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On game chromatic number analogues of Mycielsians and Brooks' Theorem

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ABSTRACT. The vertex coloring game is a two-player game on a graph with given color set in which the first player attempts to properly color the graph and the second attempts to prevent a proper coloring from being achieved. The smallest number of colors for which the first player can win no matter how the second player plays is called the game chromatic number of the graph. In this paper we initiate the study of game chromatic number for Mycielskians and a game chromatic number analogue of Brooks' Theorem (which characterizes graphs for which chromatic number is at most the maximum degree of the graph). In particular, we determine the game chromatic number of Mycielskians of complete graphs, complete bipartite graphs, and cycles. In the direction of Brooks' Theorem, we show that if there are few vertices of maximum degree or if all vertices of maximum degree are at least three edges apart, then the game chromatic number is at most the maximum degree of the graph.

1. Introduction

A *simple graph* is an ordered pair (V, E) where V is a finite nonempty set (the "vertices") and E consists of two–element subsets of V (the "edges"). We say vertices $v, w \in V$ are *adjacent* if $\{v, w\} \in E(G)$. The *(open) neighborhood* of $v \in V$, denoted $N(v)$, is the set of vertices adjacent to V and for $U \subseteq V$, define $N(U) = \cup_{u \in U} N(u)$. The *degree* of $v \in V$, denoted $d(v)$, is $|N(v)|$ and $\Delta(G)$ is the maximum degree among all vertices in V. A *proper coloring* is an assignment of color to each vertex so that no edge joins two vertices of the same color. The *chromatic number* of G, denoted $\chi(G)$, is the minimum number of colors needed to properly color G.

This paper concerns the *vertex coloring game* which is defined as follows. The vertex coloring game is a two-player game between Alice and Bob on a graph G with color set C. The players alternate turns (with Alice playing first) selecting a vertex from G and coloring it with a color from C; when coloring a vertex $v \in V$, a player must select a color v is not currently adjacent to. Alice wins the game if every vertex of G is colored and Bob wins the game otherwise. The *game chromatic number* of G, denoted $\chi_g(G)$, is the smallest $|C|$ so that Alice can win the vertex coloring game on G with color set C no matter how Bob plays.

Game chromatic number is a generalization of chromatic number in that if Bob plays cooperatively with Alice, then exactly $\chi(G)$ colors will be used. Of course, since in the vertex coloring game Bob plays as an adversary of Alice, generally more colors are needed compared to chromatic number. Game chromatic number was introduced in Gardner (1981) and has been well-studied since. Particular attention has been paid to determining bounds for the game chromatic number of planar graphs (see Kierstead and Trotter (1994); Dinski and Zhu (1999); Zhu (1999)) and trees (see

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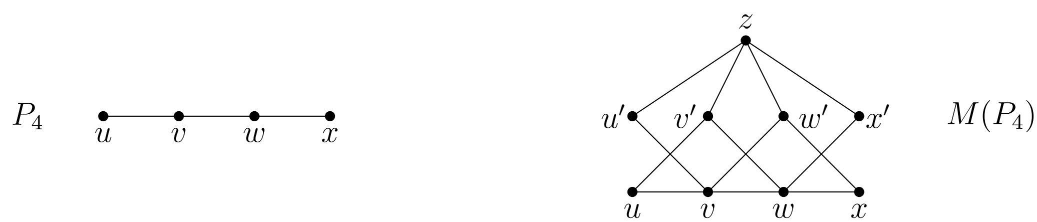
Faigle et al. (1993); Dunn et al. (2015)). More recently the game chromatic number of Cartesian products of graphs and outerplanar graphs has been studied; see Bartnicki et al. (2008); Raspaud and Wu (2009) and Guan and Zhu (1999), respectively. For a brief survey of this topic we refer the reader to Bartnicki et al. (2007).

In this paper we consider game analogues of two classical topics in graph coloring: Mycielskians and Brooks' Theorem. Given a graph $G = (\{v_1, \ldots, v_n\}, E)$, the *Mycielskian* of G, denoted $M(G)$, is the graph with vertex set $\{v_1, \ldots, v_n, v_1', \ldots, v_n', z\}$ with $E(M(G)) = E(G) \cup \{\{v_i, v_j'\} : \{v_i, v_j\} \in E(G)\} \cup \{\{z, v_i'\} : i = 1, \ldots n\}$. The figure below shows a path on four vertices and the Mycielskian of a path on four vertices.



A key, well-known property of Mycielskians is that for any graph G, $\chi(M(G)) = \chi(G) + 1$. In Section 3 we determine the game chromatic number of Mycielskians of complete graphs, complete bipartite graphs, and cycles (c.f. Propositions 2, 5, and 6 in Destacamento et al. (2014)).

The other inspiration for this paper is Brooks' Theorem which we now state.

Theorem 1.1 (Brooks' Theorem)**.** *If G is a connected graph which is not an odd cycle or complete graph, then $\chi(G) \leq \Delta(G)$.*

The natural game analogue for this theorem would be to identify $\{G : \chi_g(G) \leq \Delta(G)\}$. Characterizing this set appears to be quite a lofty goal. In Section 4 we prove two sufficient conditions for membership in this set. The conditions are: the number of vertices of maximum degree is bounded by $\lceil \Delta(G)/2 \rceil$; the distance between all vertices of maximum degree is at least three.

The remainder of the paper is organized as follows. In Section 2 we introduce some more terminology and a few lemmas. In Section 5 we conclude by mentioning some open problems related game analogues of Mycielskians and Brooks' Theorem.

2. Preliminaries

In this section we will gather a few lemmas and some terminology before proceeding to the proofs. The complete graph, denoted K_n, is the graph on n vertices with all possible edges. A complete bipartite graph, denoted $K_{m,n}$, is the graph whose vertex set is partitioned into $A \cup B$ with $|A| = m$, $|B| = n$, and whose edge set is all pairs with one vertex in A and the other in B. For $n \geq 3$, we refer to $K_{1,n}$ as a star. A cycle on n vertices, denoted C_n, is the graph with vertex set $\{v_1, v_2, \ldots, v_n\}$ and edge set $\{v_i, v_{i+1}\}$ with indices reduced modulo n.

We next state a few lemmas which will be used throughout the paper. Each is straightforward to verify based on the definition of game chromatic number and so their proofs are omitted.

Lemma 2.1. *For any graph G, $\chi(G) \leq \chi_g(G)$.*

Lemma 2.2. *For any graph G, $\chi_g(G) \leq \Delta(G) + 1$.*

Lemma 2.3. *If G satisfies $\chi_g(G) = 2$, then G is a disjoint union of one or more stars, any number of isolated edges, and any number of isolated vertices.*

In order to have concise proofs, we will also make use of the following terminology. Let G be a graph on V and suppose Alice and Bob play the vertex coloring game with k colors. We say Bob's move creates a *double-threat* at v if Bob colored v, there are distinct $x, y \in N(v)$ satisfying: both x and y are uncolored, both $N(x) \setminus N(y)$ and $N(y) \setminus N(x)$ contain an uncolored vertex, and both x and y are adjacent to all but one color which can be used on one of their uncolored neighbors. Notice in particular, that if a double-threat is created, then no matter which vertex Alice colors Bob can win the vertex coloring game with his next move. We say Bob's move at v is *forcing* if v is adjacent to an uncolored vertex w, $N(w)$ contains all but one color, and $N(w)$ contains an uncolored vertex which may be assigned the last color. Observe that if Bob makes a forcing move at v, then, using the notation in the previous sentence, Alice must color a vertex in $\{w\} \cup N(w)$ or else Bob can win with his next move.

3. Mycielskian Results

In this section we will state and prove our results about the game chromatic number of Mycielskians of complete graphs, complete bipartite graphs, and cycles.

Theorem 3.1. *For $n \geq 3$, we have $\chi_g(M(K_n)) = n + 1$.*

Proof. Suppose Alice and Bob play the vertex coloring game on $M(K_n)$ for some $n \geq 3$ with $n+1$ colors. Let $\{v_1, v_2, \ldots, v_n\}$ be the vertices of K_n and $\{v'_1, v'_2, \ldots, v'_n\}$ be the twin vertices. Alice's strategy is as follows: Alice will color z with 1 as her first move and will proceed to color the twin vertex for each of Bob's moves choosing her color to match the color of Bob's previous move if possible and if not, to pick the smallest unused color available. The high degree of symmetry of $M(K_n)$ along with Alice's strategy ensures that on each of Bob's turns, every uncolored v_i "looks like" every other uncolored v_j and every uncolored v'_i looks like every other uncolored v'_j. This means we may assume that the vertices of $M(K_n)$ are colored in the order: z, $\{v_1, v'_1\}$, $\{v_2, v'_2\}, \ldots, \{v_n, v'_n\}$.
Claim: At most $t + 1$ colors are used on $z, v_1, v'_1, \ldots, v_t, v'_t$.

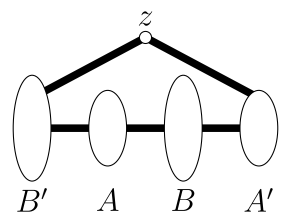
To verify the claim, we will proceed inductively on t. First observe that after z, v_1, v'_1 are colored, for $i \geq 2$, v_i and v'_i are adjacent to exactly two of z, v_1, v'_1 meaning that each is adjacent to at most two colors. Now suppose the claim holds for $t - 1$ and that v_t and v'_t have been colored. Because Alice colors greedily, the pair $\{v_t, v'_t\}$ receives at most one previously unused color which verifies the claim.

In particular the claim gives that after the pair $\{v_{n-1}, v'_{n-1}\}$ is colored, at most n colors have been used and so each of v_n and v'_n is adjacent to at most n colors. Since $n+1$ colors are available, both may be colored and so this is a winning strategy for Alice; that is, $\chi_g(M(K_n)) \leq n + 1$.

For the reverse inequality, observe that $\chi(M(K_n)) = n + 1$ and apply Lemma 2.1. □

Theorem 3.2. *If $m \leq n$, then* $\begin{cases} \chi_g(M(K_{m,n})) = 3 & \text{if } m = 1 \text{ or } m + n \text{ is even,} \\ \chi_g(M(K_{m,n})) = 4 & \text{if } m \neq 1 \text{ and } m + n \text{ is odd.} \end{cases}$

Proof. Let $K_{m,n}$ have parts A and B with $|A| = m$, $|B| = n$, and $m \leq n$. First note that $M(K_{m,n})$ is not a union of stars and so by Lemma 2.3 we have $\chi_g(M(K_{m,n})) \geq 3$. If $m = 1$, then Alice has a winning strategy with three colors by coloring z, the twin of the vertex from A, and the vertex from A within her first three turns. No matter what Bob's moves are, each will be adjacent to at most two colors just before each is colored. Moreover, once these three vertices are colored, every vertex in $B \cup B'$ is adjacent to at most two colors and so can be colored the third color for the remainder of the game. Altogether this gives that $\chi_g(M(K_{1,n})) \leq 3$. Now suppose that $m \geq 2$, $m+n$ is even, and suppose Alice and Bob play the vertex coloring game on $M(K_{m,n})$ with 3 colors. For reference, $M(K_{m,n})$ looks as follows and consists of the "pods" $\{z\}$, A, B, A', and B'.



Alice will first color z with 1. There are a few cases to consider.

- If Bob colors a vertex from A with 2, then Alice will color a vertex from B with 3. By doing so, vertices in A' will never be adjacent to 2 and vertices in B' will never be adjacent to 3. Moreover, every vertex in A, resp. B, may be colored 2, resp. 3, for the duration of the game. This, therefore, is a winning position for Alice. Alice will make a similar play if Bob colors a vertex in B with 2.
- If Bob colors a vertex in A' with 2, then Alice will color a vertex in B with 1. Now if Bob colors in $A \cup B'$, Alice will give a color to the remaining pod which is a winning position for her. If Bob colors a vertex in B with 3, Alice will color a vertex in A with 2. Similar to the above, vertices in A' cannot be adjacent to 3 for the duration of the game and so this is a winning position for Alice. If neither of these happen, then Alice and Bob will alternate coloring vertices in $A' \cup B$. Since $m+n$ is even and Bob played in $A' \cup B$ first, Alice will be the last to play in $A' \cup B$. So Bob will color a vertex in $A \cup B'$ which is covered by a previous analysis.
- If Bob colors a vertex from A, resp. B, with 1, then Alice will color a vertex in B', resp. A', with 2. From here, this case will proceed in an identical fashion to the second bullet.

In each of these cases, Alice obtains a winning position and therefore $\chi_g(M(K_{m,n})) \leq 3$, which completes this case.

Now suppose $m+n$ is odd and suppose Alice and Bob play the vertex coloring game on $M(K_{m,n})$ with 3 colors. If Alice colors a vertex in A first, Bob will color another vertex from A with a different color. Now vertices in B and B' are adjacent to two colors. Depending on what Alice does, Bob will either color z or a vertex in A' with the third color which is a winning move. A similar analysis applies if Alice first colors a vertex other than z. Suppose, then, that Alice colors z with 1. Bob will color a vertex from A with 1. Alice cannot color vertices from $B \cup A'$ or else Bob will color another vertex from the same pod with the third color. Additionally, Alice cannot color vertices of A with 2 or 3 or else Bob will color a vertex in A' with the third color. So as long as Alice colors vertices from A with 1 or vertices from B', Bob will mimic Alice's moves. Since $m+n$ is odd and Bob played in $A \cup B'$ first, Alice will make the first play in $B \cup A'$. However, as discussed this case leads to a win for Bob. Therefore, if $m+n$ is odd, then $\chi_g(M(K_{m,n})) \geq 4$.

For the reverse inequality, Alice will color z with 1. Suppose that Bob plays in A or B'. Then Alice will play in the other. If Bob plays in $A' \cup B$, then Alice will play in the other pod which is a winning position for her. If not, then Alice will play in B; note that each vertex in B is adjacent to at most two colors on this play. Now no matter where Bob plays, vertices in A' are adjacent to at most three colors and so Alice can color an A' vertex which is a winning position. Therefore, in this case $\chi_g(M(K_{m,n})) \leq 4$ and thus overall $\chi_g(M(K_{m,n})) = 4$. □

Theorem 3.3. *For $n \geq 7$, we have $\chi_g(M(C_n)) = 4$.*

Proof. Let $M(C_n)$ be the Mycelskian of C_n with $n \geq 7$. If n is odd, then we have $\chi(M(C_n)) = 4$ and so by Lemma 2.1, we have $\chi_g(M(C_n)) \geq 4$. So we want to show that $\chi_g(M(C_n)) \geq 4$ for even n. Let $\{v_1, v_2, ..., v_n\}$ be the vertices of C_n and let $\{v'_1, v'_2, ..., v'_n\}$ be the twin vertices in $M(C_n)$. To show $\chi_g(M(C_n)) \geq 4$, we will show that Bob has a winning strategy with three colors available. To do so, consider the following cases. If Alice colors v_i, then Bob creates a double-threat by coloring v'_i with a different color. If Alice colors v'_i, then Bob creates a double-threat by coloring v_i with a different color. If Alice colors z, then Bob creates a double-threat by coloring v_1 with a different color. Note that here we are using that $n \geq 7$ to ensure that there is no common neighbor of the threatened vertices. So no matter which opening move Alice makes, Bob will win with three colors which completes the claim that $\chi_g(M(C_n)) \geq 4$ for $n \geq 7$.

We will now show that Alice has a winning strategy with four colors (i.e. $\chi_g(M(C_n)) \leq 4$ for $n \geq 7$). Alice will begin by coloring z with 1 and will proceed to color the twin vertex for each of Bob's moves (i.e. if Bob colors v_i, then Alice will color v'_i and vice versa). Alice will choose her color for each move according to the following and for ease of reading, we will suppose Bob colored a vertex from $\{v_i, v'_i\}$. If Bob's color can be copied (that is, the color used is not present in $N(v_i) \cup N(v'_i)$), Alice will do so. If Bob's move cannot be copied, then he either colored v_i with 1 or colored v'_i with a color found on v'_{i-1} or v'_{i+1}. We further divide this case depending on whether zero or one of $\{v_{i-1}, v'_{i-1}\}$, $\{v_{i+1}, v'_{i+1}\}$ has been colored. (We further remark that if both $\{v_{i-1}, v'_{i-1}\}$ and $\{v_{i+1}, v'_{i+1}\}$ have been colored, then Alice can make a legal move so long as she adheres to the coloring strategy being presented.) So in the zero case (i.e. if Bob colored v_i with 1 and no vertex from $N(v_i)$ has been colored), Alice will check if either pair $\{v_{i-2}, v'_{i-2}\}$ or $\{v_{i+2}, v'_{i+2}\}$ has been colored. If neither has, Alice will color v'_i with 2. If exactly one has been colored, then Alice will color v'_i so that that pair along with $\{v_i, v'_i\}$ contains at most three colors. This is possible since v'_i is only adjacent to color 1 and so can mimic at least one the color found on that pair. If all of $v_{i-2}, v'_{i-2}, v_{i+2}, v'_{i+2}$ are colored, then either $\{v_{i-2}, v'_{i-2}\}$ and $\{v_{i+2}, v'_{i+2}\}$ have a common color other than 1 or at least one of these pairs contains the color 1. If v_{i-2} is colored 1, then Alice will mimic a color v'_i with a color found on $\{v_{i+2}, v'_{i+2}\}$. In the case of a common color other than 1, Alice will color v'_i with a common color. In each of these cases, both $N(v_{i-1})$ and $N(v_{i+1})$ contain at most three colors.

Now suppose that Bob colors a vertex from $\{v_i, v'_i\}$; v_{i-1} and v'_{i-1} have been colored; and v_{i+1} and v'_{i+1} are uncolored. If v_{i+2} and v'_{i+2} are uncolored, Alice will color the other vertex from $\{v_i, v'_i\}$ using any legal color. If v_{i+2} and v'_{i+2} are colored, then the state of the game prior to Alice's move is one of the following:

In either case, if $\gamma = \delta$, then Alice will use any legal color on v'_i, resp. v_i, and if $\gamma \neq \delta$, then Alice will use whichever of those two are not the same as β on v'_i, resp. v_i. Observe that by coloring in this way $N(v_{i+1})$ contains at most three colors. In particular this means that for the duration of the game no v_i will be adjacent to four colors and so Alice will win the vertex coloring game with four colors; that is, we have $\chi_g(M(C_n)) \leq 4$. □

Note that although they are not covered by the statement of the previous theorem, $\chi_g(M(C_n))$ for $n = 3, 4, 5$ are given by Theorems 3.1, 3.2, and Lemma 2.1 along with the upper bound given in of Theorem 3.3, respectively. To complete the picture, we also have $\chi_g(M(C_6)) = 4$ with the lower bound found in a similar fashion to, although more ad hoc than, the proof of Theorem 3.3 and upper bound following the proof directly.

We conclude this section with the following theorem which demonstrates that the game chromatic number of the Mycielskian of a graph does not necessarily have larger game chromatic number as compared to the graph which contrasts with the fact that for any graph G we have $\chi(M(G)) = \chi(G) + 1$.

Theorem 3.4. *Let P_4 be the path on four vertices. Then we have $\chi_g(M(P_4)) = \chi_g(P_4) = 3$.*

Proof. Let $M(P_4)$ be a graph with input vertices labeled v_1, v_2, v_3, v_4 from left to right and twin vertices labeled v'_1, v'_2, v'_3, v'_4 from left to right. With 3 colors, Alice's first move is to color z with 1. Thereafter, if Bob colors v_i, Alice will use the same color on v_{i+2} and if Bob colors v'_i, Alice will use the same color on v'_{i+2} in both cases with the index reduced modulo four. A straightforward exhaustive search shows that this is a winning strategy for Alice and so $\chi_g(M(P_4)) \le 3$. Observe that $\chi(M(P_4)) = 3$ and so we may apply Lemma 2.1 for the reverse inequality. Proposition 1 of Destacamento et al. (2014) gives that $\chi_g(P_4) = 3$ which completes the proof. □

4. Toward Brooks' Theorem for Game Chromatic Number

In this section we will state and prove our results concerning a game analogue of Brooks' Theorem. Throughout this section, for a graph $G = (V, E)$, we will let $V_t := \{v \in V \mid d(v) \ge t\}$ and will let $\Delta := \Delta(G)$. Our first result is similar in spirit to the proof that $\chi_g(M(K_{1,n})) = 3$. For that proof, we gave a strategy for Alice which was too "fast" for Bob in that the vertices at which Bob could have won were colored before Bob had a chance to surround any of them. In the context of a game analogue of Brooks' Theorem, this theorem is that Alice can win with Δ colors provided that there are few enough vertices of degree Δ.

Theorem 4.1. *For a graph G, if $|V_\Delta| \le \lceil \Delta/2 \rceil$, then $\chi_g(G) \le \Delta$.*

Proof. Let G be a graph with $|V_\Delta| \le \lceil \Delta/2 \rceil$ and suppose Alice and Bob play the vertex coloring game on G with Δ colors. Notice that with this many colors, Bob can only win at vertices in V_Δ. Alice's strategy is to colors vertices in V_Δ before Bob can threaten any of them and so her strategy is to color one on every turn. To see why this is a winning strategy for Alice, observe that after Bob's j^{th} turn any vertex in G can be adjacent to at most $2j$ colors. Furthermore, after Bob's j^{th} turn, at least j vertices in V_Δ are colored due to Alice's strategy. Therefore, on Alice's $|V_\Delta|^{\text{th}}$ move, the last vertex in V_Δ is adjacent to at most $2 \cdot (|V_\Delta| - 1)$ colors. By assumption, $|V_\Delta| \le \lceil \Delta/2 \rceil$ and so $2 \cdot (|V_\Delta| - 1) < \Delta$. Thus using this strategy Alice wins the game and so $\chi_g(G) \le \Delta$. □

Our next result is essentially that if Bob is too "slow" in pressuring vertices in V_Δ, then Alice will win with Δ colors. To state the next result, we introduce the following terminology and notation. The *distance* between $v, w \in V$, denoted $\rho(v, w)$, is the smallest number of edges one must move along in G to get from v to w.

Theorem 4.2. *If G is a graph with maximum degree $\Delta \ge 2$ such that for all distinct vertices $v, w \in V_\Delta$ we have $\rho(v, w) \ge 3$, then $\chi_g(G) \le \Delta$.*

Proof. Let G be a graph with maximum degree $\Delta \ge 2$ such that for all distinct $v, w \in V_\Delta$ we have $\rho(v, w) \ge 3$. Alice and Bob will play the vertex coloring game on G with Δ colors. As in the previous proof we have that Bob can only win at vertices in V_Δ. Let W_j be the set of colored vertices after turn j. Alice's strategy is to color a vertex in V_Δ on every turn chosen by the following protocol. For her first turn, she will choose one vertex from V_Δ arbitrarily. Thereafter, her move will be decided by the following two possibilities. If on Bob's i^{th} turn (so turn $2i$ overall) he colors a vertex in $N(V_\Delta \setminus W_{2i-1})$, then Alice will color the vertex in V_Δ that is adjacent to Bob's i^{th} move. We note that this is well-defined since no vertex is adjacent to more than one vertex in V_Δ because of the distance condition. Otherwise, Alice will arbitrarily color an uncolored vertex in V_Δ. Notice that Alice's strategy is such that when she colors $v \in V_\Delta$, at most one vertex in $N(v)$ is colored (namely the vertex colored by Bob on the previous turn). Since $\Delta \geq 2$, this strategy ensures that Alice can color each vertex in V_Δ (and hence she wins the game). □

Before stating the next result, we need to introduce a bit of terminology. We say G is *homeomorphic* to H if we can obtain H from G by replacing some number of edges of G with vertex-disjoint paths. The following is a corollary of Theorem 4.2 and serves as a game analogue of the fact that every graph is homeomorphic to graph with chromatic number two.

Corollary 4.3. *Every graph G with at least one edge is homeomorphic to a graph G' with $\chi_g(G') = 3$.*

Proof. Let G be a graph and let G' be the graph resulting from replacing every edge of G with a path consisting of three edges. Let V be the set of vertices in G' corresponding to vertices in G and $W = V(G') \setminus V$. Suppose Alice and Bob play the vertex coloring game with 3 colors. On every turn Alice will color a vertex in V. She will decide which of these to color based upon Bob's move as follows: if Bob colors a vertex in V, Alice will arbitrarily color an uncolored vertex in V and if Bob colors a vertex in W, Alice will color the closest vertex in V to Bob's move (which is well-defined by the definition of G').

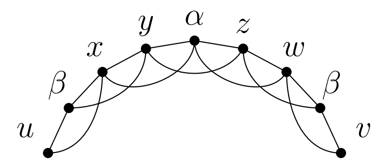
In particular, notice that by using this strategy, when Alice colors a vertex in V it is adjacent to at most one colored vertex and so, with three colors available, she will win the game. We should also note that no vertex in W can prevent a proper coloring since each has degree two. Therefore, using this strategy Alice will win with three colors; that is, $\chi_g(G') \leq 3$. Also observe that $\chi_g(G') > 2$ since G' fails the condition of Lemma 2.3 and so $\chi_g(G') = 3$. □

We next remark that although the distance condition in Theorem 4.2 is rather restrictive, it is the best global distance condition which gives the conclusion that $\chi_g(G) \leq \Delta(G)$. This is so because cycles are 2-regular and have game chromatic number three. Additionally we will show in Theorem 4.5 that there are graphs whose maximum degree vertices have distance two and for which $\chi_g(G) \not\leq \Delta(G)$. The next two theorems give game chromatic numbers of families of graphs which highlight the possible need to have vertices of V_Δ separated in order to conclude that $\chi_g(G) \leq \Delta(G)$.

Theorem 4.4. *If $n \geq 14$, then $\chi_g(C_n^2) = 5$.*

Proof. Suppose Alice and Bob play the vertex coloring game on C_n^2 for $n \geq 14$ with 4 colors and that $V(C_n^2)$ is labeled $v_1 \dots v_n$ clockwise. Since the graph is vertex transitive, we may assume that Alice's first move is to color v_1 with 1. Bob will color v_4 with 2. From here, there are three cases to consider.

- Suppose Alice next colors a vertex from $\{v_7 \dots v_{n-2}\}$. If Alice colored a vertex from $\{v_{10} \dots v_{n-2}\}$, then Bob will color v_7 with 1 and otherwise Bob will color v_{n-2} with 2. Note that the restriction on n is to ensure that Alice's second move cannot be adjacent to both v_{n-2} and v_7. Whichever Bob colors, there is now an interval of length nine as in the figure below where $\{\alpha, \beta\} = \{1, 2\}$.



Notice that Alice's third turn cannot be to color x or y because if she does, Bob will win by coloring z or w, respectively. By symmetry, Alice cannot color z or w on her third turn either. If Alice colors u from $\{3, 4\}$, Bob will color y the remaining color to surround x; the same holds if Alice colors v from $\{3, 4\}$. It remains, therefore, to consider the cases where Alice colors so that $c(u) = \alpha$, $c(v) = \alpha$, or u and v are uncolored. Bob will address these cases by coloring z with 3, y with 3, or z with 3, respectively. In each of these cases, Bob has created a double-threat and therefore wins the game.

- Suppose Alice colors a vertex from $\{v_6, v_{n-1}\}$ with the color $\gamma \in \{3, 4\}$. Bob will color v_3 or v_2, respectively, with the fourth color. Doing so creates a double-threat and so Bob will win in this case.
- It remains to consider the case where Alice colors v_5 or v_n. Given the symmetry of the state of the graph, we may assume that Alice colors v_5 with 1 (if $c(v_5) \neq 1$, then Bob will win with his second move). Now Bob will color v_8 with 3 which forces Alice to color v_7 with 2. On each of Bob's subsequent turns, he will color along the sequence of vertices $v_{11}, v_{14}, v_{17}, \ldots$ with color sequence 1, 3, 1, 3, … until he colors a vertex from $\{v_{n-5}, v_{n-4}, v_{n-3}\}$. Note that by inserting the color 4 into the sequence of colors as needed, Bob can ensure that the vertex he colors from $\{v_{n-5}, v_{n-4}, v_{n-3}\}$ receives the color 1. Also notice that these moves force Alice to color $v_{10}, v_{13}, v_{16}, \ldots$ each with the color 2. If Bob colored v_{n-3}, he will next color v_n with 3 which creates a double-threat. If Bob colored v_{n-4}, he will next color v_{n-2} with 3. Alice has three moves which prevent her from losing immediately which are: color v_{n-3} with 4, color v_{n-1} with 2, color v_n with 4. In response, Bob will: color v_n with 2, color v_2 with 4, color v_3 with 3. In each case, Bob will have surrounded a vertex with four colors. If Bob colored v_{n-5}, he will next color v_n with 3 which forces Alice to color v_2 or v_3. Bob will next color v_{n-2} with 4. No matter what Alice does, Bob will win from here since in order to have a proper coloring of C_n^2 from here, we must have all of $c(v_{n-1}) = 2$, $c(v_{n-4}) = 3$, and $c(v_{n-3}) \in \{2, 3\}$ but these cannot all happen simultaneously.

Since Bob has a winning strategy on C_n^2 with four colors available, we have $\chi_g(C_n^2) \geq 5$. The reverse inequality follows from Lemma 2.2 since $\Delta(C_n^2) = 4$ which completes the proof. □

We conclude this section by calculating the game chromatic number of a graph homeomorphic to a generalized Petersen graph. Before stating the next theorem, we will need one last term with its notation. The *closed neighborhood* of $v \in V$, denoted $N[v]$, is $N(v) \cup \{v\}$ with $N[U] = \cup_{v \in U} N[v]$ for $U \subseteq V$.

Theorem 4.5. *If G is the graph resulting from subdividing every edge of $GP(8, 3)$ once, then $\chi_g(G) = 4$.*

Proof. Let G be the graph resulting from subdividing every edge of $GP(8, 3)$ once. For ease of exposition, we assign the following labels to the vertices of G:

- the outer C_8 are $v_1 \ldots v_8$ starting at the top left and moving clockwise,

- the inner eight vertices are $w_1 \dots w_8$ so that each subscript matches the outer counterpart,
- the subdivision vertex between v_i and v_{i+1} is v'_i with indices reduced modulo eight,
- the subdivision vertex between v_i and w_i is u_i,
- we order the inner C_8 as $w_1, w_4, w_7, w_2, w_5, w_8, w_3, w_6$ and the subdivision vertices along the cycle in this order are $w'_1, w'_4, w'_7, w'_2, w'_5, w'_8, w'_3, w'_6$, respectively.

Suppose Alice and Bob play the vertex coloring game on G with three colors. We will show that Bob has a winning strategy. Without loss of generality, Alice's first move is to color a vertex from $\{v_1, v'_1, u_1\}$ 1; we will refer to the vertex she colored as a_1. Then Bob will then color v'_5 with 1.

Before examining Alice's second move, we will establish the following claim.

Claim: If there is a path on an odd number of vertices in G such that:

- the path starts and ends at degree two vertices;
- the endpoints are colored and other than those, each closed neighborhood of the interior vertices along the path is completely uncolored;
- it is Bob's turn,

then Bob will win the game.

Proof of the Claim: Suppose the vertices of the path are labeled $p_1 \dots p_k$ so that p_1 and p_k are colored. Bob will color p_3 so that $c(p_1) \neq c(p_3)$. This is a forcing move and so Alice must color a vertex in $N[p_2]$. Supposing she does so, Bob will next color p_5 so that $c(p_3) \neq c(p_5)$ which forces Alice to play in $N[p_4]$. Bob will proceed to color in this fashion until he creates a double-threat at p_{k-2} which wins the game for Bob.

For Alice's second move there are cases to consider.

(1) If Alice colors a vertex from $V \setminus (\{a_1\} \cup N[v_5] \cup N[v_6])$, then at least one of u_8, w'_4, and u_3 has the property that Bob can color it to create two disjoint paths on seven vertices (between it and v'_5) which satisfy the conditions of the claim. By coloring that vertex, no matter what Alice does next, Bob will have at least one path as in the claim for his next turn and hence he will win.

(2) If Alice colors a vertex from $\{v'_4, v'_6, u_5, u_6\}$, then there is a path on eleven vertices which starts at v'_5 and ends at the Alice's second vertex which satisfies the conditions of the claim and hence Bob will win in this case.

(3) If Alice colors v_5, then Bob will color u_3 with 1 and if Alice colors v_6, then Bob will color u_8 with 1. We will analyze the former of these and note that the argument for the latter is identical. Now notice that Alice must color a vertex from $\{w_3, w'_8, w'_3, w_6, w'_6, u_6, v_6, v'_6\}$ because otherwise the path $v'_5, v_6, u_6, w_6, w'_3, w_3, u_8$ satisfies the conditions of the claim. If Alice colors any vertex from $\{w_3, w'_8, w'_3, w_6, w'_6\}$, then Bob will color w'_4 with 1 which creates two disjoint paths on seven vertices (from v'_5 to w'_4 and from u_3 to w'_4) which satisfy the conditions of the claim and hence Bob will win in this case. If Alice colors a vertex from $\{u_6, v'_6\}$, then that vertex along with u_3 creates a path as in the claim and, again, Bob will win in this case. Finally, if Alice colors v_6, then Bob will color v'_3 with 2. Notice that the path $v'_3, v_4, u_4, w_4, w'_4, w_7, u_7, v_7, v'_7, v_8, u_8, w_8, w'_8, w_3, u_3$ will satisfy the conditions of the claim since Alice will need to color a vertex from $\{v'_2, v_3\}$ otherwise she will lose.

In all cases, Bob wins the game with three colors available and therefore $\chi_g(G) \geq 4$. The reverse inequality holds by Lemma 2.2 since $\Delta(G) = 3$ and so we have $\chi_g(G) = 4$. □

We would like to point out that working with a subdivision of $GP(8, 3)$ in Theorem 4.5 was largely for convenience. That is, using an analysis essentially identical to the above, it can be shown that subdividing each edge of $GP(n,k)$ with n large enough and $\gcd(n,k) = 1$ gives a graph with game chromatic number four.

5. Concluding Remarks

Throughout this paper we have considered problems related to the game chromatic number of Mycielskians of graphs and a game analogue of Brooks' Theorem. Although we have made some progress against each of these problems, many, many problems remain unresolved. First, as mentioned above, a key property of $M(G)$ as compared to G is that $\chi(M(G)) = \chi(G) + 1$. Each of our results suggests the following relationship between $\chi_g(M(G))$ and $\chi_g(G)$ which we put forth as a conjecture.

Conjecture 5.1. *Every graph G satisfies $\chi_g(M(G)) \geq \chi_g(G)$.*

Next we propose a game analogue of Brooks' Theorem.

Problem 5.2. *Identify all G for which $\chi_g(G) \leq \Delta(G)$.*

In contrast to Brooks' Theorem, it appears that this problem may be too ambitious. For one, a game-Brooks condition would need to exclude essentially all graphs with maximum degree at most three since maximum degree one, resp. two, means the graph is a matching along with isolated vertices, resp. cycles along with a matching and isolated vertices. Since each of these has game chromatic number one larger than the maximum degree, all would need to be excluded. Additionally, in Frieze et al. (2013) it was shown that almost all 3-regular graphs have game chromatic number four which means that essentially all of these need to be excluded as well. To further the characterization of graphs satisfying $\chi_g(G) \leq \Delta(G)$, it would be worthwhile to determine the game chromatic number of a typical 4-regular graph.

Another potential direction of research is to restrict the class of graphs to be examined. One reasonable restriction follows from the observation that it is possible for a bipartite graph to have $\chi_g(G) = \Delta(G) + 1$ (e.g. $K_{n,n}$ minus a perfect matching). Indeed in Theorem 4.5 the graph under consideration is a $(2,3)$-*biregular* bipartite graph which means that all vertices in one partition set have degree two and all vertices in the other have degree three. Given the delicate nature of determining the game chromatic number of a graph, we put forth the following question with the hope that the restriction on graphs under consideration is enough to gain some traction.

Question 5.3. *Is there a $(t-1,t)$-biregular bipartite graph with $\chi_g(G) = t + 1$?*

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