The four-tank benchmark: a simple solution by embedded model control

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Congzhi Huang, *Member IEEE*, Enrico Canuto, *Senior Member IEEE,* Carlo Novara, *Member IEEE*

***Abstract*—The four-tank benchmark is a multivariate and nonlinear control problem which has been widely studied in the literature. Two pairs of tanks in series are supplied by two pumps. Under certain configurations, the Embedded Model Control approach provides a simple decoupled solution by separately controlling the two output tank levels and treating the input flow as a partly unknown disturbance. Neglected dynamics in a form of unknown delays both in sensors and actuator dynamics is considered. The core of the control unit is a discrete-time embedded model consisting of unknown disturbance dynamics and partly known nonlinear interactions. The embedded model is driven by the plant command and by a feedback vector which is retrieved from the model error. The feedback is capable of keeping updated the unknown disturbance prediction, ready to be cancelled by the control law. The control gains are tuned using two sets of closed-loop eigenvalues in order to trade-off between disturbance rejection and robust stability. Simulated runs under different tank interactions prove design effectiveness.**

I. INTRODUCTION

The classical four-tank benchmark in Fig. 1, initially proposed by Johansson in [1], is a test platform for comparing control strategies. The benchmark problem is a representation of multivariable control instances [2]. The problem has been investigated with laboratory tests by several scholars [3], [4].

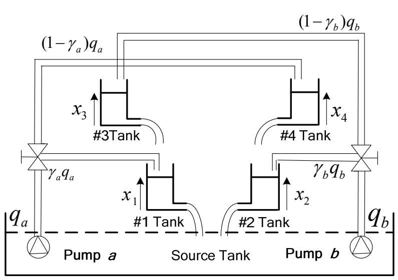

Fig. 1. Four-tank benchmark layout.

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Congzhi Huang is with School of Control and Computer Engineering, North China Electric Power University, Beijing, 102206, China. (e-mail: hcz190@ncepu.edu.cn).

Enrico Canuto and Carlo Novara are with Politecnico di Torino, Dipartimento di Automatica e Informatica, Corso Duca degli Abruzzi 24, 10129 Torino, Italy (e-mail: enrico.canuto@polito.it, carlo.novar@polito.it).

To deal with the issue of coupling and nonlinearity, various control approaches have been applied and their performance compared [5]. Two kinds of decentralized PI controllers were designed and validated in [6], [7]. To deal with minimum and non-minimum phase behavior, improved PID controllers were investigated in [8]. The four-tank benchmark problem provided a test-bed for validating model predictive control approach [9]. Different decentralized control strategies were proposed in [10]. A model predictive controller was designed and tested in [11]. A fast gradient-based distributed optimization approach was applied in [12]. A nonlinear generalized predictive control and a back-stepping approach were designed and tested in [13]. Besides the model predictive control, nonlinear control approach was also employed. A two-level control algorithm was developed in [14]. Nonlinear approach and dynamic optimization were validated in [15]. Feedback linearization with sliding mode algorithm was proposed in [16]. A fractional-order sliding mode controller was designed in [17]. A fuzzy modified model reference adaptive control approach was proposed in [18]. To deal with the multivariable dead times, decentralized integral controllability and time-domain bounds on closed loop performance were derived and discussed in [19].

Concerning perturbations in the process parameters and external disturbances, the active disturbance rejection control approach based on the concept of flatness was proposed in [20]. Other studies refer to simulation [21] and fault-diagnosis [22]. Most of the approaches focused on continuous-time control design. Modern implementation of controllers is discrete-time. An optimal discrete-time controller was proposed in [23].

The embedded model control (EMC for short, [24]) to be employed here, is a model-based approach centered on the inclusion in the control unit of a discrete-time model (embedded model, EM for short) of the controllable dynamics and of the disturbance to be rejected. The method has been applied to space control systems [25] and to other engineering applications.

The objective of the paper is to propose a simple solution to the four-tank benchmark using Embedded Model Control. We restrict to the case in which the dynamics of the four-tank benchmark is minimum phase. It corresponds to a stable zero-dynamics as pointed out in [20]. Under this restriction the controllable dynamics can be decoupled into two first order integrators (the output tanks #1 and #2 in Fig. 1), endowed with their own disturbance dynamics. The dynamics of the input tanks (#3 and #4 in Fig. 1) are included in the EM to estimate the input flow to the output tanks. The input tank level is not directly controlled, which prevents to accommodate non-minimum phase conditions. The complete solution will be subject of a future paper. Though restricted, the solution shows the simplicity of the EMC approach and how the whole range of uncertainty can be effectively treated.

The paper is organized as follows. In Section II, the four-tank state equations are recalled, and the uncertainty classes are defined. The zero-dynamic properties are obtained from a disturbance rejection control law. In Section III, by restricting to the case of a stable zero dynamics, the EM is derived as a set of four decoupled integrators each endowed with disturbance dynamics. The noise estimator is designed as a static feedback as in standard observers because of the decoupled controllable dynamics. The control law is then obtained. In Section IV, the noise estimator gains and the feedback control gains are tuned by fixing the closed loop eigenvalues. Asymptotic closed-loop transfer functions are employed to the purpose [26]. Simulation results are presented in Section V. Section VI concludes the paper.

II. Model and Uncertainty

A. State equations and Uncertainty

With reference to in Fig. 1, there are four tanks and their levels are measured by four pressure sensors. The tanks are supplied by two valves and two pumps, which are fed by the source tank. The fluid from the source tank is pumped into the #1 Tank and the #4 Tank through the pump *a*. The valve *a* is employed to separate the water into the #1 Tank with a fraction γ_a, and the #4 Tank with $1-\gamma_a$. Symmetrically, the fluid from pump *b* is fed into the #2 Tank with a fraction γ_b, and the #3 Tank with $1-\gamma_b$. Furthermore, the fluid in the #3 Tank flows into the #1 Tank, and then returns to the source tank, while the fluid in the #4 Tank discharges into the #2 Tank, and then discharges into the source tank. The variables to be controlled are the fluid levels of the #1 Tank and of the #2 Tank. Correspondingly, there are two control inputs: the flow q_a and q_b supplied by the pump *a* and pump *b*, respectively.

Based on mass balance and Bernoulli's law, the nonlinear dynamics of the four-tank process has the following affine state equations:

$$\begin{aligned}&\dot{x}(t)=Ap\left(x(t)\right)+Bq\left(t-\tau_u\right)+q_d(t),\ x(0)=x_0\\&y(t)=Cx\left(t-\tau_y\right)+e(t),\ z(t)=Fx(t)\\&0\le x(t)\le x_{\max},0\le q(t)\le q_{\max}\end{aligned}\tag{1}$$

where x is the tank fluid level (*m*), p is the tank discharge rate (*m/s*), q is the command flow (m^3/s), q_d is the unknown flow (*m/s*) whose signal class is described by a stationary stochastic process having bounded spectral density and variance, y is the measured tank level (*m*), e is the measurement error, and z is the performance variable to be controlled (the output tank level). Tank level and command flow are bounded. The zero limit of the tank level only corresponds to the level which does provide zero output flow p. Actuator and sensor delays are accounted for by τ_u and τ_y. Vector components and the expression of $p_k, k=1,2,3,4$, are given below:

$$\begin{aligned}&x=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix},p(x)=\begin{bmatrix}p_1(x_1)\\p_2(x_2)\\p_3(x_3)\\p_4(x_4)\end{bmatrix},q=\begin{bmatrix}q_a\\q_b\end{bmatrix},q_d=\begin{bmatrix}q_{d1}\\q_{d2}\\q_{d3}\\q_{d4}\end{bmatrix}\\&y=\begin{bmatrix}y_1\\y_2\\y_3\\y_4\end{bmatrix},e=\begin{bmatrix}e_1\\e_2\\e_3\\e_4\end{bmatrix},p_k(x)=\frac{a_k}{S_k}\sqrt{2gx_k}\end{aligned}\tag{2}$$

The state equation matrices are the following

$$\begin{aligned}&A=\begin{bmatrix}-1&0&1&0\\0&-1&0&1\\0&0&-1&0\\0&0&0&-1\end{bmatrix},\ C=\begin{bmatrix}c_1&0&0&0\\0&c_2&0&0\\0&0&c_3&0\\0&0&0&c_4\end{bmatrix}\\&B=\begin{bmatrix}\gamma_a/S_1&0\\0&\gamma_b/S_2\\0&(1-\gamma_b)/S_3\\(1-\gamma_a)/S_4&0\end{bmatrix},F=\begin{bmatrix}1&0&0&0\\0&1&0&0\end{bmatrix}\end{aligned}\tag{3}$$

All the parameters are uncertain but bounded. Given a generic parameter $\pi_j, j=1,\cdots,N_\pi$, the uncertainty is written as follows:

$$\pi_j=\pi_{j,nom}\left(1+\delta\pi_j\right),\left|\delta\pi_j\right|\le\delta\pi_{j,\max}\tag{4}$$

where $\pi_{j,\,nom}$ is the nominal value which is known to control designer and $\delta\pi_j$ is the bounded fractional uncertainty. The parameter ε_k in $c_k=1+\varepsilon_k$ accounts for scale factor errors.

The input relation with the discrete-time (DT) command $q(i)$ is the following conversion:

$$\begin{aligned}&q(t)=R_u\,\mathrm{int}\left(R_u^{-1}q(i)\right),R_u=\begin{bmatrix}\rho_a&0\\0&\rho_b\end{bmatrix}\\&0\le\mathrm{int}\left(R_u^{-1}q(i)\right)<\begin{bmatrix}1&1\end{bmatrix}^T N_u,\rho_h=q_{h\max}/N_u,h=a,b\end{aligned}\tag{5}$$

where ρ_h, $h=a,b$ is the quantization level obtained from the flow range $q_{h\max}$ in (1) and the integer range N_u, and i denotes a DT instant $t_i=iT$, T being the control time unit. The following conversion defines the output $y(i)$ used by the control unit:

$$\begin{aligned}&y(i)=R_y\,\mathrm{int}\left(R_y^{-1}y(t_i)\right),R_y=\mathrm{diag}\{\rho_1,\cdots,\rho_4\}\\&\rho_k=x_{k\max}/N_y,0\le\mathrm{int}\left(R_y^{-1}y(t_i)\right)<\begin{bmatrix}1&1&1&1\end{bmatrix}^T N_y\end{aligned}\tag{6}$$

Equations (5) and (6) are responsible of command and measurement quantization errors. The former ones are accounted for by the unknown disturbance q_d in (1). The latter ones by the output error e in (1).

Problem formulation. Given a set Z_{ref} of piecewise constant reference trajectories $z_{k,ref}(t)$, k=1, 2 for the fluid levels x_k,

k=1,2 guarantees that the 'true' tracking error $e_{k,ref}(t)= z_{k,ref}(t) - x_k(t)$ be bounded by:

$$\left|e_{k,ref}\left(t\right)\right| \le e_{k,\max}, \tau_s + t_j \le t < t_{j+1} \tag{7}$$

where t_j is the constant application time and τ_s is the settling time interval. The above inequality must hold within the class of uncertainty previously defined. □

B. Control Law and Zero Dynamics

In response to the above problem, it is assumed that a constant reference $x_{k,ref}(t)$, k=1, 2 is exactly tracked by x_k, k=1,2, which implies the disturbance rejection control law:

$$\begin{bmatrix} q_a \\ q_b \end{bmatrix}(t) = \begin{bmatrix} S_1\left(p_1 - p_3 - q_{d1}\right)/\gamma_a \\ S_2\left(p_2 - p_4 - q_{d2}\right)/\gamma_b \end{bmatrix}(t) \tag{8}$$

By replacing (8) in (1), a new state equation of x_k, k=3, 4, is obtained, which using

$$\dot{p}_k = \left(2S_k\sqrt{x_k}\right)^{-1} a_k\sqrt{2g}\dot{x}_k \tag{9}$$

and assuming S_k=S, can be rewritten in terms of the flow rates $p_k(x_k)$ as follows:

$$\begin{bmatrix} \dot{p}_3 \\ \dot{p}_4 \end{bmatrix} = \frac{S^2}{g}\left(-\begin{bmatrix} \frac{p_3^2}{a_3^2} & \frac{p_3 p_4}{a_3^2}\delta_b \\ \frac{p_4 p_3}{a_4^2}\delta_a & \frac{p_4^2}{a_4^2} \end{bmatrix} + \begin{bmatrix} \frac{p_3}{a_3^2}\delta_b p_1 \\ \frac{p_4}{a_4^2}\delta_a p_2 \end{bmatrix}\right) + \begin{bmatrix} d_3 \\ d_4 \end{bmatrix} \tag{10}$$

$$\delta_b = \frac{1-\gamma_b}{\gamma_b}, \delta_a = \frac{1-\gamma_a}{\gamma_a}$$

where the terms d_k are combinations of the components of q_d in (1). The following propositions from [20] help the analysis of (10).

Proposition 1. Given a flow rate ρ^* bounded as in (1) and a command u^* satisfying (8) under q_d=0, given the perturbation δ_p=$p - p^*$, the perturbation equation of (10) holds:

$$\begin{bmatrix} \delta\dot{p}_3 \\ \delta\dot{p}_4 \end{bmatrix} = -\frac{S^2}{g}\begin{bmatrix} \frac{p_3^*}{a_3^2} & \frac{p_3^*}{a_3^2}\delta_b \\ \frac{p_4^*}{a_4^2}\delta_a & \frac{p_4^*}{a_4^2} \end{bmatrix}\begin{bmatrix} \delta p_3 \\ \delta p_4 \end{bmatrix} + \frac{S^2}{g}\begin{bmatrix} \frac{p_3^*}{a_3^2}\delta_b\delta p_1 \\ \frac{p_4^*}{a_4^2}\delta_a\delta p_2 \end{bmatrix} + \begin{bmatrix} d_3 \\ d_4 \end{bmatrix} \tag{11}$$

The equation is asymptotically stable if and only if

$$\gamma_a + \gamma_b > 1\,. \tag{12}$$

Proof. The proof of (11) follows from (10) and (8). The inequality (12) is obtained from the characteristic polynomial of the state matrix. □

Proposition 2. Under the control law (8), (1) is transformed into the following normal form [27] which consists of

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}(t) = 0, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}(0) = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}, \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}(t) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}(t) \tag{13}$$

and of

$$\begin{bmatrix} \dot{p}_3 \\ \dot{p}_4 \end{bmatrix}(t) = \begin{bmatrix} f_3\left(p(x), q_d\right) \\ f_4\left(p(x), q_d\right) \end{bmatrix}, \begin{bmatrix} p_3 \\ p_4 \end{bmatrix}(0) = \begin{bmatrix} p_{30} \\ p_{40} \end{bmatrix} \tag{14}$$

In (13), z_k, k=1, 2, is the performance variable to be controlled. The first equation is linear time-invariant and can be stabilizable by adding to(8) the feedback component

$$\Delta q(t) = \begin{bmatrix} S_1F_1/\gamma_a & 0 \\ 0 & S_2F_2/\gamma_b \end{bmatrix}\begin{bmatrix} z_{1,ref} - x_1 \\ z_{2,ref} - x_2 \end{bmatrix} \tag{15}$$

The second equation is the zero dynamics. Proposition 1 and inequality (12) show that it may be tangentially unstable.

Proof. Equation (13) follows by applying (8) to (1). Stabilization of the first equation in (13) is elementary. □

The overall stabilization has been tackled in [20] by controlling a suitable combination of the tank levels, which does not coincide with the level of the output tanks as requested by the problem formulation. Here we restrict to the stable zero-dynamics case, *i.e.*, γ_a+γ_b>1, since we directly control only the output tank levels through (8) and (15). In this case the input tank levels x_k, k=3,4 may fluctuate under the action of external disturbances, whereas the output tank levels x_k, k=1, 2 are demanded to meet the requirements of the problem formulation. In the case γ_a+$\gamma_b \leqslant 1$, the input tank level may diverge reaching the lower and upper limits in (1), which in turn may demand unfeasible pump flows.

III. Embedded Model, Noise Estimator, Control Law

Each measured level y_k, k=1, ⋯,4 is associated with a third order unstable and not stabilizable state equation:

$$\begin{bmatrix} x_k \\ x_{dk} \\ x_{sk} \end{bmatrix}(i+1) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_k \\ x_{dk} \\ x_{sk} \end{bmatrix}(i) + \begin{bmatrix} b_k \\ 0 \\ 0 \end{bmatrix}u_k(i) + \begin{bmatrix} h_k(x) \\ 0 \\ 0 \end{bmatrix} + w_k(i), w_k = \begin{bmatrix} w_{xk} \\ w_{dk} \\ w_{sk} \end{bmatrix}, y_k = x_k + e_{mk} \tag{16}$$

Actuator and sensor delays are neglected. The time unit T must be designed. The overall disturbance d_k affecting the level x_k is the combination of the partly known cross-coupling $h_k(x)$ (less parametric uncertainty), of the unknown $x_{dk} + w_x$:

$$d_k(i) = x_{dk}(i) + w_x(i) + h_k(x) \tag{17}$$

The former term in (17) is the output of a second order random drift capable of encompassing a large class of stochastic and deterministic signals; the second term is an arbitrary zero-mean signal class (in the statistical framework, a DT white nose). Relations between (16) and (1) are

$$\begin{aligned} h_1(x) &= T\left(-p_{1,nom}(x_1) + p_{3,nom}(x_3)\right) \\ h_2(x) &= T\left(-p_{2,nom}(x_1) + p_{4,nom}(x_3)\right) \\ h_3(x) &= -p_{3,nom}(x_1)T,\ h_4(x) = -p_{4,nom}(x_1)T \end{aligned} \tag{18}$$

and

$$b_1 = T\gamma_{a,nom} / S_{nom},\ b_2 = T\gamma_{b,nom} / S_{nom}$$
$$b_3 = T\left(1-\gamma_{b,nom}\right) / S_{nom}, b_4 = T\left(1-\gamma_{a,nom}\right) / S_{nom}\ . \quad (19)$$
$$u_1 = u_4 = q_a,\ u_2 = u_3 = q_b$$

The subscript *nom* accounts for nominal parameters.

Since each equation (16) possesses its own measured output, the noise vector w_k can be estimated by a static feedback (noise estimator) as in standard state observers. The feedback is driven by the model error e_{mk} in (16):

$$w_k\left(i\right) = L_k e_{mk}\left(i\right), L_k = \left[l_{xk} \quad l_{dk} \quad l_{sk}\right]^T \quad (20)$$

The pair (16) and (20) is a closed-loop state predictor. The gain matrix L_k in (20) is obtained by fixing the state predictor spectrum $\Lambda_{sk}=\{1-\gamma_{sk1},\ 1-\gamma_{sk2},\ 1-\gamma_{sk3}\}$. The parameter γ_{skj} is a DT complementary eigenvalue and approximates the Fourier frequency f_{skj} (Hz) as follows

$$f_{skj} \cong \left(2\pi T\right)^{-1} \gamma_{skj} \quad (21)$$

The approximation tends to be exact as soon as $\gamma_{skj}\to 0$. The relationship between L_k and Λ_{sk} is simple and in the case of equal eigenvalues, $\gamma_{skj} = \gamma_{sk}$, j=1, 2, 3, holds.

$$l_{xk} = 3\gamma_{sk}, l_{dk} = 3\gamma_{sk}^2, l_{sk} = \gamma_{sk}^3 \quad (22)$$

By disposing of the state prediction $x_k(i+1)$ (controllable state) and $x_{dk}(i+1)$ (disturbance state entering (17)), the DT control law repeating (8) and (15) is immediate. The command $u_k(i)$, k=1, 2, which is pre-computed during the step $i-1$ holds:

$$S_k u_k\left(i\right) / \gamma_k = F_k\left(z_{k,ref}\left(i\right) - x_k\left(i\right)\right) - h_k\left(x\left(i\right)\right) - x_{dk}\left(i\right) \quad (23)$$

IV. Gain tuning

Consider a single controlled tank level x_k, k=1, 2. It has been proven in [26] that the tracking error $e_{k,ref}$ in (7) is the output of the following error equation:

$$e_k\left(i+1\right) = \begin{bmatrix} 1-l_{kx} & 1 & 0 & 0 \\ -l_{dk} & 1 & 1 & 0 \\ -l_{sk} & 0 & 1 & 0 \\ -l_{xk} & 0 & 0 & 1-F_k \end{bmatrix} e_k - \begin{bmatrix} l_{xk} \\ l_{dk} \\ l_{sk} \\ -l_{xk} \end{bmatrix}\left(e_{mk} + d_{yk}\right) \quad (24)$$
$$e_{k,ref}\left(i\right) = \begin{bmatrix}1 & 0 & 0 & -1\end{bmatrix} e_k\left(i\right) + d_{yk}\left(i\right)$$

where d_k is the total disturbance in (17) but shifted to the output (to this end it is integrated), and e_{mk} is the model error in (16). In terms of transfer functions, (24) can be shown [24] to convert into

$$e_{ref,k}\left(z\right) = -V_k\left(z\right) e_{mk}\left(z\right) + S_k\left(z\right) d_{yk}\left(z\right) \quad (25)$$

where S_k, and V_k are the sensitivity and the complementary sensitivity, respectively.

Robust stability versus neglected dynamics is guaranteed by the high-frequency decay of $|V_k(jf)|$, whereas the performance in (7) is guaranteed by the low-frequency asymptote of $|S_k(jf)|$. An asymptotic design procedure has been proved and shown in [26]. The following low high frequency asymptotes is obtained from (22):

$$\begin{aligned} V_{\infty k}\left(z\right) &= \frac{l_{dk} + F_k l_{xk}}{\left(z-1\right)^2} = 3\gamma_{sk}\frac{\gamma_{sk} + F_k}{\left(z-1\right)^2} \\ \left|V_{\infty k}\left(jf\right)\right| &\cong 3\left(f_{sk} / f\right)^2\left(1 + f_{Fk} / f_{sk}\right) \\ S_{0k}\left(z\right) &= \frac{\left(z-1\right)^3\left(F_k + l_{xk}\right)}{F_k l_{sk}} = \left(z-1\right)^3\frac{3\gamma_{sk} / F_k + 1}{\gamma_{sk}^3} \\ \left|S_{0k}\left(jf\right)\right| &\cong \left(f / f_{sk}\right)^3\left(1 + 3f_{sk} / f_{Fk}\right) \end{aligned} \quad (26)$$

In (26) f_{sk} plays the role of the state predictor BW, whereas $f_{Fk}>f_{sk}$ plays the role that of the state feedback BW. It is immediate to perceive their design roles. At first sight the time unit T does not enter (26), but the largest BW, in practice f_{Fk} must be less than the Nyquist frequency $f_{max}=0.5/T$, *i.e.* ,

$$f_{Fk} < f_{\max} / 5 \quad (27)$$

Now we assumed that in (7):

$$\tau_s < 200\ s,\ e_{k,\max} = 3\rho_k \quad (28)$$

where ρ_k has been defined in (6). The settling time τ_s is of the same order of the nominal discharge time constant:

$$\tau_k = \frac{a_k}{S_k}\sqrt{\frac{2x_{k,ref}}{g}} \approx 150 \sim 250s \quad (29)$$

The settling time τ_s is imposed by the state predictor BW f_{sk}, which implies the preliminary design scale:

$$\begin{aligned} &f_{sk} > \frac{10}{2\pi\tau_s} \approx 0.01\text{ Hz}, f_{Fk} > 5f_{sk} \cong 0.05\text{ Hz} \\ &f_{\max} > 10f_{Fk} \cong 0.5\text{ Hz} \end{aligned} \quad (30)$$

A further inequality comes from the second requirement in (28) in presence of unknown disturbances. S_{0k} in (26) being third order, may reduce third order drifts on the output to a residual white noise. For instance a third-order drift in d_{yk} having spectral density (PSD for short hereinafter):

$$S_{dy}\left(f\right) = S_{d0}\left(f_d / f\right)^3 \quad (31)$$

is reduced to a residual white noise with spectral density:

$$S_w\left(f\right) = \left|S_{ok}\left(jf\right)\right| S_{dy}\left(f\right) \cong S_{d0}\left(f_d / f_{sk}\right)^3 \quad (32)$$

Then assuming flat spectral density in (32), yields in the following design inequality:

$$f_{sk} > f_d\left(S_{d0}\sqrt{T/2} / \rho_k\right)^{1/3} \quad (33)$$

Similar inequalities can be obtained from the parametric uncertainty.

The design in (25) must be verified against the neglected dynamics. Assuming a total delay $\tau_y+\tau_u=nT$, the effect on (25) is that of a further loop which is expressed by the left hand side term as follows:

$$\begin{aligned} &\left(1 - V_k\left(z\right)\left(1 - z^{-n}\right)\right) e_{ref,k}\left(z\right) = \\ &= V_k\left(z\right) e_{mk}\left(z\right) - S_k\left(z\right) M_k\left(z\right) d_k\left(z\right) \end{aligned} \quad (34)$$

Closed stability is guaranteed (small gain theorem) if

$$\max_{|f|<f_{\max}} \left|V_k\left(jf\right)\left(1-e^{-j2\pi fnT}\right)\right| \le \eta < 1 \qquad (35)$$

where η^{-1} is the gain margin. Since the peak value of the last factor expressing the delay model error is equal to 2 and is achieved at $f \cong 0.1\, f_{max}/n$, the following inequality holds:

$$f_{Fk} < \eta f_{\max} / n \cong 0.05 \text{ Hz}, n \le 3, \ \eta \ge 0.3 \qquad (36)$$

V. SIMULATED RESULTS

The simulated parameters are listed in Table I.

TABLE I. SIMULATED PARAMETERS

Parameter	Value	Unit	Description and equation
S_{nom}	0.06	m^2	Nominal cross-section area (2)
a_k	0.9~1.5	cm^2	Discharge constant (2)
$x_{k,max}$	1.36	m	Maximum tank level (1)
γ_a, γ_b	> 0.5		Valve aperture fraction (3)
x_{0k}	0.66	m	Initial level (1)
$q_{a,max}$, $q_{b,max}$	1.1	dm^3/s	Maximum pump flow (1)
ρ_a, ρ_b	1.0	cm^3/s	Command quantization (5)
ρ_k	1.3	mm	Output quantization (6)
T	1	s	Time unit (17)
f_{max}	0.5	Hz	Nyquist frequency (28)
nT	2	s	Total delay (32)
f_d	0.5～1	mHz	Disturbance cutoff frequency (29)
S_{d0}	≈0.1	$\text{m}/\sqrt{\text{Hz}}$	Low-frequency PSD (29)
$\delta\pi_j$	<0.1		Fractional uncertainty (4)

The control parameters are listed in Table II and are derived from the design outlined in Section IV. Manual optimization has been done in order to reduce the overshoot of the sensitivity $S(z)$ defined in (25)

TABLE II. CONTROL PARAMETERS AND PERFORMANCE

Parameter	Value	Unit	Description and equation
Λ_{sk}, f_{skj}	0.008～0.016	Hz	State predictor BW (19), (28)
f_{Fk}	0.08	Hz	Feedback BW (28)
τ_s	<200	s	Settling time (26)
$e_{k,\,max}$	<1	mm	Tracking error (26)

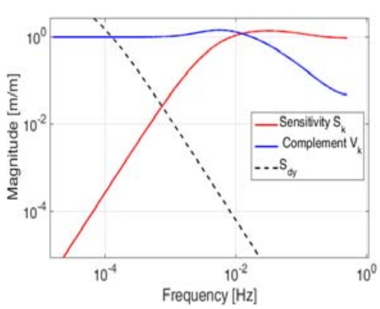

Fig. 2. Sensitivity, complementary sensitivity and disturbance spectral density.

The magnitude of the sensitivity S, of the complementary sensitivity V and of the integrated disturbance spectral density S_{dv} (shifted to the output, $\text{m}/\sqrt{\text{Hz}}$) are shown in Fig. 2. The residual flat spectral density S_w in (32) can be estimated by the product at the intersection in Fig. 2. The corresponding residual standard deviation looks in agreement with performance (7) and (28) as reported in row 4 of Table II.

Fig. 3 shows the simulated tank levels: x_1 and x_2 (output tanks) are under control and they repeat the reference levels. x_3 and x_4 are out of control and they widely fluctuate. Their fluctuation versus the output tank tracking error, which is shown in Fig. 4 and Fig. 5, prove the design effectiveness.

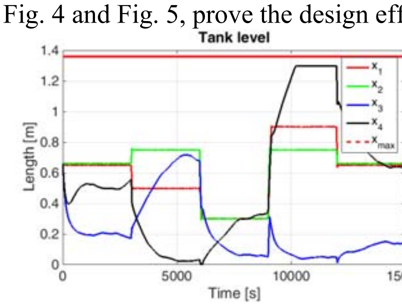

Fig. 3. Regulated and not regulated tank levels.

Fig. 4 and the enlargement in Fig. 5 show the tracking errors of the output tanks defined in (7). The enlargement shows that requirements in (28) are met. Specifically, Fig. 5 shows the limit cycle imposed by the quantization errors and the neglected dynamics (delays neglected by the EM). The magnitude is less than the output quantization ρ_k.

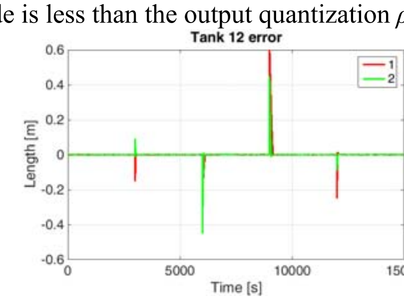

Fig. 4. Tracking error of the output tank level.

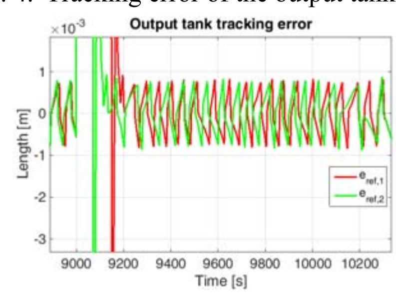

Fig. 5. Output tank tracking error (enlargement).

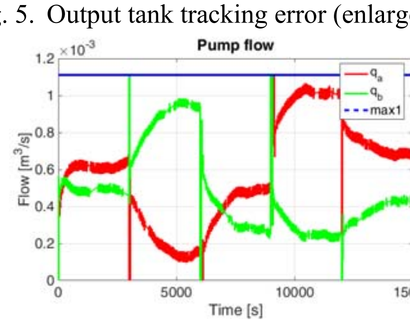

Fig. 6. Pump commanded flow.

Fig. 6 shows the pump commanded flows. They saturate during the sharp transient imposed by a stepwise reference signal. Saturation causes no detriment since the same command is dispatched to the plant and to the EM. To avoid saturation, reference profile should be smoothed.

Finally, Fig. 7 (analogous of Fig. 4) shows that a control strategy (8) and (15), restricted to the sole output tanks can not accommodate zero-dynamics instability as proved in Section II.B. Under the same conditions of the stable case, but with $\gamma_a + \gamma_b < 1$, Fig. 7 shows that the reference level cannot be always tracked with required accuracy.

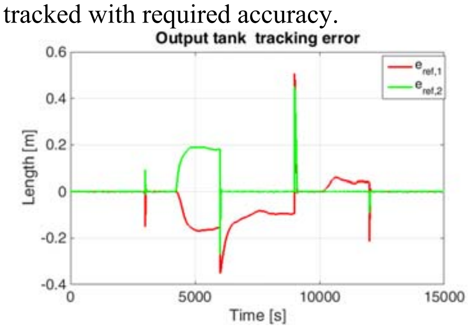

Fig. 7. Tracking error of the output tank in the unstable zero dynamics case.

VI. Conclusion

The application of EMC approach is investigated in the four- tank benchmark. The zero-dynamics is assumed to be stable in the control design, which allows a decoupled control of the sole output tanks. Under such a restricted condition, the proposed control approach is directly designed in discrete time and centered on a simple embedded model of each tank. The embedded model includes a second-order stochastic dynamics to predict and reject the unknown disturbances including parametric uncertainty. The effectiveness of the proposed approach and the achievement of the required performance are validated by the extensive simulation results. A control strategy capable of accommodating zero-dynamics instability is under study.

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