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Hartley-Ross Type Unbiased Estimators Using Ranked Set Sampling and Stratified Ranked Set Sampling

Lakhkar Khan and Javid Shabbir

ABSTRACT. This paper proposes several Hartley-Ross (HR) type unbiased estimators of the finite population mean using information on known population parameters of the auxiliary variable in ranked set sampling (RSS) and stratified ranked set sampling (S_tRSS). The variances of the proposed HR unbiased ratio-type estimators are obtained upto first degree of approximation. Theoretically, it is shown that the proposed estimators are more efficient than the usual mean estimators in RSS and S_tRSS. In simulation study, the proposed estimators are more efficient as compared to all other competing estimators.

1. Introduction

McIntyre (1952) was the first who introduced the method of ranked set sampling (RSS) as a cost efficient alternative to simple random sampling (SRS) method for those situations where measurement of the units were expensive or difficult to obtain but ranking of units according to the variable of interest were relatively simple and cheap. Takahasi and Wakimoto (1968) gave the necessary mathematical theory of RSS and showed that the sample mean estimator under RSS is an unbiased and more efficient estimator than the sample mean estimator under SRS.

Several known population parameters, namely, coefficient of variation (C_x), coefficient of kurtosis (β_{2x}), Quartiles, coefficient of correlation (ρ) etc., play a significant role in the estimation of finite population mean. For detailed study of this, see the references Kadılar and Cingi (2005), Kadılar and Cingi (2006b), Kadılar and Cingi (2006a), Kadılar et al. (2007) and Upadhyaya and Singh (1999).

Hartley and Ross (1954) were the first to propose an unbiased ratio-type estimator for finite population mean in SRS. Later Paschal (1961) proposed an unbiased ratio-type estimator in stratified random sampling. Singh et al. (2014b) and Kadılar and Cekim (2015) suggested the Hartley-Ross type unbiased estimators of finite population mean using auxiliary information such as the population coefficient of variation (C_x), coefficient of kurtosis (β_{2x}) and the coefficient of correlation (ρ) in SRS. Solanki et al. (2015) proposed some ratio-type estimators of finite population variance using known values of quartiles related to an auxiliary variable in SRS. Khan and Shabbir (2015) have also suggested a class of Hartley-Ross type unbiased estimators in RSS.

Stratified ranked set sampling (S_tRSS) was suggested by Samawi and Muttlak (1996) to obtain more efficient estimates for population mean. Using S_tRSS, the performances of the combined and separate ratio estimators were obtained by Samawi and Siam (2003). Mandowara and Mehta

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(2014) used the idea of S_tRSS to obtain efficient ratio-type estimators. Singh et al. (2014a) proposed efficient ratio and product type estimators for population mean under S_tRSS. Recently, Khan et al. (2016) proposed unbiased ratio estimators of the mean in stratified ranked set sampling.

In this paper, we suggest a new class of Hartley-Ross type unbiased estimators of finite population mean under RSS and S_tRSS schemes.

2. Ranked set sampling and notations

In ranked set sampling, we first choose a small number m as a set size such that one can easily rank the m elements of the population with sufficient accuracy. Let Y and X be the study and the auxiliary variables respectively. Then randomly select m^2 bivariate sample units from the population and allocate them into m sets, each of size m. Each sample is ranked with respect to one of the variables Y or X. Here, we assume that the perfect ranking is done on basis of the auxiliary variable X while the ranking of Y is with possible error. An actual measurement from the first sample is then taken on the unit with the smallest rank of X, together with variable Y associated with smallest rank of X. From second sample of size m, the variable Y associated with the second smallest rank of X is measured. The process is continued until the Y value associated with the highest rank of X is measured from the mth sample. This completes one cycle of sampling. The process is repeated for r cycles to obtain the desired sample of size $n = mr$ units. Thus in RSS scheme, a total of m^2r units have been drawn from the population and only mr of them are selected for analysis. To estimate population mean $(\bar{Y})$ in RSS, when using a ratio estimator, the procedure can be summarized as follows:

- **Step 1:** Randomly select m^2 bivariate sample units from the population.
- **Step 2:** Allocate these m^2 units into m sets, each of size m.
- **Step 3:** Each set is ranked with respect to the concomitant variable.
- **Step 4:** Select the ith ranked unit from the ith $(i = 1, 2, ..., m)$ set for actual magnitude.
- **Step 5:** Repeat Steps 1 through 4 for r cycles until the desired sample size $n = mr$, is obtained.

For the j^{th} cycle, the RSS is denoted by $(Y_{[1:m]j}, X_{(1:m)j})$, $(Y_{[2:m]j}, X_{(2:m)j})$, . . . , $(Y_{[m:m]j}, X_{(m:m)j})$, $(j = 1, 2, \ldots, r)$. Here $Y_{[i:m]j}$ is the i^{th} ranked unit in the i^{th} set at the j^{th} cycle of the population. To find the variances of the estimators, we define the following error terms:
Let $\bar{y}_{[rss]} = \bar{Y}(1 + \delta_0)$, $\bar{x}^{(p)}_{(rss)} = \bar{X}^{(p)}(1 + \delta_1)$, $\bar{r}^{(p)}_{(rss)} = \bar{R}^{(p)}(1 + \delta_2)$,
such that $E(\delta_s) = 0, \quad (s = 0, 1, 2)$ and
$E(\delta_0^2) = \gamma C_y^2 - D_y^2, \quad E(\delta_1^2) = \gamma C^2_{x^{(p)}} - D^2_{x^{(p)}},$
$E(\delta_0\delta_1) = \gamma C_{yx^{(p)}} - D_{yx^{(p)}}, \quad E(\delta_1\delta_2) = \gamma C_{r^{(p)}x^{(p)}} - D_{r^{(p)}x^{(p)}},$
where
$D_y^2 = \frac{1}{m^2r\bar{Y}^2}\sum_{i=1}^m \tau^2_{y[i:m]}, \quad D_{yx^{(p)}} = \frac{1}{m^2r\bar{X}^{(p)}\bar{Y}}\sum_{i=1}^m \tau_{yx^{(p)}(i:m)},$
$D^2_{x^{(p)}} = \frac{1}{m^2r\bar{X}^{(p)2}}\sum_{i=1}^m \tau^2_{x^{(p)}(i:m)}, \quad D_{r^{(p)}x^{(p)}} = \frac{1}{m^2r\bar{X}^{(p)}\bar{R}^{(p)}}\sum_{i=1}^m \tau_{r^{(p)}x^{(p)}(i:m)},$
$\tau_{y[i:m]} = (\mu_{y[i:m]} - \bar{Y}), \quad \tau_{x^{(p)}(i:m)} = (\mu_{x^{(p)}(i:m)} - \bar{X}^{(p)}),$
$\tau_{yx^{(p)}(i:m)} = (\mu_{y[i:m]} - \bar{Y})(\mu_{x^{(p)}(i:m)} - \bar{X}^{(p)}),$
$\tau_{r^{(p)}x^{(p)}(i:m)} = (\mu_{r^{(p)}(i:m)} - \bar{R}^{(p)})(\mu_{x^{(p)}(i:m)} - \bar{X}^{(p)}).$

Here, $\gamma = (\frac{1}{mr})$ and $\quad C_{yx^{(p)}} = \rho C_y C_{x^{(p)}}$, where C_y and $C_{x^{(p)}}$ are the coefficients of variation of Y and $X^{(p)}$ respectively. Also $\bar{Y}$ and $\bar{X}^{(p)}$ are the population means of Y and $X^{(p)}$ respectively. The values of $\mu_{y[i:m]}$ and $\mu_{x^{(p)}(i:m)}$ depend on order statistics from some specific distributions (see Arnold et al. (1992)).

The usual RSS sample mean estimator $(\bar{y}_{RSS})$ and its variance, are given by

$$\bar{y}_{RSS} = \frac{1}{mr}\sum_{j=1}^{r}\sum_{i=1}^{m} y_{[i:m]j}, \tag{2.1}$$

$$V(\bar{y}_{RSS}) = \bar{Y}^2\left(\gamma C_y^2 - D_y^2\right). \tag{2.2}$$

3. Proposed class of ratio-type estimators in RSS

On the lines of Kadılar and Cekim (2015), we suggest the following ratio-type estimators:

$$\bar{y}_{KS(p)} = \left[\bar{r}^{(p)}_{[rss]} + \frac{b(\bar{X} - \bar{x}_{(rss)})}{\bar{x}^{(p)}_{(rss)}}\right]\bar{X}^{(p)}, \quad p = 1, 2, \ldots, 13 \tag{3.1}$$

where $\bar{r}^{(p)}_{[rss]} = \frac{\sum_{j=1}^{r}\sum_{i=1}^{m} r^{(p)}_{[i:m]j}}{mr}$, $\quad r^{(p)}_{[i:m]j} = \frac{y_{[i:m]j}}{x^{(p)}_{(i:m)j}}$, $\quad \bar{R}^{(p)} = E(\bar{r}^{(p)}_{[rss]})$,
$x^{(1)}_{(i:m)j} = x_{(i:m)j} + C_x$, $\quad \bar{X}^{(1)} = \bar{X} + C_x$, $\quad x^{(2)}_{(i:m)j} = x_{(i:m)j} + \beta_{2(x)}$,
$\bar{X}^{(2)} = \bar{X} + \beta_{2(x)}$, $\quad x^{(3)}_{(i:m)j} = x_{(i:m)j}\beta_{2(x)} + C_x$, $\quad \bar{X}^{(3)} = \bar{X}\beta_{2(x)} + C_x$,
$x^{(4)}_{(i:m)j} = x_{(i:m)j}C_x + \beta_{2(x)}$, $\quad \bar{X}^{(4)} = \bar{X}C_x + \beta_{2(x)}$, $\quad x^{(5)}_{(i:m)j} = x_{(i:m)j}C_x + \rho$,
$\bar{X}^{(5)} = \bar{X} + C_x + \rho$, $\quad x^{(6)}_{(i:m)j} = x_{(i:m)j}\rho + C_x$, $\quad \bar{X}^{(6)} = \bar{X}\rho + C_x$,
$x^{(7)}_{(i:m)j} = x_{(i:m)j}\beta_{2(x)} + \rho$, $\quad \bar{X}^{(7)} = \bar{X}\beta_{2(x)} + \rho$, $\quad x^{(8)}_{(i:m)j} = x_{(i:m)j}\rho + \beta_{2(x)}$,
$\bar{X}^{(8)} = \bar{X}\rho + \beta_{2(x)}$, $\quad x^{(9)}_{(i:m)j} = x_{(i:m)j} + Q_1$, $\quad \bar{X}^{(9)} = \bar{X} + Q_1$,
$x^{(10)}_{(i:m)j} = x_{(i:m)j} + Q_3$, $\quad \bar{X}^{(10)} = \bar{X} + Q_3$, $\quad x^{(11)}_{(i:m)j} = x_{(i:m)j} + Q_r$,
$\bar{X}^{(11)} = \bar{X} + Q_r$, $\quad x^{(12)}_{(i:m)j} = x_{(i:m)j} + Q_d$, $\quad \bar{X}^{(12)} = \bar{X} + Q_d$,
$x^{(13)}_{(i:m)j} = x_{(i:m)j} + Q_a$, $\quad \bar{X}^{(13)} = \bar{X} + Q_a$, for each estimator.

Here, Q_i is the i^{th} quartile $(i = 1, 3)$, $Q_r = (Q_3 - Q_1)$ (inter-quartile range), $Q_d = \frac{(Q_3 - Q_1)}{2}$ (semi-quartile range), $Q_a = \frac{(Q_3 + Q_1)}{2}$ (semi-quartile average) of the auxiliary variable. Also, $b = \hat{\rho}s_y/s_x$ is the sample regression coefficient whose population regression coefficient is $\beta = \rho S_y/S_x$.

The bias of $\bar{y}_{KS(p)}$, is given by

$$B(\bar{y}_{KS(p)}) = -\left[\frac{(N-1)}{N}S_{r^{(p)}}x^{(p)}\right], \; p = 1, 2, \ldots, 13$$

where $S_{r^{(p)}x^{(p)}} = \frac{1}{N-1}\sum_{j=1}^{N}(r^{(p)}_{[i:m]j} - \bar{R}^{(p)})(x^{(p)}_{(i:m)j} - \bar{X}^{(p)})$.

An unbiased estimator of $S_{r^{(p)}x^{(p)}}$, is given by

$$s_{r^{(p)}x^{(p)}} = \frac{n}{n-1}\left(\bar{y}_{[rss]} - \bar{r}^{(p)}_{(rss)}\bar{x}^{(p)}_{(rss)}\right), \quad p = 1, 2, \ldots, 13$$

So bias of $\bar{y}_{KS(p)}$ can be estimated by

$$\hat{B}(\bar{y}_{KS(p)}) = -\left[\frac{n(N-1)}{N(n-1)}\left(\bar{y}_{[rss]} - \bar{r}^{(p)}_{[rss]}\bar{x}^{(p)}_{(rss)}\right)\right], \quad p = 1, 2, \ldots, 13. \tag{3.2}$$

Thus, a class of unbiased estimators of the population mean based on RSS, is given by

$$\bar{y}^{(u)}_{KS(p)} = \left[\left(\bar{r}^{(k)}_{[rss]} + \frac{b(\bar{X} - \bar{x}_{(rss)})}{\bar{x}^{(k)}_{(rss)}}\right)\bar{X}^{(k)} + \frac{n(N-1)}{N(n-1)}\left(\bar{y}_{[rss]} - \bar{r}^{(p)}_{[rss]}\bar{x}^{(p)}_{(rss)}\right)\right], \tag{3.3}$$

for $p = 1, 2, \ldots, 13$. In terms of $\delta' s$, we have

$$\bar{y}^{(u)}_{KS(p)} = \bar{X}^{(p)}\bar{R}^{(p)}(1+\delta_2) - b\bar{X}\delta_1(1+\delta_1)^{-1} + \ldots$$
$$\ldots + \frac{n(N-1)}{N(n-1)}\left\{\bar{Y}(1+\delta_0) - \bar{X}^{(p)}\bar{R}^{(p)}(1+\delta_1)(1+\delta_2)\right\}.$$

Assuming $\frac{n(N-1)}{N(n-1)} \cong 1$ and considering first order approximation, we get

$$(\bar{y}^{(u)}_{KS(p)} - \bar{Y}) \cong \left[\bar{Y}\delta_0 - b\bar{X}\delta_1 - \bar{X}^{(p)}\bar{R}^{(p)}\delta_2\right].$$

Taking square and then expectation, the variance of $\bar{y}^{(u)}_{KS(p)}$, is given by

$$V(\bar{y}^{(u)}_{KS(p)}) \cong \bar{Y}^2(\gamma C_y^2 - D_y^2) + \beta^2\bar{X}^2(\gamma C_x^2 - D_x^2) + \bar{X}^{(p)2}\bar{R}^{(p)2}\left(\gamma C^2_{x^{(p)}} - D^2_{x^{(p)}}\right)\ldots$$
$$\ldots - 2\bar{X}\bar{Y}\left(\gamma C_{yx} - D_{yx}\right) - 2\bar{R}^{(p)}\bar{X}^{(p)}\bar{Y}\left(\gamma C_{yx^{(p)}} - D_{yx^{(p)}}\right)\ldots$$
$$\ldots + 2\beta\bar{R}^{(p)}\bar{X}^{(p)}\bar{X}\left(\gamma C_{xx^{(p)}} - D_{xx^{(p)}}\right), \quad p = 1, 2, \ldots, 13. \tag{3.4}$$

4. A simulation study in RSS

To obtain efficiencies of the proposed Hartley-Ross type unbiased estimators, a simulation study is conducted. Ranking is performed on basis of the auxiliary variable X. Bivariate random observations (X, Y), are generated from a bivariate normal distribution with known population correlation coefficient $\rho_{yx} = 0.75$. Using $20,000$ simulations, estimates of variances for unbiased ratio estimators are computed under ranked set sampling scheme as described in Section 2. Estimators are compared in terms of relative efficiencies (REs). The simulation results are presented in Table A.1 (see Appendix A). The results show that with increase in sample size, REs increase, which is expected. We use the following expression to obtain the REs:

$$RE_{(p)} = \frac{V(\bar{y}_{(RSS)})}{V(\bar{y}^{(u)}_{KS(p)})}, \quad p = 1, 2, \ldots, 13.$$

5. Conclusion in RSS

In Table A.1, we see that the proposed unbiased ratio type estimators $\bar{y}_{KS(p)}$, $(p = 1, 2, \ldots, 13)$, have high relative efficiency in comparison to $\bar{y}_{(RSS)}$. Also, relative efficiency increases with increase in sample size. Among all the estimators, $\bar{y}_{KS(5)}$ is the most efficient. So, we conclude that the proposed Hartley-Ross type unbiased estimators are preferable than the usual RSS estimator $(\bar{y}_{(RSS)})$ under RSS scheme.

6. Stratified ranked set sampling and notations

In stratified ranked set sampling, first choose m_h independent random samples from the hth stratum of the population, each of size m_h $(h = 1, 2, ..., L)$. Rank the observations in each sample and use RSS procedure to get L independent RSS samples, each of size m_h, such that $\sum_{h=1}^{L} m_h = m$. This completes one cycle of S_tRSS. The whole process is repeated r times to get the desired sample size $n_h = m_h r$. To estimate population mean $(\bar{Y})$ in S_tRSS using a ratio estimator, the procedure can be summarized as follows:

- **Step 1:** Select m_h^2 bivariate sample units randomly from the h^{th} stratum of the population.
- **Step 2:** Arrange these selected units randomly into m_h sets, each of size m_h.
- **Step 3:** The procedure of ranked set sampling (RSS) is then applied, on each of the sets to obtain the m_h sets of ranked set samples, each of size m_h. Here ranking is done with respect to the auxiliary variable X_h. These ranked set samples are collected together to form m_h sets, each of size m_h units.
- **Step 4:** Repeat the above steps r times for each stratum to get the desired sample of size $n_h = m_h r$.

For the j^{th} cycle and the h^{th} stratum, the S_tRSS is denoted by $(Y_{h[1:m_h]j}, X_{h(1:m_h)j})$, $(Y_{h[2:m_h]j}, X_{h(2:m_h)j}), \ldots, (Y_{h[mh:m_h]j}, X_{h(mh:m_h)j})$, $(j = 1, 2, \ldots, r)$ and $h = 1, 2, \ldots, L$. Here $Y_{h[i:m_h]j}$ is the i^{th} ranked unit in the i^{th} sample at the j^{th} cycle of the h^{th} stratum. To find the variances of the estimators, we define the following error terms:

Let $\bar{y}_{h[rss]} = \bar{Y}_h(1+\delta_{0h})$, $\bar{x}^{(k)}_{h(rss)} = \bar{X}^{(k)}_h(1+\delta_{1h})$, $\bar{r}^{(k)}_{h(rss)} = \bar{R}^{(k)}_h(1+\delta_{2h})$,
such that $E(\delta_{sh}) = 0$, $(s = 0, 1, 2)$, $(h = 1, 2, ..., L)$ and
$E(\delta_{0h}^2) = \gamma_h C_{y_h}^2 - D_{y_h}^2$, $E(\delta_{1h}^2) = \gamma_h C^2_{x_h^{(k)}} - D^2_{x_h^{(k)}}$,
$E(\delta_{0h}\delta_{1h}) = \gamma_h C_{y_h x_h^{(k)}} - D_{y_h x_h^{(k)}}$, $E(\delta_{1h}\delta_{2h}) = \gamma_h C_{r_h^{(k)} x_h^{(k)}} - D_{r_h^{(k)} x_h^{(k)}}$,
where
$D^2_{y_h} = \frac{1}{m_h^2 r \bar{Y}_h^2}\sum_{i=1}^{m_h}\tau^2_{y_h[i:m_h]}$, $D_{y_h x_h^{(k)}} = \frac{1}{m_h^2 r \bar{X}_h^{(k)} \bar{Y}_h}\sum_{i=1}^{m_h}\tau_{y_h x_h^{(k)}(i:m_h)}$,
$D^2_{x_h^{(k)}} = \frac{1}{m_h^2 r \bar{X}_h^{(k)2}}\sum_{i=1}^{m_h}\tau^2_{x_h^{(k)}(i)}$, $D_{r_h^{(k)} x_h^{(k)}} = \frac{1}{m_h^2 r \bar{X}_h^{(k)} \bar{R^*}_h}\sum_{i=1}^{m_h}\tau_{r_h^{(k)} x_h^{(k)}(i:m_h)}$,
$\tau_{y_h[i:m_h]} = (\mu_{y_h[i:m_h]} - \bar{Y}_h)$, $\tau_{x_h^{(k)}(i:m_h)} = (\mu_{x_h^{(k)}(i:m_h)} - \bar{X}_h^{(k)})$,
$\tau_{y_h x_h^{(k)}(i:m_h)} = (\mu_{y_h[i:m_h]} - \bar{Y}_h)(\mu_{x_h^{(k)}(i:m_h)} - \bar{X}_h^{(k)})$,
$\tau_{r_h^{(k)} x_h^{(k)}(i:m_h)} = (\mu_{r_h^{(k)}(i:m_h)} - \bar{R}_h^{(k)})(\mu_{x_h^{(k)}(i:m_h)} - \bar{X}_h^{(k)})$ and $k = 1, 2, \ldots, 13$.

Here $\gamma_h = (\frac{1}{m_h r})$ and $C_{y_h x_h^{(k)}} = \rho C_{y_h} C_{x_h^{(k)}}$, where C_{y_h} and $C_{x_h^{(k)}}$ are the coefficients of variation of Y_h and $X_h^{(k)}$ respectively. Also $\bar{Y}_h$ and $\bar{X}_h^{(k)}$ are the population means of Y_h and $X_h^{(k)}$ respectively. The values of $\mu_{y_h[i:m_h]}$ and $\mu_{x_h^{(k)}(i:m_h)}$ depend on order statistics from some specific distributions (see Arnold et al. (1992)).

Under S_tRSS scheme, the usual sample mean estimator $(\bar{y}_{S_tRSS})$, is given by

$$\bar{y}_{S_tRSS} = \sum_{h=1}^{L} W_h \bar{y}_{h[rss]}, \tag{6.1}$$

where $\bar{y}_{h[rss]} = (1/m_h r)\sum_{j=1}^{r}\sum_{i=1}^{m_h} y_{h[i:m_h]j}$ and $W_h = N_h/N$ is the known stratum weight. The variance of $\bar{y}_{S_tRSS}$, is given by

$$V(\bar{y}_{S_tRSS}) = \sum_{h=1}^{L} W_h^2 \bar{Y}_h^2 \left(\gamma_h C^2_{y_h} - D^2_{y_h}\right). \tag{6.2}$$

7. Proposed ratio estimators in S_tRSS

On the lines of Kadılar and Cekim (2015), a class of separate ratio-type estimators in S_tRSS is given by

$$\bar{y}_{LJ(k)} = \sum_{h=1}^{L} W_h \left[\bar{r}^{(k)}_{h[rss]} + \frac{b_h(\bar{X}_h - \bar{x}_{h(rss)})}{\bar{x}^{(k)}_{h(rss)}} \right] \bar{X}^{(k)}_h, k = 1, 2, \ldots, 13 \quad (7.1)$$

where $\bar{r}^{(k)}_{h[rss]} = \frac{\sum_{j=1}^{r_h}\sum_{i=1}^{m_h} r^{(k)}_{h[i:m_h]j}}{m_h r_h}$, $\quad r^{(k)}_{h[i:m_h]j} = \frac{y_{h[i:m_h]j}}{x^{(k)}_{h(i:m_h)j}}$, $\quad \bar{R}^{(k)}_h = E(\bar{r}^{(k)}_{h[rss]})$,
$x^{(1)}_{h(i:m_h)j} = x_{h(i:m_h)j} + C_{xh}$, $\bar{X}^{(1)}_h = \bar{X}_h + C_{xh}$, $x^{(2)}_{h(i:m_h)j} = x_{h(i:m_h)j} + \beta_{2(xh)}$,
$\bar{X}^{(2)}_h = \bar{X}_h + \beta_{2(xh)}$, $x^{(3)}_{h(i:m_h)j} = x_{h(i:m_h)j}\beta_{2(xh)} + C_{xh}$, $\bar{X}^{(3)}_h = \bar{X}_h\beta_{2(xh)} + C_{xh}$,
$x^{(4)}_{h(i:m_h)j} = x_{h(i:m_h)j}C_{xh} + \beta_{2(xh)}$, $\bar{X}^{(4)}_h = \bar{X}_h C_{xh} + \beta_{2(xh)}$, $x^{(5)}_{h(i:m_h)j} = x_{h(i:m_h)j}C_{xh} + \rho_h$,
$\bar{X}^{(5)}_h = \bar{X}_h + C_{xh} + \rho_h$, $x^{(6)}_{h(i:m_h)j} = x_{h(i:m_h)j}\rho_h + C_{xh}$, $\bar{X}^{(6)}_h = \bar{X}_h\rho_h + C_{xh}$,
$x^{(7)}_{h(i:m_h)j} = x_{h(i:m_h)j}\beta_{2(xh)} + \rho_h$, $\bar{X}^{(7)}_h = \bar{X}_h\beta_{2(xh)} + \rho_h$, $x^{(8)}_{h(i:m_h)j} = x_{h(i:m_h)j}\rho_h + \beta_{2(xh)}$,
$\bar{X}^{(8)}_h = \bar{X}_h\rho_h + \beta_{2(xh)}$, $x^{(9)}_{h(i:m_h)j} = x_{h(i:m_h)j} + Q_{1h}$, $\bar{X}^{(9)}_h = \bar{X}_h + Q_{1h}$,
$x^{(10)}_{h(i:m_h)j} = x_{h(i:m_h)j} + Q_{3h}$, $\bar{X}^{(10)}_h = \bar{X}_h + Q_{3h}$, $x^{(11)}_{h(i:m_h)j} = x_{h(i:m_h)j} + Q_{rh}$,
$\bar{X}^{(11)}_h = \bar{X}_h + Q_{rh}$, $x^{(12)}_{h(i:m_h)j} = x_{h(i:m_h)j} + Q_{dh}$, $\bar{X}^{(12)}_h = \bar{X}_h + Q_{dh}$,
$x^{(13)}_{h(i:m_h)j} = x_{h(i:m_h)j} + Q_{ah}$, $\bar{X}^{(13)}_h = \bar{X}_h + Q_{ah}$, for each estimator.

Here we have, for each h: ρ_h is the coefficient of correlation, C_{xh} is the coefficient of variation and $\beta_{2(xh)}$ is coefficient of kurtosis, Q_{ih} is the i^{th} quartile $(i = 1, 3)$, $Q_{rh} = (Q_{3h} - Q_{1h})$ (inter-quartile range), $Q_{dh} = \frac{(Q_{3h} - Q_{1h})}{2}$ (semi-quartile range), $Q_{ah} = \frac{(Q_{3h} + Q_{1h})}{2}$ (semi-quartile average) of the auxiliary variable in the hth stratum. Also, $b_h = \hat{\rho}_h s_{yh}/s_{xh}$ is the sample regression coefficient in the hth stratum where population regression coefficient is $\beta_h = \rho_h S_{yh}/S_{xh}$.

The bias of the proposed class of estimators $\bar{y}_{LJ(k)}$, is given by

$$B(\bar{y}_{LJ(k)}) = -\sum_{h=1}^{L} \left[W_h \frac{(N_h - 1)}{N_h} S_{r^{(k)}_h x^{(k)}_h} \right], \quad k = 1, 2, \ldots, 13$$

where $S_{r^{(k)}_h x^{(k)}_h} = \frac{1}{N_h - 1}\sum_{j=1}^{N_h} (r^{(k)}_{h[i:m_h]j} - \bar{R}^{(k)}_h)(x^{(k)}_{h(i:m_h)j} - \bar{X}^{(k)}_h)$,
So bias of $\bar{y}_{LJ(k)}$ can be estimated by

$$\hat{B}(\bar{y}_{LK(k)}) = -\sum_{h=1}^{L} W_h \left[\frac{n_h(N_h - 1)}{N_h(n_h - 1)} \left(\bar{y}_{h[rss]} - \bar{r}^{(k)}_{h(rss)}\bar{x}^{(k)}_{h(rss)} \right) \right], \quad k = 1, 2, \ldots, 13. \quad (7.2)$$

Thus, a class of unbiased ratio-type estimators of population mean based on S_tRSS is

$$\bar{y}^{(u)}_{LJ(k)} = \sum_{h=1}^{L} W_h \left[\left(\bar{r}^{(k)}_{h[rss]} + \frac{b_h(\bar{X}_h - \bar{x}_{h(rss)})}{\bar{x}^{(k)}_{h(rss)}} \right) \bar{X}^{(k)}_h \ldots \right.$$

$$\left. \ldots + \frac{n_h(N_h - 1)}{N_h(n_h - 1)} \left(\bar{y}_{h[rss]} - \bar{r}^{(k)}_{h[rss]}\bar{x}^{(k)}_{h(rss)} \right) \right], \quad k = 1, 2, \ldots, 13 \quad (7.3)$$

In terms of $\delta' s$, we get

$$\bar{y}^{(u)}_{LJ(k)} = \sum_{h=1}^{L} W_h \Bigg[\bar{X}_h^{(k)} \bar{R}_h^{(k)} (1+\delta_{2h}) - b_h \bar{X}_h \delta_{1h} (1+\delta_{1h})^{-1} \ldots$$
$$\ldots + \frac{n_h(N_h-1)}{N_h(n_h-1)} \left\{ \bar{Y}_h(1+\delta_{0h}) - \bar{X}_h^{(k)} \bar{R}_h^{(k)} (1+\delta_{1h})(1+\delta_{2h}) \right\} \Bigg].$$

Assuming $\frac{n_h(N_h-1)}{N_h(n_h-1)} \cong 1$ and considering first order approximation, we get

$$(\bar{y}^{(u)}_{LJ(k)} - \bar{Y}) \cong \sum_{h=1}^{L} W_h \left[(\bar{Y}_h \delta_{0h} - b_h \bar{X}_h \delta_{1h} - \bar{X}_h^{(k)} \bar{R}_h^{(k)} \delta_{2h}) \right].$$

Taking square and then expectation, the variance of $\bar{y}^{(u)}_{LJ(k)}$, is given by

$$V(\bar{y}^{(u)}_{LJ(k)}) \cong \sum_{h=1}^{L} W_h^2 \Bigg[\bar{Y}_h^2 (\gamma C^2_{y_h} - D^2_{y_h}) + \beta_h^2 \bar{X}_h^2 (\gamma_h C^2_{x_h} - D^2_{x_h}) \ldots$$
$$\ldots + \bar{X}_h^{(k)2} \bar{R}_h^{(k)2} \left(\gamma_h C^2_{x_h^{(k)}} - D^2_{x_h^{(k)}} \right) - 2\bar{X}_h \bar{Y}_h \left(\gamma_h C_{y_h x_h} - D_{y_h x_h} \right) \ldots$$
$$\ldots - 2\bar{R}_h^{(k)} \bar{X}_h^{(k)} \bar{Y}_h \left(\gamma_h C_{y_h x_h^{(k)}} - D_{y_h x_h^{(k)}} \right) \ldots$$
$$\ldots + 2\beta_h \bar{R}_h^{(k)} \bar{X}_h^{(k)} \bar{X}_h \left(\gamma_h C_{x_h x_h^{(k)}} - D_{x_h x_h^{(k)}} \right) \Bigg], \quad k = 1, 2, \ldots, 13. \tag{7.4}$$

8. Efficiency comparison

We obtain the conditions under which the proposed estimators $\bar{y}^{(u)}_{KS(p)}$ and $\bar{y}^{(u)}_{LJ(k)}$ are more efficient than the usual RSS estimator $\bar{y}_{(RSS)}$ and $\bar{y}_{(S_t RSS)}$ respectively.

(i) Comparison: By (2.2) and (3.4),

$$Var(\bar{y}^{(u)}_{KS(p)}) < Var(\bar{y}_{(RSS)}), \quad \text{if}$$

$$\gamma \left[\beta \bar{X} C_x + \bar{R}^{(p)} \bar{X}^{(p)} C_{x^{(p)}} \right]^2 - \left[\beta \bar{X} D_x + \bar{R}^{(p)} \bar{X}^{(p)} D_{x^{(p)}} \right]^2 <$$
$$2\bar{Y} \left[\bar{X} \left(\gamma C_{yx} - D_{yx} \right) + \bar{R}^{(p)} \bar{X}^{(p)} \left(\gamma C_{yx^{(p)}} - D_{yx^{(p)}} \right) \right]$$

$p = 1, 2, \ldots, 13.$

(ii) Comparison: By (6.2) and (7.4),

$$Var(\bar{y}^{(u)}_{LJ(k)}) < Var(\bar{y}_{(S_t RSS)}), \quad \text{if}$$

$$\sum_{h=1}^{L} W_h^2 \left[\gamma_h \left(\beta_h \bar{X}_h C_{x_h} + \bar{R}_h^{(k)} \bar{X}_h^{(k)} C_{x_h^{(k)}}\right)^2 - \left(\beta_h \bar{X}_h D_{x_h} + \bar{R}_h^{(k)} \bar{X}_h^{(k)} D_{x_h^{(k)}}\right)^2\right] <$$
$$2\sum_{h=1}^{L} W_h^2 \bar{Y}_h \left[\bar{X}_h \left(\gamma_h C_{y_h x_h} - D_{y_h x_h}\right) + \bar{R}_h^{(k)} \bar{X}_h^{(k)} \left(\gamma_h C_{y_h x_h^{(k)}} - D_{y_h x_h^{(k)}}\right)\right]$$
$k = 1, 2, \ldots, 13.$

9. A simulation study in S_tRSS

To compare the performances of the proposed Hartley-Ross type estimators, a simulation study is conducted. Ranking is performed on the basis of the auxiliary variable X. Bivariate random observations (X_h, Y_h), $h = 1, 2, \ldots, L$ are generated from a bivariate normal population whose population parameters are $(\mu_{xh}, \mu_{yh}, \sigma_{xh}, \sigma_{yh}, \rho_{yxh})$. Using $20,000$ simulations, estimates of variances for unbiased ratio estimators are computed under stratified ranked sampling scheme as described in Section 6. Estimators are compared in terms of relative efficiencies (REs). The simulation results are presented in Tables A.2 and A.3 (see Appendix A). We use the following expression to obtain the REs:

$$RE_{(k)} = \frac{V(\bar{y}_{(S_tRSS)})}{V(\bar{y}^{(u)}_{LJ(k)})}, \quad k = 1, 2, \ldots, 13.$$

Tables A.2 and A.3 show that with decrease in correlation coefficients ρ_{yxh}, REs decrease, which are expected results. The numerical values given in the first six rows are obtained by assuming equal correlations across the strata, whereas the last four rows assume unequal correlations across the strata. Apart from correlation, it can be interesting to know how these estimators behave with changes in variability across strata. In Table A.2, the standard deviations of the bivariate normal distribution σ_{hy} and σ_{hx} remain equal across strata, while in Table A.3, they are considered to be unequal.

10. Conclusion in S_tRSS

It can be observed from the simulation results given in Tables A.2 and A.3, that the proposed unbiased ratio estimators $\bar{y}_{LJ(k)}, (k = 1, 2, \ldots, 13)$, have high relative efficiency in comparison to $\bar{y}_{(S_tRSS)}$. It is also observed that relative efficiency decreases with decrease in correlation for most of the estimators. Also, relative efficiency is high when there is equal standard deviation across strata. Among all the estimators, generally $\bar{y}_{LJ(2)}$ is more efficient as compared to the other estimators. Thus, we conclude that the proposed unbiased ratio-type estimators are preferable over $\bar{y}_{(S_tRSS)}$ under S_tRSS.

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Appendix A.

TABLE A.1. Relative efficiencies $(RE_{(p)})$ of the proposed ratio estimators with respect to usual sample mean estimator $(\bar{y}_{RSS})$ for a simulated bivariate normal distribution.

m	r	n	$RE_{(1)}$	$RE_{(2)}$	$RE_{(3)}$	$RE_{(4)}$	$RE_{(5)}$	$RE_{(6)}$	$RE_{(7)}$	$RE_{(8)}$	$RE_{(9)}$	$RE_{(10)}$	$RE_{(11)}$	$RE_{(12)}$	$RE_{(13)}$
	3	9	1.5028	1.6391	1.4161	1.5074	1.6686	1.5366	1.4503	1.6052	1.6098	1.6626	1.6209	1.6007	1.6458
3	4	12	1.5111	1.6444	1.4325	1.5149	1.6711	1.5412	1.4637	1.6199	1.6844	1.6463	1.6757	1.6037	1.6703
	5	15	1.5249	1.6695	1.4405	1.5219	1.6870	1.5573	1.4739	1.6327	1.6887	1.6445	1.6847	1.6208	1.6709
	5	20	1.4762	1.5529	1.4155	1.4319	1.5762	1.4973	1.4399	1.5210	1.5723	1.5314	1.5755	1.5397	1.5542
4	10	40	1.4765	1.5579	1.4172	1.4363	1.5800	1.4987	1.4411	1.5266	1.5768	1.5365	1.5792	1.5413	1.5592
	15	60	1.5106	1.5709	1.4520	1.4405	1.6015	1.5320	1.4757	1.5360	1.5942	1.5468	1.6014	1.5712	1.5724
	10	50	1.4087	1.4796	1.3557	1.3794	1.4973	1.4284	1.3770	1.4540	1.4949	1.4621	1.4966	1.4654	1.4806
5	15	75	1.4115	1.4861	1.3608	1.3861	1.5016	1.4302	1.3812	1.4609	1.5004	1.4689	1.5006	1.4659	1.4870
	20	100	1.4226	1.4902	1.3712	1.3870	1.5091	1.4416	1.3920	1.4638	1.5062	1.4721	1.5084	1.4772	1.4913

TABLE A.2. Relative efficiencies ($RE_{(k)}$) for separate ratio estimators with respect to usual sample mean estimator $\bar{y}_{(S_tRSS)}$ for a simulated bivariate normal distribution assuming equal standard deviation for each stratum. $L = 3$, $P_h = (.30, .30, .40)$, $m_h = (3, 4, 5)$, $r = 5$ $n_h = (15, 20, 25)$, $\mu_{xh} = (2, 3, 4)$, $\mu_{yh} = (3, 4, 6)$, $\sigma_{yh} = (1, 1, 1)$, $\sigma_{xh} = (1, 1, 1)$.

ρ_{yxh}	$RE_{(1)}$	$RE_{(2)}$	$RE_{(3)}$	$RE_{(4)}$	$RE_{(5)}$	$RE_{(6)}$	$RE_{(7)}$	$RE_{(8)}$	$RE_{(9)}$	$RE_{(10)}$	$RE_{(11)}$	$RE_{(12)}$	$RE_{(13)}$
0.90, 0.90, 0.90	1.7067	2.3993	1.5798	2.4075	2.4982	1.7223	1.6793	2.5675	2.2454	2.6117	2.0282	1.8077	2.4388
0.80, 0.80, 0.80	1.7697	2.3695	1.7242	1.7032	2.2543	1.7964	1.7377	2.4111	2.1439	2.3685	2.0023	1.8443	2.2656
0.70, 0.70, 0.70	1.8228	2.3500	1.8207	1.4511	2.1305	1.8593	1.7856	2.3221	2.1012	2.2497	1.9975	1.8793	2.1833
0.60, 0.60, 0.60	1.7903	2.2571	1.8206	1.2716	1.9819	1.8362	1.7442	2.1617	1.9899	2.0875	1.9205	1.8334	2.0450
0.50, 0.50, 0.50	1.7269	2.1679	1.7872	1.1493	1.8282	1.7820	1.6616	1.9902	1.8675	1.9301	1.8235	1.7594	1.9038
0.30, 0.30, 0.30	1.5564	1.9216	1.6150	1.0403	1.5335	1.6309	1.4278	1.6954	1.6405	1.6741	1.6274	1.5830	1.6612
0.90, 0.80, 0.70	1.8152	2.5827	1.8198	1.4392	2.2564	1.8364	1.7823	2.4636	2.1813	2.4171	2.0577	1.8912	2.3095
0.80, 0.70, 0.60	1.8431	2.5023	1.9059	1.2713	2.1191	1.8729	1.8096	2.3257	2.1044	2.2558	2.0154	1.8964	2.1881
0.70, 0.60, 0.50	1.8433	2.4382	1.9532	1.1776	2.0106	1.8798	1.8079	2.1981	2.0289	2.1307	1.9709	1.8831	2.0861
0.60, 0.50, 0.30	1.80574	2.3583	1.9824	1.0467	1.8951	1.8376	1.7673	1.9899	1.9132	1.9706	1.8867	1.8307	1.9464

TABLE A.3. Relative efficiencies $(RE_{(k)})$ for separate ratio estimators with respect to usual sample mean estimator $\bar{y}_{(S_t RSS)}$ for a simulated bivariate normal distribution assuming unequal standard deviation for each stratum. $L = 3$, $P_h = (.30, .30, .40)$, $m_h = (3, 4, 5)$, $r = 5$ $n_h = (15, 20, 25)$, $\mu_{xh} = (2, 3, 4)$, $\mu_{yh} = (3, 4, 6)$, $\sigma_{yh} = (1, 1.5, 2)$, $\sigma_{xh} = (1, 1, 1.5)$.

ρ_{yxh}	$RE_{(1)}$	$RE_{(2)}$	$RE_{(3)}$	$RE_{(4)}$	$RE_{(5)}$	$RE_{(6)}$	$RE_{(7)}$	$RE_{(8)}$	$RE_{(9)}$	$RE_{(10)}$	$RE_{(11)}$	$RE_{(12)}$	$RE_{(13)}$
0.90, 0.90, 0.90	1.4541	2.1047	1.5326	1.0929	1.7596	1.4758	1.4310	1.8722	1.6914	1.8203	1.5992	1.5010	1.7616
0.80, 0.80, 0.80	1.4679	2.0442	1.5487	1.0624	1.7097	1.4967	1.4432	1.8385	1.6708	1.7705	1.5869	1.5061	1.7255
0.70, 0.70, 0.70	1.4904	2.0055	1.5622	1.0435	1.6899	1.5279	1.4621	1.8266	1.6669	1.7521	1.6021	1.5288	1.7141
0.60, 0.60, 0.60	1.4648	1.9155	1.5292	1.0281	1.6235	1.5108	1.4307	1.7564	1.6164	1.6825	1.5606	1.4988	1.6534
0.50, 0.50, 0.50	1.4216	1.8244	1.4742	1.0175	1.5389	1.4791	1.3746	1.6789	1.5552	1.6087	1.5083	1.4535	1.5857
0.30, 0.30, 0.30	1.2912	1.6332	1.3368	1.0121	1.3112	1.3742	1.1998	1.4858	1.3941	1.4311	1.3644	1.3197	1.4158
0.90, 0.80, 0.70	1.4806	2.1108	1.5681	1.0473	1.7245	1.5070	1.4561	1.8467	1.6844	1.7883	1.6041	1.5198	1.7415
0.80, 0.70, 0.60	1.4749	2.0279	1.5628	1.0306	1.6758	1.5086	1.4488	1.7982	1.6468	1.7333	1.5855	1.5122	1.6946
0.70, 0.60, 0.50	1.4594	1.9420	1.5413	1.0297	1.6206	1.502	1.4304	1.7498	1.6111	1.6812	1.5561	1.4927	1.6502
0.60, 0.50, 0.30	1.4165	1.8513	1.51970	1.0191	1.5411	1.4587	1.3835	1.6328	1.5357	1.5864	1.4942	1.4441	1.5644

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(L. Khan) Department of Statistics, Government Degree College Toru, Khyber Pakhtunkhwa, Pakistan

Current address: Department of Statistics, Quaid-i-Azam University, Islamabad, Pakistan

E-mail address, Corresponding author: lakhkarkhan.stat@gmail.com

(J. Shabbir) Department of Statistics, Quaid-i-Azam University, Islamabad, Pakistan

E-mail address: javidshabbir@gmail.com