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(Article begins on next page)

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HOMOGENIZATION OF DISCRETE HIGH-CONTRAST ENERGIES[*]

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Abstract. This paper focuses on deriving double-porosity models from simple high-contrast atomistic interactions. Using the variational approach and Γ-convergence techniques we derive the effective double-porosity type problem and prove the convergence. We also consider the dynamical case and study the asymptotic behavior of solutions for the gradient flow of the corresponding discrete functionals.

Key words. double porosity, gamma-convergence, homogenization, discrete to continuum

AMS subject classifications. 49M25, 39A12, 39A70

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1. Introduction. Variational theories of double-porosity models can be derived by homogenization of high-contrast periodic media (see [12]). Typically, we have one or more strong phases (i.e., uniformly elliptic energies on periodic connected domains) and a weak phase with a small ellipticity constant, coupled via some lower-order term. We consider the energies that satisfy p-growth conditions and are consistent with the high-contrast structure of the medium. Our goal is to homogenize these high-contrast elliptic energies and to derive the so-called double-porosity continuous model. In the simplest case of quadratic energies, this amounts to considering energies of the form

$$(1.1) \qquad \sum_{j=1}^{N} \int_{\Omega \cap \varepsilon C_j} |\nabla u|^2 \, dx + \varepsilon^2 \int_{\Omega \cap \varepsilon C_0} |\nabla u|^2 + K \int_{\Omega} |u - u_0|^2 \, dx,$$

where ε is a geometric parameter representing the scale of the media. The strong components are modeled for $j = 1, \ldots, N$ by periodic connected Lipschitz sets C_j of $\mathbb{R}^d$ with pairwise disjoint closures; in this notation C_0 is their complement and represents the weak phase. Note that we may have $N > 1$ only in dimension $d \geq 3$, while in dimension $d = 2$ this model represents a single strong medium with weak inclusions (i.e., the set C_0 is composed of disjoint bounded components). In dimension $d = 1$ the energy trivializes since C_0 must be empty and the energy is then ε-independent. The scaling ε^2 in front of the weak phase is chosen so that the limit is nontrivial; the analyses for all other scalings are derived from this one by comparison.

If we let $\varepsilon \to 0$ these energies are approximated by their Γ-limit [9, 10], which combines the homogenized energies of each strong medium (which exist by [1, 12]) and a coupling term. Note that the energies above are not strongly coercive in L^2. They are weakly coercive in L^2, but their limit is more meaningful if computed with respect to some topology which takes into account the strong limit of the functions on each strong component (or, more precisely, of the extensions of the restrictions of

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functions on each εC_j, which are taken into account by the fundamental lemma by Acerbi et al. [1]). In this way a convergence $u_\varepsilon \to (u_1, \ldots, u_N)$ is defined; the limit then depends on these N independent functions, and takes the form

$$(1.2) \quad \sum_{j=1}^N \int_\Omega \left(\langle A_{\mathrm{hom}}^j \nabla u_j, \nabla u_j \rangle + K c_j |u_j - u_0|^2 \right) dx + \int_\Omega \varphi(u_0, u_1, \ldots, u_N) \, dx,$$

where c_j are the volume fractions of the strong components and φ is a quadratic function taking into account the interaction between the macroscopic phases. Note that the lower-order term is not continuous with respect to the convergence $u_\varepsilon \to (u_1, \ldots, u_N)$, which explains the appearance of an interaction term, whose computation in general involves a minimum problem on the weak phase C_0 (see [12] for results in the general framework of p-growth Sobolev energies, [18, 19] for perimeter energies, and [14] for free-discontinuity problems).

In this paper we derive double-porosity models from very simple atomistic interactions. Again in the case of quadratic energies, we may write the microscopic energies (in the case of the cubic lattice $\mathbb{Z}^d$) as

$$(1.3) \quad \sum_{(\alpha,\beta) \in \varepsilon \mathcal{N}_1 \cap (\Omega \times \Omega)} \varepsilon^d \left| \frac{u_\alpha - u_\beta}{\varepsilon} \right|^2$$

$$+ \varepsilon^2 \sum_{(\alpha,\beta) \in \varepsilon \mathcal{N}_0 \cap (\Omega \times \Omega)} \varepsilon^d \left| \frac{u_\alpha - u_\beta}{\varepsilon} \right|^2 + K \sum_{\alpha \in \varepsilon \mathbb{Z}^d \cap \Omega} \varepsilon^d |u_\alpha - (u_0)_\alpha|^2.$$

For explicatory purposes here we use a simplified notation with respect to the rest of the paper, and we denote by $\mathcal{N}_1$ the set of pairs in $\mathbb{Z}^d \times \mathbb{Z}^d$ between which we have strong interactions, and by $\mathcal{N}_0$ the set of pairs in $\mathbb{Z}^d \times \mathbb{Z}^d$ between which we have weak interactions; we assume that $\mathcal{N}_1$ and $\mathcal{N}_0$ are periodic sets. The energies depend on discrete functions whose values u_α are defined for $\alpha \in \varepsilon \mathbb{Z}^d$. Connected graphs of points linked by strong interactions play the role that in the continuum models is played by the sets C_j $(j \neq 0)$. In order to define a limit continuous parameter, we have to suppose that at least one such infinite connected graph exists, in which case we may take the limit of (extension of) piecewise-constant interpolations of u_α on this graph as a continuous parameter. If we have more such infinite connected graphs the limit is described again by an array $(u_1, \ldots, u_N)$. In the more precise notation of this paper below we directly define the (analogues of the) C_j and derive the corresponding strong and weak interactions accordingly. Note that weak interactions in $\mathcal{N}_0$ are due either to the existence of "weak sites" or to weak bonds between different "strong components," and, if we have more than one strong graph, the interactions in $\mathcal{N}_0$ are present also in the absence of a weak component. Under such assumptions, the limit is again of the form (1.2). In the paper we treat the general case of vector-valued u_α, where the energy densities are given by some asymptotic formulas.

It is interesting to compare the results of the paper with the existing results in [20] (see also [17] for the nonquadratic case), where the homogenization results for singular periodic structures and periodic measures have been obtained, and, in particular, for high-contrast Lagrangians and operators, in the case of critical scaling, the double porosity model has been derived. It should be noted that the 2-connectedness (or p-connectedness) condition used in [20] is replaced in the discrete case under consideration with the assumption that each strong phase has an unbounded connected component and the maximal connected component is unique.

Notice also that the contribution of each strong phase to the elliptic part of the limit Lagrangian is quasi convex. It follows from the fact that this contribution is given by the homogenized Lagrangian for the Neumann problem defined on the corresponding unbounded connected component. Quasi convexity of the effective Lagrangian for perforated domains with Neumann boundary condition is known; see [13].

From the description of Γ-limits we also derive dynamic results using the theory of minimizing movements. Under convexity assumptions, in that framework, the behavior of gradient flows of a sequence F_ε is described by the analysis of discrete trajectories $u_j^{\tau,\varepsilon}$ defined iteratively as minimizers of

$$F_\varepsilon(u) + \frac{1}{2\tau}\|u - u_{j-1}^{\tau,\varepsilon}\|^2$$

with τ a time step (in our case the norm is the L^2-norm for discrete functions). In our case, we take as F_ε the energies above without lower-order term (i.e., with $K = 0$). We first show the strong convergence of $u_j^{\tau,\varepsilon}$ as $\varepsilon \to 0$. In this way, we can treat these functions as fixed and apply the static limit results with $K = 1/(2\tau)$. We may then follow the theory for equi-coercive and convex functionals, for which gradient-flow dynamics commutes with the static limit (see [11, 4, 5]). As a result we show that the limit is described by a coupled system of PDEs (in the strong phases) and ODEs (parameterized by the weak phase). It is interesting to note that this latter parameterization is easily obtained by a discrete two-scale limit of the trajectories. Two-scale Γ-convergence was previously used in [15] for studying high-contrast Lagrangians satisfying p-growth conditions.

We finally note that in the discrete environment the topological requirements governing the interactions between the strong and weak phases are replaced with assumptions on long-range interactions. In particular, for discrete systems with second-neighbor interactions we may have a multiphase limit also in dimension one.

2. Notation. The numbers d, m, T, and N are positive integers. We introduce a T periodic *label function* $J : \mathbb{Z}^d \to \{0, 1 \ldots, N\}$ and the corresponding sets of sites

$$A_j = \{k \in \mathbb{Z}^d : J(k) = j\}, \qquad j = 0, \ldots, N.$$

Sites interact through possibly long- (but finite-) range interactions, whose range is defined through finite subsets $P_j \subset \mathbb{Z}^d$, $j = 0, \ldots, N$. Each P_j is symmetric and $0 \in P_j$.

We say that two points $k, k' \in A_j$ are P_j-*connected* in A_j if there exists a path $\{k_n\}_{n=0,\ldots,L}$ such that $k_n \in A_j$, $k_0 = k$, $k_L = k'$, and $k_n - k_{n-1} \in P_j$.

We suppose that there exists a unique infinite P_j-connected component of each A_j for $j = 1, \ldots, N$, which we denote by C_j. Note that we do not make any such assumption for A_0.

We consider the following sets of bonds between sites in $\mathbb{Z}^d$: for $j = 1, \ldots, N$

$$N_j = \{(k, k') : k, k' \in A_j, k - k' \in P_j \setminus \{0\}\};$$

for $j = 0$

$$N_0 = \{(k, k') : k - k' \in P_0 \setminus \{0\}, J(k)J(k') = 0 \text{ or } J(k) \neq J(k')\}.$$

Note that the set N_0 takes into account interactions not only among points of the set A_0, but also among pair of points in different A_j, $j = 0, \ldots, N$. A more refined model

could be introduced by defining range of interactions P_{ij} and the corresponding sets N_{ij}, in which case the sets N_j would correspond to N_{jj} for $j = 1, \ldots, N$ and N_0 the union of the remaining sets. However, for simplicity of presentation we limit our notation to a single index.

We consider interaction energy densities $f : \mathbb{Z}^d \times \mathbb{Z}^d \times \mathbb{R}^m \to \mathbb{R}$ and $g : \mathbb{Z}^d \times \mathbb{R}^m \to \mathbb{R}$. Note that the values of the function $f(k, k', z)$ will be considered only for (k, k') belonging to some N_j. The functions f and g satisfy the following conditions: $f(k, k', z) = f(k', k, z)$ (this is not a restriction up to substituting $f(k, k', z)$ with $\frac{1}{2}(f(k, k', z) + f(k', k, z))$) and there exists $p > 1$ such that

$$(2.1) \qquad c(|z|^p - 1) \le f(k, k', z) \le C(|z|^p + 1), \qquad 0 \le f(k, k', z),$$

$$(2.2) \qquad |f(k, k', z) - f(k, k', z')| \le C|z - z'|\left(1 + |z|^{p-1} + |z'|^{p-1}\right),$$

$$(2.3) \qquad f(k, k', \cdot) \text{ is positively homogeneous of degree } p \text{ if } (k, k') \in N_0,$$

$$(2.4) \qquad 0 \le g(k, u) \le C(|z|^p + 1),$$

$$(2.5) \qquad |g(k, z) - g(k, z')| \le C|z - z'|\left(1 + |z|^{p-1} + |z'|^{p-1}\right).$$

Given a bounded Lipschitz domain $\Omega \in \mathbb{R}^d$, we define the energies

$$F_\varepsilon(u) = F_\varepsilon\left(u, \frac{1}{\varepsilon}\Omega\right) = \sum_{j=1}^{N} \sum_{(k,k') \in \mathcal{N}_j^\varepsilon(\Omega)} \varepsilon^d f\left(k, k', \frac{u_k - u_{k'}}{\varepsilon}\right)$$

$$(2.6) \qquad + \sum_{(k,k') \in \mathcal{N}_0^\varepsilon(\Omega)} \varepsilon^{d+p} f\left(k, k', \frac{u_k - u_{k'}}{\varepsilon}\right) + \sum_{k \in Z^\varepsilon(\Omega)} \varepsilon^d g(k, u_k),$$

where

$$(2.7) \qquad \mathcal{N}_j^\varepsilon(\Omega) = N_j \cap \frac{1}{\varepsilon}(\Omega \times \Omega), j = 0, \ldots, N, \qquad Z^\varepsilon(\Omega) = \mathbb{Z}^d \cap \frac{1}{\varepsilon}\Omega.$$

The energy is defined on discrete functions $u : \frac{1}{\varepsilon}\Omega \cap \mathbb{Z}^d \to \mathbb{R}^m$.

The first sum in the energy takes into account all interactions between points in A_j (*hard phases*), which are supposed to scale differently than those between points in A_0 (*soft phase*) or in different phases. The latter are contained in the second sum. The third sum is a zero-order term taking into account with the same scaling all types of phases.

Note that the first sum may take into account also points in $A_j \setminus C_j$, which form "islands" of the hard phase P_j-disconnected from the corresponding infinite component. Furthermore, in this energy we may have sites that do not interact at all with hard phases.

3. Homogenization of "perforated" discrete domains. In this section we separately consider the interactions in each infinite connected component of hard phase introduced above. To that end we fix one of the indices j, $j > 0$, dropping it in the notation of this section (in particular we use the symbol C in place of C_j, etc.), and define the energies

$$(3.1) \qquad \mathcal{F}_\varepsilon(u) = \mathcal{F}_\varepsilon\left(u, \frac{1}{\varepsilon}\Omega\right) = \sum_{(k,k')\in N_C^\varepsilon(\Omega)} \varepsilon^d f\left(k, k', \frac{u_k - u_{k'}}{\varepsilon}\right),$$

where

$$(3.2) \qquad N_C^\varepsilon(\Omega) = \left\{(k, k') \in (C \times C) \cap \frac{1}{\varepsilon}(\Omega \times \Omega) : k - k' \in P, k \neq k'\right\}.$$

We also introduce the notation $C^\varepsilon(\Omega) = C \cap \frac{1}{\varepsilon}\Omega$.

DEFINITION 3.1. The *piecewise-constant interpolation* of a function $u : \mathbb{Z}^d \cap \frac{1}{\varepsilon}\Omega \to \mathbb{R}^m$, $k \mapsto u_k$, is defined as

$$u(x) = u_{\lfloor x/\varepsilon \rfloor},$$

where $\lfloor y \rfloor = (\lfloor y_1 \rfloor, \ldots, \lfloor y_d \rfloor)$ and $\lfloor s \rfloor$ stands for the integer part of s. The *convergence* of a sequence (u^ε) of discrete functions is understood as the $L^1_{\text{loc}}(\Omega)$ convergence of these piecewise-constant interpolations. Note that, since we consider local convergence in Ω, the value of $u(x)$ close to the boundary is not involved in the convergence process.

We prove an extension and compactness lemma with respect to the convergence of piecewise-constant interpolations.

LEMMA 3.2 (extension and compactness). *Let $u^\varepsilon : \frac{1}{\varepsilon}\Omega \to \mathbb{R}^m$ be a sequence such that*

$$(3.3) \qquad \sup_\varepsilon\left\{ \sum_{(k,k')\in N_C^\varepsilon(\Omega)} \varepsilon^d \left|\frac{u_k^\varepsilon - u_{k'}^\varepsilon}{\varepsilon}\right|^p + \sum_{k\in C^\varepsilon(\Omega)} \varepsilon^d |u_k^\varepsilon| \right\} < +\infty.$$

Then there exists a sequence $\widetilde{u}^\varepsilon : \frac{1}{\varepsilon}\Omega \to \mathbb{R}^m$ such that $\widetilde{u}_k^\varepsilon = u_k^\varepsilon$ if $k \in C^\varepsilon(\Omega)$ and $\text{dist}(k, \partial\frac{1}{\varepsilon}\Omega) > C(T, p, d, m)$, with $\widetilde{u}^\varepsilon$ converging up to subsequences to $u \in W^{1,p}(\Omega)$ in $L^1_{\text{loc}}(\Omega)$.

Proof. It suffices to treat the scalar case $m = 1$, up to arguing componentwise.

With fixed $i \in \mathbb{Z}^d$ we consider a periodicity cell $Y_i = iT + Y$, where $Y = [0, T)^d \cap \mathbb{Z}^d$. If we consider $k \in C \cap Y_i$ and $k' \in C \cap Y_i'$, where Y_i' is either Y_i or a neighboring periodicity cell, then the minimal path in C connecting k and k' lies in a periodicity cube $\widetilde{Y}_i = iT + [-DT, (D+1)T)^d$ for some positive integer D. We suppose that such $\widetilde{Y}_i$ is contained in $\frac{1}{\varepsilon}\Omega$. This holds if

$$(3.4) \qquad \text{dist}\left(Y_i, \partial\frac{1}{\varepsilon}\Omega\right) > C(T)$$

for some $C(T)$.

We define

$$\widetilde{u}_k^\varepsilon = \frac{1}{\#(C \cap Y)} \sum_{l\in C\cap Y_i} u_l^\varepsilon \qquad \text{for } k \in Y_i \setminus C.$$

For $k \in Y_i$ and $|k - k'| = 1$ (in the notation above $k' \in Y_i'$) we have

$$\varepsilon^d \left| \frac{\widetilde{u}_k^\varepsilon - \widetilde{u}_{k'}^\varepsilon}{\varepsilon} \right|^p \leq \varepsilon^{d-p} \left| \max_{Y_i \cup Y_i'} u^\varepsilon - \min_{Y_i \cup Y_i'} u^\varepsilon \right|^p = \varepsilon^{d-p} |u_l^\varepsilon - u_{l'}^\varepsilon|^p$$

for some $l, l' \in Y_i \cup Y_i'$. We then may take a path $\{l_n\}_{n=1...L}$ in C connecting l and l' lying in $\widetilde{Y}_i$. We then have

$$\varepsilon^d \left| \frac{\widetilde{u}_k^\varepsilon - \widetilde{u}_{k'}^\varepsilon}{\varepsilon} \right|^p \leq C \sum_{n=1}^{L} \varepsilon^d \left| \frac{u_{l_n}^\varepsilon - u_{l_{n-1}}^\varepsilon}{\varepsilon} \right|^p \leq C \sum_{j-j' \in P, j,j' \in \widetilde{Y}_i \cap C} \varepsilon^d \left| \frac{u_j^\varepsilon - u_{j'}^\varepsilon}{\varepsilon} \right|^p.$$

Summing up in k, k' we obtain

$$\sum_{|k-k'|=1, k \in Y_i} \varepsilon^d \left| \frac{\widetilde{u}_k^\varepsilon - \widetilde{u}_{k'}^\varepsilon}{\varepsilon} \right|^p \leq C T^d \sum_{j-j' \in P, j,j' \in \widetilde{Y}_i \cap C} \varepsilon^d \left| \frac{u_j^\varepsilon - u_{j'}^\varepsilon}{\varepsilon} \right|^p$$

and

(3.5)
$$\sum_{|k-k'|=1, k,k' \in \frac{1}{\varepsilon} \widetilde{\Omega}_\varepsilon} \varepsilon^d \left| \frac{\widetilde{u}_k^\varepsilon - \widetilde{u}_{k'}^\varepsilon}{\varepsilon} \right|^p \leq C D^d T^d \sum_{(j,j') \in N_C^\varepsilon(\Omega)} \varepsilon^d \left| \frac{u_j^\varepsilon - u_{j'}^\varepsilon}{\varepsilon} \right|^p,$$

where

$$\widetilde{\Omega}_\varepsilon = \bigcup \left\{ \varepsilon Y_i : (3.4) \text{ holds} \right\}.$$

Trivially, we also have the estimate

$$\sum_{k \in Y_i} |\widetilde{u}_k^\varepsilon| \leq \sum_{k \in Y_i \cap C} |u_k^\varepsilon| + \sum_{k \in Y_i \setminus C} |\widetilde{u}_k^\varepsilon| = \frac{T^d}{\#(C \cap Y)} \sum_{k \in Y_i \cap C} |u_k^\varepsilon|.$$

These two estimates ensure the precompactness of $\widetilde{u}^\varepsilon$ in $L^1(\Omega')$ for all $\Omega' \subset\subset \Omega$ and that every cluster point is in $W^{1,p}(\Omega)$ by the uniformity of the estimates (3.5) (see [2, Proposition 3.4]).   □

THEOREM 3.3 (homogenization on discrete perforated domains). *The energies $\mathcal{F}_\varepsilon$ defined in (3.1) Γ-converge with respect to the $L_{\mathrm{loc}}^1(\Omega; \mathbb{R}^m)$ topology to the energy*

(3.6)
$$\mathcal{F}_{\mathrm{hom}}(u) = \int_\Omega f_{\mathrm{hom}}(\nabla u) \, dx,$$

defined on $W^{1,p}(\Omega; \mathbb{R}^m)$, where the energy density f_{hom} satisfies

(3.7)
$$f_{\mathrm{hom}}(\xi) = \lim_{K \to +\infty} \inf \left\{ K^{-d} \mathcal{F}_1(\xi x + v, (0, K)^d) : v_k = 0 \text{ in a neighborhood of } \partial(0, K)^d \right\}.$$

Proof. The proof follows the one in the case $C = \mathbb{Z}^d$ contained in [2], and therefore we have the coerciveness condition $f(k, k', z) \geq C(|z|^p - 1)$ whenever $|k - k'| = 1$. That condition is used only to obtain precompactness of sequences with bounded energy and is substituted by the previous lemma.

The proof can also be obtained by directly using the homogenization result of [2] applied to $\mathcal{F}_\varepsilon^\eta = \mathcal{F}_\varepsilon + \eta\, G$, where

$$G(u) = \sum_{|k-k'|=1, k,k' \in \frac{1}{\varepsilon}\Omega} \varepsilon^d \left| \frac{u_k - u_{k'}}{\varepsilon} \right|^p,$$

obtaining limit energies

$$\mathcal{F}_{\text{hom}}^\eta(u) = \int_\Omega f_{\text{hom}}^\eta(\nabla u)\, dx.$$

By comparison we obtain the existence of the desired Γ-limit and the equality

$$\mathcal{F}_{\text{hom}}(u) = \inf_{\eta>0} \mathcal{F}_{\text{hom}}^\eta(u) = \int_\Omega \inf_{\eta>0} f_{\text{hom}}^\eta(\nabla u)\, dx.$$

Once this integral representation is shown to hold, standard arguments allow US to conclude the validity of formula (3.7) (see [13, section 14.1]). ☐

4. Definition of the interaction term. The homogenization result in Theorem 3.3 will describe the contribution of the hard phases to the limiting behavior of energies F_ε. We now characterize their interactions with the soft phase.

For all M positive integer and $z_1, \ldots, z_N \in \mathbb{R}^m$ we define the minimum problem

(4.1)
$$\varphi_M(z_1, \ldots, z_N) = \frac{1}{M^d} \inf\left\{ \sum_{(k,k')\in N_0(Q_M)} f(k,k',v_k - v_{k'}) + \sum_{k\in Z(Q_M)} g(k,v_k) : v \in \mathcal{V}_M \right\},$$

where

(4.2) $\quad Q_M = \left[-\frac{M}{2}, \frac{M}{2} \right)^d, \qquad N_0(Q_M) = N_0 \cap (Q_M \times Q_M), \qquad Z(Q_M) = \mathbb{Z}^d \cap Q_M,$

and the infimum is taken over the set $\mathcal{V}_M = \mathcal{V}_M(z_1, \ldots, z_N)$ of all v that are constant on each connected component of $A_j \cap Q_M$ and $v = z_j$ on C_j for $j = 1, \ldots N$.

PROPOSITION 4.1. *There exists the limit φ of φ_M uniformly on compact subsets of $\mathbb{R}^{mN}$.*

Proof. Note preliminarily that by the positive homogeneity condition for f we have

$$|f(k,k',z) - f(k,k',z')| \le C|z-z'|(|z|^{p-1} + |z'|^{p-1})$$

for $(k,k') \in N_0$. Let v be a test function for $\varphi_M(z_1, \ldots, z_N)$. In order to estimate $\varphi_M(z_1', z_2, \ldots, z_N)$ we use as a test function

$$v_k' = \begin{cases} z_1' & \text{if } k \in C_1, \\ v_k & \text{otherwise.} \end{cases}$$

We then have

$$\left| \sum_{(k,k')\in N_0(Q_M)} f(k,k',v_k' - v_{k'}') + \sum_{k\in Z(Q_M)} g(k,v_k') \right.$$
$$\left. - \sum_{(k,k')\in N_0(Q_M)} f(k,k',v_k - v_{k'}) - \sum_{k\in Z(Q_M)} g(k,v_k) \right|$$

$$\leq 2 \sum_{(k,k')\in N_0(Q_M), k\in C_1} \left| f(k, k', v'_k - v'_{k'}) - f(k, k', v_k - v_{k'}) \right|$$

$$+ \sum_{k\in C_1\cap Q_M} \left| g(k, z_1) - g(k, z'_1) \right|$$

$$\leq 2 \sum_{(k,k')\in N_0(Q_M), k\in C_1} \left| f(k, k', z'_1 - v_{k'}) - f(k, k', z_1 - v_{k'}) \right|$$

$$+ \sum_{k\in C_1\cap Q_M} \left| g(k, z_1) - g(k, z'_1) \right|.$$

By (2.5) the second sum can be simply estimated by $CM^d|z_1 - z'_1|(1 + |z_1|^{p-1} + |z'_1|^{p-1})$. As for the first sum, we have

$$\sum_{(k,k')\in N_0(Q_M), k\in C_1} \left| f(k, k', z'_1 - v_{k'}) - f(k, k', z_1 - v_{k'}) \right|$$

$$\leq \sum_{(k,k')\in N_0(Q_M), k\in C_1} C|z_1 - z'_1|\left(|z'_1 - z_1|^{p-1} + |v_k - v_{k'}|^{p-1} \right)$$

$$\leq CM^d|z_1 - z'_1|^p + C|z_1 - z'_1| \sum_{(k,k')\in N_0(Q_M), k\in C_1} |v_k - v_{k'}|^{p-1}$$

$$\leq CM^d|z_1 - z'_1|^p + C|z_1 - z'_1|M^{\frac{d}{p}}\left(\sum_{(k,k')\in N_0(Q_M), k\in C_1} |v_k - v_{k'}|^p \right)^{\frac{p-1}{p}}$$

$$\leq CM^d|z_1 - z'_1|^p + C|z_1 - z'_1|M^{\frac{d}{p}}\left(\sum_{(k,k')\in N_0(Q_M), k\in C_1} f(k, k', v_k - v_{k'}) \right)^{\frac{p-1}{p}}.$$

By the arbitrariness of v, taking infima we conclude that

$$|\varphi_M(z'_1, \ldots, z_N) - \varphi_M(z_1, \ldots, z_N)|$$
$$\leq C|z_1 - z'_1|\left(|z_1 - z'_1|^{p-1} + (\varphi_M(z_1, \ldots, z_N))^{\frac{p-1}{p}} + \left(1 + |z_1|^{p-1} + |z'_1|^{p-1}\right) \right).$$

Furthermore, by taking as a test function $v = 0$ on the complement of the $\bigcup_j C_j$ we have the estimate

$$\varphi_M(z_1, \ldots, z_N) \leq C\left(1 + \sum_j |z_j|^p\right).$$

These estimates give equiboundedness and equicontinuity of the family φ_M on bounded subsets. By the Ascoli–Arzelà theorem, to conclude it suffices to show that the whole sequence φ_M converges pointwise. To this end, we note that for integer K and M we have

(i) $\varphi_{KM} \geq \varphi_M$;
(ii) $M^d\varphi_M \leq K^d\varphi_K$ if $M \leq K$.
By (i), with fixed M the sequence φ_{M2^k} is increasing, and in particular

(4.3) $$\varphi_{M2^k} \geq \varphi_M$$

for all k.

Let k be fixed; for all K let $L_K = \lfloor K/M2^k \rfloor$, so that

$$0 \leq K - L_K M2^k \leq M2^k.$$

Then, by (ii)

$$(L_K M 2^k)^d \varphi_{L_K M 2^k} \leq K^d \varphi_K,$$

and by (4.3)

$$\varphi_K \geq \left(\frac{L_K M 2^k}{K}\right)^d \varphi_{L_K M 2^k} \geq \left(\frac{L_K M 2^k}{K}\right)^d \varphi_{M 2^k} \geq \left(\frac{L_K M 2^k}{K}\right)^d \varphi_M.$$

By taking first the liminf in K and then the limsup in M we obtain

$$\liminf_K \varphi_K \geq \limsup_M \varphi_M,$$

that is, the desired claim. □

Remark 4.2. Let $u^M \in \mathcal{V}_M$ be a sequence such that

$$\lim_M \frac{1}{M^d} \Big(\sum_{(k,k') \in N_0(Q_M)} f(k, k', u_k^M - u_{k'}^M) + \sum_{k \in Z(Q_M)} g(k, u_k^M) \Big) = \varphi(z_1, \ldots, z_N);$$

then for every sequence of constants $R_M = o(M)$ we have

$$\lim_M \frac{1}{M^d} \sum_{k,k' \in Q_M \setminus Q_{M-R_M} : k-k' \in P_0} |u_k^M - u_{k'}^M|^p = 0.$$

Indeed, otherwise taking u^M as test function for the problem defining φ_{M-R_M} $(z_1, \ldots, z_N)$, we would obtain

$$\limsup_M \varphi_{M-R_M}(z_1, \ldots, z_N) < \varphi(z_1, \ldots, z_N),$$

which is a contradiction.

We now prove that the function φ introduced in Proposition 4.1 can be defined through minimum problems with additional boundary data. This will be useful in the computation of the upper bound for the Γ-limit. We then define the boundary set of Q_M as follows: we consider R a fixed constant such that for any two points k and $k' \in Q_{M-R}$ connected in terms of P_0-interactions there exists a path of P_0-interacting points contained in Q_M, that starts at k and ends at k', and R larger than twice the diameter of each bounded connected component of any A_j for $j = 1, \ldots, N$. The existence of such $R > 0$ is ensured by the following.

PROPOSITION 4.3. *There exists $R > 0$ such that for all $M > 4R$ and for any k and $k' \in Q_{M-R}$ connected in terms of P_0-interactions, there exists a P_0 connected path in Q_M that goes from k to k'.*

Proof. For each $x \in \mathbb{Z}^d$ denote by $\mathcal{S}_x$ the maximal connected component that contains x and is defined in terms of P_0-connectedness. Clearly, each such a component is either a finite or periodic unbounded set. In the latter case, the diameter of a periodicity cell is not greater than $(cT)^d$. Let R_x be a positive number such that for any $z_1, z_2 \in \mathcal{S}_x$ that are situated in the same periodicity cell or neighboring periodicity cells, there exists a P_0-connected path that starts at z_1, ends at z_2, and belongs to $z_1 + [-R_x, R_x]^d \cap \mathbb{Z}^d$. Denote $\widetilde{R} = \max_x R_x$, where the maximum is taken over all $x \in [0, T)^d \cap \mathbb{Z}^d$. For any k and k' from $\mathcal{S}_x$ consider a path of neighboring cells situated along the segment $[k, k']$. Choosing in each such cell an element of $\mathcal{S}_x$ and recalling the definition of $\widetilde{R}$ we conclude that there exists a P_0-connected path that starts at

k, ends at k', and belongs to the R-neighborhood of the segment $[k, k']$, where R is equal to the sum of $\widetilde{R}$ and the diameter of a periodicity cell. This implies the desired statement. □

We define B_M as

$$B_M = \left((Q_M \setminus Q_{M-R}) \setminus \bigcup_{j=1}^{N} A_j \right)$$

$$\cup \bigcup \{ B : B \subset Q_M \setminus Q_{M-R} \text{ bounded connected component of } A_j \cap Q_M, j = 1, \ldots, N \}.$$

With this definition, we can set

(4.4)
$$\begin{aligned}
\widetilde{\varphi}_M(z_1, \ldots, z_N) &= \tfrac{1}{M^d} \inf \Big\{ \sum_{(k,k') \in N_0(Q_M)} f(k, k', v_k - v_{k'}) \\
&\quad + \sum_{k \in Z(Q_M)} g(k, v_k) : v \in \mathcal{V}_M, v = 0 \text{ on } B_M \Big\}.
\end{aligned}$$

PROPOSITION 4.4. *There exists the limit*

$$\lim_M \widetilde{\varphi}_M(z_1, \ldots, z_N) = \varphi(z_1, \ldots, z_N)$$

uniformly on bounded subsets of $\mathbb{R}^{mN}$, where φ is defined in Proposition 4.1.

Proof. By the same argument as in Proposition 4.1 we may show that the sequence is equibounded and equicontinuous on bounded sets. It is then sufficient to show the existence of the pointwise limit and that this coincides with that of φ_M. To this end we will estimate $\widetilde{\varphi}_M$ in terms of φ_M.

Note that we may write $\varphi_M(z_1, \ldots, z_N)$ as the sum of two independent minimum problems, the first one where only k and k' connected with $\bigcup_{j=1}^{N} C_j$ in Q_M are taken into account, and the second one where the summation is done over all other indices (disconnected with $\bigcup_{j=1}^{N} C_j$). Note that the first one is actually a minimum, of which we choose a minimizer v^M, while the second one may be only an infimum. The latter infimum can be further decomposed into a sum of disjoint infimum problems over bounded connected components, the ones intersecting Q_{M-R} being $T\mathbb{Z}^d$-translations of a finite family $\{I_l\}$ of subsets of $\mathbb{Z}^d$ by our choice of R; i.e., their value is

(4.5)
$$\inf \Big\{ \sum_{(k,k') \in N_0(I_l)} f(k, k', v_k - v_{k'}) + \sum_{k \in I_l} g(k, v_k) : v : I_l \to \mathbb{R}^m \Big\},$$

where the infimum is taken on those v that are constant on each component of $A_j \cap I_l$ for $j = 1, \ldots, N$. This value is independent of M and $z_1, \ldots, z_N$. We denote by w^l a $\frac{1}{M}$-almost minimizer of problem (4.5), that is, $w^l : I_l \to \mathbb{R}^m$ is such that the infimum in (4.5) differs from $\sum_{(k,k') \in N_0(I_l)} f(k, k', w_k^l - w_{k'}^l) + \sum_{k \in I_l} g(k, w_k^l)$ not more than $1/M$.

We define $\widetilde{v}^M \in \mathcal{V}_M$ with $\widetilde{v}^M = 0$ in B_M by setting

$$\widetilde{v}_k^M = \begin{cases} 0 & \text{if } k \in B_M, \\ w_{k-K}^l & \text{if } k \in K + I_l \text{ for some } K \in T\mathbb{Z}^d \text{ and } K + I_l \cap Q_{M-R} \neq \emptyset, \\ v_k^M & \text{otherwise.} \end{cases}$$

Using $\widetilde{v}^M$ as a test function we can estimate, recalling (2.3), (2.1), and (2.4),

$$\widetilde{\varphi}_M(z_1, \ldots, z_N) \le \varphi_M(z_1, \ldots, z_N)$$
$$+ \frac{C}{M^d} \Big(\sum_{k \text{ or } k' \in B_M, k-k' \in P_0} |\widetilde{v}_k^M - \widetilde{v}_{k'}^M|^p + \sum_{k \in B_M} g(k, 0) + M^{d-1} + \#(A_j \cap B_M) \Big)$$
$$\le \varphi_M(z_1, \ldots, z_N) + \frac{C}{M^d} \Big(\sum_{k \notin B_M : \exists k' \in B_M, k-k' \in P_0} |\widetilde{v}_k^M|^p + \#(B_M) \Big).$$

By the Poincaré inequality the sum can be estimated as

$$\sum_{k \notin B_M : \exists k' \in B_M, k-k' \in P_0} |\widetilde{v}_k^M|^p \le C \Big(\#(B_M) + \sum_{k, k' \in Q_M \setminus Q_{M-2R} : k-k' \in P_0} |v_k^M - v_{k'}^M|^p \Big).$$

Since this last sum tends to 0 as $M \to +\infty$ by Remark 4.2, we obtain

$$\widetilde{\varphi}_M(z_1, \ldots, z_N) \le \varphi_M(z_1, \ldots, z_N) + o(1).$$

Since the opposite inequality $\widetilde{\varphi}_M(z_1, \ldots, z_N) \ge \varphi_M(z_1, \ldots, z_N)$ trivially holds, we get that

$$\lim_M (\widetilde{\varphi}_M(z_1, \ldots, z_N) - \varphi_M(z_1, \ldots, z_N)) = 0$$

as desired. □

5. Statement of the convergence result. We now have all the ingredients to characterize the asymptotic behavior of F_ε.

Thanks to the compactness Lemma 3.2, we may define the convergence

$$u^\varepsilon \to (u_1, \ldots, u_N) \tag{5.1}$$

as the $L^1_{\mathrm{loc}}(\Omega; \mathbb{R}^m)$ convergence $\widetilde{u}_j^\varepsilon \to u_j$ of the extensions of the restrictions of u^ε to C_j, which is a compact convergence as ensured by that lemma.

The total contribution of the hard phases will be given separately by the contribution on the infinite connected components and the finite ones. The first one is obtained by computing independently the limit relative to each component,

$$F_\varepsilon^j(u) = \sum_{(k,k') \in N_j^\varepsilon(\Omega)} \varepsilon^d f\Big(k, k', \frac{u_k - u_{k'}}{\varepsilon}\Big), \tag{5.2}$$

where

$$N_j^\varepsilon(\Omega) = \Big\{ (k, k') \in (C_j \times C_j) \cap \frac{1}{\varepsilon}(\Omega \times \Omega) : k - k' \in P_j, k \ne k' \Big\}, \tag{5.3}$$

which is characterized by Theorem 3.3 as

$$F_{\mathrm{hom}}^j(u) = \int_\Omega f_{\mathrm{hom}}^j(\nabla u)\, dx. \tag{5.4}$$

In order to characterize the contribution of the finite connected components of A_j, we can write

$$A_j \setminus C_j = \bigcup_{l \in I_j} (A_l^j + T\mathbb{Z}^d), \tag{5.5}$$

where, due to the periodicity of the media, l runs over a finite set of indices I_j, and $A_l^j + T\mathbb{Z}^d$ and $A_{l'}^j + T\mathbb{Z}^d$ are P_j-disconnected if $l \neq l'$. To each such A_l^j we associate the minimum value

$$(5.6) \qquad m_l^j = \min\Big\{ \sum_{k,k' \in A_l^j, k-k' \in P_j} f(k, k', z_k - z_{k'}) : z : A_l^j \to \mathbb{R}^m \Big\}.$$

Note that we have no boundary conditions for the test functions z. The total contribution of the disconnected components will simply give the additive constant $m|\Omega|$, where

$$(5.7) \qquad m = \frac{1}{T^d} \sum_{j=1}^{N} \sum_{l \in I_j} m_l^j.$$

In the previous section we have introduced the energy density φ, which describes the interactions between the hard phases. Taking all contribution into account, we may state the following convergence result.

THEOREM 5.1 (double-porosity homogenization). *Let Ω be a Lipschitz bounded open set, and let F_ε be defined by (2.6) with the notation of section 2. Then there exists the Γ-limit of F_ε with respect to the convergence (5.1) and it equals*

$$(5.8) \qquad F_{\mathrm{hom}}(u_1, \ldots, u_N) = \sum_{j=1}^{N} \int_\Omega f_{\mathrm{hom}}^j(\nabla u_j)\, dx + m|\Omega| + \int_\Omega \varphi(u_1, \ldots, u_N)\, dx$$

on functions $u = (u_1, \ldots, u_N) \in (W^{1,p}(\Omega; \mathbb{R}^m))^N$, where φ is defined in Proposition 4.1, f_{hom}^j are defined by (3.7), and m is given by (5.7).

The proof of this result will be subdivided into a lower and an upper bound in the next sections.

Remark 5.2 (nonhomogeneous lower-order term). In our hypotheses the lower-order term g depends on the fast variable k, which is integrated out in the limit. We may easily include a measurable dependence on the slow variable εk by assuming $g = g(x, k, z)$ a Carathéodory function (this covers in particular the case $g = g(x, z)$ and substitute the last sum in (2.6) by

$$\sum_{k \in Z^\varepsilon(\Omega)} \varepsilon^d g(\varepsilon k, k, u_k).$$

Correspondingly, in Theorem 5.1 the integrand in the last term in (5.8) must be substituted by $\varphi(x, u_1, \ldots, u_N)$, where the definition of this last function is the same but taking $g(x, k, z)$ in place of $g(k, z)$, so that x simply acts as a parameter.

Example 5.3 (simple one-dimensional energies). We give two examples of one-dimensional energies with a nontrivial double-porosity limit due to next-to-nearest neighbor interactions.

(1) We consider $d = 1$, $\Omega = (0, 1)$ and the energies

$$\sum_{i=1}^{\lfloor 1/\varepsilon \rfloor - 1} \varepsilon \left| \frac{u_{i+1} - u_{i-1}}{\varepsilon} \right|^2 + \varepsilon^2 \sum_{i=1}^{\lfloor 1/\varepsilon \rfloor} \varepsilon \left| \frac{u_i - u_{i-1}}{\varepsilon} \right|^2.$$

In this case C_1 and C_2 are the sets of even and odd integers, and $C_0 = \emptyset$. We have $g = 0$ and the definition of φ is trivial; the limit is

$$F_{\mathrm{hom}}(u_1, u_2) = 2 \int_{(0,1)} |u_1'|^2\, dx + 2 \int_{(0,1)} |u_2'|^2\, dx + \int_{(0,1)} |u_1 - u_2|^2\, dx$$

(note the abuse of notation for u_i). Note that the second sum of the discrete energy can be interpreted as the L^2-norm of the difference between even and odd interpolations of u.

(2) We consider $d = 1$, $\Omega = (0, 1)$ and the energies

$$\sum_{i=0}^{\frac{1}{2}\lfloor 1/\varepsilon \rfloor - 1} \varepsilon \left| \frac{u_{2i+2} - u_{2i}}{\varepsilon} \right|^2 + \varepsilon^2 \sum_{i=1}^{\lfloor 1/\varepsilon \rfloor} \varepsilon \left| \frac{u_i - u_{i-1}}{\varepsilon} \right|^2 + \sum_{i=1}^{\lfloor 1/\varepsilon \rfloor} \varepsilon |u_i - u_i^0|^2.$$

In this case, C_1 is the set of even integers, C_0 is the set of odd integers, and we may take $g(x, z) = |z - u_0(x)|^2$ (we take u_0 a fixed L^2-function and $\{u_i^0\}$ an interpolation strongly converging to u_0). Correspondingly,

$$\varphi(x, u) = \frac{5}{6} |u - u_0(x)|^2,$$

and the limit is

$$F_{\text{hom}}(u) = 2 \int_{(0,1)} |u'|^2 \, dx + \frac{5}{6} \int_{(0,1)} |u - u_0(x)|^2 \, dx$$

(in this case we only have one parameter in the continuum).

6. Lower bound. Let u^ε be such that $\sup_\varepsilon F_\varepsilon(u^\varepsilon) < +\infty$ and $u^\varepsilon \to u = (u_1, \ldots, u_N)$ with respect to convergence (5.1).

We may then rewrite

$$F_\varepsilon(u^\varepsilon) \geq \sum_{j=1}^N F_\varepsilon^j(u^\varepsilon) + \sum_{j=1}^N \sum_{\substack{A_l^j \subset \frac{1}{\varepsilon}\Omega}} \sum_{\substack{k,k' \in A_l^j \\ k-k' \in P_j}} \varepsilon^d f\left(k, k', \frac{u_k^\varepsilon - u_{k'}^\varepsilon}{\varepsilon}\right)$$

$$(6.1) \qquad + \sum_{Q_M^i \subset \frac{1}{\varepsilon}\Omega} \varepsilon^d \left(\sum_{(k,k') \in N_0(Q_M^i)} \varepsilon^p f\left(k, k', \frac{u_k^\varepsilon - u_{k'}^\varepsilon}{\varepsilon}\right) + \sum_{k \in Z(Q_M^i)} g(k, u_k^\varepsilon) \right),$$

where

$$Q_M^i = Q_M + Mi, \qquad N_0(Q_M^i) = N_0 \cap (Q_M^i \times Q_M^i), \qquad Z(Q_M^i) = \mathbb{Z}^d \cap Q_M^i,$$

for $i \in \mathbb{Z}^d$.

The second term in (6.1) is estimated by taking the minimum over all z_k in the place of $u_k^\varepsilon/\varepsilon$, obtaining

$$\sum_{j=1}^N \sum_{\substack{A_l^j \subset \frac{1}{\varepsilon}\Omega}} \sum_{\substack{k,k' \in A_l^j \\ k-k' \in P_j}} \varepsilon^d f\left(k, k', \frac{u_k^\varepsilon - u_{k'}^\varepsilon}{\varepsilon}\right) \geq \varepsilon^d \sum_{j=1}^N \sum_{\substack{A_l^j \subset \frac{1}{\varepsilon}\Omega}} m_l^j$$

$$(6.2) \qquad = \varepsilon^d \sum_{j=1}^N \frac{|\Omega|}{\varepsilon^d T^d} \sum_{l \in I_J} m_l^j + o(1) = m|\Omega| + o(1).$$

In order to estimate the last term in (6.1) we estimate separately

$$\sum_{(k,k') \in N_0(Q_M^i)} \varepsilon^p f\left(k, k', \frac{u_k^\varepsilon - u_{k'}^\varepsilon}{\varepsilon}\right) + \sum_{k \in Z(Q_M^i)} g(k, u_k^\varepsilon)$$

for each fixed i. To this end, we consider the function $u^{\varepsilon,i}$ defined by

$$u_k^{\varepsilon,i} = \frac{1}{\#(C_j \cap Q_M^i)} \sum_{l \in C_j \cap Q_M^i} u_l^\varepsilon =: u^{\varepsilon,i,j} \qquad \text{if } k \in C_j \cap Q_M^i,$$

$$u_k^{\varepsilon,i} = \frac{1}{\#(A_l^j \cap Q_M^i)} \sum_{l \in A_l^j \cap Q_M^i} u_l^\varepsilon =: u_l^{\varepsilon,i,j} \qquad \text{if } k \in A_l^j \cap Q_M^i$$

for $j = 1, \ldots, N$ and $l \in I_j$, and $u_k^{\varepsilon,i} = u_k^\varepsilon$ if $k \in Q_M^i \setminus \bigcup_{j=1}^N A_j$.

We can now use Lemma 9.1 with $u = u^\varepsilon$ and v equal to the function defined by $u^{\varepsilon,i}$ on Q_M^i and note that

$$\sum_{k \in A_j} \varepsilon^d |u_k^\varepsilon - v_k|^p = \sum_i \left(\sum_{k \in C_j \cap Q_M^i} \varepsilon^d |u_k^\varepsilon - v_k|^p + \sum_{k \in (A_j \setminus C_j) \cap Q_M^i} \varepsilon^d |u_k^\varepsilon - v_k|^p \right)$$

$$= \sum_i \left(\sum_{k \in C_j \cap Q_M^i} \varepsilon^d |u_k^\varepsilon - u^{\varepsilon,i,j}|^p + \sum_{l \in I_j} \sum_{k \in A_l^j \cap Q_M^i} \varepsilon^d |u_k^\varepsilon - u_l^{\varepsilon,i,j}|^p \right)$$

$$\leq CM^p \varepsilon^p \sum_i \left(\sum_{k \in C_j \cap Q_M^i, k - k' \in P_j} \varepsilon^d \left| \frac{u_k^\varepsilon - u_{k'}^\varepsilon}{\varepsilon} \right|^p \right.$$

$$+ \left. \sum_{l \in I_j} \sum_{k \in A_l^j \cap Q_M^i} \varepsilon^d \left| \frac{u_k^\varepsilon - u_{k'}^\varepsilon}{\varepsilon} \right|^p \right)$$

$$\leq CM^p \varepsilon^p F_\varepsilon(u^\varepsilon).$$

We then have

$$\sum_{Q_M^i \subset \frac{1}{\varepsilon}\Omega} \varepsilon^d \left(\sum_{(k,k') \in N_0(Q_M^i)} \varepsilon^p f\left(k, k', \frac{u_k^\varepsilon - u_{k'}^\varepsilon}{\varepsilon}\right) + \sum_{k \in Z(Q_M^i)} g(k, u_k^\varepsilon) \right)$$

$$= \sum_{Q_M^i \subset \frac{1}{\varepsilon}\Omega} \varepsilon^d \left(\sum_{(k,k') \in N_0(Q_M^i)} f(k, k', u_k^\varepsilon - u_{k'}^\varepsilon) + \sum_{k \in Z(Q_M^i)} g(k, u_k^\varepsilon) \right)$$

$$\geq \sum_{Q_M^i \subset \frac{1}{\varepsilon}\Omega} \varepsilon^d \left(\sum_{Q_M^i \subset \frac{1}{\varepsilon}\Omega} \sum_{(k,k') \in N_0(Q_M^i)} f\left(k, k', u_k^{\varepsilon,i} - u_{k'}^{\varepsilon,i}\right) + \sum_{k \in Z(Q_M^i)} g(k, u_k^{\varepsilon,i}) \right) + o(1)$$

as $\varepsilon \to 0$

Since (a translation of) $u^{\varepsilon,i}$ can be used as a test function for $\varphi_M(u^{\varepsilon,i,1}, \ldots, u^{\varepsilon,i,N})$ we have

$$\sum_{Q_M^i \subset \frac{1}{\varepsilon}\Omega} \left(\sum_{(k,k') \in N_0(Q_M^i)} f\left(k, k', u_k^{\varepsilon,i} - u_{k'}^{\varepsilon,i}\right) + \sum_{k \in Z(Q_M^i)} g(k, u_k^{\varepsilon,i}) \right)$$

$$\geq M^d \varphi_M(u^{\varepsilon,i,1}, \ldots, u^{\varepsilon,i,N}).$$

We define the piecewise-constant functions $u_M^{\varepsilon,j}$ to be equal to $u^{\varepsilon,i,j}$ on each $Q_M^i \subset \frac{1}{\varepsilon}\Omega$ and to 0 otherwise. We then obtain

$$\sum_{Q_M^i \subset \frac{1}{\varepsilon}\Omega} \varepsilon^d \Big(\sum_{(k,k') \in N_0(Q_M^i)} \varepsilon^p f\Big(k, k', \frac{u_k^\varepsilon - u_{k'}^\varepsilon}{\varepsilon}\Big) + \sum_{k \in Z(Q_M^i)} g(k, u_k^\varepsilon)\Big)$$
$$\geq \int_\Omega \varphi_M(u_M^{\varepsilon,1}(x), \dots, u_M^{\varepsilon,N}(x))dx + o(1)$$

as $\varepsilon \to 0$.

Since

$$u^{\varepsilon,i,j} = \frac{1}{\#(C_j \cap Q_M^i)} \sum_{l \in C_j \cap Q_M^i} (\widetilde{u}_j^\varepsilon)_l,$$

where $\widetilde{u}_j^\varepsilon$ converges strongly to u_j in $L_{\text{loc}}^1(\Omega; \mathbb{R}^m)$, so that also $u_M^{\varepsilon,j}$ converges strongly to u_j for all M. By the Lebesgue dominated convergence theorem we get

$$(6.3) \qquad \lim_{\varepsilon \to 0} \int_\Omega \varphi_M(u_M^{\varepsilon,1}(x), \dots, u_M^{\varepsilon,N}(x))dx = \int_\Omega \varphi_M(u_1(x), \dots, u_N(x))dx.$$

Summing up the liminf inequalities for all F_ε^j, (6.2) and (6.3), we get

$$\liminf_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) \geq \sum_{j=1}^N \liminf_{\varepsilon \to 0} F_\varepsilon^j(u_\varepsilon) + m|\Omega| + \int_\Omega \varphi_M(u_1, \dots, u_N)\, dx$$
$$\geq \sum_{j=1}^N \int_\Omega f_{\text{hom}}^j(\nabla u_j)\, dx + m|\Omega| + \int_\Omega \varphi_M(u_1, \dots, u_N)\, dx,$$

from which (5.8) follows taking the limit as $M \to +\infty$ and using Lebesgue's theorem once again.

7. Upper bound. We prove the upper bound for a linear target function

$$u(x) = (\xi^1 x, \dots, \xi^N x),$$

the proof for an affine function following in the same way. For piecewise-affine functions the same argument applies locally, while for an arbitrary target function we proceed by approximation (see [12]).

A recovery sequence for u can be constructed as follows:

• For all $j = 1, \dots, N$ we choose a recovery sequence $u_\varepsilon^j \to \xi^j x$ for F_ε^j; we may regard u_ε^j as defined in the whole $\mathbb{Z}^d$. We set

$$(7.1) \qquad\qquad u_k^\varepsilon = (u_\varepsilon^j)_k \qquad \text{on } C_j.$$

• For each fixed M let Q_M^i be the corresponding partition of $\mathbb{Z}^d$. For all i we define

$$u^{\varepsilon,i,j} = \frac{1}{\#(C_j \cap Q_M^i)} \sum_{l \in C_j \cap Q_M^i} (u_\varepsilon^j)_l$$

for $j = 1, \ldots, N$ and take a minimum point $v^{\varepsilon,i}$ for $\widetilde{\varphi}_M(u^{\varepsilon,i,1}, \ldots, u^{\varepsilon,i,N})$. We define

$$u_k^\varepsilon = v_{k-iM}^{\varepsilon,i} \qquad \text{on } Q_M^i \setminus \bigcup_{j=1}^N A_j.$$

Notice that the function $u^{\varepsilon,i,j} - u_j^\varepsilon$ is of order εM on C_j, and thus, by Lemma 9.1, the difference

$$
\begin{aligned}
(7.2) \qquad & \sum_{Q_M^i \subset \frac{1}{\varepsilon}\Omega} \varepsilon^d \sum_{(k,k') \in N_0(Q_M^i)} f(k, k', u_k^\varepsilon - u_{k'}^\varepsilon) \\
& - \sum_{Q_M^i \subset \frac{1}{\varepsilon}\Omega} \varepsilon^d \sum_{(k,k') \in N_0(Q_M^i)} f(k, k', \hat{u}_k^\varepsilon - \hat{u}_{k'}^\varepsilon) = o(1)
\end{aligned}
$$

as $\varepsilon \to 0$; here $\hat{u}_k^\varepsilon$ stands for the function equal to $u^{\varepsilon,i,j}$ on $C_j \cap Q_M^i$ and to u_k^ε on $\frac{1}{\varepsilon}\Omega \setminus \bigcup_{j=1}^N C_j$.

• For any connected component A_l^j of $A_j \setminus C_j$ with $A_l^j \subset Q_M^i$ define

$$(7.3) \qquad u_k^\varepsilon = v_{k-iM}^{\varepsilon,i} + \varepsilon z_k^{j,l},$$

$z^{j,l}$ being a minimizer of (5.6). Note that $v_{k-iM}^{\varepsilon,i}$ is a constant function on A_l^j, so that u_k^ε is still minimizing.

With this definition of u^ε we have a recovery sequence for u. In order to check that, we introduce an outer approximation of the set Ω as $\Omega_{\varepsilon,M}$ defined by

$$\Omega_{\varepsilon,M} = \bigcup_{i \in I_\varepsilon^M} \varepsilon Q_M^i, \qquad I_\varepsilon^M = \left\{ i \in \mathbb{Z}^d : Q_M^i \cap \frac{1}{\varepsilon}\Omega \neq \emptyset \right\}.$$

In this way we have

$$
\begin{aligned}
F_\varepsilon(u^\varepsilon) &\leq F_\varepsilon\left(u^\varepsilon, \frac{1}{\varepsilon}\Omega_{\varepsilon,M} \right) \\
&\leq \sum_{j=1}^N F_\varepsilon^j\left(u^\varepsilon, \frac{1}{\varepsilon}\Omega_{\varepsilon,M} \right) + \sum_{\substack{j,l: A_l^j \cap \frac{1}{\varepsilon}\Omega_{\varepsilon,M} \neq \emptyset}} \sum_{\substack{k,k' \in A_l^j \\ k-k' \in P_j}} \varepsilon^d f\left(k, k', \frac{u_k^\varepsilon - u_{k'}^\varepsilon}{\varepsilon} \right) \\
&\quad + \sum_{i \in I_\varepsilon^M} \varepsilon^d \left(\sum_{\substack{k,k' \in Q_M^i \\ k-k' \in P_0}} f(k, k', u_k^\varepsilon - u_{k'}^\varepsilon) + \sum_{k \in Q_M^i} g(k, u_k^\varepsilon) \right) \\
&\quad + \sum_{i \in I_\varepsilon^M} \sum_{\substack{k \in Q_M^i, k' \notin Q_M^i \\ k-k' \in P_0}} \varepsilon^d f(k, k', u_k^\varepsilon - u_{k'}^\varepsilon),
\end{aligned}
$$

where we have separated the estimates for the contribution of the infinite components of the hard phases, the isolated islands of hard phases, the contributions of the soft-phase energy and the potential g inside each cube Q_M^i, and the contributions of the soft-phase interactions at the boundary of each cube.

We separately examine each term. By (7.1) and the limsup inequality for F_ε^j we have

$$(7.4) \quad F_\varepsilon^j\left(u^\varepsilon, \frac{1}{\varepsilon}\Omega_{\varepsilon,M}\right) = F_\varepsilon^j\left(u_\varepsilon^j, \frac{1}{\varepsilon}\Omega_{\varepsilon,M}\right) \le F_\varepsilon^j\left(u_\varepsilon^j, \frac{1}{\varepsilon}\Omega'\right) \le F_{\text{hom}}^j(\xi^j x, \Omega') + o(1)$$

for all fixed $\Omega' \supset\supset \Omega_{\varepsilon,M}$.

As for the second term, we have two cases:

- $A_l^j \subset Q_M^i$ for some $i \in I_\varepsilon^M$. In this case by (7.3) we have

$$(7.5) \quad \sum_{\substack{k,k'\in A_l^j \\ k-k'\in P_j}} f\left(k, k', \frac{u_k^\varepsilon - u_{k'}^\varepsilon}{\varepsilon}\right) = \sum_{\substack{k,k'\in A_l^j \\ k-k'\in P_j}} f(k, k', z_k^{j,l} - z_{k'}^{j,l}) = m_l^j,$$

  so that

$$(7.6)$$

$$\sum_{i\in I_\varepsilon^M} \sum_{j,l:A_l^j\subset Q_M^i} \sum_{\substack{k,k'\in A_l^j \\ k-k'\in P_j}} \varepsilon^d f\left(k, k', \frac{u_k^\varepsilon - u_{k'}^\varepsilon}{\varepsilon}\right) \le \sum_{j,l:A_l^j\cap\frac{1}{\varepsilon}\Omega_{\varepsilon,M}\neq\emptyset} \varepsilon^d m_l^j \le m|\Omega| + o(1).$$

- For the other A_l^j we have $u_k^\varepsilon - u_{k'}^\varepsilon = 0$ for all k, k', so that their total contribution is $O(1/M)$.

By (7.2) the third term is estimated by

$$\sum_{i\in I_\varepsilon^M} \varepsilon^d \left(\sum_{\substack{k,k'\in Q_M^i \\ k-k'\in P_0}} f(k, k', u_k^\varepsilon - u_{k'}^\varepsilon) + \sum_{k\in Q_M^i} g(k, u_k^\varepsilon) \right)$$

$$\le \sum_{i\in I_\varepsilon^M} \varepsilon^d \left(\sum_{\substack{k,k'\in Q_M^i \\ k-k'\in P_0}} f(k, k', \widehat{u}_k^\varepsilon - \widehat{u}_{k'}^\varepsilon) + \sum_{k\in Q_M^i} g(k, \widehat{u}_k^\varepsilon) \right) + o(1)$$

$$= \sum_{i\in I_\varepsilon^M} \varepsilon^d M^d \widetilde{\varphi}_M(u^{\varepsilon,i,1}, \ldots, u^{\varepsilon,i,N}) + o(1)$$

$$(7.7) \qquad \le \int_{\Omega'} \widetilde{\varphi}_M(u_1^{\varepsilon,M}, \ldots, u_N^{\varepsilon,M})\, dx + o(1),$$

where $u_j^{\varepsilon,M}$ is the above-defined piecewise-constant function with value $u^{\varepsilon,i,j}$ on Q_M^i. Note that

$$(7.8) \qquad\qquad u_j^{\varepsilon,M} \to \xi^j x \text{ in } L^p(\Omega'; \mathbb{R}^m)$$

as $\varepsilon \to 0$ for all j and M.

As for the last term, we note that the difference $u_k^\varepsilon - u_{k'}^\varepsilon$ is equal either to 0 (if both k and k' do not belong to in any C_j $j = 1, \ldots, N$), to $(u_j^\varepsilon)_k$ if $k \in C_j$ and $k' \notin \bigcup_j C_j$, or to $(u_j^\varepsilon)_k - (u_{j'}^\varepsilon)_{k'}$ if $k \in C_j$ and $k' \in C_{j'}$ with $j \neq j'$. In any case, we can estimate the total contribution by

$$(7.9) \quad C \sum_{i \in I_M^\varepsilon} \sum_{j=1}^N \sum_{k \in C_j \cap (iM+(Q_M \setminus Q_{M-R}))} \varepsilon^d (1 + |(u_j^\varepsilon)_k|^p)$$

$$= C \sum_{i \in I_M^\varepsilon} \sum_{j=1}^N \sum_{k \in C_j \cap (iM+(Q_M \setminus Q_{M-R}))} \varepsilon^d (1 + |(\widetilde{u}_j^\varepsilon)_k|^p).$$

Note that since $\widetilde{u}_j^\varepsilon$ are equi-integrable the latter term vanishes as $M \to +\infty$ uniformly in ε. In fact, it can be written as an integral over a set of measure of order $1/M$.

Taking into account this last estimate, together with (7.4), (7.5), and (7.7), we get

$$(7.10) \quad \limsup_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) \leq \sum_{j=1}^N F_{\mathrm{hom}}^j(\xi^j x, \Omega') + m|\Omega'| + \int_{\Omega'} \widetilde{\varphi}_M(u) \, dx + o(1)$$

as $M \to +\infty$. We can then let $M \to +\infty$ and use Lebesgue's theorem to obtain

$$(7.11) \quad \limsup_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon) \leq \sum_{j=1}^N F_{\mathrm{hom}}^j(\xi^j x, \Omega') + m|\Omega'| + \int_{\Omega'} \varphi(u) \, dx.$$

Eventually we obtain the desired inequality by the arbitrariness of $\Omega' \supset\supset \Omega$.

8. The dynamical case. We consider the asymptotic behavior of solutions for the gradient flow with respect to the L^2-metric of the functionals

$$F_\varepsilon(u) = F_\varepsilon\left(u, \frac{1}{\varepsilon}\Omega\right) = \sum_{j=1}^N \sum_{(k,k') \in N_j^\varepsilon(\Omega)} \varepsilon^d f\left(k, k', \frac{u_k - u_{k'}}{\varepsilon}\right)$$

$$(8.1) \quad + \sum_{(k,k') \in N_0^\varepsilon(\Omega)} \varepsilon^{d+p} f\left(k, k', \frac{u_k - u_{k'}}{\varepsilon}\right),$$

i.e., functionals (2.6) with $g = 0$, with given initial data functions $u_0^\varepsilon : \mathbb{Z}^d \cap \frac{1}{\varepsilon}\Omega \to \mathbb{R}^m$ converging to some $u_0 : \Omega \to \mathbb{R}^m$. (Note that in this notation $0 \in \mathbb{N}$ has the meaning of an initial time and should not be confused with an index $0 \in \mathbb{Z}^d$ as in the notation labeling the values of discrete functions.) To that end, we will apply the minimizing-movement scheme along a sequence of functionals (see [11, 5]): with fixed $\tau > 0$ we define recursively, for $l \in \mathbb{N}$, $l \geq 1$, $u^{\varepsilon,l}$ as the minimizers of

$$(8.2) \quad v \mapsto F_\varepsilon(v) + \frac{1}{2\tau}\|v - u^{\varepsilon,l-1}\|^2,$$

where $u^{\varepsilon,0} = u_0^\varepsilon$. We want to characterize the limits u^l of these minimizers as $\varepsilon \to 0$ as the minimizers obtained by recursively applying the same scheme to a Γ-limit F_0, i.e., to show that u^l is a minimizer of

$$(8.3) \quad v \mapsto F_0(v) + \frac{1}{2\tau}\|v - u^{l-1}\|^2.$$

The norm in these formulas is the L^2-norm in Ω.

Note that this characterization does not follow trivially from the fundamental theorem of Γ-convergence since the additional term may not be a continuous perturbation, depending on the topology chosen (e.g., the one used in Theorem 5.1).

In order to have a topology for which the last term gives a continuous perturbation, and the sequences $u^{\varepsilon,l}$ are still precompact, we cannot use the description in Theorem 5.1. We need to describe the effect of oscillations on the soft phase, as we cannot integrate out their contribution. This can be done, at the expense of introducing a larger number of variables, using two-scale convergence adapted to the discrete setting. In the continuous setting this was already done in [15].

Everywhere in this section we assume that the sites that do not interact at all with infinite components of the hard phases do not contribute to the energy functional. In other words,

$$(8.4) \quad \text{for any } k \in \bigcup_{j=0}^{N} A_j \text{ there exists } k' \in \bigcup_{j=1}^{N} C_j \text{ such that } k \text{ and } k' \text{ are connected;}$$

i.e., either $k = k'$ or there exists a path $\{k_n\}_{n=0,\ldots,K}$ such that $k_0 = k$, $k_K = k'$ and $(k_n, k_{n-1}) \in \bigcup_{j=0}^{N} N_j$.

8.1. Γ-limits with respect to discrete two-scale convergence. Let $v^\varepsilon : Z^\varepsilon(\Omega) \to \mathbb{R}$ be a sequence bounded in $L^2(\Omega)$. We say that v^ε *weakly* (respectively, *strongly*) *(discrete) two-scale converges* to the family $\{v^y\}$ for $y \in Y := \{1, \ldots, T\}^d$ with $v^y \in L^2(\Omega)$ if for all $y \in Y$ the sequence $v^{\varepsilon,y}$ of discrete functions obtained by considering only the values v_k^ε with $k = y$ modulo Y weakly (respectively, strongly) converges to the corresponding v^y; more precisely, we define $v^{\varepsilon,y}$ on $T\mathbb{Z}^d$ as

$$v_j^{\varepsilon,y} = v_{y+j}^\varepsilon$$

for $j \in T\mathbb{Z}^d$ and require that its piecewise-constant interpolation weakly converges in $L^2(\Omega)$ to v^y.

It can be checked that the definition corresponds to that of two-scale convergence as in [16, 3, 15], i.e., (for weak convergence) that for all functions $\{\varphi_k(x)\}_{k\in\mathbb{Z}^d,\, x\in\Omega}$ being T-periodic in k and smooth in x we have

$$(8.5) \quad \lim_{\varepsilon\to 0} \sum_{k\in Z^\varepsilon(\Omega)} \varepsilon^d v_k^\varepsilon \varphi_k(\varepsilon k) = \frac{1}{T^d} \sum_{y\in Y} \int_\Omega v^y(x)\varphi_y(x)\,dx.$$

Note that this is equivalent to

$$(8.6) \quad \lim_{\varepsilon\to 0} \int_\Omega v^\varepsilon(x)\varphi_{\lfloor \frac{x}{\varepsilon}\rfloor}(x)\,dx = \frac{1}{T^d} \sum_{y\in Y} \int_\Omega v^y(x)\varphi_y(x)\,dx$$

upon identification of v^ε with its piecewise-constant interpolation.

We can compute the Γ-limit of

$$G_\varepsilon(u) = F_\varepsilon(u) + \sum_{k\in Z^\varepsilon(\Omega)} \varepsilon^d g(u_k - w_k^\varepsilon)$$

with respect to the weak two-scale convergence $u^\varepsilon \to \{u^y\}$, where $g : \mathbb{R}^m \to \mathbb{R}$ is a continuous function and w^ε strongly two-scale converges to $\{w^y\}$.

THEOREM 8.1. *The Γ-limit of G_ε with respect to weak discrete two-scale convergence is*

$$G_0(\{u^y\}) = \sum_{j=1}^{N} \frac{1}{\#(C_j \cap Y)} \sum_{y \in C_j \cap Y} \int_{\Omega} f_{\mathrm{hom}}^{j}(\nabla u^y) \, dx$$

(8.7)
$$+ \frac{1}{T^d} \sum_{y \in Y \cap \bigcup_{j=1}^{N} C_j} \int_{\Omega} g(u^y(x) - w^y(x)) \, dx + \int_{\Omega} \varphi_g(x, \{u^y(x)\}) \, dx$$

with the constraint that u^y is independent of y on each C_j, and φ_g is given by

$$\varphi_g(x, \{u^y\}) = \lim_{M \to +\infty} \frac{1}{T^d M^d} \inf \Big\{ \sum_{(k,k') \in N_0(Q_{TM})} f(k, k', v_k - v_{k'})$$

(8.8)
$$+ \sum_{k \in Z_0(Q_{TM})} g(v_k - w^k(x)) : \sum_{k \in Q_{TM} \cap (y + T\mathbb{Z}^d)} v_k = M^d u^y \Big\},$$

where each test function v is extended by TM-periodicity.

Proof. The proof follows that of Theorem 5.1, with a different characterization of the interaction energy density φ_g in terms of the variables $\{u^y\}$. The changes follow the ones for the corresponding theorem in the continuum [12] section 7.2. $\square$

PROPOSITION 8.2. *If f and g are convex, then*

$$\varphi_g(x, \{u^y\}) = \frac{1}{T^d} \Big(\sum_{(k,k') \in N_0^{\#}(Q_T)} f(k, k', u^k - u^{k'}) + \sum_{k \in Z(Q_T)} g(u^k - w^k(x)) \Big),$$

where

$$N_0^{\#}(Q_T) = \{(k, k') \in N_0 : k \in Q_T\}.$$

Proof. The proof follows by a classical argument for periodic convex minimization problems (see [13, section 14.3]), noting that by Jensen's inequality we may take $M = 1$ and a test function v replaced by its mean value on each y. By the average constraint in the definition of φ_g this argument fixes exactly the value equal to u^y on each y. The definition of $N_0^{\#}(Q_T)$ is given so as to avoid double counting in the computation of the interactions. $\square$

Example 8.3. In order to illustrate the difference with Theorem 5.1 we consider Example 5.3(2). In that case $C_0 \cap Y$ is the only point 1, so that weak discrete two-scale convergence reduces to the separate weak convergence of even and odd interpolations and then, by the coerciveness on even interpolations, to the strong convergence of even interpolations and the weak convergence of odd interpolations. The Γ-limit is then expressed by

$$G_0(u^1, u^2) = 2 \int_{(0,1)} |(u^2)'|^2 \, dx + \int_{(0,1)} |u^2 - u^1|^2 \, dx + \frac{1}{2} \int_{(0,1)} |u^1 - u_0|^2 \, dx$$
$$+ \frac{1}{2} \int_{(0,1)} |u^2 - u_0|^2 \, dx,$$

where u^1 is the limit of odd interpolations and u^2 the limit of even interpolations. Note that the computation of the minimum

$$\min \Big\{ \frac{1}{2} |u^1 - u_0|^2 + |u^2 - u^1|^2 : u^1 \in \mathbb{R} \Big\}$$

gives the integrand in the limit of Example 5.3(2).

LEMMA 8.4. *Let $g_k^\varepsilon(u) = C|u - w_k^\varepsilon|^2$ with w^ε strongly two-scale converging to w^y and*

$$\sup_\varepsilon F_\varepsilon(w^\varepsilon) < +\infty.$$

Then the recovery sequences for G_0 converge strongly.

Proof. Take u^ε a recovery sequence for $\{u^y\}$. Note first that since w^ε converges strongly, then $|u^\varepsilon|^2\, dx$ cannot concentrate on the boundary of Ω. In order to check this, we can consider the localized version of G_ε:

$$G_\varepsilon(u, A) = F_\varepsilon\left(u, \frac{1}{\varepsilon}A\right) + \sum_{k \in Z^\varepsilon(A)} \varepsilon^d g(u_k - w_k^\varepsilon),$$

which Γ-converges to the corresponding $G_0(\{u^y\}, A)$, defined as in (8.7) with A in the place of Ω. Note that if A is an open set with boundary of zero Lebesgue measure, then u^ε is a recovery sequence also for $G_\varepsilon(u, A)$ at $\{u^y\}$. Indeed suppose by contradiction that

$$\limsup_{\varepsilon \to 0} G_\varepsilon(u^\varepsilon, A) > G_0(\{u^y\}, A);$$

then we have (note that the first inequality simply follows from the positiveness of f and g)

$$\limsup_{\varepsilon \to 0} G_\varepsilon(u^\varepsilon, \Omega) \geq \limsup_{\varepsilon \to 0}\left(G_\varepsilon(u^\varepsilon, A) + G_\varepsilon(u^\varepsilon, \Omega \setminus \overline{A})\right)$$

$$\geq \limsup_{\varepsilon \to 0} G_\varepsilon(u^\varepsilon, A) + \liminf_{\varepsilon \to 0} G_\varepsilon(u^\varepsilon, \Omega \setminus \overline{A})$$

$$> G_0(\{u^y\}, A) + G_0(\{u^y\}, \Omega \setminus \overline{A}) = G_0(\{u^y\}, \Omega),$$

which contradicts the limsup inequality for u^ε on Ω. Since we have (identifying as usual discrete functions with piecewise-constant interpolations)

$$\int_A |u^\varepsilon|^2\, dx \leq 2\left(\int_A |w^\varepsilon|^2\, dx + \int_A |u^\varepsilon - w^\varepsilon|^2\right) \leq 2\sup_\varepsilon \int_A |w^\varepsilon|^2\, dx + \frac{2}{C}G_\varepsilon(u^\varepsilon, A'),$$

where A' is any open set with $A \subset\subset A'$ in Ω, we obtain that $\int_A |u^\varepsilon|^2\, dx$ is arbitrarily small if we take A a small neighborhood of $\partial\Omega$.

We then have to show strong convergence in the interior of Ω.

Let $\{Q_\delta\}$ be a family of disjoint cubes of size δ contained in Ω. We can then write

(8.9)

$$G_0(\{u^y\}) = \lim_{\varepsilon \to 0}\left(F_\varepsilon(u^\varepsilon) + C \sum_{k \in Z^\varepsilon(\Omega)} \varepsilon^d |u_k^\varepsilon - w_k^\varepsilon|^2\right)$$

$$\geq \sum_{\{Q_\delta\}} \liminf_{\varepsilon \to 0}\left(F_\varepsilon(u^\varepsilon, Q_\delta) + C \sum_{k \in Z^\varepsilon(Q_\delta)} \varepsilon^d |u_k^\varepsilon - w_k^\varepsilon|^2\right)$$

$$\geq \sum_{\{Q_\delta\}}\left(\sum_{j=1}^N \frac{1}{\#(C_j \cap Y)} \sum_{y \in C_j \cap Y} \int_{Q_\delta} f_{\mathrm{hom}}^j(\nabla u^y)\, dx\right.$$

$$+ \frac{C}{T^d} \sum_{y \in \bigcup_{j=1}^N C_j} \int_\Omega |u^y(x) - w^y(x)|^2\, dx\bigg)$$

$$+ \sum_{\{Q_\delta\}} \liminf_{\varepsilon \to 0} \varepsilon^d\left(\sum_{(k,k') \in N_0(Q_{\delta/\varepsilon})} f(k, k', u_k^\varepsilon - u_{k'}^\varepsilon) + \sum_{k \in Z(Q_{\delta/\varepsilon})} C|u_k^\varepsilon - w_k^\varepsilon|^2\right).$$

In order to estimate the last term, for all $y \in Y$ and k with $k - y \in T\mathbb{Z}^d$ we substitute u_k^ε with the average $u^{\varepsilon,y}$ over all $k' \in Q_{\delta/\varepsilon}$ with $k' - y \in T\mathbb{Z}^d$. Note that we may suppose that $\delta/\varepsilon \in T\mathbb{Z}$, up to a vanishing error in the computation of these averages as $\varepsilon \to 0$, so that

$$u^{\varepsilon,y} = \frac{T^d \varepsilon^d}{\delta^d} \sum_{k' \in Q_{\delta/\varepsilon} \cap (y + T\mathbb{Z}^d)} u_{k'}^\varepsilon.$$

In the following for all k we indicate by $y = y_k$ the (unique) point in $Y \cap (k + T\mathbb{Z}^d)$.

With fixed η, by using the Young inequality and the convexity inequality on the first term, we then obtain

(8.10)
$$\varepsilon^d \sum_{\{Q_\delta\}} \Big(\sum_{(k,k') \in N_0(Q_{\delta/\varepsilon})} f(k, k', u_k^\varepsilon - u_{k'}^\varepsilon) + \sum_{k \in Z(Q_{\delta/\varepsilon})} C|u_k^\varepsilon - w_k^\varepsilon|^2 \Big)$$

$$\geq \varepsilon^d \sum_{\{Q_\delta\}} \Big(\sum_{(k,k') \in N_0(Q_{\delta/\varepsilon})} f(k, k', u_k^\varepsilon - u_{k'}^\varepsilon) + \sum_{k \in Z(Q_{\delta/\varepsilon})} C(1 - \eta)|u_k^\varepsilon - w^{\varepsilon,y}|^2$$

$$- C\Big(\frac{1}{\eta} - 1\Big) \sum_{k \in Z(Q_{\delta/\varepsilon})} |w_k^\varepsilon - w^{\varepsilon,y}|^2 \Big)$$

$$\geq \varepsilon^d (1 - \eta) \sum_{\{Q_\delta\}} \Big(\sum_{(k,k') \in N_0(Q_{\delta/\varepsilon})} f(k, k', u_k^\varepsilon - u_{k'}^\varepsilon) + \sum_{k \in Z(Q_{\delta/\varepsilon})} C|u_k^\varepsilon - w^{\varepsilon,y}|^2$$

$$- \sum_{k \in Z(Q_{\delta/\varepsilon})} C|u^{\varepsilon,y} - w^{\varepsilon,y}|^2 + \sum_{k \in Z(Q_{\delta/\varepsilon})} C|u^{\varepsilon,y} - w^{\varepsilon,y}|^2 - \frac{1}{\eta} \sum_{k \in Z(Q_{\delta/\varepsilon})} C|w_k^\varepsilon - w^{\varepsilon,y}|^2 \Big)$$

$$= \varepsilon^d (1 - \eta) \sum_{\{Q_\delta\}} \Big(\sum_{(k,k') \in N_0(Q_{\delta/\varepsilon})} f(k, k', u_k^\varepsilon - u_{k'}^\varepsilon) + \sum_{k \in Z(Q_{\delta/\varepsilon})} C\big(|u_k^\varepsilon|^2 - |u^{\varepsilon,y}|^2\big)$$

$$+ \frac{\delta^d}{\varepsilon^d T^d} \sum_{y \in Y} C|u^{\varepsilon,y} - w^{\varepsilon,y}|^2 - \frac{1}{\eta} \sum_{k \in Z(Q_{\delta/\varepsilon})} C|w_k^\varepsilon - w^{\varepsilon,y}|^2 \Big)$$

$$\geq (1 - \eta) \frac{\delta^d}{T^d} \sum_{\{Q_\delta\}} \Big(\sum_{(y,y') \in N_0^{\#}(Y)} f(y, y', u^{\varepsilon,y} - u^{\varepsilon,y'}) + \sum_{y \in Y} C|u^{\varepsilon,y} - w^{\varepsilon,y}|^2 \Big) - C' \frac{\varepsilon}{\delta} \delta^d$$

$$+ \varepsilon^d (1 - \eta) \sum_{\{Q_\delta\}} \sum_{k \in Z(Q_{\delta/\varepsilon})} C(|u_k^\varepsilon|^2 - |u^{\varepsilon,y}|^2) - \frac{(1 - \eta)}{\eta} \varepsilon^d \sum_{\{Q_\delta\}} \sum_{k \in Z(Q_{\delta/\varepsilon})} C|w_k^\varepsilon - w^{\varepsilon,y}|^2.$$

Note that by taking into account only interactions with $(k, k') \in N_0(Q_{\delta/\varepsilon})$ we have neglected some interactions "through the boundary" of $Q_{\delta/\varepsilon}$, which introduce an error on the boundary of the hard components. After a proper adjustment of the position of Q_δ this error can be estimated from below by $-C' \varepsilon \delta^{d-1}$ using the convexity and the Poincaré inequality as follows. By (2.2) and the Poincaré inequalities on the first hard phase C_1 we have

$$\varepsilon^d \sum_{\{Q_\delta\}} \sum_{k \in C_1 \cap Z(Q_{\delta/\varepsilon})} |u_k^\varepsilon - C^\varepsilon|^p \leq C$$

for some constant C^ε, and we can take C^ε equal to the average of u_k^ε over $\bigcup_{\{Q_\delta\}} (C_1 \cap Z(Q_{\delta/\varepsilon}))$. Denote by $\hat{C}_1$ the set of $k \in \mathbb{Z}^d$ that are connected to C_1. Combining the

last estimate with the energy bound and considering (2.2) we get

$$\varepsilon^d \sum_{\{Q_\delta\}} \sum_{k \in \hat{\mathcal{C}}_1 \cap Z(Q_{\delta/\varepsilon})} |u_k^\varepsilon - C^\varepsilon|^p \leq C.$$

Next, we choose $\mathcal{R}$ such that any two points do not interact if the distance between them is greater than or equal to $\mathcal{R}$. For each $\varepsilon > 0$ one can adjust the position of the cubes $Q_{\delta/\varepsilon}$ in such a way that

$$\varepsilon^d \sum_{\{Q_\delta\}} \sum_{k \in \hat{\mathcal{C}}_1 \cap Z(Q_{\delta/\varepsilon}^{\mathcal{R}})} |u_k^\varepsilon - C^\varepsilon|^p \leq C_{\mathcal{R}} \frac{\varepsilon}{\delta},$$

where

$$Z(Q_{\delta/\varepsilon}^{\mathcal{R}}) = \{k \in Z(Q_{\delta/\varepsilon}) : \operatorname{dist}(k, \partial Q_{\delta/\varepsilon}) \leq \mathcal{R}\}.$$

Setting $\widehat{N}_0(Q_{\delta/\varepsilon}) = N_0(Q_{\delta/\varepsilon}) \cap (\mathcal{C}_1 \times \mathcal{C}_1)$, with the help of Jensen's inequality we obtain

$$\varepsilon^d \sum_{\{Q_\delta\}} \sum_{(k,k') \in \widehat{N}_0(Q_{\delta/\varepsilon})} f(k, k', u_k^\varepsilon - u_{k'}^\varepsilon)$$

$$= \varepsilon^d \sum_{\{Q_\delta\}} \sum_{(k,k') \in \widehat{N}_0(Q_{\delta/\varepsilon})} f(k, k', (u_k^\varepsilon - C^\varepsilon) - (u_{k'}^\varepsilon - C^\varepsilon))$$

$$\geq \frac{\delta^d}{T^d} \sum_{\{Q_\delta\}} \sum_{(y,y') \in (N_0^\#(Y) \cap (\mathcal{C}_1 \times \mathcal{C}_1))} f(y, y', (u^{\varepsilon,y} - C^\varepsilon) - (u^{\varepsilon,y'} - C^\varepsilon)) - C' \frac{\varepsilon}{\delta} \delta^d$$

$$= \frac{\delta^d}{T^d} \sum_{\{Q_\delta\}} \sum_{(y,y') \in (N_0^\#(Y) \cap (\mathcal{C}_1 \times \mathcal{C}_1))} f(y, y', u^{\varepsilon,y} - u^{\varepsilon,y'}) - C' \frac{\varepsilon}{\delta} \delta^d.$$

Considering (8.4) and summing up over all the connected components yields

$$\varepsilon^d \sum_{\{Q_\delta\}} \sum_{(k,k') \in N_0(Q_{\delta/\varepsilon})} f(k, k', u_k^\varepsilon - u_{k'}^\varepsilon)$$

$$\geq \frac{\delta^d}{T^d} \sum_{\{Q_\delta\}} \sum_{(y,y') \in N_0^\#(Y)} f(y, y', u^{\varepsilon,y} - u^{\varepsilon,y'}) - C' \frac{\varepsilon}{\delta} \delta^d.$$

Passing now in (8.10) to the limit as $\varepsilon \to 0$, we obtain the estimate

$$\liminf_{\varepsilon \to 0} \varepsilon^d \sum_{\{Q_\delta\}} \Big(\sum_{(k,k') \in N_0(Q_{\delta/\varepsilon})} f(k, k', u_k^\varepsilon - u_{k'}^\varepsilon) + \sum_{k \in Z(Q_{\delta/\varepsilon})} C|u_k^\varepsilon - w_\varepsilon^k|^2 \Big)$$

$$\geq (1-\eta) \frac{\delta^d}{T^d} \sum_{\{Q_\delta\}} \Big(\sum_{(y,y') \in N_0(Y)} f(y, y', u_\delta^y - u_\delta^{y'}) + \sum_{k \in Y} C|u_\delta^y - w_\delta^y|^2 \Big)$$

$$+ C(1-\eta) \liminf_{\varepsilon \to 0} \varepsilon^d \sum_{\{Q_\delta\}} \Big(\sum_{k \in Z(Q_{\delta/\varepsilon})} (|u_k^\varepsilon|^2 - |u_\delta^y|^2) - \frac{1}{\eta} \sum_{k \in Z(Q_{\delta/\varepsilon})} |w_k^\varepsilon - w^{\varepsilon,y}|^2 \Big),$$

where the subscript δ indicates the average on Q_δ.

Note that, using Proposition 8.2,

(8.11)
$$\int_{Q_\delta} \varphi_g(x, \{u_\delta^y\}) \, dx$$

$$= \frac{1}{T^d} \int_{Q_\delta} \Big(\sum_{(y,y')\in N_0^\#(Y)} f(y, y', u_\delta^y - u_\delta^{y'}) + \sum_{y\in Y} C|u_\delta^y - w^y(x)|^2 \Big) \, dx$$

$$= \frac{\delta^d}{T^d} \Big(\sum_{(y,y')\in N_0^\#(Y)} f(y, y', u_\delta^y - u_\delta^{y'}) + \sum_{y\in Y} C|u_\delta^y - w_\delta^y|^2 \Big) + O\Big(\int_{Q_\delta} |w_\delta^y - w^y(x)|^2 \, dx \Big).$$

Comparing (8.9) and (8.11), by the arbitrariness of the partition $\{Q_\delta\}$ and $\eta > 0$, and noting that $\sum \chi_{Q_\delta}\{u_\delta^y\}$ converge to $\{u^y\}$ as $\delta \to 0$ we then get

$$\liminf_{\varepsilon\to 0} \int_\Omega (|u^{\varepsilon,y}|^2 - |u^y|^2) \le 0,$$

which implies the strong convergence for all $y \in Y$.   □

8.2. Minimizing movements. We now fix initial data u_0^ε strongly converging to $\{u_0^y\}$ and with $\sup_\varepsilon F_\varepsilon(u_\varepsilon^0) < +\infty$. Given $\tau > 0$ we define iteratively the functions $u_{\tau,n}^\varepsilon$ as the unique minimizers of the problems

$$\min\Big\{ F_\varepsilon(v) + \frac{1}{2\tau} \sum_{k\in Z^\varepsilon(\Omega)} \varepsilon^d |v_k - (u_{\tau,n-1}^\varepsilon)_k|^2 \Big\},$$

where we have set $u_{\tau,0}^\varepsilon = u_0^\varepsilon$.

THEOREM 8.5. *Suppose that f and all f_{hom}^j are continuously twice differentiable. For all choices of infinitesimal sequences ε and τ, the functions $u^{\tau,\varepsilon}(x,t)$ defined by*

$$u^{\tau,\varepsilon}(x,t) = (u_{\tau,\lfloor t/\tau\rfloor}^\varepsilon)_{\lfloor x/\varepsilon\rfloor}$$

converge in $C^{1/2}((0,+\infty); L^2(\Omega))^{T^d}$ to a vector function $\{u^y\}$ with $y \in Y$. The components of this function are independent of y on each $C_j \cap Y$, so that we equivalently use the notation u_j for their common value. With this notation and setting

$$c_j = \frac{\#(C_j \cap Y)}{T^d}$$

for all $j = 0,\ldots,N$, $\{u^y\}$ is characterized as the solution of the coupled system

$$c_j \frac{\partial u_j}{\partial t} = \operatorname{div}\Big(\nabla f_{\mathrm{hom}}^j(\nabla u_j) \Big) - \frac{1}{T^d} \sum_{(y,y')\in N_0(Y), y\in C_j} \frac{\partial}{\partial u} f(y, y', u_j - u^{y'})$$

(8.12)
$$+ \frac{1}{T^d} \sum_{(y',y)\in N_0(Y), y\in C_j} \frac{\partial}{\partial u} f(y', y, u^{y'} - u_j), \qquad j = 1,\ldots,N,$$

$$\frac{\partial u^y}{\partial t} = - \sum_{(y,y') \in N_0(Y)} \frac{\partial}{\partial u} f(y, y', u^y - u^{y'})$$

$$+ \sum_{(y',y) \in N_0(Y)} \frac{\partial}{\partial u} f(y', y, u^{y'} - u^y), \qquad y \in Y \cap C_0,$$

with u_j satisfying Neumann boundary conditions

$$\nabla f^j_{\text{hom}}(\nabla u_j) \cdot \nu = 0$$

on $\partial\Omega \times (0, +\infty)$, and u^y the coupling condition

(8.13) $$u^y = u_j \text{ if } y \in C_j \cap Y$$

and the initial conditions

$$u^y(0, x) = u_0^y(x).$$

This limit function also coincides with the limit of gradient flows of F_ε.

Proof. By the convexity of the functionals we can use the stability for minimizing movements along F_ε. The results will follow by applying Theorem 11.2 in [11], provided that we have strong convergence of minimizing sequences (see [11, Remark 11.2]). This follows from Lemma 8.4 applied iteratively with

$$g_k^\varepsilon(u) = \frac{1}{2\tau}|u - (u^\varepsilon_{\tau,n-1})_k|^2,$$

so that all sequences $u^\varepsilon_{\tau,n}$ are strongly converging as $\varepsilon \to 0$ (thanks to the strong convexity of the Γ-limit). If we denote by $\{u^y_{\tau,n}\}$ their two-scale limit, by the fundamental theorem of Γ-convergence they solve iteratively an analogous minimization scheme with $u^{\tau,0}_\varepsilon = \{u_0^y\}$, and $\{u^y_{\tau,n}\}$ being the unique minimizer of

(8.14)

$$\min\Big\{\sum_{j=1}^N \frac{1}{\#(C_j \cap Y)} \sum_{y \in C_j \cap Y} \int_\Omega f^j_{\text{hom}}(\nabla v^y) \, dx$$

$$+ \frac{1}{T^d} \sum_{y \in Y} \frac{1}{2\tau} \int_\Omega |v^y(x) - u^y_{\tau,n-1}(x)|^2 \, dx + \frac{1}{T^d} \sum_{(y,y') \in N_0(Y)} \int_\Omega f(y, y', v^y - v^{y'}) \, dx\Big\},$$

with the constraint that v^y is constant on each component C_j.

Under the assumption that f and f^j_{hom} are C^2 we can derive the Euler–Lagrange equations for $\{u^y_{\tau,n}\}$. It is convenient to separate the hard and soft phases by introducing the functions

(8.15) $$u_j^{\tau,n} = u^y_{\tau,n} \text{ if } y \in C_j \cap Y$$

for $j = 1, \ldots, N$ and the set of indices $C_0 = Y \setminus \bigcup_{j=1}^N C_j$.

For $j = 1, \ldots, N$ we obtain

$$-\text{div}\nabla f^j_{\text{hom}}(\nabla u_j^{\tau,n}) + c_j \frac{u_j^{\tau,n} - u_j^{\tau,n-1}}{\tau} + \sum_{(y,y') \in N_0(Y), y \in C_j} \frac{\partial}{\partial u} f(y, y', u_j^{\tau,n} - u^{y'}_{\tau,n})$$

$$- \sum_{(y',y) \in N_0(Y), y \in C_j} \frac{\partial}{\partial u} f(y', y, u^{y'}_{\tau,n} - u_j^{\tau,n}) = 0$$

with Neumann boundary condition, which reads

$$\nabla f_{\text{hom}}^{j}(\nabla u_{j}^{\tau,n}) \cdot \nu = 0 \qquad \text{on } \partial\Omega, \quad j = 1, \ldots, N,$$

where ν stands for the exterior normal on $\partial\Omega$.

For fixed $y \in C_0$ we obtain instead

$$\frac{u_{y}^{\tau,n} - u_{y}^{\tau,n-1}}{\tau} + \sum_{(y,y') \in N_0(Y)} \frac{\partial}{\partial u} f(y, y', u_{\tau,n}^{y} - u_{\tau,n}^{y'})$$

$$- \sum_{(y',y) \in N_0(Y)} \frac{\partial}{\partial u} f(y', y, u_{\tau,n}^{y'} - u_{\tau,n}^{y}) = 0.$$

Note the coupling condition (8.15).

We define the piecewise-constant trajectories

$$u_{j}^{\tau}(t,x)) = u_{j}^{\tau,\lfloor t\tau \rfloor}(x)$$

for $j = 1, \ldots, N$ and

$$u_{\tau}^{y}(t,x) = u_{\tau,\lfloor t\tau \rfloor}^{y}(x)$$

for $y \in C_0$, which converge uniformly in $[0, +\infty)$ as $\tau \to 0$ to functions $u_j(t,x)$ and $u^y(t,x)$, respectively. By passing to the limit in the Euler–Lagrange equations we obtain system (8.12). □

Remark 8.6. The limit system is not decoupled also if $C_0 = \emptyset$, in which case we have the system of partial differential equations for u_j only,

$$c_j \frac{\partial u_j}{\partial t} = \text{div}\left(\nabla f_{\text{hom}}^{j}(\nabla u_j)\right)$$

$$- \frac{1}{T^d} \sum_{j' \neq j} \Big(\sum_{(y,y') \in N_0(Y), y \in C_j, y' \in C_{j'}} \frac{\partial}{\partial u} f(y, y', u_j - u_{j'})$$

$$+ \frac{1}{T^d} \sum_{(y',y) \in N_0(Y), y \in C_j, y' \in C_{j'}} \frac{\partial}{\partial u} f(y', y, u_{j'} - u_j) \Big).$$

Example 8.7. In the case of the energies in Example 5.3(2) the limit $(u_1(t,x), u_2(t,x))$ satisfies

$$\begin{cases} \dfrac{1}{2} \dfrac{\partial u_2}{\partial t} = 4 \dfrac{\partial^2 u_2}{\partial x^2} - 2u_2 + 2u_1, \\[2mm] \dfrac{\partial u_1}{\partial t} = 4(u_2 - u_1), \\[2mm] u_1(x,0) = u_2(x,0) = u^0(x). \end{cases}$$

Note that we may solve the ODE and obtain the integro-differential problem satisfied by $u = u_2$ only,

$$\begin{cases} \dfrac{1}{2} \dfrac{\partial u(x,t)}{\partial t} = 4 \dfrac{\partial^2 u(x,t)}{\partial x^2} - 2u(x,t) + 2u^0(x)e^{-4t} + 2 \displaystyle\int_0^t e^{4(s-t)} u(x,s)\, ds, \\[2mm] u(x,0) = u^0(x). \end{cases}$$

Consider also an example of two-dimensional energies.

Example 8.8. On the unit square $[0, 1] \times [0, 1]$ we define the energy by

$$
F_\varepsilon(u) = \sum_{i=1}^{\lfloor \frac{1}{2\varepsilon} \rfloor - 1} \sum_{j=1}^{\lfloor \frac{1}{\varepsilon} \rfloor - 1} \varepsilon^2 \left| \frac{u_{2i,j} - u_{2i,j-1}}{\varepsilon} \right|^2 + \sum_{i=1}^{\lfloor \frac{1}{\varepsilon} \rfloor - 1} \sum_{j=1}^{\lfloor \frac{1}{2\varepsilon} \rfloor - 1} \varepsilon^2 \left| \frac{u_{i,2j} - u_{i-1,2j}}{\varepsilon} \right|^2
$$

$$
+ \varepsilon^2 \sum_{i=1}^{\lfloor \frac{1}{2\varepsilon} \rfloor - 1} \sum_{j=1}^{\lfloor \frac{1}{\varepsilon} \rfloor - 1} \varepsilon^2 \left| \frac{u_{2i-1,j} - u_{2i-1,j-1}}{\varepsilon} \right|^2
$$

$$
+ \varepsilon^2 \sum_{i=1}^{\lfloor \frac{1}{\varepsilon} \rfloor - 1} \sum_{j=1}^{\lfloor \frac{1}{2\varepsilon} \rfloor - 1} \varepsilon^2 \left| \frac{u_{i,2j-1} - u_{i-1,2j-1}}{\varepsilon} \right|^2.
$$

In this case $T = 2$ that is the period is equal to 2 in each coordinate direction. There is one strong phase. The discrete periodicity cell consists of four points, three of them belonging to the strong phase and one to the weak phase. Denote the limit function on the weak and strong phases by $u_1(t, x)$ and $u_2(t, x)$, respectively. They satisfy the following system:

$$
\begin{cases}
\dfrac{3}{4} \dfrac{\partial u_2}{\partial t} = \Delta u_2 + 2(u_1 - u_2), \\[2mm]
\dfrac{\partial u_1}{\partial t} = 8(u_2 - u_1), \\[2mm]
u_1(x, 0) = u_2(x, 0) = u^0(x);
\end{cases}
$$

on the boundary of the square the function u_2 satisfies the homogeneous Neumann boundary condition.

9. Appendix.

LEMMA 9.1. *Let* $u_k = v_k$ *if* $k \notin \bigcup_{j=1}^{N} A_j$. *Then we have*

$$
\sum_{(k,k') \in N_0(\Omega)} \varepsilon^d |f(k, k', u_k - u_{k'}) - f(k, k', v_k - v_{k'})|
$$

$$
(9.1) \qquad \leq C \sum_{j=1}^{N} \left(\sum_{k \in A_j} \varepsilon^d |u_k - v_k|^p \right)^{1/p} \left(F_\varepsilon(u) + F_\varepsilon(v) \right)^{p-1/p}.
$$

Proof. We estimate

$$
\sum_{(k,k') \in N_0(\Omega)} \varepsilon^d |f(k, k', u_k - u_{k'}) - f(k, k', v_k - v_{k'})|
$$

$$
\leq C \sum_{(k,k') \in N_0(\Omega)} \varepsilon^d |(u_k - u_{k'}) - (v_k - v_{k'})|(|u_k - u_{k'}|^{p-1} + |v_k - v_{k'}|^{p-1})
$$

$$
\leq C \sum_{j=1}^{N} \sum_{(k,k') \in N_0(\Omega), k \in A_j} \varepsilon^d |u_k - v_k|(|u_k - u_{k'}|^{p-1} + |v_k - v_{k'}|^{p-1})
$$

$$
\leq C \sum_{j=1}^{N} \left(\sum_{(k,k') \in N_0(\Omega), k \in A_j} \varepsilon^d |u_k - v_k|^p \right)^{1/p}
$$

$$
\left(\sum_{(k,k') \in N_0(\Omega), k \in A_j} (|u_k - u_{k'}|^p + |v_k - v_{k'}|^p) \right)^{(p-1/p)}
$$

$$\leq C \sum_{j=1}^{N} \Big(\sum_{k \in A_j} \varepsilon^d |u_k - v_k|^p \Big)^{1/p} \Big(\sum_{(k,k') \in N_0(\Omega)} \varepsilon^d \big(|u_k - u_{k'}|^p + |v_k - v_{k'}|^p \big) \Big)^{(p-1/p)}.$$

The required claim then follows by (2.2) and (2.3). $\quad\square$

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