The Linear Elasticity Tensor of Incompressible Materials

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Dedicated to Prof. Raymond W. Ogden, on occasion of his 70th birthday

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Abstract

With a universally accepted abuse of terminology, materials having much larger stiffness for volumetric than for shear deformations are called *incompressible*. This work proposes two approaches to the evaluation of the correct form of the linear elasticity tensor of the so-called incompressible materials, both stemming from the non-linear theory. In the approach of strict incompressibility, one imposes the kinematical constraint of isochoric deformation. In the approach of quasi-incompressibility, which is often employed to enforce incompressibility in numerical applications such as the Finite Element Method, one instead assumes a decoupled form of the elastic potential (or strain energy), which is written as the sum of a function of the volumetric deformation only and a function of the distortional deformation only, and then imposes that the bulk modulus be much larger than all other moduli. The conditions which the elasticity tensor has to obey for both strict incompressibility and quasi-incompressibility have been derived, regardless of the material symmetry. The representation of the linear elasticity tensor for the quasi-incompressible case differs from that of the strictly incompressible case by one parameter, which can be conveniently chosen to be the bulk modulus. Some important symmetries have been studied in detail, showing that the linear elasticity tensor for the cases of isotropy, transverse isotropy and orthotropy is characterised by 1, 3, 6 independent parameters, respectively, for the case of strict incompressibility, and 2, 4, 7 independent parameters, respectively, for the case of quasi-incompressibility, as opposed to the 2, 5, 9 parameters, respectively, of the general compressible case.

Keywords: covariant representation, Elasticity, elasticity tensor, incompressibility, quasi-incompressibility, incompressible, quasi-incompressible, nearly-incompressible, material symmetry, anisotropy.

1 Introduction

In several contexts of Continuum Mechanics, and particularly for materials such as elastomers and soft biological tissues, whose stiffness under volumetric compression is usually several orders of magnitude higher than the stiffness in shear, the mechanical behaviour of materials is studied under the assumption of either strict incompressibility or quasi-incompressibility. The constraint of isochoric (i.e., volume-preserving) deformation is often employed to approximate the behaviour of incompressible materials. To be more precise, we recall here that an idealised material body is said to be strictly incompressible when the substantial derivative of its mass density vanishes identically [1], i.e., when $D_t \varrho = 0$, with ϱ being the mass density of the body, and D_t the substantial derivative operator. In the case in which the mass of the body is locally conserved, the mass balance law of the body reads

$$D_t \varrho + \varrho \, \mathrm{div}(\boldsymbol{v}) = 0, \tag{1}$$

where $\boldsymbol{v}$ is the velocity. Thus, setting $D_t \varrho$ equal to zero implies that Equation (1) reduces to

$$\mathrm{div}(\boldsymbol{v}) = 0, \tag{2}$$

in which case the velocity field is said to be divergence-free. Since the divergence of the velocity field is related to the time derivative of the volume ratio $J = \det \boldsymbol{F}$ (where $\boldsymbol{F}$ is the deformation gradient) by

$$\dot{J} = J \, \mathrm{div}(\boldsymbol{v}), \tag{3}$$

the vanishing of $\mathrm{div}(\boldsymbol{v})$ implies that the volume ratio J is constant in time, and therefore isochoric motions are compatible with the requirement of incompressibility.

The assumption of strict incompressibility, however, yields both theoretical and computational issues. Within the framework of the Finite Element Method, it requires the development of robust and efficient numerical schemes that prevent from mesh locking (e.g., the Lagrange multiplier method, penalty methods, the Hu-Washizu variational principle, methods based on higher-order shape functions), while ensuring flexibility and containing computational cost [2, 3, 4]. Granted hyperelastic material behaviour from some given natural configuration of a body, these schemes generally express the elastic potential into a part depending solely on the volumetric deformation and a part depending solely on the distortional deformation. Whereas the former one depends solely on parameters that, in the linear theory, reduce to the bulk modulus (this can be either the true one or some suitably chosen constant, as in penalty methods), the latter one strongly depends on the body's material symmetries (see, e.g., [5, 6]) and, in non-linear theories, is either obtained by fitting experimental data or given from the outset. In any case, the total non-linear elastic potential should lead to an elasticity tensor that is consistent with its linear counterpart [7], which is, in principle, always measurable experimentally. In linear elasticity, the usual approach to study incompressibility is based on the compliance elasticity tensor, the inverse of the (stiffness) elasticity tensor $\mathbb{L}$, and on making the bulk modulus diverge (e.g., [8]). In this way, the (stiffness) elasticity tensor diverges and is thus not defined.

Here, we propose a rigorous framework for determining the correct form of the linear elasticity tensor of incompressible and quasi-incompressible materials, starting from the theory of Non-Linear Elasticity. We shall start by performing a full inverse Piola transform of the standard material elasticity tensor, so to derive the standard spatial elasticity tensor, and then evaluate the latter at zero strain, to finally obtain the (spatial) linear elasticity tensor. This approach can be exploited to enforce that the non-linear elastic material is consistent with its linearised counterpart [7]. We had previously [9] worked out the calculations for the case of isotropic quasi-incompressible materials and now aim at giving the general expression of the elasticity tensor for the strictly incompressible and the quasi-incompressible cases, regardless of material symmetry, and then retrieve the important particular cases of isotropy, transverse isotropy and orthotropy. For the case of strict incompressibility, we show that the number of independent elastic constants decreases from 2 to 1 for isotropy, from 5 to 3 for transverse isotropy, and from 9 to 6 for orthotropy. For the case of quasi-incompressibility, the bulk modulus is an additional independent elastic constant in all cases. The framework we propose also allows to conveniently check for the positive semi-definiteness (strictly incompressible case) or definiteness (quasi-incompressible case) of the elasticity tensor. Positive definiteness or semi-definiteness determine the strict convexity or convexity, respectively, of the quadratic potential of the linear theory, and influence the mathematical properties of the solutions, such as existence, uniqueness, smoothness, etc. (see, e.g., [10]).

This work is motivated by the importance that the elasticity tensor has in general, in both linear and non-linear Elasticity. Indeed, the elasticity tensor plays an essential role in Computational Mechanics, as it is the main "ingredient" defining the large stiffness matrices that are then employed by the solver modules of Finite Element packages. In Non-Linear Elasticity, the choice of the form of the elasticity tensor is various, depending on the choice of objective stress rate and measure of rate of deformation (see, e.g., [11, 12]). In contrast, in the small-strain theory, since all measures of stress converge to the Cauchy stress, and all measures of strain converge to the infinitesimal strain, also all possible elasticity tensors converge to the "classical" elasticity tensor of Linear Elasticity.

The paper is structured in six sections (including Introduction and Discussion). Section 2 introduces the notation, reports some results from Tensor Algebra that are relevant to our purposes, and recalls the expressions of the elasticity tensor of the non-linear and linear theory for the general compressible case when the volumetric-distortional decomposition of the deformation [13, 14, 15] is used. Section 3 deals with incompressible elasticity and includes our results on the representation of the elasticity tensor in the non-linear and the linear theory, *regardless* of material symmetry. Section 4 consists of the study of the linear elasticity tensor for incompressible materials for the case of isotropy, transverse isotropy and orthotropy. Section 5 is devoted to a discussion about the issues of invertibility and positive definiteness of the linear elasticity tensor for the cases of strict incompressibility and quasi-incompressibility.

2 Theoretical Background

Here, we briefly introduce the general notation employed in this work, report some results from Tensor Algebra that are related to the Theory of Elasticity [16], and recall the representation of the material, spatial and linear elasticity tensors when the volumetric-distortional decomposition of the deformation is used [17, 16]. Furthermore, we also refer to Walpole's formalism for the representation of fourth-order tensors [18] in all possible symmetries, although here we limit ourselves to the most common cases: isotropy, transverse isotropy, orthotropy. Walpole had introduced this formalism in an earlier work [19], which we used extensively in the past (see. e.g., [20]). The newer representation devised by Walpole [18], which we employ here, introduces a very convenient matrix-based formalism. With respect to, e.g., Spencer's [21] representation (which has several convenient features on which we shall not elaborate here), one of the greatest advantages of Walpole's representation [18] is that it makes it extremely easy to check for the positive definiteness or invertibility of a fourth-order tensor, seen as an operator between spaces of second-order tensors.

Although differentiable manifolds are the most general and appropriate theatre for the description of Mechanics [22, 23], we restrict ourselves to the (much) simpler case of a three-dimensional affine space, which avoids the long series of theoretical intricacies brought about by high-level Differential Geometry. Roughly speaking, an affine space is a vector space in which any point can be a "local origin", thereby allowing vectors to be attached at any point. More rigorously, an affine space is given by a set $\mathcal{S}$, called the point space, considered together with a vector space $\mathcal{V}$, called the modelling space, and a map $\mathcal{F} : \mathcal{S} \times \mathcal{S} \to \mathcal{V}$ that, for every pair of points x, y of $\mathcal{S}$, yields a vector of $\mathcal{V}$ denoted $\mathcal{F}(x, y) = y - x = \boldsymbol{v}$, called the oriented segment from x to y. This map has to satisfy anti-commutativity, i.e., $[x - y] = -[y - x]$, the triangle rule, i.e., $y - x = [y - z] + [z - x]$, and the axiom of arbitrary origin, i.e., for every $x \in \mathcal{S}$ and $\boldsymbol{v} \in \mathcal{V}$ there exists one, and only one, $y \in \mathcal{S}$, such that $y - x = \boldsymbol{v}$. Given any point $x \in \mathcal{S}$, the axiom of arbitrary origin permits to define the set $T_x\mathcal{S} = \{\boldsymbol{v}_x = y - x : y \in \mathcal{S}\}$ of all the vectors emanating from x. The space $T_x\mathcal{S}$ and its dual space $T_x^\star\mathcal{S}$ are called the tangent space and the cotangent space, respectively, at point x, and their elements are called tangent vectors and tangent covectors, respectively, at point x. The disjoint union of all tangent spaces $T_x\mathcal{S}$ for all $x \in \mathcal{S}$ is called the tangent bundle of $\mathcal{S}$, and is denoted by $T\mathcal{S}$; the cotangent bundle $T^\star\mathcal{S}$ is defined analogously. A thorough introduction to affine spaces is given, e.g., by Epstein [23].

The structure of affine space is the minimal structure needed for Differential Calculus, since a derivative is in fact a tangent vector. This is immediately reflected in the description of Classical Physics, where the structure of affine space allows for attaching a vector representing a given physical quantity at any point of space. The prime example is that of the velocity, which, being the time derivative of a trajectory, is in fact a tangent vector in the sense of affine spaces, aside from being also tangent to the trajectory of the particle. The modelling space used in the definition of the physical affine space $\mathcal{S}$ of Classical Physics is the familiar $\mathbb{R}^3$. This space $\mathcal{S}$ is indeed very similar to $\mathbb{R}^3$ and one barely sees the difference, as long as vectors from the same tangent space are involved. Therefore, in many works in the literature (among which some of our past works), the affine space $\mathcal{S}$ of Classical Physics is simply denoted $\mathbb{R}^3$. However, following a didactical approach, we prefer to keep the distinction between the affine space $\mathcal{S}$ and its modelling space $\mathbb{R}^3$.

Throughout this work, we employ the covariant formalism, i.e., we keep the distinction between a vector space and its dual space or, equivalently, between vectors and covectors. Aside from the fact that this allows for introducing general curvilinear coordinates, and for accounting for geometrical non-linearities, it is of fundamental importance to clarify the transformation laws that each physical quantity obeys. Indeed, vectors and covectors obey different transformation laws, and therefore the pull-back and push-forward operations, crucial in Continuum Mechanics, are performed in a different way (see Section 2.1). Furthermore, as has also been remarked by Marsden and Hughes [22], the operations of pull-back/push-forward and of index raising/lowering *do not* commute, which means that even extra care must be taken when transforming vectorial or covectorial objects. The covariant formalism helps avoid errors, since it makes this non-commutativity evident.

In conclusion, we deem the small additional pain of using the structure of affine space and the covariant formalism worth it for the exposition of our results. The notation in this and in some previous works [16, 24], to which we shall extensively refer, mostly follows the classical treatise by Marsden and Hughes [22], with some relatively small variations.

2.1 General Notation

Lowercase symbols and indices are reserved to spatial quantities in the natural three-dimensional space $\mathcal{S}$ of Classical Mechanics. Uppercase symbols and indices denote material quantities in the reference configuration $\mathcal{B}_R \subset \mathcal{S}$ (or in the body manifold $\mathcal{B}$, if no particular reference configuration is chosen [22, 25, 23]). At each point $x \in \mathcal{S}$, the tangent and cotangent spaces are denoted $T_x\mathcal{S}$ and $T^\star_x\mathcal{S}$, respectively. The tangent and cotangent bundles are denoted $T\mathcal{S}$ and $T^\star\mathcal{S}$, respectively. Similarly, one defines the tangent and cotangent spaces $T_X\mathcal{B}_R$ and $T^\star_X\mathcal{B}_R$ at $X \in \mathcal{B}_R$, and the tangent and cotangent bundles $T\mathcal{B}_R$ and $T^\star\mathcal{B}_R$. The spaces of spatial and material tensors of order $m = r + s$, with r vector feet and s covector feet (i.e., with r contravariant indices and s covariant indices) are denoted $[T\mathcal{S}]^r{}_s$ and $[T\mathcal{B}_R]^r{}_s$, respectively. The simple contraction of two tensors such that the last foot of the first tensor is a vector and the first foot of the second tensor is a covector (or vice versa) is indicated by simply juxtaposing the two tensors, e.g., for $\boldsymbol{a} \in [T\mathcal{S}]^2_0$ and $\boldsymbol{c} \in [T\mathcal{S}]^0_2$, the contraction $\boldsymbol{a}\,\boldsymbol{c}$ has components $a^{ab}c_{bc}$. The double contraction of two tensors is similar to the simple contraction, except that the last two feet of the first tensor and the first two feet of the second tensor contract, and is denoted by a colon, e.g., for $\mathbb{T} \in [T\mathcal{S}]^2{}_2$ and $\boldsymbol{a} \in [T\mathcal{S}]^2_0$, the contraction $\mathbb{T} : \boldsymbol{a}$ has components $\mathrm{T}^{ab}{}_{cd}a^{cd}$.

The spaces $T\mathcal{S}$ and $T\mathcal{B}_R$ are assumed to be equipped with metric tensors $\boldsymbol{g}$ and $\boldsymbol{G}$, respectively. The scalar products induced by the metric tensors $\boldsymbol{g}$ and $\boldsymbol{G}$ are denoted by the symbol $\langle\,\cdot\,,\,\cdot\,\rangle$ for tensors of any order. For vectors or covectors, this is replaced by a simple low dot, e.g., for the case of spatial vectors, $\boldsymbol{g}(\boldsymbol{u},\boldsymbol{v}) = \boldsymbol{u}\,\boldsymbol{g}\,\boldsymbol{v} = \langle\boldsymbol{u},\boldsymbol{v}\rangle = \boldsymbol{u}.\boldsymbol{v}$. For the case of higher-order tensors (of the same type), each couple of homologous indices has to be contracted with the appropriate metric tensor, e.g., for the case of spatial "contravariant" fourth-order tensors (i.e., tensors in $[T\mathcal{S}]^4_0$), we have $\langle\mathbb{A},\mathbb{B}\rangle = \mathrm{A}^{abcd}\,g_{ai}\,g_{bj}\,g_{ck}\,g_{dl}\,\mathrm{B}^{ijkl}$. Note that we employ the usual identification $g^{ab} \equiv (\boldsymbol{g}^{-1})^{ab}$ throughout. The metric tensor $\boldsymbol{g}$ lowers contravariant indices, e.g., for the case of a vector $\boldsymbol{v}$, it gives the associated covector $\boldsymbol{v}^\flat = \boldsymbol{g}\boldsymbol{v}$, with components $v_a = g_{ab}v^b$. Analogously, the inverse metric tensor $\boldsymbol{g}^{-1}$ raises covariant indices, e.g., for the case of a covector $\boldsymbol{\varphi}$, it gives the associated vector $\boldsymbol{\varphi}^\sharp = \boldsymbol{g}^{-1}\boldsymbol{\varphi}$, with components $\varphi^a = g^{ab}\varphi_b$. Moreover, we use a single low dot to indicate that the metric tensor (or its inverse) is involved in the contraction of two tensors such that the last foot of the first tensor and the first foot of the second tensor are of the same type. For instance, if $\boldsymbol{a},\boldsymbol{b} \in\; [T\mathcal{S}]^2_0$, the expression $\boldsymbol{a}.\boldsymbol{b}$ stands for $\boldsymbol{a}\,\boldsymbol{g}\,\boldsymbol{b}$, which has components $a^{ab}\,g_{bc}\,b^{cd}$.

The deformation, $\chi : \mathcal{B}_R \to \mathcal{S}$, maps material points $X \in \mathcal{B}_R$ into spatial points $x = \chi(X) \in \mathcal{S}$, and its tangent map, the deformation gradient $\boldsymbol{F} : T\mathcal{B}_R \to T\mathcal{S}$, maps material tangent vectors $\boldsymbol{W} \in T\mathcal{B}_R$ into spatial tangent vectors $\boldsymbol{w} = \boldsymbol{F}\boldsymbol{W} \in T\mathcal{S}$, such that the directional derivative of χ with respect to $\boldsymbol{W}$ at point X is $(\partial_{\boldsymbol{W}}\chi)(X) = \boldsymbol{F}(X)\boldsymbol{W}$, and the components of $\boldsymbol{F}$ are $F^a{}_A = \chi^a{}_{,A}$. Given a material tensor field $\mathbb{P}$ valued in $[T\mathcal{B}_R]^r{}_s$, its push-forward $\chi_*[\mathbb{P}] = \mathbb{p}$ is the tensor field valued in $[T\mathcal{S}]^r{}_s$ obtained by contracting each contravariant index with $\boldsymbol{F}$ and each covariant index with $\boldsymbol{F}^{-T}$, which in components reads $\mathrm{p}^{a\cdots}{}_{\dots b} = F^a{}_A \dots (\boldsymbol{F}^{-T})_b{}^B\,\mathrm{P}^{A\cdots}{}_{\dots B}$. Analogously, given a spatial tensor field $\mathbb{q}$ valued in $[T\mathcal{S}]^r{}_s$, its pull-back $\chi^*[\mathbb{q}] = \mathbb{Q}$ is the tensor field valued in $[T\mathcal{B}_R]^r{}_s$ obtained by contracting each contravariant index by $\boldsymbol{F}^{-1}$ and each covariant index by $\boldsymbol{F}^T$, i.e., $\mathrm{Q}^{A\cdots}{}_{\dots B} = (\boldsymbol{F}^{-1})^A{}_a \dots (\boldsymbol{F}^T)_B{}^b\,\mathrm{q}^{a\cdots}{}_{\dots b}$. Note that the operations of pull-back/push-forward and of index raising/lowering *do not* commute: indeed, e.g., $\chi^*[\boldsymbol{v}^\flat] = \boldsymbol{F}^T[\boldsymbol{g}\boldsymbol{v}] \neq \boldsymbol{G}[\boldsymbol{F}^{-1}\boldsymbol{v}] = [\chi^*[\boldsymbol{v}]]^\flat$ (see, e.g., [22]).

The right and left Cauchy-Green deformation tensors are the pull-back $\boldsymbol{C} = \boldsymbol{F}^T.\boldsymbol{F} = \boldsymbol{F}^T\boldsymbol{g}\,\boldsymbol{F}$ of the spatial metric $\boldsymbol{g}$, and the push-forward $\boldsymbol{b} = \boldsymbol{F}.\boldsymbol{F}^T = \boldsymbol{F}\,\boldsymbol{G}^{-1}\boldsymbol{F}^T$ of the inverse material metric $\boldsymbol{G}^{-1}$, respectively. Their inverses $\boldsymbol{B} = \boldsymbol{C}^{-1} = \boldsymbol{F}^{-1}.\boldsymbol{F}^{-T} = \boldsymbol{F}^{-1}\boldsymbol{g}^{-1}\boldsymbol{F}^{-T}$ and $\boldsymbol{c} = \boldsymbol{b}^{-1} = \boldsymbol{F}^{-T}.\boldsymbol{F}^{-1} = \boldsymbol{F}^{-T}\boldsymbol{G}\boldsymbol{F}^{-1}$ are the pull-back of the inverse spatial metric $\boldsymbol{g}^{-1}$

and the push-forward of the material metric $\boldsymbol{G}$, respectively. The Green-Lagrange strain, comparing the pull-back $\boldsymbol{C}$ of the spatial metric $\boldsymbol{g}$ to the material metric $\boldsymbol{G}$, is given by $\boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - \boldsymbol{G})$. The volume ratio can be defined as $J = \det \boldsymbol{F} \equiv \sqrt{\det \boldsymbol{C}} = \sqrt{\det \boldsymbol{b}}$ [24] and its time derivative is $\dot{J} = J \operatorname{div}(\boldsymbol{v}) = J\, \boldsymbol{B} : \dot{\boldsymbol{E}} = \frac{1}{2} J\, \boldsymbol{B} : \dot{\boldsymbol{C}}$ [3]. In the volumetric-distortional decomposition of the deformation [13, 14, 15], we have $\boldsymbol{F} = J^{1/3}\bar{\boldsymbol{F}}$, $\boldsymbol{C} = J^{2/3}\bar{\boldsymbol{C}}$, where $\bar{\boldsymbol{C}} = \bar{\boldsymbol{F}}^T.\bar{\boldsymbol{F}}$, and $\boldsymbol{E} = J^{2/3}\bar{\boldsymbol{E}} + \frac{1}{2}(J^{2/3} - 1)\boldsymbol{G}$, where $\bar{\boldsymbol{E}} = \frac{1}{2}(\bar{\boldsymbol{C}} - \boldsymbol{G})$.

2.2 Identity, Spherical, Deviatoric Operators; Tensor Basis for Isotropy

In the space $[T\mathcal{S}]^2{}_2$ of symmetric fourth-order tensors (symmetric in the sense of metric transposition [16]) with the first two feet being vectorial and the second two being covectorial (in terms of components, with the first two indices being contravariant and the second two being covariant), the symmetric identity, spherical, and deviatoric operators [16], defined by using the special tensor products $\underline{\otimes}$ and $\overline{\otimes}$ introduced by Curnier et al. [26], read

$$\mathbb{I} = \tfrac{1}{2}(\boldsymbol{i} \,\underline{\otimes}\, \boldsymbol{i} + \boldsymbol{i} \,\overline{\otimes}\, \boldsymbol{i}), \tag{4a}$$
$$\mathbb{K} = \tfrac{1}{3}\, \boldsymbol{g}^{-1} \otimes \boldsymbol{g}, \tag{4b}$$
$$\mathbb{M} = \mathbb{I} - \mathbb{K}, \tag{4c}$$

and have components

$$I^{ab}{}_{cd} = \tfrac{1}{2}(\delta^a{}_c \delta^b{}_d + \delta^a{}_d \delta^b{}_c), \tag{5a}$$
$$\mathrm{K}^{ab}{}_{cd} = \tfrac{1}{3}\, g^{ab} g_{cd}, \tag{5b}$$
$$\mathrm{M}^{ab}{}_{cd} = \tfrac{1}{2}(\delta^a{}_c \delta^b{}_d + \delta^a{}_d \delta^b{}_c) - \tfrac{1}{3}\, g^{ab} g_{cd}. \tag{5c}$$

When applied to a symmetric second-order tensor $\boldsymbol{a} \in [T\mathcal{S}]^2_0$, $\mathbb{K}$ and $\mathbb{M}$ yield the spherical and deviatoric parts of $\boldsymbol{a}$, respectively, i.e.,

$$\operatorname{sph}(\boldsymbol{a}) = \mathbb{K} : \boldsymbol{a} = \tfrac{1}{3} \operatorname{tr}(\boldsymbol{a})\, \boldsymbol{g}^{-1}, \quad \operatorname{dev}(\boldsymbol{a}) = \mathbb{M} : \boldsymbol{a} = \boldsymbol{a} - \tfrac{1}{3} \operatorname{tr}(\boldsymbol{a})\, \boldsymbol{g}^{-1}, \tag{6}$$

where $\operatorname{tr}(\,\cdot\,)$ is the natural trace operator, such that $\operatorname{tr}(\boldsymbol{a}) = \boldsymbol{g}^{-1} : \boldsymbol{a} = g^{ab} a_{ab}$. Furthermore, $\{\mathbb{K}, \mathbb{M}\}$ is the canonical basis of the subspace of symmetric isotropic tensors in $[T\mathcal{S}]^2{}_2$, where isotropy is defined as the symmetry (i.e., invariance) with respect to arbitrary rotations. The spherical and deviatoric operators enjoy the properties of idempotence and orthogonality [19, 18, 16], i.e.,

$$\mathbb{K} : \mathbb{K} = \mathbb{K}, \qquad \mathbb{M} : \mathbb{M} = \mathbb{M}, \tag{7a}$$
$$\mathbb{K} : \mathbb{M} = \mathbb{O}, \qquad \mathbb{M} : \mathbb{K} = \mathbb{O}, \tag{7b}$$

where $\mathbb{O}$ is the null fourth-order tensor in $[T\mathcal{S}]^2{}_2$.

Stiffness and compliance elasticity tensors belong to $[T\mathcal{S}]^4_0$ and $[T\mathcal{S}]^0_4$, respectively and, for our purposes, it is important to recall the expressions of the identity, spherical and deviatoric operators in these spaces. These are obtained by raising and lowering all indices of the tensors in Equation (4), respectively, to obtain [16]

$$\mathbb{I}^\sharp = \tfrac{1}{2}(\boldsymbol{g}^{-1} \,\underline{\otimes}\, \boldsymbol{g}^{-1} + \boldsymbol{g}^{-1} \,\overline{\otimes}\, \boldsymbol{g}^{-1}), \tag{8a}$$
$$\mathbb{K}^\sharp = \tfrac{1}{3}\, \boldsymbol{g}^{-1} \otimes \boldsymbol{g}^{-1}, \tag{8b}$$
$$\mathbb{M}^\sharp = \mathbb{I}^\sharp - \mathbb{K}^\sharp, \tag{8c}$$

and

$$
\begin{aligned}
\mathbb{I}^\flat &= \tfrac{1}{2}(\boldsymbol{g} \,\underline{\otimes}\, \boldsymbol{g} + \boldsymbol{g} \,\overline{\otimes}\, \boldsymbol{g}), &\text{(9a)}\\
\mathbb{K}^\flat &= \tfrac{1}{3}\, \boldsymbol{g} \otimes \boldsymbol{g}, &\text{(9b)}\\
\mathbb{M}^\flat &= \mathbb{I}^\flat - \mathbb{K}^\flat, &\text{(9c)}
\end{aligned}
$$

which have component expressions

$$
\begin{aligned}
\mathrm{I}^{abcd} &= \tfrac{1}{2}(g^{ac}g^{bd} + g^{ad}g^{bc}), &\text{(10a)}\\
\mathrm{K}^{abcd} &= \tfrac{1}{3}\, g^{ab}g^{cd}, &\text{(10b)}\\
\mathrm{M}^{abcd} &= \tfrac{1}{2}(g^{ac}g^{bd} + g^{ad}g^{bc}) - \tfrac{1}{3}\, g^{ab}g^{cd}, &\text{(10c)}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{I}_{abcd} &= \tfrac{1}{2}(g_{ac}\, g_{bd} + g_{ad}\, g_{bc}), &\text{(11a)}\\
\mathrm{K}_{abcd} &= \tfrac{1}{3}\, g_{ab}\, g_{cd}, &\text{(11b)}\\
\mathrm{M}_{abcd} &= \tfrac{1}{2}(g_{ac}\, g_{bd} + g_{ad}\, g_{bc}) - \tfrac{1}{3}\, g_{ab}\, g_{cd}. &\text{(11c)}
\end{aligned}
$$

Again, $\{\mathbb{K}^\sharp, \mathbb{M}^\sharp\}$ and $\{\mathbb{K}^\flat, \mathbb{M}^\flat\}$ are the canonical bases of the subspaces of symmetric isotropic tensors in $[T\mathcal{S}]^4_0$ and $[T\mathcal{S}]^0_4$, respectively. Also the tensors $\mathbb{K}^\sharp$ and $\mathbb{M}^\sharp$, and the tensors $\mathbb{K}^\flat$ and $\mathbb{M}^\flat$ enjoy idempotence and orthogonality, and a thorough analysis can be found in a previous work [16], which reports the results obtained by Walpole [19, 18] in the covariant formalism also adopted here.

Note that a symmetric isotropic fourth-order tensor in $[T\mathcal{S}]^2{}_2$ (or $[T\mathcal{S}]^4_0$ or $[T\mathcal{S}]^0_4$) is positive definite if, and only if, its components in the basis $\{\mathbb{K}, \mathbb{M}\}$ (or $\{\mathbb{K}^\sharp, \mathbb{M}^\sharp\}$ or $\{\mathbb{K}^\flat, \mathbb{M}^\flat\}$, respectively), are strictly positive, and invertible if, and only if, its components are both different from zero.

In the Theory of Elasticity, the pulled-back material counterparts of the spatial operators in $[T\mathcal{S}]^2{}_2$, $[T\mathcal{S}]^4_0$ and $[T\mathcal{S}]^0_4$ are of particular relevance, and we recall them here [16], omitting the component forms, which can be deduced by analogy with those of the spatial operators [16]. The pull-back of the operators in $[T\mathcal{S}]^2{}_2$ yields the operators in $[T\mathcal{B}_R]^2{}_2$,

$$
\begin{aligned}
\mathbb{I} &= \tfrac{1}{2}(\boldsymbol{I} \,\underline{\otimes}\, \boldsymbol{I} + \boldsymbol{I} \,\overline{\otimes}\, \boldsymbol{I}), &\text{(12a)}\\
\mathbb{K}^* &= \tfrac{1}{3}\, \boldsymbol{B} \otimes \boldsymbol{C}, &\text{(12b)}\\
\mathbb{M}^* &= \mathbb{I}^* - \mathbb{K}^*, &\text{(12c)}
\end{aligned}
$$

where we note that the pull-back $\mathbb{I}^*$ coincides with the material identity $\mathbb{I}$. When applied to a symmetric second-order tensor $\boldsymbol{A} \in [T\mathcal{B}_R]^2_0$, $\mathbb{K}^*$ and $\mathbb{M}^*$ yield the pulled-back spherical and deviatoric parts of $\boldsymbol{A}$, respectively, evaluated with respect to the pulled-back metric $\boldsymbol{C} = \chi^*[\boldsymbol{g}]$, i.e.,

$$
\mathrm{Sph}^*(\boldsymbol{A}) = \mathbb{K}^* : \boldsymbol{A} = \tfrac{1}{3}\,\mathrm{Tr}^*(\boldsymbol{A})\,\boldsymbol{B}, \quad \mathrm{Dev}^*(\boldsymbol{A}) = \mathbb{M}^* : \boldsymbol{A} = \boldsymbol{A} - \tfrac{1}{3}\,\mathrm{Tr}^*(\boldsymbol{A})\,\boldsymbol{B}, \tag{13}
$$

where $\mathrm{Tr}^*(\,\cdot\,)$ is the material pulled-back trace operator [16], i.e., the trace evaluated with respect to the pulled-back metric $\boldsymbol{C} = \chi^*[\boldsymbol{g}]$, such that $\mathrm{Tr}^*(\boldsymbol{A}) = \boldsymbol{C} : \boldsymbol{A} = C_{AB}A^{AB}$. The pull-back of the operators in $[T\mathcal{S}]^4_0$ yields the operators in $[T\mathcal{B}_R]^4_0$

$$
\begin{aligned}
\mathbb{I}^{\sharp *} &= \tfrac{1}{2}(\boldsymbol{B} \,\underline{\otimes}\, \boldsymbol{B} + \boldsymbol{B} \,\overline{\otimes}\, \boldsymbol{B}), &\text{(14a)}\\
\mathbb{K}^{\sharp *} &= \tfrac{1}{3}\, \boldsymbol{B} \otimes \boldsymbol{B}, &\text{(14b)}\\
\mathbb{M}^{\sharp *} &= \mathbb{I}^{\sharp *} - \mathbb{K}^{\sharp *}, &\text{(14c)}
\end{aligned}
$$

and the pull-back of the operators in $[T\mathcal{S}]_4^0$ yields the operators in $[T\mathcal{B}_R]_4^0$

$$\mathbb{I}^{\flat *} = \tfrac{1}{2}(\boldsymbol{C} \,\underline{\otimes}\, \boldsymbol{C} + \boldsymbol{C} \,\overline{\otimes}\, \boldsymbol{C}), \tag{15a}$$
$$\mathbb{K}^{\flat *} = \tfrac{1}{3}\,\boldsymbol{C} \otimes \boldsymbol{C}, \tag{15b}$$
$$\mathbb{M}^{\flat *} = \mathbb{I}^{\flat *} - \mathbb{K}^{\flat *}. \tag{15c}$$

We recall that $\boldsymbol{C}$ is the right Cauchy-Green deformation and $\boldsymbol{B} = \boldsymbol{C}^{-1}$ is its inverse.

2.3 Tensor Basis For Transverse Isotropy

Let $\boldsymbol{m} \in T\mathcal{S}$ be a unit vector with respect to the metric $\boldsymbol{g}$, i.e., such that its Euclidean norm is unitary:

$$\|\boldsymbol{m}\|^2 = \boldsymbol{m}.\boldsymbol{m} = \boldsymbol{m}\,\boldsymbol{g}\,\boldsymbol{m} = 1. \tag{16}$$

Transverse isotropy with respect to $\boldsymbol{m}$ is defined as the symmetry (i.e., the invariance) with respect to rotations about $\boldsymbol{m}$. The direction identified by $\boldsymbol{m}$ is called symmetry axis and the class of equivalence of the planes orthogonal to $\boldsymbol{m}$ is called transverse plane.

The basis of all second-order tensors in $[T\mathcal{S}]_0^2$ with transverse isotropy with respect to direction $\boldsymbol{m}$ is given by

$$\boldsymbol{a} = \boldsymbol{m} \otimes \boldsymbol{m}, \tag{17a}$$
$$\boldsymbol{t} = \boldsymbol{g}^{-1} - \boldsymbol{a}. \tag{17b}$$

Note that $\boldsymbol{t}$ is the complement of tensor $\boldsymbol{a}$ to $\boldsymbol{g}^{-1}$ (the "contravariant identity", i.e., the identity in the tensor space $[T\mathcal{S}]_0^2$), and that both $\boldsymbol{a}$ and $\boldsymbol{t}$ are invariant under the transformation mapping $\boldsymbol{m}$ into $-\boldsymbol{m}$, i.e., the sense of $\boldsymbol{m}$ is irrelevant. Tensor $\boldsymbol{a}$ is often called structure tensor or fabric tensor of direction $\boldsymbol{m}$. By means of the metric tensor $\boldsymbol{g}$, it is possible to contract tensors $\boldsymbol{a}$ and $\boldsymbol{t}$ with a vector $\boldsymbol{v}$, and to obtain the axial and transverse components of $\boldsymbol{v}$:

$$v_{\parallel} = \boldsymbol{a}.\boldsymbol{v}, \tag{18a}$$
$$v_{\perp} = \boldsymbol{t}.\boldsymbol{v}. \tag{18b}$$

By means of suitable tensor products, Walpole [18] derived a basis for fourth-order tensors with transverse isotropy with respect to $\boldsymbol{m}$, which we report for tensors in $[T\mathcal{S}]_0^4$:

$$\mathbb{U}_{11} = \boldsymbol{a} \otimes \boldsymbol{a}, \tag{19a}$$
$$\mathbb{U}_{22} = \tfrac{1}{2}\,\boldsymbol{t} \otimes \boldsymbol{t}, \tag{19b}$$
$$\mathbb{U}_{12} = \tfrac{\sqrt{2}}{2}\,\boldsymbol{a} \otimes \boldsymbol{t}, \tag{19c}$$
$$\mathbb{U}_{21} = \tfrac{\sqrt{2}}{2}\,\boldsymbol{t} \otimes \boldsymbol{a}, \tag{19d}$$
$$\mathbb{V}_{1} = \tfrac{1}{2}\,(\boldsymbol{t} \,\underline{\otimes}\, \boldsymbol{t} + \boldsymbol{t} \,\overline{\otimes}\, \boldsymbol{t} - \boldsymbol{t} \otimes \boldsymbol{t}), \tag{19e}$$
$$\mathbb{V}_{2} = \tfrac{1}{2}\,(\boldsymbol{a} \,\underline{\otimes}\, \boldsymbol{t} + \boldsymbol{a} \,\overline{\otimes}\, \boldsymbol{t} + \boldsymbol{t} \,\underline{\otimes}\, \boldsymbol{a} + \boldsymbol{t} \,\overline{\otimes}\, \boldsymbol{a}). \tag{19f}$$

In this basis, a tensor $\mathbb{T} \in [T\mathcal{S}]_0^4$, transversely isotropic with respect to $\boldsymbol{m}$, is expressed as

$$\mathbb{T} = \tilde{\mathbb{T}}^{pr}\mathbb{U}_{pr} + \tilde{\mathbb{T}}^{\alpha}\mathbb{V}_{\alpha}, \tag{20}$$

where we call the collection $\{\tilde{\mathbb{T}}\}$ of Walpole's components $\tilde{\mathbb{T}}^{pr}$ and $\tilde{\mathbb{T}}^{\alpha}$ *Walpole's representation* of $\mathbb{T}$ [20]. Since the tensors $\mathbb{U}_{pr}$ constitute an algebra isomorphic to that of 2×2 matrices, Walpole's components can be grouped as [18]

$$\{\tilde{\mathbb{T}}\} = \left\{ \begin{bmatrix} \tilde{\mathbb{T}}^{11} & \tilde{\mathbb{T}}^{12} \\ \tilde{\mathbb{T}}^{21} & \tilde{\mathbb{T}}^{22} \end{bmatrix}, \tilde{\mathbb{T}}^{1}, \tilde{\mathbb{T}}^{2} \right\}, \tag{21}$$

and all operations on transversely isotropic tensors $\mathbb{T}$ can be performed by working on their representations $\{\tilde{\mathrm{T}}\}$. The four $\tilde{\mathrm{T}}^{pr}$ and the two $\tilde{\mathrm{T}}^{\alpha}$ are obtained by the scalar product of $\mathbb{T}$ with each of the basis tensors, with some normalisation constants:

$$\tilde{\mathrm{T}}^{pr} = \langle \mathbb{T}, \mathbb{U}_{pr} \rangle, \qquad \tilde{\mathrm{T}}^{\alpha} = \tfrac{1}{2} \langle \mathbb{T}, \mathbb{V}_{\alpha} \rangle. \tag{22}$$

Since $\mathbb{U}_{12}^T = \mathbb{U}_{21}$, tensor $\mathbb{T}$ possesses diagonal symmetry if, and only if, $\tilde{\mathrm{T}}^{12} = \tilde{\mathrm{T}}^{21}$, in which case it has only 5, rather than 6, independent components.

Given an orthonormal basis $\{\boldsymbol{e}_a\}_{a=1}^3$, such that $\boldsymbol{e}_1 = \boldsymbol{m}$, the components $\{\tilde{\mathrm{T}}\}$ are related to the conventional components T^{abcd} of $\mathbb{T}$ by

$$\{\tilde{\mathrm{T}}\} = \left\{ \begin{bmatrix} \mathrm{T}^{1111} & \sqrt{2}\,\mathrm{T}^{1122} \\ \sqrt{2}\,\mathrm{T}^{2211} & 2\,\mathrm{T}^{2222} - 2\,\mathrm{T}^{2323} \end{bmatrix}, 2\,\mathrm{T}^{2323},\, 2\,\mathrm{T}^{1212} \right\}. \tag{23}$$

Note that the full-symmetric "contravariant" fourth-order identity, spherical and deviatoric operators in $[T\mathcal{S}]^4_0$, defined in Equation (8), have Walpole's representations

$$\{\tilde{\mathrm{I}}^{\sharp}\} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, 1,\, 1 \right\}, \tag{24a}$$

$$\{\tilde{\mathrm{K}}^{\sharp}\} = \left\{ \begin{bmatrix} \frac{1}{3} & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & \frac{2}{3} \end{bmatrix}, 0,\, 0 \right\}, \tag{24b}$$

$$\{\tilde{\mathrm{M}}^{\sharp}\} = \left\{ \begin{bmatrix} \frac{2}{3} & -\frac{\sqrt{2}}{3} \\ -\frac{\sqrt{2}}{3} & \frac{1}{3} \end{bmatrix}, 1,\, 1 \right\}. \tag{24c}$$

It is very important to notice that *all* the associated tensors obtained from a tensor in $[T\mathcal{S}]^4_0$ by lowering any of its indices (i.e., by transforming any of its vector feet into covector feet by means of the metric tensor $\boldsymbol{g}$) share *the same* Walpole representation, as it becomes clear by looking at the scalar products (22) in components (e.g., $\mathrm{T}^{abcd}\, g_{ai}\, g_{bj}\, g_{ck}\, g_{dl}\, [\mathbb{U}_{pr}]^{ijkl}$), manipulating the metric tensors, and exploiting identities of the type $g^{hm} g_{mn} = \delta^h{}_n$. In practice, the transformation is entirely shifted onto the basis tensors, leaving Walpole's components untouched. This allows for exploiting the isomorphism between transversely isotropic fourth-order tensors and their Walpole's representation to perform any operation. For example, the double contraction of a tensor in $[T\mathcal{S}]^2{}_2$ and one in $[T\mathcal{S}]^4_0$ can be performed by multiplying the matrix of the former with the matrix of the latter, and the individual scalars of the former with those of the latter, without worrying about which indices are contravariant and which covariant, as this is all taken into account by the basis tensors.

For the case of transverse isotropy, a tensor $\mathbb{T}$ is positive definite if its Walpole's representation $\{\tilde{\mathrm{T}}\}$ is such that the 2×2 matrix $[\tilde{\mathrm{T}}^{pq}]$ is positive definite, and the two scalars $\tilde{\mathrm{T}}^{\alpha}$ are strictly positive. Similarly, $\mathbb{T}$ is invertible if $[\tilde{\mathrm{T}}^{pq}]$ is invertible and the two scalars $\tilde{\mathrm{T}}^{\alpha}$ are different from zero, and the inverse $\mathbb{T}^{-1}$ (which belongs to $[T\mathcal{S}]^0_4$, if $\mathbb{T}$ belongs to $[T\mathcal{S}]^4_0$) has Walpole's representation

$$\{\tilde{\mathrm{T}}^{-1}\} = \left\{ \begin{bmatrix} \tilde{\mathrm{T}}^{11} & \tilde{\mathrm{T}}^{12} \\ \tilde{\mathrm{T}}^{21} & \tilde{\mathrm{T}}^{22} \end{bmatrix}^{-1}, 1/\tilde{\mathrm{T}}^{1},\, 1/\tilde{\mathrm{T}}^{2} \right\}. \tag{25}$$

2.4 Tensor Basis For Orthotropy

Let $\{\boldsymbol{m}_p\}_{p=1}^3$ be a basis for $T\mathcal{S}$, satisfying the condition of orthonormality with respect to the metric $\boldsymbol{g}$, i.e.,

$$\boldsymbol{m}_p . \boldsymbol{m}_q = \boldsymbol{m}_p\, \boldsymbol{g}\, \boldsymbol{m}_q = \delta_{pq}, \tag{26}$$

Given such a basis, the inverse metric tensor can be expressed as

$$\boldsymbol{g}^{-1} = \textstyle\sum_{p=1}^{3} \boldsymbol{m}_p \otimes \boldsymbol{m}_p. \tag{27}$$

Orthotropy with respect to the basis $\{\boldsymbol{m}_p\}_{p=1}^{3}$ is defined as the symmetry (i.e., invariance) under reflection of any of the three $\boldsymbol{m}_p$.

The orthonormal basis $\{\boldsymbol{m}_p\}_{p=1}^{3}$ can be used to construct the basis for the space of second-order tensors in $[T\mathcal{S}]_0^2$ as

$$\boldsymbol{z}_{pq} = \boldsymbol{m}_p \otimes \boldsymbol{m}_q, \tag{28}$$

and the basis for the space of fourth-order tensors in $[T\mathcal{S}]_0^4$ as

$$\mathbb{Z}_{pqrs} = \boldsymbol{z}_{pq} \otimes \boldsymbol{z}_{rs} = \boldsymbol{m}_p \otimes \boldsymbol{m}_q \otimes \boldsymbol{m}_r \otimes \boldsymbol{m}_s. \tag{29}$$

The basis for the subspace of the space of $[T\mathcal{S}]_0^2$ with orthotropy with respect to $\{\boldsymbol{m}_p\}_{p=1}^{3}$ is obtained by defining the three tensors [18]

$$\boldsymbol{a}_p = \boldsymbol{z}_{pp} = \boldsymbol{m}_p \otimes \boldsymbol{m}_p, \quad \textit{no} \text{ sum on } p, \tag{30}$$

which are often called structure tensors or fabric tensors of the directions $\boldsymbol{m}_p$. It is immediate to verify that the tensors (30) are invariant for reflections of the $\boldsymbol{m}_p$ (transformations mapping $\boldsymbol{m}_p$ into $-\boldsymbol{m}_p$), i.e., are orthotropic with respect to $\{\boldsymbol{m}_p\}_{p=1}^{3}$, linearly independent, and generate the space of orthotropic tensors with respect to $\{\boldsymbol{m}_p\}_{p=1}^{3}$. The corresponding basis for the subspace of the space of fourth-order tensors in $[T\mathcal{S}]_0^4$ with orthotropy with respect to $\{\boldsymbol{m}_p\}_{p=1}^{3}$ was obtained by Walpole [18] as

$$\mathbb{U}_{pr} = \mathbb{Z}_{pprr}, \quad \forall p, r \in \{1,2,3\}, \ \textit{no} \text{ sum on } p \text{ and } r, \tag{31a}$$
$$\mathbb{V}_1 = \tfrac{1}{2}\left[\mathbb{Z}_{2323} + \mathbb{Z}_{3232}\right], \tag{31b}$$
$$\mathbb{V}_2 = \tfrac{1}{2}\left[\mathbb{Z}_{1313} + \mathbb{Z}_{3131}\right], \tag{31c}$$
$$\mathbb{V}_3 = \tfrac{1}{2}\left[\mathbb{Z}_{1212} + \mathbb{Z}_{2121}\right]. \tag{31d}$$

A fourth-order tensor $\mathbb{T} \in [T\mathcal{S}]_0^4$, orthotropic with respect to $\{\boldsymbol{m}_p\}_{p=1}^{3}$, can be thus written as

$$\mathbb{T} = \tilde{\mathrm{T}}^{pr}\, \mathbb{U}_{pr} + \tilde{\mathrm{T}}^{\alpha}\, \mathbb{V}_{\alpha}, \tag{32}$$

where we call the collection $\{\tilde{\mathrm{T}}\}$ of Walpole's components $\tilde{\mathrm{T}}^{pr}$ and $\tilde{\mathrm{T}}^{\alpha}$ *Walpole's representation* of the tensor $\mathbb{T}$. Similarly to the case of transverse isotropy, Walpole [18] showed that the basis tensors $\mathbb{U}_{pr}$ constitute an algebra isomorphic to that of 3×3 matrices and that the components $\tilde{\mathrm{T}}^{pr}$ and $\tilde{\mathrm{T}}^{\alpha}$ can be grouped as

$$\{\tilde{\mathrm{T}}\} = \left\{[\tilde{\mathrm{T}}^{pr}], \tilde{\mathrm{T}}^{1}, \tilde{\mathrm{T}}^{2}, \tilde{\mathrm{T}}^{3}\right\}. \tag{33}$$

The nine $\tilde{\mathrm{T}}^{pr}$ and the three $\tilde{\mathrm{T}}^{\alpha}$ are obtained as the scalar product of $\mathbb{T}$ with each of the basis tensors:

$$\tilde{\mathrm{T}}^{pr} = \langle \mathbb{T}, \mathbb{U}_{pr} \rangle, \qquad \tilde{\mathrm{T}}^{\alpha} = \tfrac{1}{2}\, \langle \mathbb{T}, \mathbb{V}_{\alpha} \rangle. \tag{34}$$

Since $\mathbb{U}_{pr} = \mathbb{U}_{rp}^{T}$, diagonal symmetry of $\mathbb{T}$ is attained if, and only if, the matrix $[\tilde{\mathrm{T}}^{pr}]$ is symmetric. In this case, $\mathbb{T}$ has 9, rather than 12, independent components.

Note that the relation of Walpole's components $\tilde{\mathrm{T}}^{pr}$ and $\tilde{\mathrm{T}}^{\alpha}$ with the conventional components T^{abcd} of $\mathbb{T}$ is quite more straightforward in the case of orthotropy compared to the case of transverse isotropy, indeed:

$$\{\tilde{\mathrm{T}}\} = \left\{ \begin{bmatrix} \mathrm{T}^{1111} & \mathrm{T}^{1122} & \mathrm{T}^{1133} \\ \mathrm{T}^{2211} & \mathrm{T}^{2222} & \mathrm{T}^{2233} \\ \mathrm{T}^{3311} & \mathrm{T}^{3322} & \mathrm{T}^{3333} \end{bmatrix}, 2\,\mathrm{T}^{2323},\ 2\,\mathrm{T}^{1313},\ 2\,\mathrm{T}^{1212} \right\}. \tag{35}$$

The full-symmetric "contravariant" fourth-order identity, spherical and deviatoric operators in $[T\mathcal{S}]^4_0$ of Equation (8) have Walpole's representations

$$\{\tilde{\mathbb{I}}^\sharp\} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, 1,\, 1,\, 1 \right\}, \tag{36a}$$

$$\{\tilde{\mathbb{K}}^\sharp\} = \left\{ \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}, 0,\, 0,\, 0 \right\}, \tag{36b}$$

$$\{\tilde{\mathbb{M}}^\sharp\} = \left\{ \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}, 1,\, 1,\, 1 \right\}. \tag{36c}$$

Analogously to the case of transverse isotropy, *all* the associated tensors obtained from a tensor in $[T\mathcal{S}]^4_0$ by lowering any of its indices by means of the metric tensor $\boldsymbol{g}$ share *the same* Walpole representation, as the transformation is entirely ascribed to the basis tensors. Again, this allows for orthotropic fourth-order tensors to be inverted, summed and double-contracted by working on their Walpole's representations.

positive definiteness and invertibility of an orthotropic fourth-order tensor $\mathbb{T}$ are checked analogously to the case of transverse isotropy. $\mathbb{T}$ is positive definite if the 3×3 matrix $[\tilde{\mathbb{T}}^{pq}]$ in its Walpole's representation $\tilde{\mathbb{T}}$ is positive definite, and the three scalars $\tilde{\mathbb{T}}^\alpha$ are strictly positive, and invertible if $[\tilde{\mathbb{T}}^{pq}]$ is invertible and the three scalars $\tilde{\mathbb{T}}^\alpha$ are different from zero. The Walpole representation of the inverse is analogous to that of the transversely isotropic case seen in Equation (25).

2.5 Hyperelasticity and Volumetric-Distortional Decomposition

Within a purely mechanical framework, the dissipation density D per unit volume of the undeformed configuration of a body comprised of a simple material is defined by [27]

$$D = -\dot{W} + \boldsymbol{S} : \dot{\boldsymbol{E}} \geq 0. \tag{37}$$

In the inequality (37), which has to hold at all points X of $\mathcal{B}_R$ and at all times, W is the stored energy function per unit volume of $\mathcal{B}_R$, $\boldsymbol{S}$ is the second Piola-Kirchhoff stress tensor, and $\boldsymbol{E}$ is the Green-Lagrange strain tensor. For the case of a hyperelastic material, W and $\boldsymbol{S}$ are expressed as constitutive functions of $\boldsymbol{E}$, such that

$$W = \hat{W}(\boldsymbol{E}), \quad \boldsymbol{S} = \hat{\boldsymbol{S}}(\boldsymbol{E}), \tag{38}$$

and W is referred to as the elastic potential (or strain energy) density. We remark that the constitutive functions may depend explicitly on the position X, in which case the material is inhomogeneous, but we omit indicating this dependence for the sake of a lighter notation. Substituting (38) into (37) yields

$$\begin{aligned} D &= -\frac{\partial \hat{W}}{\partial \boldsymbol{E}}(\boldsymbol{E}) : \dot{\boldsymbol{E}} + \hat{\boldsymbol{S}}(\boldsymbol{E}) : \dot{\boldsymbol{E}} \\ &= \left[\hat{\boldsymbol{S}}(\boldsymbol{E}) - \frac{\partial \hat{W}}{\partial \boldsymbol{E}}(\boldsymbol{E}) \right] : \dot{\boldsymbol{E}} \geq 0. \end{aligned} \tag{39}$$

The inequality (39) implies that D is a function of $\boldsymbol{E}$ and $\dot{\boldsymbol{E}}$, i.e. $D = \hat{D}(\boldsymbol{E}, \dot{\boldsymbol{E}})$. Since $\dot{\boldsymbol{E}}$ is neither an independent nor a dependent constitutive variable, $\hat{D}$ depends linearly on $\dot{\boldsymbol{E}}$

(in particular, $\hat{D}(\boldsymbol{E}, \mathbf{O}) = 0$), which therefore can be varied arbitrarily. Consequently, in order to ensure that the inequality is always respected, it must hold that

$$\left[\hat{\boldsymbol{S}}(\boldsymbol{E}) - \frac{\partial \hat{W}}{\partial \boldsymbol{E}}(\boldsymbol{E}) \right] : \dot{\boldsymbol{E}} = 0, \tag{40}$$

which implies that the second Piola-Kirchhoff stress is given by the derivative of the elastic potential with respect to the Green-Lagrange strain:

$$\boldsymbol{S} = \hat{\boldsymbol{S}}(\boldsymbol{E}) = \frac{\partial \hat{W}}{\partial \boldsymbol{E}}(\boldsymbol{E}). \tag{41}$$

The second derivative of the elastic potential is the material elasticity tensor

$$\mathbb{C} = \hat{\mathbb{C}}(\boldsymbol{E}) = \frac{\partial^2 \hat{W}}{\partial \boldsymbol{E}^2}(\boldsymbol{E}), \tag{42}$$

which, evaluated at zero strain, yields the material linear elasticity tensor

$$\mathbb{L} = \hat{\mathbb{C}}(\mathbf{O}) = \frac{\partial^2 \hat{W}}{\partial \boldsymbol{E}^2}(\mathbf{O}). \tag{43}$$

The inverse Piola transform of the material elasticity tensor $\mathbb{C}$ is the spatial elasticity tensor

$$\mathbb{c} = J^{-1} \, \chi_*[\mathbb{C}], \quad \mathrm{c}^{abcd} = J^{-1} F^a{}_A F^b{}_B F^c{}_C F^d{}_D \, \mathrm{C}^{ABCD}, \tag{44}$$

which, evaluated at zero strain, yields the spatial linear elasticity tensor $\mathbb{l}$. Equivalently, the spatial linear elasticity tensor $\mathbb{l}$ can be obtained as the inverse Piola transform of the material linear elasticity tensor $\mathbb{L}$ performed in the undeformed state, when $J = 1$ and $\boldsymbol{F} = \mathbf{1}$, where $\mathbf{1}$ is the shifter [1, 22], i.e., in components,

$$\mathrm{l}^{abcd} = \mathbf{1}^a{}_A \mathbf{1}^b{}_B \mathbf{1}^c{}_C \mathbf{1}^d{}_D \, \mathrm{L}^{ABCD}. \tag{45}$$

Physically, the shifter parallel transports tangent vectors from a material point to a spatial point and, in the most general case, its representing matrix is orthogonal, which means that the components of $\mathbb{L}$ and $\mathbb{l}$ differ merely by a rigid rotation. Moreover, for the particular case of collinear Cartesian coordinates in $\mathcal{B}_R$ and $\mathcal{S}$, the components of the shifter $\mathbf{1}$ are simply $\mathbf{1}^a{}_A = \delta^a{}_A$, and therefore the components of the material and spatial linear elasticity tensors coincide. For this reason, in Linear Elasticity, it is practically equivalent to speak about the material or the spatial linear elasticity tensor. Therefore, it is indifferent to speak about material symmetries in the material or in the spatial picture, and this is why, in Sections 2.2, 2.3, 2.4, we reported the tensor bases in the spatial picture only. We remark that, in the general non-linear case, the material symmetries of a body are studied in the *material* picture of Mechanics (e.g., [28, 1, 22, 15]).

When the volumetric-distortional decomposition of the deformation [13, 14] is employed, the elastic potential is written as a function

$$\hat{W}(\boldsymbol{E}) = \hat{\Psi}(J(\boldsymbol{E}), \bar{\boldsymbol{E}}(\boldsymbol{E})) \tag{46}$$

of the determinant J of the deformation gradient $\boldsymbol{F}$ and the distortional Green-Lagrange strain $\bar{\boldsymbol{E}}$, which are both regarded as explicit functions of the "full" Green-Lagrange strain $\boldsymbol{E}$. Note the slight abuse of notation in writing $J = J(\boldsymbol{E}) = \sqrt{\det(2\boldsymbol{E} + \boldsymbol{G})} = \sqrt{\det \boldsymbol{C}}$

and $\bar{\boldsymbol{E}} = \bar{\boldsymbol{E}}(\boldsymbol{E})$. It has been shown [17, 16] that, with the decomposition (46), the material elasticity tensor reads

$$\begin{aligned}\mathbb{C} = &-J\,p\,[3\,\mathbb{K}^{\sharp *} - 2\,\mathbb{I}^{\sharp *}] + 3\,J^2\,\mathrm{K}\,\mathbb{K}^{\sharp *} + \\ &+ J^{1/3}[\boldsymbol{B} \otimes (\mathbb{M}^* : \boldsymbol{Y}) + (\mathbb{M}^* : \boldsymbol{Y}) \otimes \boldsymbol{B}] + \\ &+ J^{-4/3}\,\mathbb{M}^* : \tilde{\mathbb{C}} : \mathbb{M}^{*T} + \\ &+ \tfrac{2}{3} J^{-2/3}\,\mathrm{Tr}^*(\tilde{\boldsymbol{S}})\,\mathbb{M}^{\sharp *} - \\ &- \tfrac{2}{3}\big[\boldsymbol{B} \otimes \mathrm{Dev}^*(\boldsymbol{S}) + \mathrm{Dev}^*(\boldsymbol{S}) \otimes \boldsymbol{B}\big],\end{aligned} \tag{47}$$

where $p = -\partial\hat{\Psi}/\partial J$ is the hydrostatic pressure, $\tilde{\boldsymbol{S}} = \partial\hat{\Psi}/\partial\bar{\boldsymbol{E}}$ is the second Piola-Kirchhoff pseudo stress, $\mathrm{K} = \partial^2\hat{\Psi}/\partial J^2$ is the (large strain) bulk modulus, $\boldsymbol{Y} = \partial^2\hat{\Psi}/\partial J\partial\bar{\boldsymbol{E}}$ is the coupling tensor (or interaction tensor), $\tilde{\mathbb{C}} = \partial^2\hat{\Psi}/\partial\bar{\boldsymbol{E}}^2$ is the pseudo elasticity tensor, $\mathrm{Tr}^*(\tilde{\boldsymbol{S}}) = \boldsymbol{C} : \tilde{\boldsymbol{S}}$ is the pulled-back trace of $\tilde{\boldsymbol{S}}$, $\mathrm{Dev}^*(\boldsymbol{S}) = \mathbb{M}^* : \boldsymbol{S} = J^{-2/3}\mathbb{M}^* : \tilde{\boldsymbol{S}}$ is the pulled-back deviatoric part of $\boldsymbol{S}$. The spatial elasticity tensor reads

$$\begin{aligned}\mathbbm{c} = &-p\,[3\mathbbm{k}^{\sharp} - 2\,\mathbbm{i}^{\sharp}] + 3\,J\,\mathrm{K}\,\mathbbm{k}^{\sharp} + \\ &+ J^{1/3}[\boldsymbol{g}^{-1} \otimes (\mathbbm{m} : \boldsymbol{y}) + (\mathbbm{m} : \boldsymbol{y}) \otimes \boldsymbol{g}^{-1}] + \\ &+ J^{-4/3}\,\mathbbm{m} : \tilde{\mathbbm{c}} : \mathbbm{m}^{T} + \\ &+ \tfrac{2}{3} J^{-2/3}\,\mathrm{tr}(\tilde{\boldsymbol{\sigma}})\,\mathbbm{m}^{\sharp} - \\ &- \tfrac{2}{3}\big[\boldsymbol{g}^{-1} \otimes \mathrm{dev}(\boldsymbol{\sigma}) + \mathrm{dev}(\boldsymbol{\sigma}) \otimes \boldsymbol{g}^{-1}\big],\end{aligned} \tag{48}$$

where $\tilde{\mathbbm{c}} = J^{-1}\chi_*[\tilde{\mathbb{C}}]$, $\tilde{\boldsymbol{\sigma}} = J^{-1}\chi_*[\tilde{\boldsymbol{S}}]$, $\mathrm{dev}(\boldsymbol{\sigma}) = J^{-1}\chi_*[\mathrm{Dev}^*(\boldsymbol{S})]$, and $\boldsymbol{y} = J^{-1}\chi_*[\boldsymbol{Y}]$ are the inverse Piola transforms of $\tilde{\mathbb{C}}$, $\tilde{\boldsymbol{S}}$, $\mathrm{Dev}^*(\boldsymbol{S})$, and $\boldsymbol{Y}$, respectively, and $\mathrm{tr}(\tilde{\boldsymbol{\sigma}}) = J^{-1}\mathrm{Tr}^*(\tilde{\boldsymbol{S}})$.

If the undeformed configuration, achieved when $\boldsymbol{E}$ vanishes and J is identically one, is also stress-free, then both p and $\boldsymbol{\sigma}$ vanish identically, and the linear elasticity tensor is obtained from Equation (48) as

$$\mathbbm{l} = 3\kappa\,\mathbbm{k}^{\sharp} + \boldsymbol{g}^{-1} \otimes [\mathbbm{m} : \boldsymbol{\alpha}] + [\mathbbm{m} : \boldsymbol{\alpha}] \otimes \boldsymbol{g}^{-1} + \mathbbm{m} : [\tilde{\mathbbm{l}} + 2\beta\,\mathbbm{m}^{\sharp}] : \mathbbm{m}^{T}, \tag{49}$$

where the (linear elasticity) bulk modulus κ, $\boldsymbol{\alpha}$, β and $\tilde{\mathbbm{l}}$ are the values of K, $\boldsymbol{y}$, $\frac{1}{3}\mathrm{tr}(\tilde{\boldsymbol{\sigma}})$ and $\tilde{\mathbbm{c}}$, respectively, in the undeformed configuration.

It has also been shown [17] that, in the purely algebraic decomposition of the linear elasticity tensor, obtained by premultiplying $\mathbbm{l}$ by the identity $\mathbbm{i}$, post multiplying by $\mathbbm{i}^T$, and decomposing the identity into $\mathbbm{k} + \mathbbm{m}$, i.e.,

$$\begin{aligned}\mathbbm{l} &= \mathbbm{i} : \mathbbm{l} : \mathbbm{i}^T = (\mathbbm{k} + \mathbbm{m}) : \mathbbm{l} : (\mathbbm{k} + \mathbbm{m})^T \\ &= \mathbbm{k} : \mathbbm{l} : \mathbbm{k}^T + \mathbbm{k} : \mathbbm{l} : \mathbbm{m}^T + \mathbbm{m} : \mathbbm{l} : \mathbbm{k}^T + \mathbbm{m} : \mathbbm{l} : \mathbbm{m}^T,\end{aligned} \tag{50}$$

the identities

$$\mathbbm{k} : \mathbbm{l} : \mathbbm{k}^T = 3\kappa\,\mathbbm{k}^{\sharp}, \tag{51a}$$
$$\mathbbm{k} : \mathbbm{l} : \mathbbm{m}^T = \boldsymbol{g}^{-1} \otimes [\mathbbm{m} : \boldsymbol{\alpha}], \tag{51b}$$
$$\mathbbm{m} : \mathbbm{l} : \mathbbm{k}^T = [\mathbbm{m} : \boldsymbol{\alpha}] \otimes \boldsymbol{g}^{-1}, \tag{51c}$$
$$\mathbbm{m} : \mathbbm{l} : \mathbbm{m}^T = \mathbbm{m} : [\tilde{\mathbbm{l}} + 2\beta\,\mathbbm{m}^{\sharp}] : \mathbbm{m}^{T}, \tag{51d}$$

hold, implying that the expression (49) of the linear elasticity tensor, obtained by use of the decomposition of the deformation, is term-by-term equivalent to the purely algebraic decomposition (50). In Equations (51), the term (51a) is *purely spherical*, the terms (51b) and (51c) are *mixed*, and the term (51d) is *purely deviatoric.* Equations (51) are the key result in the evaluation of the linear elasticity tensor of strictly incompressible and quasi-incompressible materials.

It is very important to note that, because of the orthogonality of the spherical and deviatoric operators, each of the four terms (51) is orthogonal to the other three in the scalar product induced by the metric $\boldsymbol{g}$ in the space $[T\mathcal{S}]_0^4$ of fourth-order "contravariant" tensors. In particular, we note that, since $\mathbb{K} : \mathbb{L} : \mathbb{K}^T = 3\kappa\,\mathbb{K}^\sharp$ is orthogonal to the other three terms, and $\langle \mathbb{K}^\sharp, \mathbb{K}^\sharp \rangle = 1$, it is possible to obtain the bulk modulus as

$$\kappa = \tfrac{1}{3}\,\langle \mathbb{K}^\sharp, \mathbb{L} \rangle = \tfrac{1}{3}\,\langle \mathbb{K}^\sharp, \mathbb{K} : \mathbb{L} : \mathbb{K}^T \rangle = \tfrac{1}{3}\,\langle \mathbb{K}^\sharp, 3\kappa\,\mathbb{K}^\sharp \rangle = \tfrac{1}{9}\,g_{ab}\,g_{cd}\,\mathrm{L}^{abcd}. \tag{52}$$

3 Incompressibile Hyperelasticity

This section is dedicated to the derivation of the conditions that the linear elasticity tensor must obey for the cases of strict incompressibility and quasi-incompressibility. Strict incompressibility is a kinematical constraint on the volumetric deformation $J = \det \boldsymbol{F}$, whereas quasi-incompressibility is obtained by requiring that a very large elastic energy is needed to make the volumetric deformation J change from its initial value of 1.

3.1 Strict Incompressibility

When the deformation is isochoric (strict incompressibility), $\dot{\boldsymbol{E}}$ in Equation (40) is no longer arbitrary. Rather, it is subjected to the constraint

$$\dot{J} = J\,\mathrm{div}(\boldsymbol{v}) = J\,\boldsymbol{B} : \dot{\boldsymbol{E}} = \tfrac{1}{2} J\,\boldsymbol{B} : \dot{\boldsymbol{C}} = 0, \tag{53}$$

which states that the only admissible deformations are those such that $\boldsymbol{B} = \boldsymbol{C}^{-1}$ is orthogonal to $\dot{\boldsymbol{C}}$ in the sense of (53), i.e., $\boldsymbol{B} : \dot{\boldsymbol{C}} = B^{AB}\dot{C}_{AB} = 0$. Since the constraint (53) is holonomic, it can be put into algebraic form by direct integration with respect to time. Setting, with the usual abuse of notation, $J = J(\boldsymbol{E})$, and performing the integration under the condition that J is equal to one in the undeformed configuration leads to $J = J(\boldsymbol{E}) = 1$.

Combining Equations (40) and (53) one obtains

$$\boldsymbol{S} - \frac{\partial \hat{W}}{\partial \boldsymbol{E}}(\boldsymbol{E}) = \lambda\,J\,\boldsymbol{B}, \tag{54}$$

where λ is an arbitrary scalar, Lagrange multiplier arising from the kinematical constraint of isochoric motion. If we denote the hydrostatic pressure by π, in order to distinguish it from the "constitutive" hydrostatic pressure $p = -\partial\hat{\Psi}/\partial J$ introduced in the previous section, and recall the definition of hydrostatic pressure as the scalar of the spherical part (hydrostatic stress) of the Cauchy stress $\boldsymbol{\sigma}$,

$$-\pi\,\boldsymbol{g}^{-1} = \mathbb{K} : \boldsymbol{\sigma} = \tfrac{1}{3}\,\mathrm{tr}(\boldsymbol{\sigma})\,\boldsymbol{g}^{-1} = \tfrac{1}{3}\,(\boldsymbol{g} : \boldsymbol{\sigma})\,\boldsymbol{g}^{-1}, \tag{55}$$

and its full Piola transform,

$$-J\,\pi\,\boldsymbol{B} = J\,\mathbb{K}^* : \boldsymbol{S} = \tfrac{1}{3}\,J\,\mathrm{Tr}^*(\boldsymbol{S})\,\boldsymbol{B} = \tfrac{1}{3}\,J\,(\boldsymbol{C} : \boldsymbol{S})\,\boldsymbol{B}, \tag{56}$$

involving the second Piola-Kirchhoff stress $\boldsymbol{S}$, it can be shown that $\lambda = -\pi$ if, and only if,

$$\mathrm{Tr}^*\left(\frac{\partial \hat{W}}{\partial \boldsymbol{E}}(\boldsymbol{E})\right) = \frac{\partial \hat{W}}{\partial \boldsymbol{E}}(\boldsymbol{E}) : \boldsymbol{C} = 0, \tag{57}$$

where we recall that Tr^* is the pulled-back trace operator [16] described in Section 2.2. If $\hat{W}$ is regarded a function of $\boldsymbol{C}$ rather than of $\boldsymbol{E}$, Equation (57) means that the potential has to be a homogeneous function of order zero in $\boldsymbol{C}$. Exploiting Euler's theorem on homogeneous functions (see [3]) and going back to the argument $\boldsymbol{E}$, one shows that the potential $\hat{W}$ must have the form

$$\hat{W}(\boldsymbol{E}) = \hat{W}_d(\bar{\boldsymbol{E}}(\boldsymbol{E})), \tag{58}$$

i.e., $\hat{W}$ must be given by an explicit function $\hat{W}_d$ of the distortional strain $\bar{\boldsymbol{E}}$, called distortional potential. The second Piola-Kirchhoff stress reads

$$\boldsymbol{S} = \hat{\boldsymbol{S}}(\boldsymbol{E}) = \mathrm{Sph}^*(\boldsymbol{S}) + \mathrm{Dev}^*(\boldsymbol{S}) = -J\,\pi\,\boldsymbol{B} + \frac{\partial \hat{W}}{\partial \boldsymbol{E}}(\boldsymbol{E}), \tag{59}$$

where we recall that Sph^* and Dev^* are the pulled-back spherical and deviatoric operators associated with $\mathbb{K}^*$ and $\mathbb{M}^*$, respectively (Equation (12)). The material elasticity tensor is evaluated as in Equation (60), keeping in mind that all derivatives of $\hat{W}(\boldsymbol{E}) = \hat{W}_d(\bar{\boldsymbol{E}}(\boldsymbol{E}))$ with respect to J vanish identically:

$$\begin{aligned}\mathbb{C} = &-J\,\pi\,[3\,\mathbb{K}^{\sharp *} - 2\,\mathbb{I}^{\sharp *}] + \\ &+ J^{-4/3}\,\mathbb{M}^* : \tilde{\mathbb{C}} : \mathbb{M}^{*T} + \\ &+ \tfrac{2}{3} J^{-2/3}\,\mathrm{Tr}^*(\tilde{\boldsymbol{S}})\,\mathbb{M}^{\sharp *} - \\ &- \tfrac{2}{3}\big[\boldsymbol{B} \otimes \mathrm{Dev}^*(\boldsymbol{S}) + \mathrm{Dev}^*(\boldsymbol{S}) \otimes \boldsymbol{B}\big].\end{aligned} \tag{60}$$

The spatial elasticity tensor is therefore

$$\begin{aligned}\mathbb{c} = &-\pi\,[3\mathbb{k}^{\sharp} - 2\,\mathbb{i}^{\sharp}] + \\ &+ J^{-4/3}\,\mathbb{m} : \tilde{\mathbb{c}} : \mathbb{m}^{T} + \\ &+ \tfrac{2}{3} J^{-2/3}\,\mathrm{tr}(\tilde{\boldsymbol{\sigma}})\,\mathbb{m}^{\sharp} - \\ &- \tfrac{2}{3}\big[\boldsymbol{g}^{-1} \otimes \mathrm{dev}(\boldsymbol{\sigma}) + \mathrm{dev}(\boldsymbol{\sigma}) \otimes \boldsymbol{g}^{-1}\big],\end{aligned} \tag{61}$$

and the linear elasticity tensor reduces to

$$\mathbb{l} = \mathbb{m} : [\tilde{\mathbb{l}} + 2\beta\,\mathbb{m}^{\sharp}] : \mathbb{m}^{T}. \tag{62}$$

Comparing Equations (62) and (51) we conclude that the linear elasticity tensor $\mathbb{l}$ of a strictly incompressible material must obey the three conditions

$$\mathbb{k} : \mathbb{l} : \mathbb{k}^{T} = \mathbb{o}, \quad \mathbb{k} : \mathbb{l} : \mathbb{m}^{T} = \mathbb{o}, \quad \mathbb{m} : \mathbb{l} : \mathbb{k}^{T} = \mathbb{o}, \tag{63}$$

i.e., it must not contain spherical or mixed terms, but exclusively the deviatoric one. This result is valid in general, *regardless* of the material symmetry.

3.2 Quasi-Incompressibility

In this case, the elastic potential admits the particular *decoupled* form

$$\hat{W}(\boldsymbol{E}) = \hat{\Psi}(J(\boldsymbol{E}), \bar{\boldsymbol{E}}(\boldsymbol{E})) = \hat{U}(J(\boldsymbol{E})) + \hat{W}_d(\bar{\boldsymbol{E}}(\boldsymbol{E})). \tag{64}$$

The mixed derivative $\boldsymbol{Y} = \partial^2\hat{\Psi}/\partial J \partial \bar{\boldsymbol{E}}$ vanishes identically, which yields the material elasticity tensor

$$\begin{aligned}\mathbb{C} = &-J\,p\,[3\,\mathbb{K}^{\sharp *} - 2\,\mathbb{I}^{\sharp *}] + 3\,J^2\,\mathrm{K}\,\mathbb{K}^{\sharp *} + \\ &+ J^{-4/3}\,\mathbb{M}^* : \tilde{\mathbb{C}} : \mathbb{M}^{*T} + \\ &+ \tfrac{2}{3} J^{-2/3}\,\mathrm{Tr}^*(\tilde{\boldsymbol{S}})\,\mathbb{M}^{\sharp *} - \\ &- \tfrac{2}{3}\big[\boldsymbol{B} \otimes \mathrm{Dev}^*(\boldsymbol{S}) + \mathrm{Dev}^*(\boldsymbol{S}) \otimes \boldsymbol{B}\big],\end{aligned} \tag{65}$$

the spatial elasticity tensor

$$
\begin{aligned}
\mathbb{C} = & -p\,[3\,\mathbb{K}^\sharp - 2\,\mathbb{I}^\sharp] + 3\,J\,\mathrm{K}\,\mathbb{K}^\sharp + \\
& + J^{-4/3}\,\mathbb{M} : \tilde{\mathbb{C}} : \mathbb{M}^T + \\
& + \tfrac{2}{3} J^{-2/3}\,\mathrm{tr}(\tilde{\boldsymbol{\sigma}})\,\mathbb{M}^\sharp - \\
& - \tfrac{2}{3}\big[\boldsymbol{g}^{-1} \otimes \mathrm{dev}(\boldsymbol{\sigma}) + \mathrm{dev}(\boldsymbol{\sigma}) \otimes \boldsymbol{g}^{-1}\big],
\end{aligned}
\tag{66}
$$

and the linear elasticity tensor

$$
\mathbb{L} = 3\kappa\,\mathbb{K}^\sharp + \mathbb{M} : [\tilde{\mathbb{L}} + 2\beta\,\mathbb{M}^\sharp] : \mathbb{M}^T. \tag{67}
$$

Comparing Equations (67) and (51), we deduce that the linear elasticity tensor of a quasi-incompressible material must obey the two conditions

$$
\mathbb{K} : \mathbb{L} : \mathbb{M}^T = \mathbb{O}, \quad \mathbb{M} : \mathbb{L} : \mathbb{K}^T = \mathbb{O}, \tag{68}
$$

i.e., it must contain no mixed terms, but only the spherical and the deviatoric ones. By comparing Equations (62) and (67) and recalling (end of Section 2.5) that the term $\mathbb{K} : \mathbb{L} : \mathbb{K}^T = 3\kappa\,\mathbb{K}^\sharp$ is orthogonal to the other three terms in Equation (51), it is evident that, as one would expect, the linear elasticity tensor for the quasi-incompressible case has one additional parameter with respect to the strictly incompressible case. It is convenient to identify this one additional parameter with the bulk modulus κ, obtained in Equation (52). We emphasise again that this is valid *regardless* of the material symmetry.

Remark 3.1. We take this chance to remark that the decoupled potential (64) can be used *exclusively* for quasi-incompressible materials, and yields inconsistent material behaviour in the general compressible case: this has been reported a few decades ago by Musgrave [29] in the context of crystal elasticity, and demonstrated in a previous work [17] with the same methodology used here, i.e., by linearising the spatial elasticity tensor of the non-linear theory. Indeed, if the potential (64) were used for a compressible material, the linear elasticity tensor would be subjected to the conditions (68), which would reduce the number of independent elastic constants with respect to the general case. Therefore, one would find a compressible material with a given symmetry having *less* independent constants than expected (e.g., 4 rather than 5 for transverse isotropy, and 7, rather than 9, for orthotropy, as we shall show in Section 4 for quasi-incompressible materials). Whereas nothing, in principle, prevents conditions (68) from occurring for a compressible material, such material cannot certainly be considered a general case. Indeed, a first consequence of the adoption of (64) for the compressible case would be that an anisotropic material would not undergo distortional deformations under a hydrostatic stress, which is contrary to experimental observation (this has also been remarked in a recent paper by Vergori et al. [30]).

4 Some Particular Material Symmetries

The conditions (63) for strict incompressibility and (68) for quasi-incompressibility are general, and hold regardless of material symmetry. When a material symmetry is given, conditions (63) and (68) can be employed to find the number of independent components of the elasticity tensor. As we shall show, the case of isotropy is trivial. For the cases of transverse isotropy and orthotropy, it is convenient to enforce conditions (63) and (68) within Walpole's formalism [18], which, due to the isomorphism between fourth-order tensors and the corresponding Walpole's representations, allows for evaluating the double contractions of tensors in conditions (63) and (68) by means of the matrix multiplication of the matrix parts and regular multiplication of scalars of the scalar parts of the corresponding Walpole's representations of the tensors.

4.1 Isotropy

The linear elasticity tensor of a generic isotropic material is a fourth-order tensor in $[T\mathcal{S}]_0^4$ with the form

$$\mathbb{L} = 3\kappa\,\mathbb{K}^\sharp + 2\mu\,\mathbb{M}^\sharp, \tag{69}$$

where κ is the bulk modulus and μ is the shear modulus. If strict incompressibility is enforced, conditions (63) impose the that the linear elasticity tensor has only one independent elastic modulus, the shear modulus μ, and representation

$$\mathbb{L}_{\text{strict}} = 2\mu\,\mathbb{M}^\sharp. \tag{70}$$

In contrast, the quasi-incompressibility conditions (68), are always identically verified, and therefore the elasticity tensor keeps two independent elastic constants, as in the general compressible case, and reads

$$\mathbb{L}_{\text{quasi}} = 3\kappa\,\mathbb{K}^\sharp + 2\mu\,\mathbb{M}^\sharp, \tag{71}$$

where the bulk modulus κ is much larger than the shear modulus μ.

Remark 4.1. Note that a quite common representation for isotropic elasticity tensors is in the form $\mathbb{L} = 3\lambda\,\mathbb{K}^\sharp + 2\mu\,\mathbb{I}^\sharp$, where λ and μ are called *Lamé's constants*, and μ is still the shear modulus. This representation is very useful in several circumstances, such as, for example, in computations based on the Finite Element Method, where the term $2\mu\,\mathbb{I}^\sharp$ generates the symmetric, positive definite modified stiffness operator relating the nodal displacements with the nodal pressures and the external generalised forces (cf., e.g., [31]). Nevertheless, we believe that there are cases in which the representation $\mathbb{L} = 3\kappa\,\mathbb{K}^\sharp + 2\mu\,\mathbb{M}^\sharp$ is more advantageous and physically sound. Indeed, the algebraic computations involving the elasticity tensor are easier (and their physical meaning becomes clearer), since $\mathbb{K}^\sharp$ and $\mathbb{M}^\sharp$ form an orthogonal basis [18, 20, 17, 16] (in contrast, $\mathbb{K}^\sharp$ and $\mathbb{I}^\sharp$ do not). Moreover, the constants κ and μ, which must be both strictly positive, have a direct physical meaning. For this reason, we prefer the representation terms of in $\mathbb{K}^\sharp$ and $\mathbb{M}^\sharp$.

4.2 Transverse Isotropy

Using Walpole's formalism (Section 2.3), the linear elasticity tensor $\mathbb{L}$ of a generic transversely isotropic material has representation

$$\{\tilde{\mathbb{L}}\} = \left\{ \begin{bmatrix} n & \sqrt{2}\,l \\ \sqrt{2}\,l & 2c \end{bmatrix}, 2\mu_t,\, 2\mu_a \right\}, \tag{72}$$

where n is the elastic modulus in uniaxial strain (compare with the Young's modulus, which is the modulus in uniaxial stress), c is the plane-strain bulk modulus (in the transverse plane), l is called cross modulus, μ_t is the shear modulus in the transverse plane, and μ_a is the shear modulus in any plane containing the symmetry axis.

The three strict incompressibility conditions (63) reduce to the two scalar conditions

$$n + 4\,(c + l) = 0, \qquad n - 2c + l = 0, \tag{73}$$

where we note that $\mathbb{K} : \mathbb{L} : \mathbb{M}^T = \mathbb{O}$ and $\mathbb{M} : \mathbb{L} : \mathbb{K}^T = \mathbb{O}$ both yield $n - 2c + l = 0$. These two conditions state that only one of n, c and l is independent. Mathematically, electing any of the three as the independent parameter is indifferent. However, looking at the physical meaning of each, we note that the most appropriate choice is

$$\alpha = -l. \tag{74}$$

Indeed, both uniaxial strain and plane strain, to which n and c refer, respectively, are strain states that *cannot* be attained under the constraint of isochoric motion. The parameter l, instead, can be thought to be related to a triaxial state of strain that is compatible with isochoric motion. The cross-modulus l is the transversely isotropic equivalent of the first Lamé's modulus λ of isotropic elasticity, to which it reduces in the limit case, as it can be easily verified with Spencer's representation [21]. Note that, in general, similarly to λ, l can be negative, and must indeed be negative to ensure positive semi-definiteness and therefore convexity for the case of strict incompressibility, as we shall see in Section 5. With this choice, the linear elasticity tensor for strict incompressibility is represented by

$$\{\tilde{\mathbb{L}}_{\text{strict}}\} = \left\{ \begin{bmatrix} 2\alpha & -\sqrt{2}\,\alpha \\ -\sqrt{2}\,\alpha & \alpha \end{bmatrix}, 2\mu_t,\, 2\mu_a \right\}, \tag{75}$$

with only three (from the original five) independent elastic constants.

The quasi-incompressibility conditions (68) yield the single scalar condition

$$n - 2c + l = 0, \tag{76}$$

meaning that only two of n, c and l are independent. Here we choose, as independent parameters, the bulk modulus

$$\kappa = \tfrac{1}{3}\,\langle \mathbb{K}^\sharp, \mathbb{L}\rangle = \tfrac{1}{9}\, g_{ab}\, g_{cd}\, \mathrm{L}^{abcd} = \tfrac{1}{9}\,[n + 4\,(c + l)], \tag{77}$$

which is a linear combination of n, c and l, obtained by applying Equation (52) to the case of transverse isotropy, and

$$\alpha' = \kappa - l. \tag{78}$$

With this choice, the linear elasticity tensor for the transversely isotropic quasi-incompressible case reads

$$\{\tilde{\mathbb{L}}_{\text{quasi}}\} = \left\{ \begin{bmatrix} \kappa + 2\alpha' & \sqrt{2}\,(\kappa - \alpha') \\ \sqrt{2}\,(\kappa - \alpha') & 2\,\kappa + \alpha' \end{bmatrix}, 2\mu_t,\, 2\mu_a \right\}, \tag{79}$$

with four independent elastic constants: one more than for the case of strict incompressibility. Recalling Walpole's transversely isotropic representation of $\mathbb{K}^\sharp$ (Equations (24)), the elasticity tensor can be written as

$$\{\tilde{\mathbb{L}}_{\text{quasi}}\} = 3\kappa\,\{\tilde{\mathbb{K}}\} + \{\tilde{\mathbb{L}}'_{\text{strict}}\}, \tag{80}$$

where $\{\tilde{\mathbb{L}}'_{\text{strict}}\}$ has the same form as $\{\tilde{\mathbb{L}}_{\text{strict}}\}$ of Equation (75), except for α being replaced by α'. Equation (80) emphasises that the quasi-incompressible case has one additional independent elastic constant with respect to the strictly incompressible case.

4.3 Orthotropy

Using Walpole's formalism (Section 2.4), the linear elasticity tensor $\mathbb{L}$ of a generic orthotropic material has representation

$$\{\tilde{\mathbb{L}}\} = \left\{ \begin{bmatrix} \mathrm{L}^{1111} & \mathrm{L}^{1122} & \mathrm{L}^{1133} \\ \mathrm{L}^{1122} & \mathrm{L}^{2222} & \mathrm{L}^{2233} \\ \mathrm{L}^{1133} & \mathrm{L}^{2233} & \mathrm{L}^{3333} \end{bmatrix}, 2\mu_{23}, 2\mu_{13}, 2\mu_{12} \right\}, \tag{81}$$

where the diagonal elements of the symmetric 3×3 matrix are the moduli in uniaxial strain in the three orthotropic directions, the off-diagonal elements are the cross moduli, and μ_{pq} are the shear moduli in the pq-planes.

Conditions (63) for strict incompressibility reduce to the three independent scalar conditions

$$
\begin{aligned}
\mathrm{L}^{1111} + \mathrm{L}^{2222} + \mathrm{L}^{3333} + 2\,\mathrm{L}^{2233} + 2\,\mathrm{L}^{1133} + 2\,\mathrm{L}^{1122} &= 0, && (82a)\\
2\,\mathrm{L}^{1111} - \mathrm{L}^{2222} - \mathrm{L}^{3333} - 2\,\mathrm{L}^{2233} + \mathrm{L}^{1133} + \mathrm{L}^{1122} &= 0, && (82b)\\
-\mathrm{L}^{1111} + 2\,\mathrm{L}^{2222} - \mathrm{L}^{3333} + \mathrm{L}^{2233} - 2\,\mathrm{L}^{1133} + \mathrm{L}^{1122} &= 0, && (82c)
\end{aligned}
$$

which imply that only three of the six L^{ppqq} (no sum on p and q) are independent. Supported by arguments analogical to those made for the case of transverse isotropy, we choose, as independent parameters, the negatives of the cross moduli, i.e.,

$$
\alpha_{pq} = -\mathrm{L}^{ppqq}, \quad p \neq q, \text{ no sum on } p \text{ and } q, \tag{83}
$$

and obtain the representation

$$
\{\tilde{\mathrm{L}}_{\text{strict}}\} = \left\{ \begin{bmatrix} \alpha_{12} + \alpha_{13} & -\alpha_{12} & -\alpha_{13} \\ -\alpha_{12} & \alpha_{12} + \alpha_{23} & -\alpha_{23} \\ -\alpha_{13} & -\alpha_{23} & \alpha_{13} + \alpha_{23} \end{bmatrix}, 2\mu_{23}, 2\mu_{13}, 2\mu_{12} \right\}, \tag{84}
$$

with six independent elastic constants (from the original nine).

For the case of orthotropy, the quasi-incompressibility conditions (68) yield the two scalar conditions

$$
\begin{aligned}
2\,\mathrm{L}^{1111} - \mathrm{L}^{2222} - \mathrm{L}^{3333} - 2\,\mathrm{L}^{2233} + \mathrm{L}^{1133} + \mathrm{L}^{1122} &= 0, && (85a)\\
-\mathrm{L}^{1111} + 2\,\mathrm{L}^{2222} - \mathrm{L}^{3333} + \mathrm{L}^{2233} - 2\,\mathrm{L}^{1133} + \mathrm{L}^{1122} &= 0, && (85b)
\end{aligned}
$$

meaning that only four of the six L^{ppqq} (no sum on p and q) are independent. If the independent parameters are chosen to be the bulk modulus

$$
\begin{aligned}
\kappa &= \tfrac{1}{3}\,\langle \mathbb{K}^\sharp, \mathbb{L} \rangle = \tfrac{1}{9}\, g_{ab}\, g_{cd}\, \mathrm{L}^{abcd} = \\
&= \tfrac{1}{9}\,(\mathrm{L}^{1111} + \mathrm{L}^{2222} + \mathrm{L}^{3333} + 2\,\mathrm{L}^{2233} + 2\,\mathrm{L}^{1133} + 2\,\mathrm{L}^{1122}),
\end{aligned} \tag{86}
$$

obtained by applying equation (52) to the case of orthotropy, and

$$
\alpha'_{pq} = \kappa - \mathrm{L}^{ppqq}, \quad p \neq q, \text{ no sum on } p \text{ and } q, \tag{87}
$$

the linear elasticity tensor for the orthotropic quasi-incompressible case reads

$$
\{\tilde{\mathrm{L}}_{\text{quasi}}\} = \left\{ \begin{bmatrix} \kappa + \alpha'_{12} + \alpha'_{13} & \kappa - \alpha'_{12} & \kappa - \alpha'_{13} \\ \kappa - \alpha'_{12} & \kappa + \alpha'_{12} + \alpha'_{23} & \kappa - \alpha'_{23} \\ \kappa - \alpha'_{13} & \kappa - \alpha'_{23} & \kappa + \alpha'_{13} + \alpha'_{23} \end{bmatrix}, 2\mu_{23}, 2\mu_{13}, 2\mu_{12} \right\}, \tag{88}
$$

with seven independent elastic constants: again, one more than for the case of strict incompressibility. Similarly to what has been done for the case of transverse isotropy, considering the orthotropic representation of $\mathbb{K}^\sharp$ (Equations (36)), the elasticity tensor can be written

$$
\{\tilde{\mathrm{L}}_{\text{quasi}}\} = 3\kappa\,\{\tilde{\mathrm{K}}\} + \{\tilde{\mathrm{L}}'_{\text{strict}}\}, \tag{89}
$$

where $\{\tilde{\mathrm{L}}'_{\text{strict}}\}$ has the same form as $\{\tilde{\mathrm{L}}_{\text{strict}}\}$ of Equation (84), except for the parameters α_{pq} being replaced by α'_{pq}.

5 Positive Definiteness and Invertibility

As already remarked at the end of Section 3.2, by comparing Equations (62) and (67), we deduce that the strictly incompressible and the quasi-incompressible cases differ from each other because of the presence of the bulk modulus κ as an additional parameter in the latter case. Here we would like to show that, for this reason, the linear elasticity tensor is positive semi-definite for the case of strict incompressibility and positive definite for the case of quasi-incompressibility. For the case of quasi-incompressibility, the positive definiteness of the elasticity tensor implies its invertibility. For the case of strict incompressibility, the positive semi-definiteness of the elasticity tensor, implying its non-invertibility, mathematically translates the physical impossibility to have an infinite bulk modulus. This can be shown by looking at the examples of isotropy, transverse isotropy, and orthotropy reported in Section 4.

For the isotropic quasi-incompressible case (but this is identical for the general compressible case), the inverse of the elasticity tensor $\mathbb{L}_{\text{quasi}} = 3\kappa\,\mathbb{K}^\sharp + 2\mu\,\mathbb{M}^\sharp$ is given by $\mathbb{L}_{\text{quasi}}^{-1} = (3\kappa)^{-1}\mathbb{K}^\flat + (2\mu)^{-1}\mathbb{M}^\flat$, as is immediately verifiable by evaluating $\mathbb{L}_{\text{quasi}} : \mathbb{L}_{\text{quasi}}^{-1} = \mathbb{I}$ in components, or by accounting for the orthogonality and idempotence of $\mathbb{K}$ and $\mathbb{M}$ [19, 18, 16]. Moreover, $\mathbb{L}_{\text{quasi}}$ is positive definite if, and only if, both κ and μ are positive. For the isotropic strictly incompressible case, it is evident that $\mathbb{L}_{\text{strict}} = 2\mu\,\mathbb{M}^\sharp$ is not invertible, and therefore it is only positive semi-definite, provided that μ is positive.

Exploiting Walpole's formalism [18], the transversely isotropic and orthotropic cases are treated in a similar way. In Walpole's representation of the elasticity tensors for transverse isotropy (Equations (75) and (79)) and orthotropy (Equations (84) and (88)), the individual scalars (shear moduli) must be positive and the matrix must be positive definite to ensure positive definiteness of the tensor. The positive definiteness of the matrix parts can be checked by evaluating their eigenvalues.

For transverse isotropy, the eigenvalues of the 2×2 matrix are

$$0, \quad 3\alpha, \tag{90}$$

for strict incompressibility (positive semi-definiteness attained for $\alpha > 0$, i.e., $l < 0$), and

$$3\kappa, \quad 3\alpha' = 3(\kappa - l), \tag{91}$$

for quasi-incompressibility (positive definiteness attained for $\kappa > 0$ and $\kappa > l$).

For orthotropy, the eigenvalues of the 3×3 matrix are

$$0, \quad (\alpha_{23} + \alpha_{13} + \alpha_{12}) \pm \sqrt{(\alpha_{23} + \alpha_{13} + \alpha_{12})^2 - 3(\alpha_{23}\alpha_{13} + \alpha_{13}\alpha_{12} + \alpha_{12}\alpha_{23})}, \tag{92}$$

for strict incompressibility (positive semi-definiteness attained for $(\alpha_{23} + \alpha_{13} + \alpha_{12}) > 0$, i.e., $(\mathrm{L}^{2233} + \mathrm{L}^{1133} + \mathrm{L}^{1122}) < 0$, as the symmetry of the matrix ensures that the eigenvalues are all real, and the term under square root is positive and smaller than $(\alpha_{23} + \alpha_{13} + \alpha_{12})$, in absolute value), and

$$3\kappa, \quad (\alpha'_{23} + \alpha'_{13} + \alpha'_{12}) \pm \sqrt{(\alpha'_{23} + \alpha'_{13} + \alpha'_{12})^2 - 3(\alpha'_{23}\alpha'_{13} + \alpha'_{13}\alpha'_{12} + \alpha'_{12}\alpha'_{23})}, \tag{93}$$

for quasi-incompressibility (positive definiteness attained for $\kappa > 0$ and $(\alpha'_{23} + \alpha'_{13} + \alpha'_{12}) > 0$, i.e., $\kappa > \frac{1}{3}(\mathrm{L}^{2233} + \mathrm{L}^{1133} + \mathrm{L}^{1122})$).

We conclude noting that, regardless of the material symmetry, if the term $3\kappa\,\mathbb{K}^\sharp$, with $\kappa > 0$, is added to $\mathbb{L}_{\text{strict}}$ (which is equivalent to referring to the corresponding quasi-incompressible material), the resulting fourth-order tensor can be inverted. Then, the strictly incompressible case is retrieved by performing the limit for $\kappa \to \infty$.

6 Discussion

In order to retrieve the correct expression of the linear elasticity tensor for incompressible materials, we followed the path dictated by the non-linear Theory of Elasticity, and modelled incompressibility in two ways. In the strict incompressibility approach, one imposes the kinematical constraint of isochoric motion, and treats the hydrostatic pressure as the associated Lagrange multiplier. In the quasi-incompressibility approach, one uses the bulk modulus as a penalty number to keep volumetric deformations very small. We derived the algebraic conditions for a fourth-order tensor to represent the elasticity tensor of strictly incompressible and quasi-incompressible materials, regardless of the material symmetry. This constitutes a rigorous framework for the determination of the correct form of the linear elasticity tensor of incompressible materials, which can be used to enforce the physical requirement of compatibility of a non-linear elastic material with its linear counterpart [7, 17, 16].

By using the elegant formalism introduced by Walpole [18], we studied the cases of isotropy, transverse isotropy and orthotropy. We proved that the linear elasticity tensor for the case of isotropy, transverse isotropy and orthotropy is characterised by 1, 3, 6 independent material parameters, respectively, in the strictly incompressible case (i.e. when the kinematically admissible deformations are isochoric), and by 2, 4, 7 independent material parameters, respectively, in the quasi-incompressible case (i.e. when the volumetric-deviatoric decoupling of the strain energy function is considered), from the original 2, 5, 9 parameters, respectively, of compressible linear elasticity. Walpole's formalism makes the study of the positive definiteness of the elasticity tensor extremely simple: a tensor is positive definite if its Walpole's representation is such that the matrix part is positive definite, and all the scalars are positive (note that if the tensor is positive definite then it is invertible, and that positive semi-definiteness is treated analogously).

An immediate application of the results presented here is for all those elastic potentials defined in terms of the linear elasticity tensor. This is the case of Fung-type potentials [32, 33, 34], which are monotonic functions of a quadratic form in the Green-Lagrange strain $\boldsymbol{E}$, i.e., take the form $\hat{W}(\boldsymbol{E}) = a\,\varphi(\frac{1}{2}\,\boldsymbol{E} : \mathbb{Q} : \boldsymbol{E})$, where a is a material constant, $\mathbb{Q}$ is a symmetric, positive definite (or positive semi-definite) fourth-order tensor in $[T\mathcal{B}_R]^4_0$, and φ is a convex, monotonic function (Fung's original potential is exponential, with $\varphi = \exp - \mathrm{id}$, where id is the identity in $\mathbb{R}$). It has been shown [35] that, in order to ensure convexity of the potential, the fourth-order tensor $\mathbb{Q}$ of the quadratic form must be related to the material linear elasticity $\mathbb{L}$ by $\mathbb{Q} = a^{-1}\,\mathbb{L}$. Therefore, for a strictly incompressible or quasi-incompressible Fung-type potential, since the spatial linear elasticity tensor $\mathbb{L}$ must obey the algebraic conditions (63) or (68), respectively, so must the material linear elasticity tensor $\mathbb{L}$ (see Equation (45)), and therefore so must the tensor $\mathbb{Q}$ of the quadratic form (with the appropriate spherical and deviatoric operators: in this case the material operators $\mathbb{K}$ and $\mathbb{M}$, which are analogical to the spatial $\mathbb{K}$ and $\mathbb{M}$, and whose expression is reported in [16]). An application of incompressible Fung-type potentials can be found in the work by Bellini et al. [36].

We note that results equivalent to those presented here have been found, for the case of transverse isotropy, by deBotton and Ponte-Castañeda [37] based on the earlier - but equivalent - version of Walpole's formalism [19]. Based on the spectral decomposition of the compliance tensor, Itskov and Aksel [38] introduced a procedure to study the admissible values of the elastic constants for the cases of strict and quasi-incompressibility, and found a closed expression of the elasticity tensor without explicit use of the eigenvalue problem solution. Moreover, our results for the case of quasi-incompressibility coincide with those recently reported by Vergori et al. [30], who also proved that, for the case of monoclinic symmetry, the number of independent parameters reduce from 13 to 10. A natural extension of our work could be to include the monoclinic symmetry to retrieve the results by Vergori et al. [30] in the quasi-incompressible case, and to study the strictly incompressible case. Finally, it is an open problem to understand how the results presented in this work should be generalised to the case of second-gradient continua [39, 40], particularly when used to describe fibre-reinforced composites or porous media [41, 42, 43, 44] saturated with incompressible fluids, or in the case of N-th grade continua [45], or in the case of beams, plates and shells [46, 47].

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