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Constructing Galois 2-extensions of the 2-adic Numbers

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ABSTRACT. Let $\mathbf{Q}_2$ denote the field of 2-adic numbers, and let G be a group of order 2^n for some positive integer n. We provide an implementation in the software program GAP of an algorithm due to Yamagishi that counts the number of nonisomorphic Galois extensions $K/\mathbf{Q}_2$ whose Galois group is G. Furthermore, we describe an algorithm for constructing defining polynomials for each such extension by considering quadratic extensions of Galois 2-adic fields of degree 2^{n-1}. While this method does require that some extensions be discarded, we show that this approach considers far fewer extensions than the best general construction algorithm currently known, which is due to Pauli-Sinclair based on the work of Monge. We end with an application of our approach to completely classify all Galois 2-adic fields of degree 16, including defining polynomials, ramification index, residue degree, valuation of the discriminant, and Galois group.

1. Introduction

Let p be a prime number and denote by $\mathbf{Q}_p$ the field of p-adic numbers. A consequence of Krasner's Lemma is the following: for a fixed positive integer n, there are only finitely many nonisomorphic extensions of $\mathbf{Q}_p$ of degree n. It is therefore natural to ask for a formula which counts extensions; or at least a formula which counts extensions with specified invariants, such as ramification index, residue degree, valuation of the discriminant, Galois group, etc.

Some results are well known. For example, there is a unique unramified extension of $\mathbf{Q}_p$ of degree n; i.e., where the residue degree is n. Tamely ramified extensions are also well understood; that is, extensions where p does not divide the ramification index. In particular, fix a positive integer e not divisible by p and a positive integer f. Then the number of nonisomorphic tamely ramified extensions of $\mathbf{Q}_p$ of degree $n = ef$ with ramification index e and residue degree f is given by

$$\sum_{d|f} \phi(f/d) \cdot \gcd(e, p^d - 1),$$

where ϕ is Euler's totient function.

It is also possible to count nonisomorphic extensions of wildly ramified extensions (where p divides the ramification index) using a formula due to Monge (2011). For example, Table 1.1 gives the number of nonisomorphic extensions of $\mathbf{Q}_2$ of degree $2n$ for $1 \leq n \leq 16$. The case $n = 1$ is well known. The cases $n = 2, 3, 4, 5$ are classified in Jones and Roberts (2006, 2008), where defining polynomials and Galois groups are given. A similar classification for the cases $n = 6, 7$ appears in Awtrey et al. (2016, 2015a); Awtrey and Shill (2013); Awtrey et al. (2015c). No other cases have been completely classified.

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TABLE 1.1. The number of nonisomorphic extensions of $\mathbf{Q}_2$ of degree $2n$ for $1 \leq n \leq 16$.

Degree	Number
2	7
4	59
6	47
8	1823
10	158
12	5493
14	590
16	890111

Degree	Number
18	2991
20	314543
22	8942
24	43488949
26	35294
28	19173718
30	209854
32	114799238127

The first general algorithm for constructing defining polynomials for all nonisomorphic extensions is due to Pauli-Roblot and appears in Pauli and Roblot (2001). Based on recent work in Monge (2014), a faster method is developed in Pauli and Sinclair (2017) for constructing totally ramified extensions of unramified p-adic fields. In both cases, the algorithms construct a large collection of polynomials some subset of which is guaranteed to define all nonisomorphic extensions. In particular, it is necessary to discard isomorphic extensions. Though we note that the number of polynomials that must be discarded is far smaller in the Pauli-Sinclair method when compared with the Pauli-Roblot method.

If we are interested only in Galois extensions, we can use either of these algorithms. But not only must we discard isomorphic extensions, we also need to discard non-Galois extensions. The purpose of this paper is to offer an alternative approach to constructing defining polynomials of Galois extensions of $\mathbf{Q}_2$ of degree 2^n for some positive integer n. Our method constructs these extensions as quadratic extensions of Galois 2-adic fields of degree 2^{n-1}.

The remainder of the paper is organized as follows. In Section 2 we describe an algorithm in Yamagishi (1995) that counts the number of Galois 2-adic fields of degree 2^n with a given 2-group G as its Galois group. This result is useful for us for two reasons: it allows us to verify when we have found all defining polynomials in our algorithm, and it allows us to demonstrate why our method considers fewer polynomials than the Pauli-Sinclair method (see Table 3.1 for a comparison). In Section 3 we describe our algorithm for constructing Galois extensions of $\mathbf{Q}_2$ of degree 2^n for some positive integer n. In Section 4, we apply our results to completely classify Galois 2-adic fields of degree 16. We compute defining polynomials for each of the 251 nonisomorphic extensions, the ramification index, residue degree, valuation of the discriminant, and Galois group of each extension. The raw data can be obtained by emailing the first author. We provide summary tables which count the number of extensions by ramification index, discriminant exponent, and Galois group. Our final section discusses the computational feasibility of constructing Galois 2-adic fields of degrees 2^n for $n \geq 5$.

2. Counting Galois 2-extensions of $\mathbf{Q}_2$

In this section, we describe an algorithm in Yamagishi (1995) that counts the number of Galois extensions of $\mathbf{Q}_2$ of degree 2^n whose Galois group is some specified 2-group G. We include our implementation of this algorithm for the software program GAP (2013). We then use our algorithm

TABLE 2.1. The number of Galois extensions of $\mathbf{Q}_2$ of degree 2^n for $1 \leq n \leq 6$.

Degree	**Number**
2	7
4	19
8	67
16	251
32	915
64	4131

to count Galois extensions of $\mathbf{Q}_2$ of degree 2^n for $1 \leq n \leq 6$, giving a complete description of the counts for each possible Galois group in the case $n = 4$.

As mentioned in Yamagishi (1995), the Galois group of the maximal pro-2-extension of $\mathbf{Q}_2$ is a one-relator group whose relation is given in Labute (1967). Combining this characterization with a well-known enumeration argument, Yamagishi proves the following result. Note, we are restricting Yamagishi's results to the case where the base field is $\mathbf{Q}_2$.

Theorem 2.1 (Yamagishi)**.** *Let G be a finite 2-group. The number of nonisomorphic Galois 2-adic fields with Galois group G is*

$$\frac{1}{\#\mathit{Aut}(G)} \sum_{H \leq G} \mu_G(H)\alpha(H),$$

where $\#\mathit{Aut}(G)$ denotes the size of the automorphism group of G and where μ_G and α are defined as follows:

(1) If $H \leq G$ with $[G : H] = 2^i$, then

$$\mu_G(H) = \begin{cases} (-1)^i 2^{i(i-1)/2} & \text{if } H \geq \Phi(G) \\ 0 & \text{otherwise} \end{cases},$$

where $\Phi(G)$ denotes the Frattini subgroup of G.

(2) For irreducible complex characters χ of H, we have

$$\alpha(H) = \sum_{\chi} \sum_{g,h \in H} \chi(g^2h^3)\chi(h).$$

Using Theorem 2.1, we can count the number of Galois extensions of $\mathbf{Q}_2$ of degree 2^n by running over all possible groups G of order 2^n. For example, Table 2.1 contains this data for $1 \leq n \leq 6$.

If we focus on the degree 16 Galois extensions of $\mathbf{Q}_2$, Table 2.2 gives more refined information about the 251 extensions by showing how many have a given Galois group. Up to isomorphism, there are 14 groups of order 16. One way to access these groups in GAP is via the TransitiveGroup library. In the table, we identify each group in two ways: (1) by its T-number (accessed by typing `TransitiveGroup(16,T)` in GAP), and (2) by a more descriptive name that indicates its structure. For the descriptive name, C_n denotes the cyclic group of order n, E_n the elementary abelian group of order n, D_n the dihedral group of order $2n$, Q_n the generalized quaternion group of order n, $\times$ a direct product, and $\rtimes$ a semidirect product. In the case of $C_8 \rtimes C_2$, there are different mappings from C_2 into Aut(C_8). These give rise to two distinct groups of order 16 (other than D_8). One is defined by the mapping $x \mapsto x^3$, and the other is given by $x \mapsto x^5$. We distinguish these two cases in the obvious way: by $C_8 \rtimes_3 C_2$ and $C_8 \rtimes_5 C_2$, respectively.

TABLE 2.2. For each of the 14 groups of order 16, column **T** gives the group's transitive number, **Name** gives a descriptive name of the group, and **Number** gives the number of nonisomorphic Galois extensions of $\mathbf{Q}_2$ of degree 16 with the specified group as its Galois group.

T	**Name**	**Number**
1	C_{16}	48
2	$E_4 \times C_4$	3
3	E_{16}	0
4	$C_4 \times C_4$	4
5	$C_8 \times C_2$	36
6	$C_8 \rtimes_5 C_2$	36
7	$Q_8 \times C_2$	3
8	$C_4 \rtimes C_4$	12
9	$D_4 \times C_2$	9
10	$E_4 \rtimes C_4$	12
11	$Q_8 \rtimes C_2$	16
12	$C_8 \rtimes_3 C_2$	36
13	D_8	16
14	Q_{16}	20

We computed the data in Table 2.2 with the software program GAP (2013). For example, here are several functions we wrote for GAP which will accomplish this task. The main function is `Count2adicFields`, which takes one input, a 2-group G (called either from GAP's TransitiveGroup library or SmallGroup library). The other functions are merely auxiliary functions which support the main function, including many for working with character tables to help compute $\alpha(G)$.

```
---------------------------
IsEl := function(g,cc)
  return(IsSubset(Elements(cc),[g]));
end;

---------------------------
FindIndex := function(g,cc)
  local j;
  for j in [1..Size(cc)] do
    if
      IsEl(g,cc[j]) then return(j); fi; od;
  return(0);
end;

---------------------------
EvalChi := function(g,chi,cc)
  local ind;
    ind := FindIndex(g,cc);
  return(chi[ind]);
end;
```

```
---------------------------
OneChiSum := function(g,H,chi,cc)
  local lis;
    lis := List(H,j->EvalChi(g^2*j^3,chi,cc)*EvalChi(j,chi,cc));
  return(Sum(lis));
end;

---------------------------
MyChiSum := function(G,H,chi,cc)
  local lis;
    lis := List(G,j->OneChiSum(j,H,chi,cc));
  return(Sum(lis));
end;

---------------------------
MyAlpha := function(g)
  local tbl,cc,c,chi,lis;
    tbl := CharacterTable(g);
    cc := ConjugacyClasses(tbl);
    c := Size(cc);
    chi := Irr(tbl);
    lis := List(chi,j->MyChiSum(g,g,j,cc));
  return(Sum(lis));
end;

---------------------------
MyMu := function(g,h)
  local f,rc,i;
    f := FrattiniSubgroup(g);
      if
        IsSubgroup(h,f) then
        rc := RightCosets(g,h);
        i := LogInt(Size(rc),2);
        return((-1)^i*2^(1/2*i*(i-1)));
      else return(0); fi;
end;
---------------------------
Count2adicFields := function(g)
  local aut,as,lis;
    aut := AutomorphismGroup(g);
    as := AllSubgroups(g);
    lis := List(as,j->MyMu(g,j)*MyAlpha(j));
  return(1/Size(aut)*Sum(lis));
end;
---------------------------
```

3. Our Algorithm

In this section, we describe our algorithm for producing polynomials defining all nonisomorphic Galois extensions of $\mathbf{Q}_2$ of degree 2^n for some positive integer n.

Our next result forms the basis of our algorithm for constructing Galois 2-extensions of $\mathbf{Q}_2$. In particular, we show that every Galois extension of $\mathbf{Q}_2$ of degree 2^n can be constructed as a quadratic extension of a Galois extension of $\mathbf{Q}_2$ of degree 2^{n-1}. This result is an immediate consequence of the following proposition.

Proposition 3.1. *Let p be a prime number and $K/\mathbf{Q}_p$ a Galois extension of degree p^n with Galois group G. There exists a subfield F of K with $[K : F] = p$ such that $F/\mathbf{Q}_p$ is Galois. Thus K can be realized as a cyclic degree p extension of a Galois p-adic field of degree p^{n-1}.*

Proof. Since G is a p-group, its center $Z(G)$ is nontrivial. Since $Z(G)$ is also a p-group, it contains an element of order p and hence a subgroup H of order p. Since $H \leq Z(G)$, H is a normal subgroup of G. Let F be the subfield fixed by H. Since H has order p, the Galois correspondence implies that K/F is a Galois extension of degree p with Galois group H; hence K/F is a cyclic degree p extension. Furthermore, it also follows that $[F : \mathbf{Q}_p] = p^{n-1}$. Since H is normal, $F/\mathbf{Q}_p$ is Galois; in fact, its Galois group is isomorphic to G/H. □

Letting $p = 2$ in Proposition 3.1, we see that every Galois extension of $\mathbf{Q}_2$ of degree 2^n can be realized as a quadratic extension of a Galois extension of $\mathbf{Q}_2$ of degree 2^{n-1}. How many such quadratic extensions are there? The answer is $2^{2+2^{n-1}} - 1$.

To see this, let K be any extension of $\mathbf{Q}_2$ of degree m. The quadratic extensions of K are in one-to-one correspondence with the nontrivial representatives of K^*/K^{*2}. Let O denote the ring of integers of K, P its unique prime ideal, π a uniformizer, f the residue degree of K, U the units of K in O and U_1 the units congruent to 1 modulo P. Note that U_1 has a natural O-module structure, being finitely generated of rank m. Now, we can decompose K^* in the following way:

$$K^* \simeq \pi^{\mathbf{Z}} \times U \simeq \mathbf{Z} \times \mu_K \times U_1 \simeq \mathbf{Z} \times \mu_K \times \mu_{K,1} \times \mathbf{Z}_2^m,$$

where μ_K denotes the $(2^f - 1)$-st roots of unity in U, and $\mu_{K,1}$ denotes roots of unity congruent to 1 modulo P. Since μ_K has odd order and $\mu_{K,1}$ has even order, it follows that

$$K^*/K^{*2} \simeq C_2 \times C_2 \times C_2^m \simeq C_2^{2+m}.$$

In Table 3.1, we compare the number of extensions that our method produces with the number produced by the Pauli-Sinclair algorithm. As the data show, our approach requires much less filtering as the degree of the extensions increase.

We point out that quadratic extensions of 2-adic fields are quickly constructed using the algorithm in Pauli and Roblot (2001).

We now describe our algorithm for computing Galois extensions of $\mathbf{Q}_2$ of degree 2^n.

Algorithm 3.2. *Let n be a positive integer, and let L be the set of Galois extensions of $\mathbf{Q}_2$ of degree 2^{n-1}. If $n = 1$, take $L = \{\mathbf{Q}_2\}$. This algorithm produces polynomials defining all nonisomorphic Galois extensions of $\mathbf{Q}_2$ of degree 2^n.*

(1) For each $K \in L$, form all quadratic extensions of K (using Pauli and Roblot (2001) for example) and produce polynomials defining each field as an extension over $\mathbf{Q}_2$. There should be a total of $2^{2+2^{n-1}} - 1$ such polynomials.

TABLE 3.1. For $1 \leq n \leq 5$, Column **Alg** counts the number of extensions produced by Algorithm 3.2 for computing Galois extensions of $\mathbf{Q}_2$ of degree 2^n, **P-S** counts the number of extensions produced by the Pauli-Sinclair method, and **Number** gives the number of nonisomorphic Galois extensions.

Degree	**Number**	**Alg**	**P-S**
2	7	7	7
4	19	105	89
8	67	1,197	4,945
16	251	68,541	4,777,313
32	915	65,797,893	1,221,308,375,461

TABLE 3.2. Quadratic extensions of $\mathbf{Q}_2$, including a defining polynomial, ramification index e, residue degree f, valuation of the discriminant c, and Galois group G.

Poly	**e**	**f**	**c**	**G**
$x^2 + x + 1$	1	2	0	C_2
$x^2 + 2x + 2$	2	1	2	C_2
$x^2 + 2x + 6$	2	1	2	C_2
$x^2 + 2$	2	1	3	C_2
$x^2 + 6$	2	1	3	C_2
$x^2 + 10$	2	1	3	C_2
$x^2 + 14$	2	1	3	C_2

(2) *Discard all extensions that are not Galois. For example, compute the extension's automorphism group and check if the automorphism group has size 2^n (the automorphisms can be represented as roots of linear factors after factoring the polynomial over its stem field).*
(3) *For each group G of order 2^n, use Theorem 2.1 to count the number of nonisomorphic extensions of $\mathbf{Q}_2$ of degree 2^n with Galois group G.*
(4) *Partition all extensions by their automorphism groups. For each set in the partition, determine if the number of extensions in the set matches the computation from Step (3). If not, discard isomorphic extensions (using the Root-Finding Algorithm in Pauli and Roblot (2001), for example).*
(5) *Return the list of all remaining polynomials along with their automorphism groups identified either as a transitive subgroup of S_{2^n} or using the SmallGroup libraries located in Bosma et al. (1997) or GAP.*

In Tables 3.2–3.5 we list the outputs of our algorithm for values of n between 1 and 3. We also include each extension's ramification index, residue degree, and discriminant valuation. Note, we also lowered coefficients of these defining polynomials in an effort to produce polynomials whose coefficients were either 0 or as small as possible.

TABLE 3.3. Galois quartic extensions of $\mathbf{Q}_2$, including a defining polynomial, ramification index e, residue degree f, valuation of the discriminant c, and Galois group G.

Poly	e	f	c	G
x^4+x+1	1	4	0	C_4
$x^4+2x^3+9x^2+13$	2	2	4	E_4
$x^4+8x^3+5x^2+6x+1$	2	2	4	C_4
$x^4+14x^3+11x^2+14x+27$	2	2	6	C_4
$x^4+2x^3+5x^2+22x+1$	2	2	6	C_4
$x^4+18x^3+5x^2+2x+17$	2	2	6	E_4
$x^4+6x^3+23x^2+10x+7$	2	2	6	E_4
$x^4+2x^2+4x+10$	4	1	8	E_4
x^4+2x^2+4x+2	4	1	8	E_4
$x^4+4x^3+2x^2+4x+6$	4	1	8	E_4
$x^4+4x^3+2x^2+4x+14$	4	1	8	E_4
$x^4+8x^3+4x^2+10$	4	1	11	C_4
$x^4+8x^3+4x^2+2$	4	1	11	C_4
x^4+4x^2+18	4	1	11	C_4
x^4+4x^2+26	4	1	11	C_4
$x^4+8x^3+4x^2+26$	4	1	11	C_4
x^4+4x^2+2	4	1	11	C_4
x^4+4x^2+10	4	1	11	C_4
$x^4+8x^3+4x^2+18$	4	1	11	C_4

4. Application to Galois 2-adic Fields of Degree 16 and Higher

The data in Tables 3.2–3.5 was already known due to Jones and Roberts (2006, 2008). In this section, we use Algorithm 3.2 to construct all Galois extensions of $\mathbf{Q}_2$ of degree 16. To our knowledge, this data has not been computed before; though we do note that one polynomial for each possible Galois group does appear in Awtrey et al. (2015b). The entire set of defining polynomials, along with the extension's ramification index, residue degree, discriminant exponent, and Galois group can be obtained by contacting the first author. Instead, we provide summary tables where we count the number of extensions by ramification index, discriminant exponent, and Galois group. See Tables 4.1–4.4.

The entire computation took approximately 6 hours and 21 minutes on a machine with a 3 GHz Intel Core i7 processor and 8 GB of RAM. By comparison, the total time to compute degree 8 fields was approximately 7 minutes, while the algorithm finished in about 30 seconds for degree 4 fields. Quadratic extensions of $\mathbf{Q}_2$ are well known, and our algorithm took about 1/100th of a second.

We have not carried out the computation for the next highest degree (32) in full. But we have done one case, where we constructed all quadratic extensions of the unramified extension of $\mathbf{Q}_2$ of degree 16. For this case, the defining polynomial we chose was $f(x) = x^{16} + x^5 + x^3 + x + 1$. Let $F/\mathbf{Q}_2$ be the extension defined by $f(x)$. There are a total of $2^{18} - 1 = 262143$ nonisomorphic quadratic extensions of F. Using Pauli and Sinclair (2017), we obtained polynomials defining these extensions in about 2 hours and 27 minutes. We then computed the automorphism groups of

TABLE 3.4. Galois octic extensions of $\mathbf{Q}_2$, including a defining polynomial, ramification index e, residue degree f, valuation of the discriminant c, and Galois group G.

Poly	**e**	**f**	**c**	**G**
$x^8+x^4+x^3+x+1$	1	8	0	C_8
$x^8+6x^7+11x^6+4x^5+10x^4+14x^3+12x^2+14x+1$	2	4	8	C_8
$x^8+12x^7+5x^6+8x^5+10x^4+12x^3+8x^2+4x+9$	2	4	8	$C_4\times C_2$
$x^8+8x^7+24x^6+2x^5+14x^4+24x^3+x^2+6x+11$	2	4	12	C_8
$x^8+12x^7+8x^6+10x^5+22x^4+16x^3+27x^2+14x+1$	2	4	12	$C_4\times C_2$
$x^8+28x^7+16x^6+14x^5+24x^4+24x^3+29x^2+10x+17$	2	4	12	$C_4\times C_2$
$x^8+4x^7+14x^6+18x^5+26x^4+12x^3+21x^2+2x+31$	2	4	12	C_8
$x^8+20x^7+24x^6+28x^5+17x^4+16x^3+18x^2+20x+1$	4	2	16	E_8
$x^8+14x^6+8x^5+29x^4+28x^3+16x^2+24x+17$	4	2	16	$C_4\times C_2$
$x^8+16x^7+2x^6+12x^5+5x^4+24x^3+16x^2+20x+1$	4	2	16	$C_4\times C_2$
$x^8+28x^7+31x^4+24x^3+8x^2+28x+25$	4	2	16	$C_4\times C_2$
$x^8+12x^7+28x^6+11x^4+12x^2+20x+1$	4	2	16	D_4
$x^8+28x^7+12x^6+24x^5+3x^4+24x^3+4x^2+28x+1$	4	2	16	D_4
$x^8+14x^7+8x^5+15x^4+4x^2+2x+5$	4	2	12	D_4
$x^8+12x^6+2x^5+15x^4+14x^3+14x^2+2x+13$	4	2	12	D_4
$x^8+36x^7+62x^6+24x^5+33x^4+40x^3+22x^2+20x+51$	4	2	22	D_4
$x^8+20x^7+62x^6+24x^5+39x^4+56x^3+50x^2+4x+15$	4	2	22	C_8
$x^8+12x^7+46x^6+8x^5+41x^4+40x^3+46x^2+44x+1$	4	2	22	$C_4\times C_2$
$x^8+52x^7+50x^6+40x^5+13x^4+40x^3+2x^2+20x+33$	4	2	22	Q_8
$x^8+4x^7+46x^6+32x^5+57x^4+16x^3+62x^2+60x+59$	4	2	22	D_4
$x^8+44x^7+2x^6+48x^5+49x^4+56x^3+6x^2+44x+55$	4	2	22	C_8
$x^8+12x^7+26x^6+48x^5+9x^4+24x^3+22x^2+60x+39$	4	2	22	C_8
$x^8+12x^7+22x^6+45x^4+16x^3+14x^2+36x+63$	4	2	22	C_8
$x^8+20x^7+30x^6+16x^5+15x^4+48x^3+50x^2+44x+63$	4	2	22	$C_4\times C_2$
$x^8+4x^7+2x^6+8x^5+21x^4+32x^3+30x^2+60x+35$	4	2	22	Q_8
$x^8+60x^7+10x^6+56x^5+25x^4+40x^3+10x^2+28x+17$	4	2	22	$C_4\times C_2$
$x^8+4x^7+14x^6+8x^5+19x^4+32x^3+46x^2+20x+7$	4	2	22	$C_4\times C_2$
$x^8+8x^5+2x^4+8x^3+4x^2+8x+26$	8	1	24	D_4
$x^8+8x^7+4x^6+2x^4+8x^3+4x^2+8x+30$	8	1	24	$C_4\times C_2$
$x^8+8x^7+2x^4+4x^2+8x+26$	8	1	24	$C_4\times C_2$
$x^8+8x^7+8x^5+2x^4+8x^3+4x^2+8x+18$	8	1	24	D_4
$x^8+4x^6+2x^4+4x^2+8x+6$	8	1	24	Q_8
$x^8+8x^7+4x^6+2x^4+4x^2+8x+30$	8	1	24	Q_8
$x^8+4x^6+2x^4+8x^3+4x^2+8x+6$	8	1	24	$C_4\times C_2$

each of these extensions, a computation which took approximately 24 hours and 41 minutes. In addition to the unramified extension, we found a total of 6 Galois extensions; 3 with C_{32} as Galois group and 3 with $C_2\times C_{16}$ as Galois group. Let $u\in F$ be a root of $f(x)$. Defining polynomials for these six extensions (as quadratic polynomials over F) are as follows:

- With Galois group C_{32}:

TABLE 3.5. Octic extensions of $\mathbf{Q}_2$ (cont).

Poly	e	f	c	G
$x^8+8x^7+4x^6+2x^4+8x^3+4x^2+8x+14$	8	1	24	$C_4\times C_2$
$x^8+2x^4+4x^2+8x+2$	8	1	24	$C_4\times C_2$
$x^8+4x^6+2x^4+8x^3+4x^2+8x+22$	8	1	24	$C_4\times C_2$
$x^8+4x^6+2x^4+4x^2+8x+22$	8	1	24	Q_8
$x^8+2x^4+4x^2+8x+18$	8	1	24	$C_4\times C_2$
$x^8+8x^5+2x^4+8x^3+4x^2+8x+10$	8	1	24	D_4
$x^8+8x^7+2x^4+4x^2+8x+10$	8	1	24	$C_4\times C_2$
$x^8+8x^7+8x^5+2x^4+8x^3+4x^2+8x+2$	8	1	24	D_4
$x^8+8x^7+4x^6+2x^4+4x^2+8x+14$	8	1	24	Q_8
$x^8+4x^7+10x^4+4x^2+6$	8	1	22	D_4
$x^8+4x^7+4x^6+10x^4+4x^2+8x+10$	8	1	22	D_4
$x^8+4x^7+2x^4+4x^2+14$	8	1	22	D_4
$x^8+4x^7+10x^4+4x^2+14$	8	1	22	D_4
$x^8+4x^7+4x^6+10x^4+4x^2+8x+2$	8	1	22	D_4
$x^8+4x^7+4x^6+2x^4+4x^2+8x+10$	8	1	22	D_4
$x^8+4x^7+4x^6+2x^4+4x^2+8x+2$	8	1	22	D_4
$x^8+4x^7+2x^4+4x^2+6$	8	1	22	D_4
$x^8+8x^6+16x^5+4x^4+50$	8	1	31	C_8
$x^8+16x^7+16x^5+4x^4+16x^3+26$	8	1	31	C_8
$x^8+16x^5+4x^4+16x^3+10$	8	1	31	C_8
$x^8+8x^6+16x^5+4x^4+34$	8	1	31	C_8
$x^8+16x^7+8x^6+16x^5+4x^4+34$	8	1	31	C_8
$x^8+16x^7+16x^5+4x^4+16x^3+10$	8	1	31	C_8
$x^8+8x^6+16x^5+4x^4+18$	8	1	31	C_8
$x^8+16x^5+4x^4+16x^3+26$	8	1	31	C_8
$x^8+16x^7+8x^6+16x^5+4x^4+2$	8	1	31	C_8
$x^8+16x^7+8x^6+16x^5+4x^4+18$	8	1	31	C_8
$x^8+8x^6+16x^5+4x^4+2$	8	1	31	C_8
$x^8+16x^5+4x^4+16x^3+42$	8	1	31	C_8
$x^8+16x^7+16x^5+4x^4+16x^3+42$	8	1	31	C_8
$x^8+16x^7+16x^5+4x^4+16x^3+58$	8	1	31	C_8
$x^8+16x^5+4x^4+16x^3+58$	8	1	31	C_8
$x^8+16x^7+8x^6+16x^5+4x^4+50$	8	1	31	C_8

  - $x^2+8u^{13}+2$
  - $x^2+2x+4u^{13}+2$
  - $x^2+4x+8u^{13}+2$
- With Galois group $C_2\times C_{16}$:
  - x^2+2
  - x^2+2x+2
  - x^2+4x+2

TABLE 4.1. The number of Galois extensions of $\mathbf{Q}_2$ of degree 16 with ramification index 2.

G	**c = 16**	**24**	**Total**
$\mathbf{C_{16}}$	1	2	3
$\mathbf{C_8 \times C_2}$	1	2	3
Total	2	4	6

TABLE 4.2. The number of Galois extensions of $\mathbf{Q}_2$ of degree 16 with ramification index 4.

G	**c = 32**	**36**	**40**	**Total**
$\mathbf{C_{16}}$			4	4
$\mathbf{E_4 \times C_4}$		1		1
$\mathbf{C_4 \times C_4}$			2	2
$\mathbf{C_8 \times C_2}$		3	2	5
$\mathbf{C_8 \rtimes_5 C_2}$	1	1	2	4
$\mathbf{C_4 \rtimes C_4}$			2	2
$\mathbf{E_4 \rtimes C_4}$	1	1		2
Total	2	6	12	20

TABLE 4.3. The number of Galois extensions of $\mathbf{Q}_2$ of degree 16 with ramification index 8.

G	**c = 32**	**36**	**40**	**44**	**48**	**62**	**Total**
$\mathbf{C_{16}}$						8	8
$\mathbf{E_4 \times C_4}$					2		2
$\mathbf{C_4 \times C_4}$					2		2
$\mathbf{C_8 \times C_2}$					4	8	12
$\mathbf{C_8 \rtimes_5 C_2}$			4		4	8	16
$\mathbf{Q_8 \times C_2}$					3		3
$\mathbf{C_4 \rtimes C_4}$					2		2
$\mathbf{D_4 \times C_2}$		2		4	3		9
$\mathbf{E_4 \rtimes C_4}$		2					2
$\mathbf{Q_8 \rtimes C_2}$			4	4	8		16
$\mathbf{C_8 \rtimes_3 C_2}$	2	2			4	4	12
$\mathbf{D_8}$					2	2	4
$\mathbf{Q_{16}}$	2	2			2	2	8
Total	4	8	8	8	36	32	96

Extrapolating the total time for this computation to the other 250 cases, we estimate that constructing all Galois 2-adic fields of degree 32 would take approximately 283 days of computing time, running on a single core. This can of course be sped up, if computations are run in parallel. The degree 64 case, on the other hand, is most likely not tractable without using many cores. In this case, we need to construct all quadratic extensions of the 915 Galois 2-adic fields of degree 32; in each case there are $2^{34} - 1 = 17179869183$ such extensions. We then need to compute automorphism groups; a task that takes about 1/2 second for each extension. Thus we would need approximately 250000 years of computing time on a single core.

TABLE 4.4. The number of Galois extensions of $\mathbf{Q}_2$ of degree 16 with ramification index 16.

G	c = 54	58	64	79	Total
$\mathbf{C_{16}}$				32	32
$\mathbf{C_8 \times C_2}$			16		16
$\mathbf{C_8 \rtimes_5 C_2}$			16		16
$\mathbf{C_4 \rtimes C_4}$	8				8
$\mathbf{E_4 \rtimes C_4}$	8				8
$\mathbf{C_8 \rtimes_3 C_2}$		16	8		24
$\mathbf{D_8}$		8	4		12
$\mathbf{Q_{16}}$		8	4		12
Total	16	32	48	32	128

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