x\,\partial_y}\xi_2 - \mathrm{R}^{\widetilde{\widetilde{\nabla}}}_{\partial_x\partial_y}\widetilde{\nabla}_{\partial_x}\xi_2 \\
&= (-\kappa_y\lambda)_x\xi_3 - \kappa_y\lambda\widetilde{\nabla}_{\partial_x}\xi_3 - \frac{\lambda_x}{\lambda}\mathrm{R}^{\widetilde{\widetilde{\nabla}}}_{\partial_x\,\partial_y}\xi_2 - \mathrm{R}^{\widetilde{\widetilde{\nabla}}}_{\partial_x\partial_y}\widetilde{\nabla}_{\partial_x}\xi_2 \\
&= (-\kappa_y\lambda)_x\xi_3 - \kappa_y\lambda\widetilde{\nabla}_{\partial_x}\xi_3 + \lambda_x\kappa_y\xi_3 - \mathrm{R}^{\widetilde{\widetilde{\nabla}}}_{\partial_x\partial_y}\widetilde{\nabla}_{\partial_x}\xi_2 \\
&= (-\kappa_y\lambda)_x\xi_3 + \kappa_y\lambda\xi_2 + \lambda_x\kappa_y\xi_3 - \mathrm{R}^{\widetilde{\widetilde{\nabla}}}_{\partial_x\partial_y}\widetilde{\nabla}_{\partial_x}\xi_2 \\
&= \kappa_y\lambda\xi_2 - \kappa_{xy}\lambda\xi_3 - \mathrm{R}^{\widetilde{\widetilde{\nabla}}}_{\partial_x\partial_y}\widetilde{\nabla}_{\partial_x}\xi_2 \\
&= \kappa_y\lambda\xi_2 - \kappa_{xy}\lambda\xi_3 + \mathrm{R}^{\widetilde{\widetilde{\nabla}}}_{\partial_x\partial_y}\left(\frac{\lambda_y}{2\lambda}\xi_1 + \frac{\lambda_x}{2\lambda}\xi_2 + \kappa\lambda\xi_3\right) \\
&= \kappa_y\lambda\xi_2 - \kappa_{xy}\lambda\xi_3 + \mathrm{R}^{\widetilde{\widetilde{\nabla}}}_{\partial_x\partial_y}\left(\frac{\lambda_y}{2\lambda}\xi_1 + \frac{\lambda_x}{2\lambda}\xi_2\right) \\
&= \kappa_y\lambda\xi_2 - \kappa_{xy}\lambda\xi_3 - \frac{\lambda_y\kappa_x}{2}\xi_3 - \frac{\lambda_x\kappa_y}{2}\xi_3 \\
&= \kappa_y\lambda\xi_2 - \left(\kappa_{xy}\lambda + \frac{\lambda_y\kappa_x}{2} + \frac{\lambda_x\kappa_y}{2}\right)\xi_3 \\
(\widetilde{\nabla}_{\partial_x}\mathrm{R}^{\widetilde{\nabla}})_{\partial_x\partial_y}\xi_3 &= \widetilde{\nabla}_{\partial_x}(\mathrm{R}^{\widetilde{\widetilde{\nabla}}}_{\partial_x\partial_y}\xi_3) - \mathrm{R}^{\widetilde{\widetilde{\nabla}}}_{\nabla_{\partial_x}\partial_x\,\partial_y}\xi_3 - \mathrm{R}^{\widetilde{\widetilde{\nabla}}}_{\partial_x\,\nabla_{\partial_x}\partial_y}\xi_3 - \mathrm{R}^{\widetilde{\widetilde{\nabla}}}_{\partial_x\partial_y}\widetilde{\nabla}_{\partial_x}\xi_3 \\
&= -\mathrm{R}^{\widetilde{\widetilde{\nabla}}}_{\partial_x\partial_y}\widetilde{\nabla}_{\partial_x}\xi_3 = \mathrm{R}^{\widetilde{\widetilde{\nabla}}}_{\partial_x\partial_y}\xi_2 = -\kappa_y\lambda\xi_3
\end{aligned}
$$

The lemma follows by taking into account the above computations. □

7.2 Proof of Theorem 3

Proof. Since the domain U of the Killing vector field X is assumed to be dense, it is enough to show that X can be extended about any point of the boundary of U. If the Killing vector field is zero then the lemma is trivial. So, we will assume that X is not zero; recall that this implies that the zero set of X is discrete. Let p_0 be a boundary point of U. Then Lemma 2 implies that either $(\widetilde{\nabla}_{\partial_x}\mathrm{R}^{\widetilde{\nabla}})_{\partial_x\partial_y}$ or $(\widetilde{\nabla}_{\partial_y}\mathrm{R}^{\widetilde{\nabla}})_{\partial_x\partial_y}$ has rank 2 in a small disk B_{p_0} of p_0. Assume that $(\widetilde{\nabla}_{\partial_x}\mathrm{R}^{\widetilde{\nabla}})_{\partial_x\partial_y}$ has rank 2 on B_{p_0}. Then the kernel of $(\widetilde{\nabla}_{\partial_x}\mathrm{R}^{\widetilde{\nabla}})_{\partial_x\partial_y}$ defines a smooth real line subbundle $\mathcal{L}$ of the restriction of $T\mathbb{R}^2 \oplus \mathfrak{so}(T\mathbb{R}^2)$ to B_{p_0}.

We claim that $\mathcal{L}$ is a flat parallel line bundle w.r.t. the Kostant connection $\widetilde{\nabla}$.

In fact, let ξ be a generator of $\mathcal{L}$ and Y any vector field of B_{p_0}. First we show that

$$\xi \wedge \widetilde{\nabla}_Y \xi \equiv 0. \tag{7}$$

Observe that, on the intersection $B_{p_0} \cap U$, the parallel section σ induced by the Killing vector field X must take values in $\mathcal{L}$. Since the zero set of X is discrete, the equality (7) holds on $B_{p_0} \cap U$, hence it holds on B_{p_0} in view of the fact we assumed that U is dense. This shows that any covariant derivative of the generator ξ is in $\mathcal{L}$, so $\mathcal{L}$ is $\widetilde{\nabla}$-parallel.

It follows that $\mathcal{L}$ is flat since σ is a parallel section taking values on $\mathcal{L}|_{B_{p_0}\cap U}$ with U a dense subset. Then the section σ can be extended to a parallel section of $\mathcal{L}$ on the whole of B_{p_0} as B_{p_0} is simply connected. This shows that X extends to a Killing vector field of $U \cup B_{p_0}$. □

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