The Hereditary Approach to Time Dependent Modeling with Fractional Exponential Function

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ABSTRACT. For polymer based composites, the creep and relaxation parameters are examined using the hereditary approach to time dependent mechanical behavior. Specifically, methods of optimization are utilized to determine the optimal parameter estimates for the fractional exponential functions. The relative advantages of the Rabatnov kernel over the Abel kernel have been examined. Finally, the results of this parameter estimation using the Rabatnov kernel (i.e. fractional exponential) are compared to results obtained from experimental data.

1. Introduction

Boltzmann's work in the middle of the 19th century involved work on the hereditary mechanics accounting for the time dependent stress-strain relationship also known as the memory effect (Boltzman, 1876). This work was later developed in Volterra's research on integral equations. The modeling of the deformation, or creep, processes in viscoelastic solids with memory of the history of loading is practical because it has many engineering applications. These applications include quasistatic loading, ranging loading conditions like short and long-term creep, and cyclic deformation for a wide range of polymer-based composites and nanocomposites. In the model, ε denotes strain (%), σ denotes load stress (MPa), and t denotes elapsed time (hrs). The solids being discussed have "memory" because the load stress previously applied manifests as present load stress, so that past stress can affect present stress. Introduction of the memory effect relates back to Volterra's research, and it leads to analysis of Volterra integral equations to model the relationship between stress and strain (Volterra, 1913).

It is a common practice to use a completely different set of parameters and kernel functions to describe the direct creep and inverse relaxation process (the main reason being the complexity of obtaining the inverse of the kernel function). The usage of a fractional exponential function, i.e. Rabotnov function, as a kernel function to predict creep has one main advantage - a simply obtained inverse known as Abel's type kernel. This allows for significantly simpler modeling of relaxation if the creep parameters are optimized and calculated. The objective of this paper is to verify the above described approach for the experimental creep and relaxation data obtained for three types of nanocomposite materials.

2. Methods

It is a common practice to model the stress-strain relationship for viscoelastic (i.e. hereditary) media by means of differential equations involving high order derivatives of stress, σ, and strain. When working with the differential equation, a large number of terms are typically needed in order to get an estimate that is accurate. Similarly, a large number of material parameters are needed, and these can be difficult to determine experimentally. Thus, it is often helpful to replace the differential equation (Suvorova et al., 2003)

$$a_0\sigma + a_1\frac{d\sigma}{dt} + ... + a_n\frac{d^n\sigma}{dt^n} = b_0\varepsilon + b_1\frac{d\varepsilon}{dt} + ... + b_m\frac{d^m\varepsilon}{dt^n} \tag{2.1}$$

by the integral equation (Rabotnov, 1969)

$$\sigma = E\varepsilon - \int_0^t \Gamma(t-\tau)\,\varepsilon(\tau)\,d\tau \tag{2.2}$$

where

$$\Gamma(t) = \sum_{i=1}^m A_1\varepsilon^{-\alpha_i(t-\tau)} \tag{2.3}$$

is the relaxation function and α_i are the order of the fractional derivatives found in the intermediate steps in reformulating the constitutive equations. E is the modulus of elasticity and τ is the time elapsed from the start of leading process until the current moment, t. Using the integral equation (2.2) it can be shown (Rabotnov, 1969) that its inverse is given by:

$$\varepsilon = B\sigma + \int_0^t K(t-\tau)\,\sigma(\tau)\,d\tau \tag{2.4}$$

where $K(t-\tau)$ is the direct creep kernel. From these inverse equations, a more general stress/creep relationship can be written as follows:

$$\varepsilon = \frac{1}{E}\left[\sigma + \int_0^t K(t-\tau)\,\sigma(\tau)\,d\tau\right],$$
$$\sigma = E\left[\varepsilon - \int_0^t \Gamma(t-\tau)\,\varepsilon(\tau)\,d\tau\right]. \tag{2.5}$$

Furthermore, the Rabotnov function (i.e. fractional exponential function); which will be used in this model because of advantages discussed later, uses a fractional exponential kernel defined by the following series:

$$K(t) = \lambda\sum_{n=0}^\infty \frac{-\beta^n t^{n(1-\alpha)}}{\Gamma\big((1-\alpha)(n+1)\big)}. \tag{2.6}$$

Substituting this kernel into (2.5) the following equation is obtained

$$\varepsilon(t) = \varepsilon_0\left[1 + \lambda\sum_{n=0}^\infty \frac{(-\beta)^n\,t^{(n+1)(1-\alpha)}}{\Gamma\big((1-\alpha)(n+1)+1\big)}\right] \tag{2.7}$$

where α, β, λ, and ε_0 are parameters of the fractional exponential kernel and constitutive equation obtained by optimization of experimental data. In the application being discussed, the Rabotnov kernel is more efficient than the Abel kernel for several reasons. The first is that the Rabotnov

kernel predicts long-term creep better than Abel. This is because kernels of the Abel type have weakly defined singularities. Second, the Rabotnov kernel has a simple inverse (i.e. Abel type). This is important because it allows one to describe relaxation with the same set of parameters that one obtains for creep which can be proved through the use of the Neumann series for the inverse of the Volterra equation and the multiplication theorem for fractional exponential operators (Rabotnov, 1969). If Abel's kernel is defined as

$$I_\alpha\left(t-\tau\right) = \frac{\left(t-\tau\right)^\alpha}{\Gamma\left(1+\alpha\right)}, \tag{2.8}$$

then the fractional exponential operator $\ni_\alpha^*(\beta)$ known as the Rabotnov operator, is defined from the equation

$$\frac{1}{1-\beta I_\alpha^{*2}} = 1 + \beta \ni_\alpha^*(\beta). \tag{2.9}$$

Thus, the Rabotnov kernel is the inverse of Abel's, which is shown by Neumann's series expansion and is defined by the infinite sum

$$\ni_\alpha^*\left(\beta, t-\tau\right) = \left(t-\tau\right)^\alpha \sum_0^\infty \frac{\beta^n \left(t-\tau\right)^{n(1-\alpha)}}{\Gamma\left[\left(n+1\right)\left(1+\alpha\right)\right]}. \tag{2.10}$$

Consequently the relaxation can be analyzed with the same set of parameters estimates that is obtained through optimization for creep model since the Abel kernel is the inverse of the Rabotnov kernel.

3. Results

The objective is to obtain the optimal parameter estimates for the parameters $p = [\epsilon_0, \lambda, \beta]$. The kernel $K\left(p, t\right)$ is the exponential operator of arbitrary order which has several important features. First, the initial moment singularity $(t = 0)$ is integratable. Second, as t approaches infinity, the operator has asymptotic exponential behavior.

In the t domain, the Rabotnov kernel is defined as an infinite series, so the Laplace transform is applied which leads to the problem of optimization in the complex domain. It has been shown (Viktorova et al., 2013) that the obtained parameter set is equivalent to the optimal parameter set in the real domain.

The following equation can be obtained from (2.7) by using power regressions for the creep strain

$$at^b = \varepsilon_0 \left[1 + \lambda \sum_{n=0}^\infty \frac{\left(-\beta\right)^n t^{(n+1)(1-\alpha)}}{\Gamma\left(\left(1-\alpha\right)\left(n+1\right)+1\right)}\right]. \tag{3.1}$$

Applying the Laplace-Carson transformation to both sides of (3.1), the following expression is derived:

$$a\frac{\Gamma\left(1+b\right)}{s^b} = \varepsilon_0 \left[1 + \frac{\lambda}{s^{1-\alpha+\beta}}\right]. \tag{3.2}$$

This is a significant improvement because there is no longer a complicated infinite sum to evaluate.

The results of this optimization can be seen in Figures 4.1 and 4.2. Table 4.1 is a table of the parameter estimates. It is important to note that the results were calculated at $\sigma=0.3\sigma_y$, $\sigma=0.4\sigma_y$,

	Data Set 1		
	$\sigma = 0.3\sigma_y$	$\sigma = 0.4\sigma_y$	$\sigma = 0.5\sigma_y$
α	0.59	0.66	0.70
ε_0	4.697×10^{-2}	2.688×10^{-1}	7.548×10^{-2}
λ	2.077	1.159	7.217
β	7.431×10^{-2}	1.015×10^{-1}	1.453×10^{-1}
Obj. Val	2.644×10^{-3}	4.328×10^{-2}	1.354×10^{-2}

	Data Set 2		
	$\sigma = 0.3\sigma_y$	$\sigma = 0.4\sigma_y$	$\sigma = 0.5\sigma_y$
α	0.67	0.69	0.76
ε_0	7.476×10^{-2}	3.406×10^{-1}	6.997×10^{-2}
λ	1.226	1.164	7.457
β	1.728×10^{-2}	1.108×10^{-1}	8.300×10^{-2}
Obj. Val	6.289×10^{-3}	3.410×10^{-2}	7.330×10^{-3}

TABLE 4.1. Parameter Estimates.

and $\sigma=0.5\sigma_y$ where σ_y is the yield stress for three tested types of nanocomposites described in (Viktorova et al., 2013).

The relaxation process with the use of Abel's type kernel can be modeled by the following equations

$$\sigma = \frac{\phi(\varepsilon)}{1 + K^*} = \phi(\varepsilon)\left(1 - K^* + K^{*2} - K^{*3}\right) \tag{3.3}$$

Using the fact that, at relaxation, ε and therefore $\phi(\varepsilon) = constant$, the following relation is obtained:

$$\phi(\varepsilon) = \sigma + \int_0^t \frac{k}{(t-\tau)^\alpha}\sigma(\tau)\,d\tau. \tag{3.4}$$

Relaxation graphs can be seen in Figure 4.2. These graphs show that the calculated results are extremely close to the experimental data with the circles representing the numerical results based on the parametric estimates obtained by optimization techniques described above, while squares are the experimental data obtained for two types of nanocomposites described above.

4. Conclusion

From this research, we can conclude that the use of the fractional exponential kernel has many advantages over the Abel kernel in the problem of optimization of the creep parameters for hereditary media. This kernel allows one to easily take the Laplace transform and shift from the t to the s domain, thus eliminating the infinite sum without any loss of generality. From this, the optimal parameters for the short-term creep experiments are obtained and can then be applied to the relaxation equation. These parameters can also be used to predict long-term creep as well.

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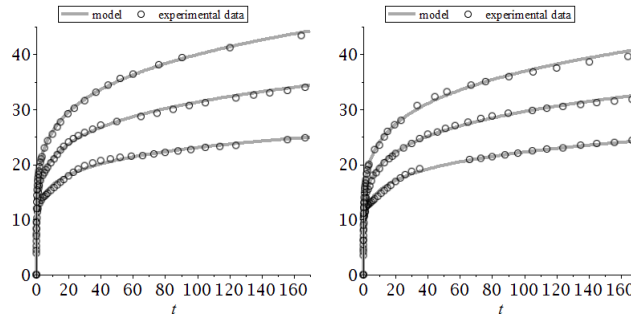Boltzman, L. (1876). Zur Theorie der elastischen Nachwirkung. *Annalen der Physik und Chemie*.

FIGURE 4.1. Strain versus time for various stress levels with model using estimated parameters. Data set 1 on the left, data set 2 on the right.
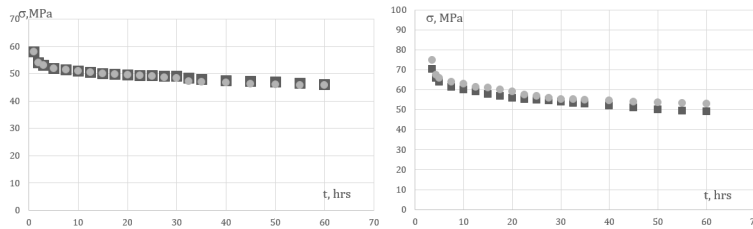

FIGURE 4.2. Relaxation graphs - circles represent calculated results using the parameters determined from the creep experiments. Squares are the experimental results. Data set 1 on the left, data set 2 on the right.

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