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Conformal bootstrap analysis for single and branched polymers

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The determinant method in the conformal bootstrap is applied for the critical phenomena of a single polymer in arbitrary D dimensions. The scale dimensions (critical exponents) of the polymer ($2 < D \leq 4$) and the branched polymer ($3 < D \leq 8$) are obtained from the small determinants. It is known that the dimensional reduction of the branched polymer in D dimensions to the Yang–Lee edge singularity in $D-2$ dimensions holds exactly. We examine this equivalence by the small determinant method.

Subject Index A13, I10

1. Introduction

The conformal field theory in arbitrary dimensions was developed a long time ago [1,2], and the modern numerical approach was initiated by Ref. [3]. Studies using this conformal bootstrap method have led to promising results for various symmetries in general dimensions D. The review article Ref. [4] includes conformal bootstrap developments where recent references may be found.

Instead of taking many relevant operators, the determinant method with small prime operators provides interesting results for the non-unitary cases. The determinant method is applied on the Yang–Lee edge singularity with considerable accuracy [5–7]. The polymer case is known as another non-unitary case. The method of finding a kink at the boundary of the unitary condition for an $O(N)$ vector model [8,9] breaks down for $N < 1$, and one needs higher operators for the polymer case, which corresponds to $N = 0$ [10].

This paper deals with two different polymers using the determinant method: the single polymer, and branched polymers in a solvent. They have different upper critical dimensions, 4 and 8 respectively. It is well known that the polymer in a solvent is equivalent to a self-avoiding walk, which was studied by the renormalization group ϵ expansion ($\epsilon = 4 - D$) for the $N \to 0$ limit of an $O(N)$ vector model [11,12].

A branched polymer in D dimensions ($3 < D < 8$) is equivalent to the Yang–Lee edge singularity in $D - 2$ dimensions, as shown by the ϵ expansion ($\epsilon = 8 - D$) [13,14] and by supersymmetry [15]. This equivalence is further proved exactly in Refs. [16,17]. Due to this rigorous proof, the dimensional reduction $D \to D - 2$ should hold for $3 < D \leq 8$ in the conformal bootstrap analysis. Since the Yang–Lee edge singularity for $1 < D \leq 6$ has been studied by the conformal bootstrap method [5–7], it is interesting to apply the determinant method to the branched polymer concerning verification of the equivalence.

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We are concerned with two issues related to polymers: (i) the critical phenomena of polymers belong to the logarithmic conformal field theory since the central charge C becomes zero [18–20], and (ii) the method of the replica limit $N \to 0$ is equivalent to the use of supersymmetry [15]. The validity of the supersymmetric arguments has been discussed for a long time for the random magnetic field the Ising model (RFIM) [21]. In RFIM, the dimensional reduction to a $(D-2)$-dimensional pure Ising model will break down at some lower critical dimensions, which has been shown rigorously [22]. Then the lower critical dimension is suggested to be around three dimensions, above which the supersymmetry argument may be valid [23]. The study of the branched polymer is theoretically interesting from the point of the validity of the supersymmetry. The conformal bootstrap method may give a clue to the relation between the supersymmetry and the replica limit.

In this paper we evaluate the scale dimensions of the single polymer and a branched polymer by the determinant method with a small number of operators. This study is an extension of a previous analysis of the Yang–Lee edge singularity [5,7] in which we had a constraint $\Delta_\phi = \Delta_\epsilon$ due to the equation of motion in ϕ^3 theory. We define the scale dimension of the energy as $\Delta_\epsilon = \Delta_{\phi^2}$, where ϕ is the order parameter of ϕ^3 theory. For the polymers, we have ϕ^4 theory by symmetry. Instead of $\Delta_\phi = \Delta_\epsilon$, we have an important constraint of the crossover exponent $\hat{\varphi}$ [24] of the $O(N)$ vector model. It is related to Δ_ϵ as $\Delta_T = \Delta_\epsilon$, where $\Delta_T = D - \hat{\varphi}/\nu$ (ν is the critical exponent of the correlation length). The scale dimension of the energy is defined generally by $\Delta_\epsilon = D - \frac{1}{\nu}$. Therefore, for polymers we have the crossover exponent $\hat{\varphi} = 1$, which leads to $\Delta_T = \Delta_\epsilon$.

Although we use these constraints in the determinant method, we extend the analysis by introducing a small difference between Δ_T and Δ_ϵ ($\Delta_T \neq \Delta_\epsilon$), which is analogous to "resolution of singularity" by "blow-up," to locate the values of the scale dimensions [7].

The bootstrap method uses the crossing symmetry of the four-point amplitude. The four-point correlation function for the scalar field $\phi(x)$ is given by

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{g(u,v)}{|x_{12}|^{2\Delta_\phi}|x_{34}|^{2\Delta_\phi}}, \tag{1}$$

and the amplitude $g(u,v)$ is expanded as the sum of conformal blocks $G_{\Delta,L}$ (L is a spin),

$$g(u,v) = 1 + \sum_{\Delta,L} p_{\Delta,L} G_{\Delta,L}(u,v). \tag{2}$$

The crossing symmetry of $x_1 \leftrightarrow x_3$ implies

$$\sum_{\Delta,L} p_{\Delta,L} \frac{v^{\Delta_\phi} G_{\Delta,L}(u,v) - u^{\Delta_\phi} G_{\Delta,L}(v,u)}{u^{\Delta_\phi} - v^{\Delta_\phi}} = 1. \tag{3}$$

In the previous paper [10], a polymer case was studied from the kink behavior at the unitary boundary. The minor method consists of the derivatives at the symmetric point $z = \bar{z} = 1/2$ of Eq. (3). By the change of variables $z = (a + \sqrt{b})/2$, $\bar{z} = (a - \sqrt{b})/2$, derivatives are taken about a and b. Since the number of equations becomes larger than the number of truncated variables Δ, we need to consider the minors to determine the values of Δ. The matrix elements of minors are expressed by

$$f^{(m,n)}_{\Delta,L} = \left(\partial_a^m \partial_b^n \frac{v^{\Delta_\phi} G_{\Delta,L}(u,v) - u^{\Delta_\phi} G_{\Delta,L}(v,u)}{u^{\Delta_\phi} - v^{\Delta_\phi}} \right)\Bigg|_{a=1,b=0}, \tag{4}$$

and the minors of 2×2 and 3×3, for instance d_{ij} and d_{ijk}, are determinants such as

$$d_{ij} = \det\left(f_{\Delta,L}^{(m,n)}\right), \quad d_{ijk} = \det\left(f_{\Delta,L}^{(m,n)}\right), \tag{5}$$

where i, j, k are numbers chosen differently from $(1, \ldots, 6)$, following the dictionary correspondence to (m, n) as $1 \to (2, 0)$, $2 \to (4, 0)$, $3 \to (0, 1)$, $4 \to (0, 2)$, $5 \to (2, 1)$, and $6 \to (6, 0)$. We use the same notations for the conformal block and for the minors as Ref. [7].

2. Replica limit $N \to 0$ for a polymer

There are many examples of critical phenomena that are non-unitary. A negative value of the coefficients of the operator product expansion (OPE) leads to the non-unitary case. For instance, this can be seen in the case of the Yang–Lee edge singularity and in polymers. For such non-unitary critical phenomena the unitarity condition does not hold, and direct application of the unitarity boundary condition does not work. For instance, an $O(N)$ vector model shows a kink behavior of the unitary bound for $N > 1$ [9], but it loses the kink behavior for $N < 1$, i.e. the kink becomes a smooth curve. One needs other conditions to determine the anomalous dimensions for polymers, which are realized in the replica limit $N \to 0$ [10].

On the other hand, the determinant method works for the non-unitary Yang–Lee edge singularity [5,7]. Therefore, it is meaningful to apply the determinant method to polymers, and that is the aim of this paper.

However, one needs to solve some difficulties in the replica limit $N \to 0$ for this application. The first one is related to the symmetric tensor operator in Eq. (8), whose anomalous dimension in Eq. (9) is related to the crossover exponent of the $O(N)$ vector model [24]. For the $O(N)$ vector model, the four point function is expressed as [2]

$$\left\langle \phi_i(x_1)\phi_j(x_2)\phi_k(x_3)\phi_l(x_4)\right\rangle \frac{g(u,v)}{|x_{12}|^{2\Delta_\phi}|x_{34}|^{2\Delta_\phi}}, \tag{6}$$

with

$$\begin{aligned} g(u,v) = 1 &+ \sum_S \delta_{ij}\delta_{kl} p_{\Delta,L} G_{\Delta,L}(u,v) \\ &+ \sum_T \left(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl} - \frac{2}{N}\delta_{ij}\delta_{kl}\right) p_{\Delta,L} G_{\Delta,L}(u,v) \\ &+ \sum_A (\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}) p_{\Delta,L} G_{\Delta,L}(u,v), \end{aligned} \tag{7}$$

where S means the singlet sector, T is a tensor sector, and A is an asymmetric tensor sector.

$$\begin{aligned} S: &\quad \epsilon(x) = \sum_{a=1}^{N} : {\phi_a}^2(x) :, \\ T: &\quad \varphi_{ab}(x) =: \phi_a(x)\phi_b(x) : -\frac{\delta_{ab}}{N}\sum_{c=1}^{N} : {\phi_c}^2(x) : . \end{aligned} \tag{8}$$

The catastrophic divergence is the factor $1/N$ in the tensor sector T in Eq. (8) in the limit of $N \to 0$. It is known that the crossover exponent ϕ of the $O(N)$ vector model becomes one in the replica limit.

Therefore, the anomalous dimension of this symmetric tensor operator T, denoted as Δ_T, becomes degenerate to $\Delta_{\phi^2} = \Delta_\epsilon$ since we have

$$\Delta_T = D - \frac{\hat{\varphi}}{\nu}, \tag{9}$$

where $\hat{\varphi}$ is a crossover exponent and ν is a critical exponent for the correlation length. We have, by definition,

$$\Delta_\epsilon = D - \frac{1}{\nu}, \tag{10}$$

and we have following degeneracy, due to $\hat{\varphi} = 1$,

$$\Delta_T = \Delta_\epsilon, \tag{11}$$

for the polymer case. This degeneracy of $\Delta_\epsilon = \Delta_T$ may solve the catastrophe of the replica limit as indicated in Ref. [29].

The second catastrophe, which is related to the central charge C, gives the logarithmic conformal field theory (CFT) [18,19]. The polymer has central charge $C = 0$. The minor method gives the values of the anomalous dimensions without knowing the OPE coefficients, as Eq. (4) indicates. The central charge C is expressed by the OPE coefficient as

$$C = \frac{(\Delta_\phi)^2}{p_{[D,2]}}, \tag{12}$$

where Δ_ϕ is an anomalous dimension. $p_{[D,2]}$ is the square of the OPE coefficient of the energy–momentum tensor, and it has a simple pole when $C = 0$. It is known in two dimensions that the vanishing central charge $C = 0$ leads to the logarithmic CFT [18,19]. For general dimensions, the situation may be same but the precise forms of the OPE coefficients are unknown.

3. Determinant method for a single polymer

3.1. $D = 4$

We consider the case of the self-avoiding walk or polymer in a solvent. For this case, which corresponds to the $N = 0$ limit of the $O(N)$ vector model, degeneracy of Δ_ϵ and Δ_T occurs since the crossover exponent $\hat{\varphi}$ becomes exactly one by ϵ expansion in all orders for $N = 0$:

$$\Delta_\epsilon = \Delta_T. \tag{13}$$

At the upper critical dimension $D = 4$, we have a free field value of $\Delta_\phi = 1.0$. The intersection of loci d_{123} and d_{124} with the $\Delta_\epsilon = \Delta_t$ line is shown in Fig. 1, in which the polymer's scale dimension becomes $\Delta_\epsilon = \Delta_T = 2.0$. In the figure, on the straight line of $\Delta_\epsilon = \Delta_T$ any point satisfies the condition, and the value of $\Delta_\epsilon = \Delta_T$ is not determined uniquely. We use the "blow-up" technique for this degeneracy by introducing the small parameter, which indicates $\Delta_\epsilon \neq \Delta_T$. The blow-up technique is known in the theory of resolution of singularities [25]. Then, the intersection of the lines will provide the value of $\Delta_\epsilon = \Delta_T$ at the intersection point. We call this procedure "blow-up." In Fig. 1, the zero loci of the minors d_{123}, d_{124}, d_{134}, and d_{234} intersect with a straight line of $\Delta_\epsilon = \Delta_T$

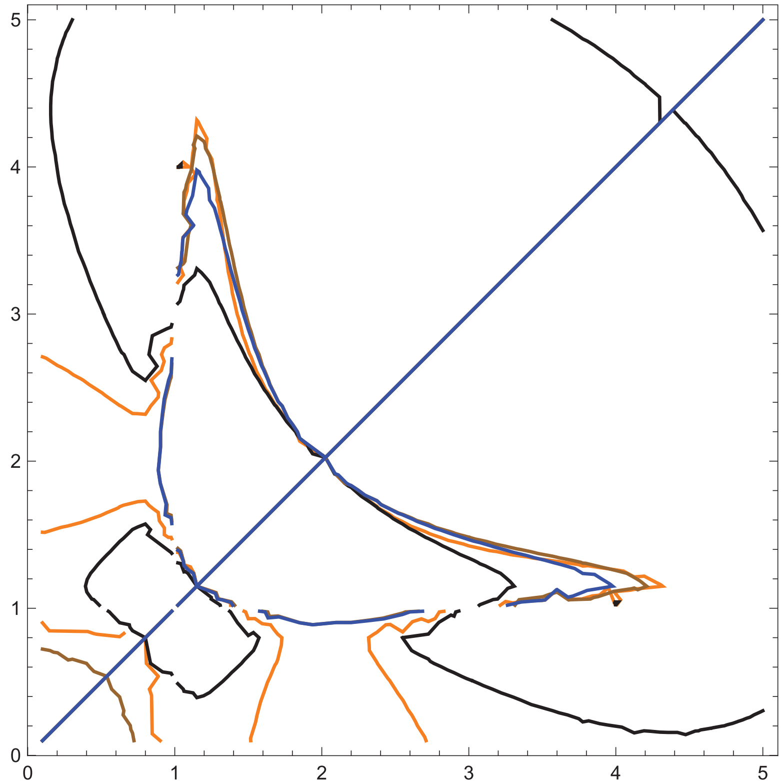

Fig. 1. $D = 4$: Scale dimensions of the polymer. The zero loci of the 3×3 minor d_{123} (red), d_{124} (brown), d_{134} (black), and d_{234} (blue) intersect in $D = 4$ for $\Delta_\phi = 1.0$ at the point of $\Delta_T = \Delta_\epsilon = 2.0$. The axes are $(x, y) = (\Delta_\epsilon, \Delta_T)$. This figure shows the blow-up of the degeneracy of $\Delta_\epsilon = \Delta_T$.

at $\Delta_\epsilon = 2$. The notation of 3×3 minors d_{ijk} is given by Eq. (5). The 3×3 minor, for instance d_{123}, is

$$d_{123} = \det \begin{pmatrix} f^{(2,0)}_{\Delta_\epsilon, L=0} & f^{(2,0)}_{(D,2)} & f^{(2,0)}_{\Delta_T, L=0} \\ f^{(4,0)}_{\Delta_\epsilon, L=0} & f^{(4,0)}_{(D,2)} & f^{(4,0)}_{\Delta_T, L=0} \\ f^{(0,1)}_{\Delta_\epsilon, L=0} & f^{(0,1)}_{(D,2)} & f^{(0,1)}_{\Delta_T, L=0} \end{pmatrix}, \tag{14}$$

where $f^{(m,n)}_{\Delta,L}$ is given by Eq. (4). We consider here only the $N = 0$ case.

3.2. $D = 3$

For three dimensions, the previous conformal bootstrap method gives the values of $\Delta_T = \Delta_\epsilon = 1.2984$ and $\Delta_\phi = 0.5141$ [10], and Monte Carlo gives $\Delta_T = 1.2982$, $\Delta_\phi = 0.5125$ [26]. The ϵ expansion gives the estimation as $\Delta_T = 1.2999$ and $\Delta_\phi = 0.5142$ [27]. The bootstrap method has an estimate of Δ_T [10] which is close to the result of the ϵ expansion.

In $D = 3$, if we adapt the value of $\Delta_\phi = 0.514$, which is taken from Ref. [10], the intersection of d_{123} (orange), d_{124} (brown), d_{134} (black), and d_{234} (blue) are shown in Fig. 2, in which $\Delta_\epsilon = \Delta_T = 1.3$ is obtained from d_{124} (the brown line).

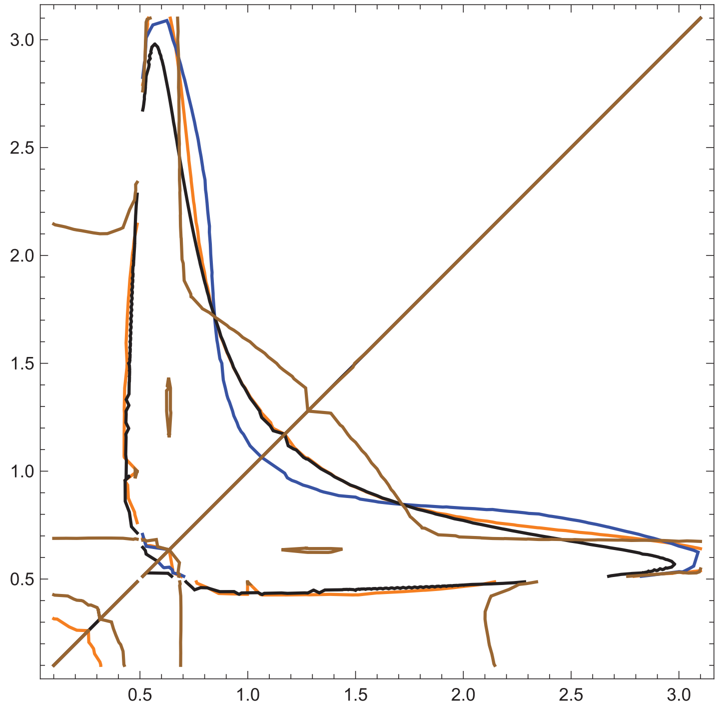

Fig. 2. $D = 3$: Scale dimensions of the polymer. The zero loci of the 3×3 minor d_{123} (orange), d_{124} (brown), d_{134} (black), and d_{234} (blue). The point $\Delta_T = \Delta_\epsilon = 1.3$ is realized in d_{124} (brown). The axes are $(x, y) = (\Delta_\epsilon, \Delta_T)$.

The ϵ expansion of Δ_ϵ ($\epsilon = 4 - D$) for the polymer case, which is obtained by the limit $N \to 0$ in the expression of the $O(N)$ vector model, is given by [11]

$$\Delta_\epsilon = 2 - \frac{3}{4}\epsilon + \frac{11}{128}\epsilon^2 + O(\epsilon^3), \tag{15}$$

which becomes $\Delta_\epsilon = 1.57$ for $D = 3.5$ ($\epsilon = 0.5$). For $D = 3$, ϵ expansion by Borel–Padé analysis gives $\Delta_\epsilon = 1.3$ [27], which is close to the values obtained by the 3×3 determinant d_{124} in Fig. 2. There are splits (or jumps) of the intersection points along the diagonal line around $\Delta_\epsilon = 1.3$. Such split behavior has also been observed in the analysis of the Yang–Lee edge singularity at the critical dimension D_c, where $\Delta_\phi = 0$ and the central charge C is estimated as $C = 0$. We took the maximum value of Δ_T of the splitting points in Fig. 2. The Yang–Lee edge singularity indicates a reasonable value of D_c by taking the maximum point of the splitting for the blow-up [7]. In the polymer case, the maximum of the split values in $D = 3$ is close to the ϵ analysis (Table 1). Further investigation of this split (or jump) behavior is required by systematic analysis, which remains as future work. The determinant method using a small-rank matrix gives a rough estimation, and the result depends on the choice of d_{ijk}. The recent article in Ref. [20] also pointed out that a special choice of the determinant is better for the estimation of the anomalous dimension Δ_ϵ. The method for the estimation of the error bar was suggested in Ref. [28].

The central charge $C = 0$ suggests the pole of the OPE coefficient of the energy momentum tensor, which leads to the logarithmic CFT behavior. The pole for $N \to 0$ in Eq. (8) is related to the degeneracy of $\Delta_\epsilon = \Delta_T$ as explained in Ref. [29] for the polymer case; the OPE coefficient of Δ_T has a pole.

Table 1. Scale dimensions of a single polymer.*

D	Δ_ϕ	$\Delta_T = \Delta_\epsilon$	Δ_ϵ(ϵ expansion, exact value)
2	0.1	0.7	0.666
3	0.514	1.3	1.299
3.5	0.75	1.57	1.57
4	1.0	2.0	2

* The value of Δ_T is obtained from the zero loci of 3×3 minors. For $D = 2$, exact values are $\Delta_\phi = 5/48$, $\Delta_\epsilon = 2/3$.

4. Branched polymer

There is a remarkable equivalence, the so-called dimensional reduction, between a branched polymer in D dimensions ($3 < D < 8$) and the Yang–Lee edge singularity in $D-2$ dimensions ($1 < D < 6$); the critical exponents become the same. The branched polymer is described by ϕ^3 theory, but the upper critical dimension is known to be 8 due to the disorder. The $\epsilon = 8-D$ expansion for the critical exponent agrees with the exponent of the Yang–Lee edge singularity in the $\epsilon = 6 - D$ expansion; for instance, the critical exponent ν becomes the same for both models [13,14].

The action of the branched polymer has branching terms in addition to the self-avoiding term (single polymer). We write this action for the pth branched polymer as N-replica field theory [19]:

$$S = \int d^D x \left(\frac{1}{2} \sum_{\alpha=1}^{N} \left[(\nabla \phi_\alpha)^2 - \sum_{p=1}^{\infty} u_p \phi_\alpha^p \right] + \lambda \left(\sum_{\alpha=1}^{N} \phi_\alpha^2 \right)^2 \right). \tag{16}$$

The term ϕ_α^p represents the pth branched polymer. After rescaling and by neglecting irrelevant terms, the following action is obtained:

$$S = \int d^D x \left(\frac{1}{2} \sum_{\alpha=1}^{N} \left[(\nabla \phi_\alpha)^2 + V(\phi_\alpha) \right] + g \sum_{\alpha,\beta=1}^{N} \phi_\alpha \phi_\beta \right), \tag{17}$$

where $V(\phi) = t\phi - \frac{1}{3}\phi^3 + O(\phi^4)$.

As before, the $N = 0$ replica limit of the $O(N)$ vector model is applied for this branched polymer. The condition of $\Delta_\epsilon = \Delta_T$ is also essential in a branched polymer problem. We find several fixed points in the blow-up plane of $\Delta_\epsilon, \Delta_T$. In the branched polymer case, we find a new triple degeneracy,

$$\Delta_\epsilon = \Delta_T = \Delta_\phi + 1. \tag{18}$$

The last term of 1 is a trivial term due to the definition of Δ_ϕ in D dimensions. The ϵ expansion of the branched polymer becomes [13,14]

$$\eta = -\frac{1}{9}\epsilon, \tag{19}$$

where $\epsilon = 8 - D$. The scaling dimension Δ_ϕ is defined by

$$\Delta_\phi(\text{branched polymer}) = \frac{D - 2 + \eta}{2}. \tag{20}$$

In above formula, if we put $D \to D-2$, and put $\epsilon = 6-D$, then we get

$$\Delta_\phi(\text{Yang–Lee edge singularity}) = 2 - \frac{5}{9}\epsilon, \tag{21}$$

where $\epsilon = 6-D$. This shows exactly the dimensional reduction relation between a branched polymer and the Yang–Lee edge singularity.

The exponent ν of the Yang–Lee edge singularity ($\epsilon = 6-D$) is

$$\frac{1}{\nu} = \frac{1}{2}(D+2-\eta) = \frac{1}{2}(8-\epsilon+\frac{1}{9}\epsilon) = 4 - \frac{4}{9}\epsilon. \tag{22}$$

This leads to the Yang–Lee edge singularity,

$$\Delta_\epsilon = D - \frac{1}{\nu} = (6-\epsilon) - (4 - \frac{4}{9}\epsilon) = 2 - \frac{5}{9}\epsilon = \Delta_\phi. \tag{23}$$

The condition $\Delta_\epsilon = \Delta_\phi$ is a necessary condition for the Yang–Lee edge singularity due to the equation of motion.

By dimensional reduction, the values of the exponents η and ν of the branched polymer become the same as the Yang–Lee edge singularity. The scale dimensions of Δ_ϵ and Δ_ϕ, however, become different since they involve the space dimension D explicitly. In a branched polymer of $D = 8$,

$$\Delta_\epsilon = 4, \quad \Delta_\phi = 3, \tag{24}$$

where, for the Yang–Lee edge singularity of $D = 6$,

$$\Delta_\epsilon = 2, \quad \Delta_\phi = 2. \tag{25}$$

In general dimension $D \leq 8$, from the equivalence to the Yang–Lee edge singularity, we have

$$\Delta_\epsilon = \Delta_\phi + 1, \tag{26}$$

as shown in Eq. (24) for $D = 8$. This relation is related to the supersymmetry, as discussed in Refs. [10,30,31].

We get the following relations:

$$\begin{aligned} &\Delta_\phi(\text{branched polymer in } D \text{ dim.}) = \Delta_\phi(\text{Yang–Lee in } D-2 \text{ dim.}) + 1, \\ &\Delta_\epsilon(\text{branched polymer in } D \text{ dim.}) = \Delta_\epsilon(\text{Yang–Lee in } D-2 \text{ dim.}) + 2. \end{aligned} \tag{27}$$

In Fig. 3, the intersection map of the loci of minors for the branched polymer is shown. The contours of zero loci for d_{123} (blue), d_{124} (brown), d_{134} (green), and d_{234} (red) are shown in different colors. The fixed point of $\Delta_\epsilon = 4$ and $\Delta_\phi = 3$ in Eq. (24) is obtained, which values are consistent with the Yang–Lee edge singularity by dimensional reduction. The parameter of Q (spin 4) is chosen as 10. Figure 3 shows the map of $(x, y) = (\Delta_\phi, \Delta_\epsilon)$, and has singular lines. The horizontal line at $\Delta_\phi = 6$ is due to the degeneracy of $\Delta_\epsilon = \Delta_T = 6$, and the other horizontal line at $\Delta_\epsilon = 3$ is due to the pole of $\Delta_\epsilon = (D-2)/2$.

We confirm the dimensional reduction to the Yang–Lee model in $D-2$ dimensions for $4 < D < 8$ by the 3×3 determinant method.

In Fig. 4, the branched polymer in $D = 8$ is considered in the blow-up map of $(\Delta_\epsilon, \Delta_T)$ with $\Delta_\phi = 2$. There is a fixed point at $\Delta_\epsilon = \Delta_T = 4.0$ for the branched polymer. This corresponds to

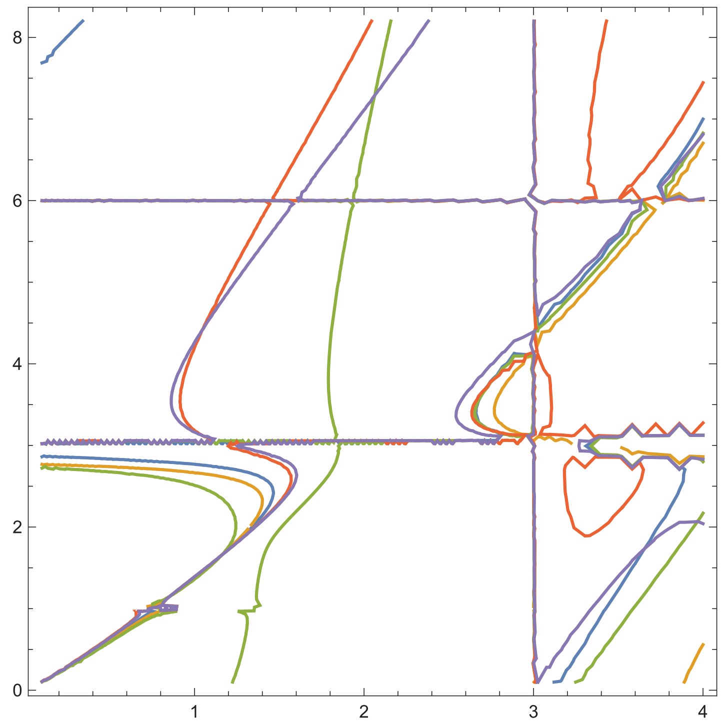

Fig. 3. Branched polymer in $D = 8$: The fixed point $\Delta_\epsilon = 4$, $\Delta_\phi = 3$ is obtained for the branched polymer in Eq. (24). These values agree with the Yang–Lee edge singularity at $D = 6$ by dimensional reduction. The axes are $(x, y) = (\Delta_\phi, \Delta_\epsilon)$.

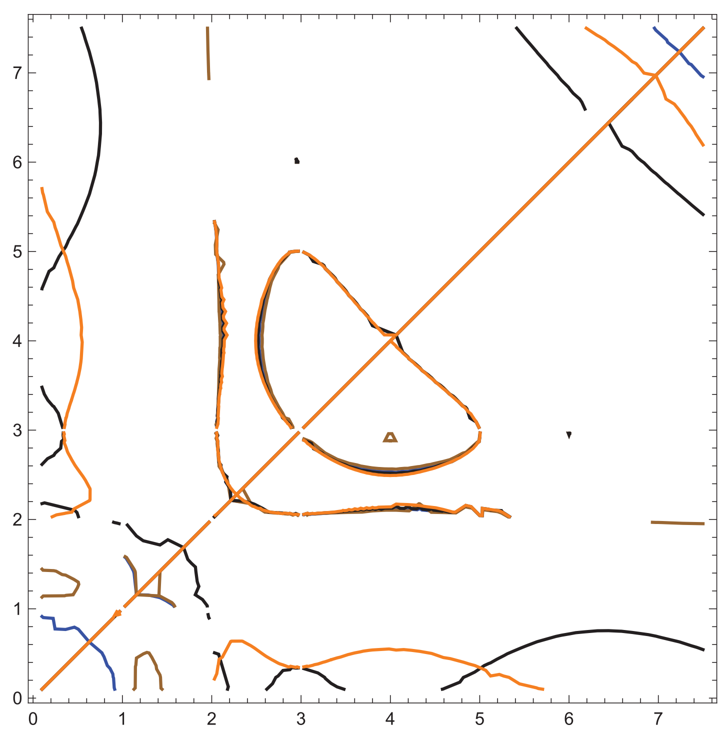

Fig. 4. Branched polymer in D=8: The zero loci of the 3×3 minor .d_{123} (black), d_{134}(brown), d_{124}(orange), d_{234}(blue) in a blow up plane ($\Delta_T \neq \Delta_\epsilon$). The fixed point appears at $\Delta_\epsilon = \Delta_T$= 4.0. The axis is (x,y) = $(\Delta_\epsilon, \Delta_T)$.

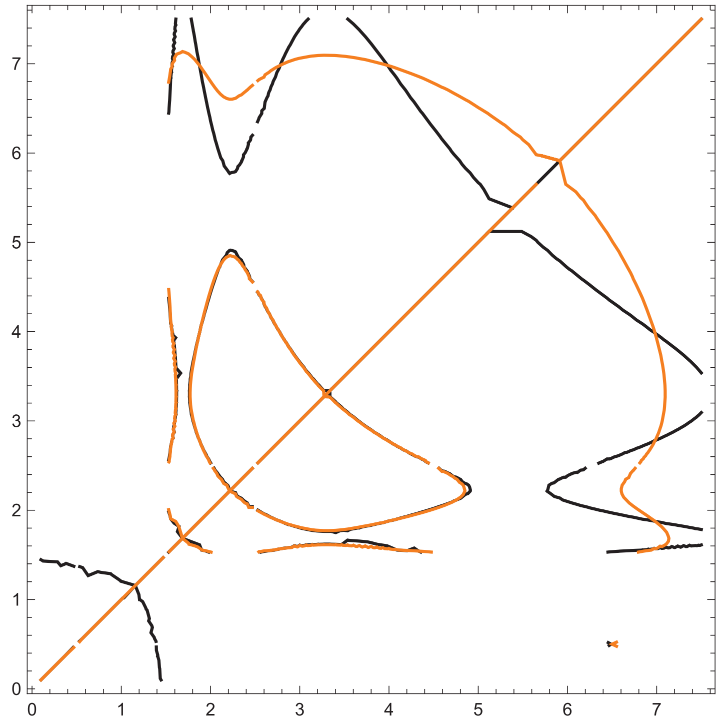

Fig. 5. Branched polymer in $D = 7$: The zero loci of the 3×3 minors d_{123} (black) and d_{124} (orange) in a blow-up plane. The axes are $(x, y) = (\Delta_\epsilon, \Delta_T)$. The intersection point in the blow-up map is $\Delta_\epsilon = 3.4$, which is consistent with the dimensional reduction relation between the branched polymer and the Yang–Lee edge singularity.

the Yang–Lee edge singularity in $D = 6$ ($\Delta_\epsilon = \Delta_\phi = 2.0$) due to the dimensional reduction. These values satisfy Eq. (27).

In Fig. 5, the branched polymer in $D = 7$ is shown with $\Delta_\phi = 1.4255$. The fixed point can be read as $\Delta_\epsilon = \Delta_T = 3.4$. This value corresponds to $\Delta_\epsilon = 1.4$ of the Yang–Lee edge singularity in $D = 5$.

5. Summary

We have analyzed a single polymer and a branched polymer, and we have found that they are characterized by the degeneracies of the primary operators, $\Delta_\epsilon = \Delta_T$, which value is obtained in a blow-up plane as an intersection point.

For a single polymer, the scaling dimension Δ_ϵ is obtained from the intersection of the zero loci of 3×3 minors with rather good accuracy (Table 1).

We find in the branched polymer case the exact relation of $\Delta_\epsilon = \Delta_\phi + 1$ in the determinant method with good numerical accuracy. This relation is consistent with the relation of the dimensional reduction between the branched polymer and the Yang–Lee edge singularity. In the Yang–Lee edge singularity, by the equation of motion we have $\Delta_\epsilon = \Delta_\phi$. The relation $\Delta_\epsilon = \Delta_\phi + 1$ is a characteristic relation in supersymmetry theory [10,30,31], where Grassmann coordinates give the dimensional reduction (-2) [15].

The validity of the dimensional reduction in a random field Ising model has long been discussed, and it is known that the reduction to a pure Ising model does not work in the lower dimensions. We will discuss this problem by the conformal bootstrap determinant method in a separate paper [23].

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