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CONSERVATIVE $P1$ CONFORMING AND NONCONFORMING GALERKIN FEMS: EFFECTIVE FLUX EVALUATION VIA A NONMIXED METHOD APPROACH*

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Abstract. Given a $P1$ conforming or nonconforming Galerkin finite element method (GFEM) solution p_h, which approximates the exact solution p of the diffusion-reaction equation $-\nabla \cdot \mathcal{K}\nabla p + \alpha p = f$ with full tensor variable coefficient $\mathcal{K}$, we evaluate the approximate flux $\mathbf{u}_h$ to the exact flux $\mathbf{u} = -\mathcal{K}\nabla p$ by a simple but physically intuitive formula over each finite element. The flux is sought in the continuous (in normal component) or the discontinuous Raviart–Thomas space. A systematic way of deriving such a formula is introduced. This direct method retains local conservation property at the element level, typical of mixed methods (finite element or finite volume type), but avoids solving an indefinite linear system. In short, the present method retains the best of the GFEM and the mixed method but without their shortcomings. Thus we view our method as a conservative GFEM and demonstrate its equivalence to a certain mixed finite volume box method. The equivalence theorems explain how the pressure can decouple basically cost free from the mixed formulation. The accuracy in the flux is of first order in the $H(\text{div};\Omega)$ norm. Numerical results are provided to support the theory.

Key words. $P1$ conforming or nonconforming, finite element, porous media, Darcy velocity, displacement method, equilibrium method, finite volume, conservative scheme, mixed method

AMS subject classifications. 65F10, 65N20, 65N30

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1. Introduction.

Consider the variable-coefficient diffusion-reaction equation in a polygonal domain $\Omega \subset \mathbb{R}^2$

$$\left\{ \begin{array}{rcll} -\nabla \cdot \mathcal{K}\nabla p + \alpha p & = & f & \text{in } \Omega, \\ p & = & 0 & \text{on } \partial\Omega, \end{array} \right. \tag{1.1}$$

where $\mathcal{K} = \mathcal{K}(\mathbf{x})$ is a symmetric positive definite matrix function such that there exist two positive constants β_1 and β_2 with

$$\beta_1 \xi^T \xi \le \xi^T \mathcal{K}(\mathbf{x}) \xi \le \beta_2 \xi^T \xi \qquad \forall \xi \in \mathbb{R}^2, \mathbf{x} \in \bar{\Omega}. \tag{1.2}$$

For ease of exposition, the function α is assumed to be a nonnegative piecewise constant function.

Introducing a flux variable $\mathbf{u} := -\mathcal{K}\nabla p$, we can write the above equation as the system of first order partial differential equations

$$\left\{ \begin{array}{rcll} \nabla \cdot \mathbf{u} + \alpha p & = & f & \text{in } \Omega, \\ \mathbf{u} + \mathcal{K}\nabla p & = & 0 & \text{in } \Omega, \\ p & = & 0 & \text{on } \partial\Omega. \end{array} \right. \tag{1.3}$$

The variable p, as a state variable, can be interpreted as concentration, displacement,

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or pressure, and the variable $\mathbf{u}$, as a flux variable, can be interpreted as diffusive flux, stress, Darcy velocity, respectively. In applications, the homogeneous Neumann boundary condition is usually used since it represents the no-flow condition. However, for expository purpose here, we use homogeneous Dirichlet boundary condition, but our results below hold for the Neumann condition case as well. We will comment about the latter case along the way.

In this paper we will interpret the system (1.3) as modeling an incompressible single phase flow in a reservoir, ignoring gravitational effects. The matrix $\mathcal{K}$ is the mobility κ/μ, the ratio of permeability tensor to viscosity of the fluid, $\mathbf{u}$ is the Darcy velocity, and p the pressure. The second equation of (1.3) is the Darcy law and the first represents conservation of mass with f standing for a source or sink term. It is desirable to have an accurate approximate Darcy velocity. While the standard Galerkin finite element method (GFEM) based on (1.1) results in "easy-to-solve" symmetric positive definite finite element systems, it does not provide accurate flux automatically and is nonconservative at the element level. On the other hand, the mixed method approach based on (1.3), well known in the finite element circle [3, 1, 20], can provide accurate flux and is locally conservative. The same approach can be applied in conjunction with the finite volume method (termed mixed finite volume or covolume methods) as well [6, 11, 12, 20, 21, 22]. In the mixed approach, one has to face solving an indefinite symmetric system resulting from the saddle point formulation. Among many approaches to effectively solving such a system let us mention the mixed hybrid Lagrange multiplier method [2] and its more recent variant [8, 9] of finding an equivalent finite element method for the Lagrange multipliers. A version of these techniques applied to finite volume methods can be found in [14]. However, in the process of finding effective iterative solvers, the physical interpretation of the variables involved seems to be blurred.

Hence it is of practical value to pose the following question. Can one compute the approximate pressure and Darcy velocity in two simple stages? First, an approximate pressure p_h is obtained via a standard conforming or nonconforming $P1$ (linear) Galerkin finite element method applied to the second order elliptic problem (1.1). Then an approximate flux $\mathbf{u}_h$ to the exact flux $\mathbf{u} \in H(\mathrm{div};\Omega) := \{\mathbf{w} : \mathbf{w} \in \mathbf{L}^2(\Omega), \nabla \cdot \mathbf{w} \in L^2(\Omega)\}$ is recovered by a generic formula such that over each element K

$$\mathbf{u}_h = -\bar{\mathcal{K}}\nabla p_h + \text{a correction term}, \tag{1.4}$$

where $\bar{\mathcal{K}}$ is a smoothed or averaged version of the possibly discontinuous tensor $\mathcal{K}$ over K. By "generic" we mean that the formula should be physically intuitive and locally conservative. Thus, for instance, to be physically intuitive we can take $\bar{\mathcal{K}} = \frac{1}{|K|}\int_K \mathcal{K} dx$, and the relation (1.4) resembles the original Darcy law $\mathbf{u} = -\mathcal{K}\nabla p$. The correction term is added to validate local conservation property at the element level, typical of mixed methods. For simplicity let us illustrate that with the diffusion equation ($\alpha = 0$). Without the correction term, one would have $\nabla \cdot \mathbf{u}_h = 0$ since the divergence of the first term on the right is zero, whereas the exact flux has $\nabla \cdot \mathbf{u} = f$ or $\int_K \nabla \cdot \mathbf{u} = \int_K f dx$ over an element K.

There are many ways to derive a correction term in the above generic formula without having to solve any large linear system. For example, one could use a finite volume approach, as done in [13]. Here in the introduction, let us take a Taylor's expansion viewpoint. Again consider the pure diffusion case. Assume that over each element K we want the approximate flux $\mathbf{u}_h$ to be linear. Expanding it about B, the barycenter of K, we have

$$\mathbf{u}_h(\mathbf{x}) = \mathbf{u}_h(B) + \mathcal{D}\mathbf{u}_h(B)(\mathbf{x} - \mathbf{x}_B), \qquad \mathbf{x} \in K, \tag{1.5}$$

where $\mathcal{D}\mathbf{u}_h$ is the Jacobian matrix of $\mathbf{u}_h$ and $\mathbf{x}_B$ is the position vector of the barycenter B. Now if we seek $\mathbf{u}_h$ in the lowest order Raviart–Thomas space restricted to a triangular element K, i.e., $\mathbf{u}_h = (a + bx, c + by)^t$ on K, $a, b \in R, \mathbf{x} = (x, y)^t$, then we have the important relation

$$\mathcal{D}\mathbf{u}_h(B)(\mathbf{x} - \mathbf{x}_B) = b(\mathbf{x} - \mathbf{x}_B) = \frac{1}{2}\nabla \cdot \mathbf{u}_h(\mathbf{x} - \mathbf{x}_B). \tag{1.6}$$

Now let us further require that the local conservation property holds:

$$\int_K \nabla \cdot \mathbf{u}_h dx = \int_K f dx = |K| f_K$$

or equivalently

$$\nabla \cdot \mathbf{u}_h = f_K,$$

where f_K is the average of f over K and $|K|$ is the area of K. Substituting these into (1.5), we finally arrive at a computable formula:

$$\mathbf{u}_h(\mathbf{x}) = -\bar{\mathcal{K}}\nabla p_h + \frac{f_K}{2}(\mathbf{x} - \mathbf{x}_B), \qquad \mathbf{x} = (x, y)^t, \tag{1.7}$$

where the first term on the right replaces $\mathbf{u}_h(B)$ (a cell-centered value concept). This formula was actually used in [13]. It is perhaps helpful here to see a one-dimensional example where the above approach reproduces the exact flux.

Example 1.1. Consider

$$\begin{aligned} -p(x)'' &= 1, \qquad && x \in (0, 1), \\ p(0) &= p(1) = 0. && \end{aligned}$$

We use the standard $P1$ conforming elements on a uniform grid with step-size h. Then it is easy to see using the Green's function and orthogonality condition that the finite element solution p_h is nothing but the piecewise linear interpolant of p at the grid points. The exact solution is $p(x) = -\frac{1}{2}x^2 + \frac{1}{2}x$ and the exact flux is $\mathbf{u} = x - \frac{1}{2}$. In this context, over each $K = [x_i, x_{i+1}] = [ih, (i+1)h]$

$$\mathbf{u}_h = -p_h'(x) + f_K(x - x_B),$$

where $p_h'(x) = [p(x_{i+1}) - p(x_i)]/(x_{i+1} - x_i)$, and it is easy to derive the result $\mathbf{u}_h = \mathbf{u}$.

Thus it should be clear from this viewpoint that for the flux in $\mathbb{R}^d, d = 1, 2$, or 3, the natural starting point is on each K

$$\mathbf{u}_h = -\bar{\mathcal{K}}\nabla p_h + \frac{f_K}{d}(\mathbf{x} - \mathbf{x}_B) + \mathbf{C}_K, \tag{1.8}$$

where $\mathbf{C}_K$ is a small constant correction term, so that the local conservation relation $\nabla \cdot \mathbf{u}_h = f_K$ still holds. In general, $\mathbf{C}_K = O(h)$, so that the last two terms of (1.8) both tend to zero in the limit. Finally, we mention in passing that the three-dimensional version of our scheme is very important for hydrology and petroleum problems, but for ease of presentation we will use one and two dimensions. All of our results below hold for three dimensions as well.

In the next section we shall derive the generic formula from another viewpoint, i.e., whether the approximate flux has a continuous normal component across interelement edges. It should be noted that in the Taylor's expansion approach there is no guarantee that the formula (1.8) produces a flux with continuous normal components across edges. It turns out that in the nonconforming GFEM case in two dimensions, the continuity can be achieved by choosing $\mathbf{C}_K$ properly. This is done in section 2 via the relation (2.14), which is also used in [18] where the diffusion tensor $\mathcal{K}$ is a piecewise constant scalar function and the right-hand side f is piecewise constant. For the conforming GFEM case, formula (1.8) produces a flux with discontinuous normal components across edges, although the local conservation property holds. (However, the one-dimensional conforming case still produces continuous fluxes.) In both cases, it is shown in Theorem 4.4 that the approximate fluxes converge, but we do not recommend using the conforming case to solve the Neumann problem since the no-flow boundary condition will be violated—the flux being discontinuous across the boundary edge. Theorems 3.2–3.4 show that each of our conservative GFEMs is also equivalent to a simple mixed finite volume box method; this extends our results in [13]. These theorems also shed some light on how easy the pressure equation can decouple from the flux in the mixed finite volume method (cf. Remarks 3.1–3.6), a feature not shared by the mixed finite element method.

After completion of this work, we learned that in the engineering circle, Gresho and Sani [16] raised the issue of how to call a GFEM "conservative" and whether finite volume method is better than FEM in the conservative properties (nodal or element conservation). The equivalence theorems in section 3 clarify and answer these questions.

We also provided numerical results to support our theory. The GFEM part of obtaining approximate pressure p_h is done by preconditioned conjugate gradient method although multigrid methods can be applied as well [4]. The flux is obtained basically cost free using formula (1.8).

Throughout the rest of this paper, we use the standard notation $W^{l,p}$ for the usual Sobolev spaces and $|\cdot|_{m,K}$, $\|\cdot\|_{m,K}$ for the semi- and full H^m norm, $m = 0, 1, 2$. We omit the subscript K when $K = \Omega$ and sometimes use $\|\cdot\|$ when writing an L^2 norm. Also $|\cdot|_{H(\mathrm{div};\Omega)}$ is the $H(\mathrm{div};\Omega)$ seminorm.

2. Construction of the flux formula. Let $\mathcal{T}_h = \{K\}$ be the usual nonoverlapping finite element triangulation of the domain $\Omega = \cup_{K\in\mathcal{T}_h} K$. Furthermore $\mathcal{T}_h$ is assumed to be regular, that is, $\min_{K\in\mathcal{T}_h} d(K)/\rho(K) \geq C$ for a constant C independent of h. Here $\rho(K)$ is the diameter of triangle K, $d(K)$ the diameter of the inscribed circle of K, and $h = \max_{K\in\mathcal{T}_h} \rho(K)$. Define the lowest order continuous (in normal component) Raviart–Thomas space [19]

$$\mathbf{V}_h = \{\mathbf{u}_h \in H(\mathrm{div};\Omega) : \mathbf{u}_h|_K \in RT_0(K) \quad \forall K \in \mathcal{T}_h\},$$

where $RT_0(K) = \{\mathbf{u} = (u^1, u^2) : u^1 = a + bx, u^2 = c + by \text{ in } K\}$ and the standard $P1$ nonconforming finite element space

$$Y_h = \{p_h|_K \in P_1(K) : p_h \text{ continuous at the middle point of each } e \in \partial K\}.$$

With reference to Figure 2.1, let λ_S be the usual nodal linear basis function associated with the vertex S of $K = K_L$: λ_S is one at S and zero at the other two

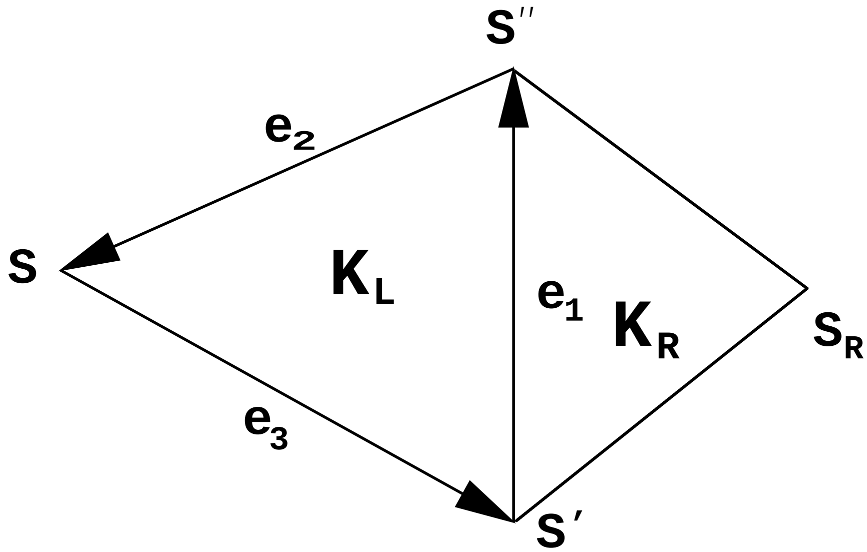

FIG. 2.1. *Local elements based on an interior edge.*

vertices. For any $q_h(\mathbf{x}) \in Y_h$, we have the local representation on K

$$q_h(\mathbf{x})|_K = \sum_{e \in \partial K} p_e \varphi_e(\mathbf{x}), \tag{2.1}$$

where $\varphi_e(\mathbf{x}) = 1 - 2\lambda_S(\mathbf{x})$ is the local basis function of the space Y_h and S is the vertex opposite to e in triangle K ($e = e_1$ in Figure 2.1). It is easy to see $\nabla\varphi_e(\mathbf{x}) = \frac{|e|}{|K|}\mathbf{n}_e = \text{const}$, where $|e|$ is the length of edge e.

Given any triangular element $K \in \mathcal{T}_h$, we always orient K counterclockwise as shown in Figure 2.1 (e.g., $K = K_L$ there). Then the three local basis functions of $\mathbf{V}_h$ associated with the three edges are as follows. For example, for the edge $e = e_1$ of $K = K_L$ in Figure 2.1, we define

$$\mathbf{P}_{K,e}(\mathbf{x}) := \frac{1}{2|K|}\begin{bmatrix} x - x_S \\ y - y_S \end{bmatrix} \qquad \forall \quad (x,y) \in K. \tag{2.2}$$

Note that for the unit exterior normal $\mathbf{n}$ to the edge e,

$$\mathbf{P}_{K,e}(\mathbf{x}) \cdot \mathbf{n} = \begin{cases} 1/|e| & \forall \quad \mathbf{x} \in e = S'S'', \\ 0 & \forall \quad \mathbf{x} \in SS', \\ 0 & \forall \quad \mathbf{x} \in SS''. \end{cases} \tag{2.3}$$

The other two basis functions $P_{K,e_i}, i = 2, 3$ are defined similarly.

For any $\mathbf{u}_h(\mathbf{x}) \in \mathbf{V}_h$, we have the local representation on K

$$\mathbf{u}_h(\mathbf{x})|_K = \sum_{e \in \partial K} u_e \mathbf{P}_{K,e}(\mathbf{x}), \tag{2.4}$$

where $u_e = \int_e \mathbf{u} \cdot \mathbf{n} ds$ is the flux across edge e.

From now on, let us assume that *the function* α *in* (1.1) *is piecewise constant with respect to* $\mathcal{T}_h$. Suppose we already have an approximate pressure p_h and let us define the approximate Darcy velocity as

$$\mathbf{u}_h(\mathbf{x}) = -\mathcal{A}_K \nabla p_h + (f_K - \alpha_K p_K)\mathbf{P}_K(\mathbf{x}) + \mathbf{C}_K \quad \forall \mathbf{x} \in K, \tag{2.5}$$

where $\mathcal{A}_K := \frac{1}{|K|}\int_K \mathcal{K} dx$, the average of the tensor $\mathcal{K}$ over K, $f_K := \frac{1}{|K|}\int_K f dx$, the average of f over K, $p_K := \frac{1}{|K|}\int_K p_h dx$, the average of p_h over K, $\alpha_K = \alpha|_K$ is a constant, and

$$\mathbf{P}_K(\mathbf{x}) = \frac{1}{2}\begin{bmatrix} x - x_B \\ y - y_B \end{bmatrix} = \frac{|K|}{3}\sum_{e \in \partial K} \mathbf{P}_{K,e}(\mathbf{x}) \qquad \forall \quad (x, y) \in K \tag{2.6}$$

with (x_B, y_B) being the coordinates of B. Here $\mathbf{C}_K$ is a constant vector on K to be determined by the continuity condition in the normal component and the first two terms on the right of (2.5) were explained in the introduction. Hence one should observe that $\nabla \cdot \mathbf{u}_h = f_K - \alpha_K p_K$ on K which is an analogue of $(1.3)_1$.

We now turn to the determination of the constant $\mathbf{C}_K$ by the continuity of the normal component across any interelement. To do this we need to be more specific about the approximate pressure p_h. So consider the original problem (1.1) whose weak formulation is to find $p \in H_0^1$ such that

$$a(p, q) = (f, q) \qquad \forall q \in H_0^1, \tag{2.7}$$

where

$$a(p, q) := \int_\Omega (\mathcal{K}\nabla p) \cdot \nabla q d\mathbf{x} + \int_\Omega \alpha p q d\mathbf{x}. \tag{2.8}$$

A $P1$ nonconforming FEM discretization is to find $p_h \in Y_{h,0}$ such that

$$a_h(p_h, q_h) = (\tilde{f}, q_h) \qquad \forall q_h \in Y_{h,0}, \tag{2.9}$$

where

$$a_h(p_h, q_h) := \sum_K \int_K (\mathcal{K}\nabla p_h) \cdot \nabla q_h d\mathbf{x} + \alpha_K \int_K p_h q_h, \tag{2.10}$$

and here

$$Y_{h,0} := \{q \in Y_h \text{ and vanishes at the midpoints of boundary edges}\}.$$

Note that we have set the right-hand side function as $\tilde{f}$ which is an L^2 function that approximates f in the sense that

$$\int_K \tilde{f} = \int_K f \tag{2.11}$$

and

$$\|\tilde{f} - f\|_{0,K} \le Ch|f|_{1,K} \quad \forall K \in \mathcal{T}_h. \tag{2.12}$$

The reason for doing so will be clear later. There are two common choices of such $\tilde{f}$: one is $\tilde{f} = f$ in the standard nonconforming FEM and one is $\tilde{f} = f_K$ [13]. Other possibilities arise from use of quadrature rules.

Given an element K, let us observe that

$$(\mathcal{K}\nabla q_h, \nabla w_h)_K = (\mathcal{A}_K \nabla q_h, \nabla w_h)_K \tag{2.13}$$

for all linear polynomials q_h, w_h on K. Here $(\cdot,\cdot)_K$ is the L^2 inner product on K. Now let K_L and K_R be two triangles with common edge e (cf. Figure 2.1) and let $\varphi \in Y_{h,0}$ be the global basis function such that $\varphi = 1$ at the midpoint of e and zero at other nodes so that locally $\varphi = 1 - 2\lambda$, with λ being the Lagrange nodal basis function (area coordinate function) corresponding to the opposite vertex of e. Thus setting $q_h = \varphi$ in (2.9) and using (2.13) we have

$$(\mathcal{A}_K \nabla p_h, \nabla\varphi)_{K_L} + (\mathcal{A}_K \nabla p_h, \nabla\varphi)_{K_R} + \int_{K_L \cup K_R} \alpha p_h \varphi dx = \int_{K_L \cup K_R} \tilde{f}\varphi dx. \tag{2.14}$$

Now use Green's formula or integration by parts and the property that φ vanishes at the midpoints on the boundary of $K_L \cup K_R$ to see that

$$|e|(\mathcal{A}_{K_L}\nabla p_h^- \cdot \mathbf{n}_L + \mathcal{A}_{K_R}\nabla p_h^+ \cdot \mathbf{n}_R) + p_h(m)(\alpha_{K_L}|K_L| + \alpha_{K_R}|K_R|)/3 = \int_{K_L \cup K_R} \tilde{f}\varphi dx,$$

where m is the midpoint of edge e. To avoid clustering of the notations we also use g^-(resp., g^+) for the g value restricting to K_L(resp., K_R). By (2.5)

$$\begin{aligned}\mathcal{A}_{K_L}\nabla p_h^- \cdot \mathbf{n}_L + \mathcal{A}_{K_R}\nabla p_h^+ \cdot \mathbf{n}_R &= [(f_{K_L} - \alpha_{K_L} p_{K_L})\mathbf{P}_{K_L}(x) + \mathbf{C}_{K_L} - \mathbf{u}_h(x)] \cdot \mathbf{n}_L \\ &\quad + [(f_{K_R} - \alpha_{K_R} p_{K_R})\mathbf{P}_{K_R}(x) + \mathbf{C}_{K_R} - \mathbf{u}_h(x)] \cdot \mathbf{n}_R\end{aligned}$$

and so at the midpoint m of the edge e

$$\begin{aligned}&\mathbf{u}_h^-(m) \cdot \mathbf{n}_L + \mathbf{u}_h^+(m) \cdot \mathbf{n}_R \\ &\quad = [(f_{K_L} - \alpha_{K_L} p_{K_L})\mathbf{P}_{K_L}(m) + \mathbf{C}_{K_L}] \cdot \mathbf{n}_L - \frac{1}{|e|}\int_{K_L} \tilde{f}\varphi dx + \frac{\alpha_{K_L}}{3|e|} p_h(m)|K_L| \\ &\quad + [(f_{K_R} - \alpha_{K_R} p_{K_R})\mathbf{P}_{K_R}(m) + \mathbf{C}_{K_R}] \cdot \mathbf{n}_R - \frac{1}{|e|}\int_{K_R} \tilde{f}\varphi dx + \frac{\alpha_{K_R}}{3|e|} p_h(m)|K_R|.\end{aligned} \tag{2.15}$$

We now *enforce* the continuity requirement at midpoint m of edge e

$$0 = \mathbf{u}_h^-(m) \cdot \mathbf{n}_L + \mathbf{u}_h^+(m) \cdot \mathbf{n}_R \tag{2.16}$$

in the following way. In triangle $K = K_L$ let $e = e_1$ with φ_1 be the basis on e_1,

$$|e_1|[(f_K - \alpha_K p_K)\mathbf{P}_K(m_1) + \mathbf{C}_K] \cdot \mathbf{n}_1 - \int_K \tilde{f}\varphi_1 dx + \frac{\alpha_K}{3} p_h(m_1)|K| = 0. \tag{2.17}$$

Similarly, on $e = e_2$ of K we set

$$|e_2|[(f_K - \alpha_K p_K)\mathbf{P}_K(m_2) + \mathbf{C}_K] \cdot \mathbf{n}_2 - \int_K \tilde{f}\varphi_2 dx + \frac{\alpha_K}{3} p_h(m_2)|K| = 0. \tag{2.18}$$

From (2.6), we have the fact that $\mathbf{P}_K \cdot \mathbf{n}_1 = \frac{|K|}{3|e_1|}$ and $\mathbf{P}_K \cdot \mathbf{n}_2 = \frac{|K|}{3|e_2|}$. Using this fact, (2.17) and (2.18) become

$$|e_1|\mathbf{n}_1 \cdot \mathbf{C}_K = \int_K \tilde{f}\varphi_1 - \frac{|K|}{3} f_K - \alpha_K \frac{|K|}{3}(p_h(m_1) - p_K), \tag{2.19}$$

$$|e_2|\mathbf{n}_2 \cdot \mathbf{C}_K = \int_K \tilde{f}\varphi_2 - \frac{|K|}{3} f_K - \alpha_K \frac{|K|}{3}(p_h(m_2) - p_K). \tag{2.20}$$

Remark 2.1. One can set up a similar equation on e_3, but it is easy to see that the equation can be obtained by summing the two above equations using $\sum |e_i|\mathbf{n}_i = \mathbf{0}$ and the assumption (2.11).

Let $R(\theta)$ be the rotation matrix through the angle θ, and noticing that $R(\frac{\pi}{2})\mathbf{n}_i = \mathbf{e}_i/|e_i|$, we can solve for $\mathbf{C}_K$ and obtain

$$\mathbf{C}_K = R(-\frac{\pi}{2})\begin{bmatrix}\mathbf{e}_1^T\\ \mathbf{e}_2^T\end{bmatrix}^{-1}\begin{bmatrix}\int_K \tilde{f}\varphi_1 - \frac{|K|}{3}f_K - \alpha_K\frac{|K|}{3}(p_h(m_1)-p_K)\\ \int_K \tilde{f}\varphi_2 - \frac{|K|}{3}f_K - \alpha_K\frac{|K|}{3}(p_h(m_2)-p_K)\end{bmatrix}. \tag{2.21}$$

Clearly with the above construction the flux is continuous in the normal direction at the midpoint of each interior side. Since $\mathbf{u}_h\cdot\mathbf{n} = $ const on each edge of K if $\mathbf{u}_h \in (RT)_0$, the flux is continuous across interior edges in the normal direction. Also it is easy to check that from (2.5) we have $\nabla\cdot\mathbf{u}_h = f_K - \alpha_K p_K$, which means local conservation of mass.

Remark 2.2. If the Neumann boundary condition $\mathbf{u}\cdot\mathbf{n} = 0$ is imposed then our choice of $\mathbf{C}_K$ automatically enforces $\mathbf{u}_h\cdot\mathbf{n} = 0$.

To see this, just apply (2.5), with either K_L or K_R void, since the absence of the Dirichlet condition enables us to use the global basis φ based at a boundary node. Arguing as before, we see that our choice of $\mathbf{C}_K$ then is equivalent to validating (2.16). If the Dirichlet condition is imposed then this step is not applicable since the φ is not available.

Remark 2.3. It should be clear that the above technique actually also works for the one-dimensional conforming case and the flux thus produced is continuous. Let us end this section by an example in one dimension where our flux formula reproduces the exact flux exactly.

Example 2.1. Consider the variational problem corresponding to the discontinuous coefficient boundary value problem

$$\begin{aligned} -(A(x)p')' &= 1, \qquad\qquad x\in(0,1),\\ p(0) = p(1) &= 0, \end{aligned}$$

where the coefficient $A(x)$ is a step function

$$A(x) = \begin{cases} 1 & \forall x\in I_1 = (0,\frac{1}{2}),\\ \frac{1}{2} & \forall x\in I_2 = (\frac{1}{2},1). \end{cases}$$

The exact solution $p(x) = -\frac{1}{2}x^2 + \frac{7}{12}x$ for $x\in I_1$ and $p(x) = -(x^2 - \frac{7}{6}x + \frac{1}{6})$ for $x\in I_2$. This time the $P1$ conforming FE solution is still the interpolant since the BVP's Green's function is still piecewise linear. Using p_h one can calculate the approximate flux $\mathbf{u}_h$ by (2.5), and once again see that the exact flux $\mathbf{u} = -A(x)p'(x)$ is exactly reproduced using only two elements I_1 and I_2. Also note that the interface continuity condition $2p'|_{I_1}(x) = p'|_{I_2}(x)$ at $x = 1/2$ is correctly produced.

3. Equivalence between conservative GFEMs and mixed box methods. Since our flux formula makes valid the local conservation property $\nabla\cdot\mathbf{u}_h = f_K - \alpha_K p_K$, the GFEM now can be considered conservative. So it makes sense to ask if the conservative GFEM is equivalent to some mixed finite volume methods. It turns out, as we will show below, that each of our conservative GFEMs is a mixed box method in disguise.

On the other hand, it is very natural for a finite volume person in favor of local conservation to use a mixed box method (see below) on the system (1.3) and then try to solve the resulting indefinite algebraic system using the mixed hybrid Lagrange multiplier method. Knowing the above equivalence (or a change of viewpoint) leads instead to the immediate decoupling of the pressure and flux. There is no need for the mixed Lagrange multiplier method, a nice feature not shared by the standard mixed FEM. We now elaborate on these points.

Let us introduce the lowest order discontinuous Raviart–Thomas space

$$\mathbf{V}_h^d = \{\mathbf{u}_h \in L^2(\Omega) : \mathbf{u}_h|_K \in RT_0(K) \quad \forall K \in \mathcal{T}_h\},$$

where $RT_0(K) = \{\mathbf{u} = (u^1, u^2) : u^1 = a + bx, u^2 = c + by \text{ in } K\}$. Thus a function in $\mathbf{V}_h^d$ doesn't need continuous normal components.

Let $\mathbb{C}$ be a piecewise constant function with respect to $\mathcal{T}_h$ and $\mathbb{C}|_K = \mathbb{C}_K\ \forall K$ in $\mathcal{T}_h$, let χ_K be the characteristic function of element K, and let $\mathbb{X}_K$ be any constant vector multiple of χ_K.

The following lemma will be used quite often in this section.

LEMMA 3.1. *Consider the mixed box method of finding* $(\mathbf{u}_h, p_h) \in \mathbf{V}_h^d \times Y_{h,0}$ *such that*

$$(\nabla \cdot \mathbf{u}_h + \alpha p_h - f, \chi_K) = 0 \quad \forall K \in \mathcal{T}_h, \tag{3.1}$$

$$(\mathbf{u}_h + \mathcal{K}\nabla p_h - \mathbb{C}, \mathbb{X}_K) = 0 \quad \forall K \in \mathcal{T}_h \tag{3.2}$$

along with the problem of finding $(\mathbf{u}_h, p_h) \in \mathbf{V}_h^d \times Y_{h,0}$ *such that over each* K *in* $\mathcal{T}_h$

$$\mathbf{u}_h = -\mathcal{A}_K \nabla p_h + (f_K - \alpha_K p_K)\mathbf{P}_K + \mathbb{C}_K. \tag{3.3}$$

Then the two above problems are equivalent.

Proof. We first show that (3.3) implies (3.1)–(3.2).

Take divergence on (3.3), recall (2.6), and integrate against the characteristic function χ_K to see that (3.3) implies (3.1). Now integrate (3.3) against $\mathbb{X}_K$ and use the fact that $(\mathbf{P}_K, \mathbb{X}_K) = 0$ (one point quadrature rule using the barycenter B) to get (3.2).

Second, we prove that (3.1)–(3.2) implies (3.3).

From (3.1) we see that on K

$$\nabla \cdot \mathbf{u}_h = f_K - \alpha_K p_K.$$

Since $\mathbf{u}_h \in \mathbf{V}_h^d$, by Taylor's expansion

$$\mathbf{u}_h = \mathbf{u}_K + \frac{1}{2}\nabla \cdot \mathbf{u}_h(\mathbf{x} - \mathbf{x}_B),$$

where $\mathbf{u}_K = \frac{1}{|K|}\int_K \mathbf{u}_h d\mathbf{x} = -\mathcal{A}_K \nabla p_h + \mathbb{C}_K$ by (3.2). So

$$\mathbf{u}_h = -\mathcal{A}_K \nabla p_h + (f_K - \alpha_K p_K)\mathbf{P}_K + \mathbb{C}_K$$

which is (3.3). ∎

Remark 3.1. Let us make two observations. First, the above two systems are both underdetermined. Secondly, the system (3.1)–(3.2) consists of two first order PDEs in weak form, while the problem (3.3) contains on the surface no second order PDEs at all. So if one is inclined to think that, in a mixed formulation of a second order PDE, one should obtain an equivalent system of two first order PDEs, the above lemma might appear strange. But formula (3.3) actually harbors or implies (but is not implied by) the following "second order" PDE in weak form:

$$\sum_K (\mathcal{K}\nabla p_h, \nabla q_h)_K + \sum_K \alpha_K (p_K, q_h)_K = \sum_K (f_K, q_h)_K - \sum_K ((\mathbf{u}_h - \mathbb{C})\cdot \mathbf{n}, q_h)_{\partial K}, \qquad q_h \in Y_{h,0}. \tag{3.4}$$

The reason why (3.4) does not imply (3.3) is that if $q_h \in Y_{h,0}$ satisfies $\mathbb{X}_K = \chi_K \xi$, $\xi = (0,1)^t$ or $(1,0)^t$, then q_h must have a support larger than K.

Let us derive (3.4) from (3.3). By Lemma 3.1 it suffices to use (3.1)–(3.2) to show the following. By (3.2), integration by parts, and (3.1)

$$\begin{aligned}\sum_K (\mathcal{K}\nabla p_h, \nabla q_h)_K &= \sum_K (-\mathbf{u}_h + \mathbb{C}_K, \nabla q_h)_K \\ &= \sum_K (\nabla\cdot \mathbf{u}_h, q_h)_K - (\mathbf{u}_h\cdot\mathbf{n}, q_h)_{\partial K} + (\mathbb{C}_K\cdot\mathbf{n}, q_h)_{\partial K} \\ &= \sum_K (f_K - \alpha_K p_K, q_h)_K - (\mathbf{u}_h\cdot\mathbf{n}, q_h)_{\partial K} + (\mathbb{C}_K\cdot\mathbf{n}, q_h)_{\partial K},\end{aligned}$$

which is (3.4).

Remark 3.2. Now if $\mathbf{u}_h - \mathbb{C} \in \mathbf{V}_h^d$ has continuous normal components (e.g., $\mathbb{C} = 0$ and $\mathbf{u}_h \in \mathbf{V}_h$), then p_h can be determined without knowledge of $\mathbf{u}_h$:

$$\sum_K (\mathcal{K}\nabla p_h, \nabla q_h)_K + \sum_K \alpha_K (p_K, q_h)_K = \sum_K (f_K, q_h)_K. \tag{3.5}$$

Here we have used the fact that $(\cdot,\cdot)_{\partial K}$ terms cancel upon summation since $(\mathbf{u}_h - \mathbb{C})\cdot\mathbf{n}$ and q_h are continuous at midpoints of edges.

Remark 3.3. Note that Lemma 3.1 and Remarks 3.1 and 3.2 are still valid if we replace the nonconforming space $Y_{h,0}$ by $X_{h,0}$, the corresponding $P1$ conforming space vanishing on the boundary.

Now choosing the constant function $\mathbb{C}$ to be identically zero, we have the following theorem.

THEOREM 3.2. *Consider the conservative GFEM: find* $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Y_{h,0}$ *such that*

$$\sum_K (\mathcal{K}\nabla p_h, \nabla q_h)_K + \sum_K \alpha_K (p_K, q_h)_K = \sum_K (f_K, q_h)_K \tag{3.6}$$

and

$$\mathbf{u}_h = -\mathcal{A}_K \nabla p_h + (f_K - \alpha_K p_K)\mathbf{P}_K. \tag{3.7}$$

Also consider the mixed finite volume method: find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Y_{h,0}$ *such that*

$$(\nabla\cdot\mathbf{u}_h + \alpha p_h - f, \chi_K) = 0, \tag{3.8}$$

$$(\mathbf{u}_h + \mathcal{K}\nabla p_h, \mathbb{X}_K) = 0. \tag{3.9}$$

Then the two problems are equivalent.

Proof. By Lemma 3.1, (3.7) is equivalent to (3.8)–(3.9) with the appended condition that $\mathbf{u}_h \in \mathbf{V}_h$. Equation (3.7) with $\mathbf{u}_h \in \mathbf{V}_h$ and $p_h \in Y_{h,0}$ is of course equivalent to (3.7) itself and its dependent relation (3.6) by (3.4) (setting $\mathbb{C} = \mathbf{0}$ and using $\mathbf{u}_h \in \mathbf{V}_h$). ∎

Remark 3.4. Unlike the two underdetermined systems in Lemma 3.1, the above conservative GFEM actually has a unique solution. In fact, the left side of (3.6) is coercive and hence p_h exists and is unique. So one can compute $\mathbf{u}_h$ by (3.7). To see that $\mathbf{u}_h$ is actually in $\mathbf{V}_h$ we can proceed as in setting up the relation (2.14) and show easily that $\mathbf{u}_h \cdot \mathbf{n}$ is continuous across interelement edges. An alternative to check that $\mathbf{u}_h \in \mathbf{V}_h$ is to observe that (3.8)–(3.9) is a square linear system and then use its equivalence to (3.7) to see that the homogeneous system has only zero solution [13].

Remark 3.4.1. The reader will find a numerical example (Example 6, section 5.2) for the method (3.6)–(3.7).

Comparing (3.6)–(3.7) with (3.4) (setting $\mathbb{C} = \mathbf{0}$), we conclude that

$$\sum_K (\mathbf{u}_h \cdot \mathbf{n}, q_h)_{\partial K} = 0. \tag{3.10}$$

In the nonconforming case, this equation is not surprising since one can also derive it quite easily noting that the normal component is continuous and that q_h is linear on e. However, the same equation becomes the appended constraint (3.15) in the counterpart of Theorem 3.2 for the conforming case.

THEOREM 3.3. *Consider the conservative GFEM: find* $(\mathbf{u}_h, p_h) \in \mathbf{V}_h^d \times X_{h,0}$ *such that*

$$\sum_K (\mathcal{K}\nabla p_h, \nabla q_h)_K + \sum_K \alpha_K (p_K, q_h)_K = \sum_K (f_K, q_h)_K \tag{3.11}$$

and

$$\mathbf{u}_h = -\mathcal{A}_K \nabla p_h + (f_K - \alpha_K p_K)\mathbf{P}_K. \tag{3.12}$$

Also consider the mixed finite volume method: find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h^d \times X_{h,0}$ *such that*

$$(\nabla \cdot \mathbf{u}_h + \alpha p_h - f, \chi_K) = 0 \quad \forall K \in \mathcal{T}_h, \tag{3.13}$$

$$(\mathbf{u}_h + \mathcal{K}\nabla p_h, \mathbb{X}_K) = 0 \quad \forall K \in \mathcal{T}_h, \tag{3.14}$$

$$\sum_K (\mathbf{u}_h \cdot \mathbf{n}, q_h)_{\partial K} = 0 \quad \forall q_h \in X_{h,0}, \tag{3.15}$$

where $\mathbf{n}$ *is the unit exterior normal to* ∂K*. Then the two problems are equivalent.*

Proof. Recall Remark 3.2 and the theorem can be easily proved along the same line as in Theorem 3.2. ∎

Remark 3.5. Due to the presence of the constraint (3.15), the Galerkin finite element method (3.11)–(3.12) is equivalent to a nontraditional finite volume box method. More importantly, the solution $\mathbf{u}_h$ of the GFEM may not have continuous normal components, but it does obey some kind of jump condition across interelements. To see this, let q_h in (3.15) be the global (pyramid) basis function based at O, then we see that

$$\sum_e [\mathbf{u}_h \cdot \mathbf{n}]|_e = 0, \tag{3.16}$$

where $[\cdot]$ stands for the jump across the edge e, $\mathbf{n}$ is a unit normal to e, and the summation is over all edges e from those $K \in \mathcal{T}_h$ having O as a common vertex.

Remark 3.6. A remark on the correct coupling of spaces is in order here. From Theorems 3.2 and 3.3, we see that the right combination of the pressure and velocity spaces is either the nonconforming pressure space with continuous Raviart–Thomas velocity space or the conforming pressure space with the discontinuous Raviart–Thomas space.

Remark 3.7. Returning to Theorem 3.2, one notes that the mixed finite volume method is equivalent to the standard nonconforming method when there is no absorption term ($\alpha = 0$). In the presence of the absorption term, it is not obvious (*Theorem* 3.2 *not applicable*) that the usual nonconforming method, which has $\sum_K(\alpha p_h, q_h)_K$ term *instead of* $\sum_K(\alpha_K p_K, q_h)_K$ *as in* (3.6)*, will produce continuous normal fluxes unless* $\alpha = 0$. For this reason we now return to developing the equivalence results along the above line for the conservative GFEM considered in the previous section.

THEOREM 3.4. *Consider the conservative GFEM: find* $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Y_{h,0}$ *such that*

$$\sum_K(\mathcal{K}\nabla p_h, \nabla q_h)_K + \sum_K(\alpha p_h, q_h)_K = \sum_K(\tilde{f}, q_h)_K \tag{3.17}$$

and

$$\mathbf{u}_h = -\mathcal{A}_K\nabla p_h + (f_K - \alpha_K p_K)\mathbf{P}_K + \mathbf{C}_K, \tag{3.18}$$

where $\mathbf{C}_K$ *is defined in* (2.21)*. Also consider the mixed finite volume method:* $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Y_{h,0}$ *such that*

$$(\nabla\cdot\mathbf{u}_h + \alpha p_h - f, \chi_K) = 0 \quad \textit{in } \Omega, \tag{3.19}$$

$$(\mathbf{u}_h + \mathcal{K}\nabla p_h - \mathbf{C}, \mathbb{X}_K) = 0 \quad \textit{in } \Omega, \tag{3.20}$$

where $\mathbf{C}$ *is the piecewise constant function such that* $\mathbf{C}|_K = \mathbf{C}_K$*. Then the two above problems are equivalent.*

Proof. Using Lemma 3.1 we immediately deduce the fact that (3.18) is equivalent to (3.19)–(3.20). Recall that (3.18) implies (3.4). So we need to show that (3.17) is the same equation as (3.4). Comparing the two equations and noting that $\sum_K(\mathbf{u}_h\cdot\mathbf{n}, q_h)_{\partial K} = 0$, we see that it suffices to show $\forall q_h \in Y_{h,0}$

$$\sum_K(\alpha_K p_K, q_h)_K - (\alpha p_h, q_h)_K = \sum_K(f_K - \tilde{f}, q_h)_K - \sum_K(\mathbf{C}_K\cdot\mathbf{n}, q_h)_{\partial K}. \tag{3.21}$$

To show this we can check its validity on each global basis function φ associated with edge e as in setting up relation (2.14). With the notations there we can easily verify the left-hand side of (3.21) is

$$\alpha_{K_L}p_{K_L}\frac{1}{3}|K_L| + \alpha_{K_R}p_{K_R}\frac{1}{3}|K_R| - \alpha_{K_L}p_h(m)\frac{1}{3}|K_L| - \alpha_{K_R}p_h(m)\frac{1}{3}|K_R|, \tag{3.22}$$

where m is the midpoint of e. Writing (2.19) in the present context, we have

$$|e|\mathbf{n}_i\cdot\mathbf{C}_{K_i} = \int_{K_i}\tilde{f}\varphi - \frac{|K_i|}{3}f_{K_i} - \alpha_{K_i}\frac{|K_i|}{3}(p_h(m) - p_{K_i}), \quad i = R, L. \tag{3.23}$$

Using this to compute the right-hand side of (3.21) and compare with its left-hand side (3.22), we see (3.21) holds. This completes the proof. ∎

4. Error estimates. In this section we show the general error estimate for $\tilde{f}$ that satisfies assumption (2.11) and (2.12) under flux formula (2.5).

We first present the error estimate for the pressure variable p. Define the seminorm

$$|q|_h := \left(\sum_{K\in\mathcal{T}_h} |q|_{1,K}^2 \right)^{1/2} \qquad \forall q \in H_0^1 \oplus Y_{h,0}. \tag{4.1}$$

It is clear $|\cdot|_h$ is a full norm on space $Y_{h,0}$ in the Dirichlet case. It is well known [7, 4, 5] that the solution p_h of system (2.9) converges to the solution p of

$$a(\tilde{p}, q) = (\tilde{f}, q) \qquad \forall q \in H_0^1, \tag{4.2}$$

that is, there exists a constant C independent of h such that

$$\|\tilde{p} - p_h\|_0 + h|\tilde{p} - p_h|_h \le Ch^2\|\tilde{p}\|_2 \tag{4.3}$$

provided that the problem data is smooth enough so that the solution $\tilde{p}$ is in H^2 and the elliptic regularity estimate $||\tilde{p}||_2 \le C||\tilde{f}||_0$ holds. For example, if $\tilde{f} \in L^2$ and $\mathcal{K} \in C^1(\bar{\Omega})$ on a convex domain Ω, then $\tilde{p} \in H^2$ is guaranteed (p. 4 of [4], [17]). Now we need only to measure the error $\|p - \tilde{p}\|_0$ and $\|p - \tilde{p}\|_h$.

LEMMA 4.1. *Assume $f \in L^2$ and locally $f \in H^1(K)$ $\forall K \in \mathcal{T}_h$; then there exists a constant C independent of h such that*

$$|p - \tilde{p}|_h \le Ch^2|f|_h, \tag{4.4}$$

$$\|p - \tilde{p}\|_0 \le Ch^2|f|_h. \tag{4.5}$$

Proof. Subtracting bilinear form (2.7) from (4.2), we get

$$a(p - \tilde{p}, q) = (f - \tilde{f}, q) \quad \forall q \in H_0^1.$$

We notice that $\int_K f - \tilde{f} = 0$, so $(f - \tilde{f}, q)_K = (f - \tilde{f}, q - q_K)_K$ where $q_K = \frac{1}{|K|}\int_K q$ is a constant function on each K. Then by Cauchy–Schwarz inequality, the assumption (2.12), and the interpolation theorem, we have

$$|(f - \tilde{f}, q)| \le Ch^2|f|_h|q|_h$$

and so

$$|a(p - \tilde{p}, q)| \le Ch^2|f|_h|q|_h.$$

Taking $q = p - \tilde{p}$, we have by the coercivity of $a(\cdot,\cdot)$,

$$|p - \tilde{p}|_h^2 \le Ch^2|f|_h|p - \tilde{p}|_h.$$

Therefore

$$|p - \tilde{p}|_h \le Ch^2|f|_h.$$

By Poincare inequality, we also get

$$\|p - \tilde{p}\|_0 \le Ch^2|f|_h. \qquad \square \tag{4.6}$$

Remark 4.1. We note that in the case of $\tilde{f} = f$, we don't need the assumption $f \in H^1$ locally for this lemma to hold. So in all the results that follow, the assumption $f \in H^1(K)$ can be removed for the $\tilde{f} = f$ case. Thus in all the results below one can drop the $\|f\|_h$ term on the right-hand side of the estimate if dealing with $\tilde{f} = f$ case.

Combining Lemma 4.1 and (4.3) we have the following error estimate for the pressure p.

THEOREM 4.2. *There exists a constant C independent of h such that*

$$\|p - p_h\|_0 + h|p - p_h|_h \le Ch^2(\|f\|_0 + |f|_h) \tag{4.7}$$

provided that the solution p is in H^2, and the L^2 function f is locally in $H^1(K)$ $\forall K \in \mathcal{T}_h$.

Before we prove the main error estimate theorem for $\mathbf{u}$, a remark is in order.

Remark 4.2. The flux formula $\mathbf{u}_h = -\mathcal{A}_K \nabla p_h + (f_K - \alpha_K p_K)\mathbf{P}_K + \mathbf{C}_K$ will be intuitively convergent in some sense to the exact flux if one observes $\mathbf{P}_K$ and $\mathbf{C}_K$ are both $O(h_K)$. We set up these relations in the next lemma.

LEMMA 4.3. *There exists a constant C independent of h such that*

$$\|\mathbf{C}_K\|_{0,K} \le Ch_K(\|\tilde{f}\|_{0,K} + \|p_h\|_{0,K}), \tag{4.8}$$

$$\|\mathbf{P}_K\|_{0,K} \le h_K^2, \tag{4.9}$$

$$|f_K| \le h_K^{-1}\|f\|_{0,K}, \tag{4.10}$$

$$|p_K| \le h_K^{-1}\|p_h\|_{0,K}. \tag{4.11}$$

Proof. We prove only the first inequality. The remaining ones are trivial. Note that

$$\|\mathbf{C}_K\|_{0,K} \le \left(\int_K \|\mathbf{C}_K\|_2^2 d\mathbf{x}\right)^{1/2} \le h_K\|\mathbf{C}_K\|_2.$$

From (2.21), the minimum angle property of partition $\mathcal{T}_h$, and Cauchy–Schwarz inequality, we have

$$\begin{aligned}
\|\mathbf{C}_K\|_2 &\le \left\| \begin{bmatrix} \mathbf{e}_1^T \\ \mathbf{e}_2^T \end{bmatrix}^{-1} \right\| \left(\left\| \begin{bmatrix} \int_K \tilde{f}\varphi_1 - \frac{|K|}{3} f_K \\ \int_K \tilde{f}\varphi_2 - \frac{|K|}{3} f_K \end{bmatrix} \right\|_2 + \alpha_K \left\| \begin{bmatrix} \int_K p_h\varphi_1 - \frac{|K|}{3} p_K \\ \int_K p_h\varphi_2 - \frac{|K|}{3} p_K \end{bmatrix} \right\|_2 \right) \\
&\le C/h_K \left(\left\| \begin{bmatrix} \int_K \tilde{f}\varphi_1 - \frac{|K|}{3} f_K \\ \int_K \tilde{f}\varphi_2 - \frac{|K|}{3} f_K \end{bmatrix} \right\|_2 + \alpha_K \left\| \begin{bmatrix} \int_K p_h\varphi_1 - \frac{|K|}{3} p_K \\ \int_K p_h\varphi_2 - \frac{|K|}{3} p_K \end{bmatrix} \right\|_2 \right) \\
&= C/h_K \left(\left\| \begin{bmatrix} \int_K \tilde{f}(\varphi_1 - 1/3) \\ \int_K \tilde{f}(\varphi_2 - 1/3) \end{bmatrix} \right\|_2 + \alpha_K \left\| \begin{bmatrix} \int_K p_h(\varphi_1 - 1/3) \\ \int_K p_h(\varphi_2 - 1/3) \end{bmatrix} \right\|_2 \right) \\
&\le C/h_K \left(\sqrt{\|\tilde{f}\|_{0,K}^2 \|\varphi_1 - 1/3\|_{0,K}^2 + \|\tilde{f}\|_{0,K}^2 \|\varphi_2 - 1/3\|_{0,K}^2} \right. \\
&\qquad \left. + \alpha_K \sqrt{\|p_h\|_{0,K}^2 \|\varphi_1 - 1/3\|_{0,K}^2 + \|p_h\|_{0,K}^2 \|\varphi_2 - 1/3\|_{0,K}^2} \right) \\
&= \sqrt{2}C/h_K(\|\tilde{f}\|_{0,K}\|\varphi_1 - 1/3\|_{0,K} + \alpha_K\|p_h\|_{0,K}\|\varphi_1 - 1/3\|_{0,K}) \\
&= \frac{2}{3}C\sqrt{|K|}/h_K(\|\tilde{f}\|_{0,K} + \alpha_K\|p_h\|_{0,K}) \\
&\le C(\|\tilde{f}\|_{0,K} + \|p_h\|_{0,K}).
\end{aligned}$$

Therefore

$$\|\mathbf{C}_K\|_{0,K} \le Ch_K(\|\tilde{f}\|_{0,K} + \|p_h\|_{0,K}). \qquad \square$$

Remark 4.3. Observe that

$$\tilde{f}_K = \frac{1}{|K|}\int_K \tilde{f} = \frac{1}{|K|}\int_K f = f_K,$$

and

$$|\tilde{f}_K| \le h_K^{-1}\|\tilde{f}\|_{0,K}.$$

Now we are ready to show the following error estimates for the flux $\mathbf{u}$.

THEOREM 4.4. *There exists a constant C independent of h such that*

$$\|\mathbf{u}-\mathbf{u}_h\|_0 \le Ch(\|f\|_{0,\Omega} + \|p\|_{0,\Omega} + |p|_{1,\Omega} + |f|_h), \tag{4.12}$$

$$|\mathbf{u}-\mathbf{u}_h|_{H(\mathrm{div})} \le Ch(|f|_h + |p|_1 + h\|f\|_{0,\Omega}) \tag{4.13}$$

provided that the solution p is in H^2, and the L^2 function f is locally in $H^1(K)$ $\forall K \in \mathcal{T}_h$.

Proof. From (2.5) we have

$$\mathbf{u}_h(\mathbf{x}) - \mathbf{u}(\mathbf{x})|_K = -\mathcal{A}_K\nabla p_h + \mathcal{K}\nabla p + (f_K - \alpha_K p_K)\mathbf{P}_K(\mathbf{x}) + \mathbf{C}_K.$$

So

$$\|\mathbf{u}_h - \mathbf{u}\|_{0,K} \le \|\mathcal{A}_K\nabla p_h - \mathcal{K}\nabla p\|_{0,K} + |f_K - \alpha_K p_K|\|\mathbf{P}_K\|_{0,K} + \|\mathbf{C}_K\|_{0,K}.$$

By the last lemma, the remark below it, and the boundedness assumption on α, we know

$$\begin{aligned}|f_K - \alpha_K p_K|\|\mathbf{P}_K\|_{0,K} + \|\mathbf{C}_K\|_{0,K} &\le (|f_K| + \alpha_K|p_K|)h_K^2 + Ch_K(\|\tilde{f}\|_{0,K} + \|p_h\|_{0,K}) \\ &\le Ch_K(\|\tilde{f}\|_{0,K} + \|p_h\|_{0,K}),\end{aligned} \tag{4.14}$$

whereas by the triangle inequality, boundedness of $\mathcal{K}$, the interpolation theorem, and the triangle inequality again

$$\begin{aligned}\|\mathcal{A}_K\nabla p_h - \mathcal{K}\nabla p\|_{0,K} &\le \|\mathcal{A}_K\nabla p_h - \mathcal{K}\nabla p_h\|_{0,K} + \|\mathcal{K}\nabla p_h - \mathcal{K}\nabla p\|_{0,K} \\ &= (\nabla p_h^T \int_K (\mathcal{A}_K - \mathcal{K})^2 d\mathbf{x}\nabla p_h)^{1/2} + \left(\int_K |\mathcal{K}(\nabla p_h - \nabla p)|^2 d\mathbf{x}\right)^{1/2} \\ &\le Ch_K|p_h|_{1,K}|\mathcal{K}|_{1,\infty,K} + Ch_K|p_h - p|_{1,K} \\ &\le Ch_K(|p_h|_{1,K} + |p_h - p|_{1,K}) \\ &\le Ch_K(|p|_{1,K} + |p_h - p|_{1,K}).\end{aligned}$$

Now combining this with (4.14), using the triangle inequality, and assumption (2.12), we have

$$\begin{aligned}\|\mathbf{u}_h - \mathbf{u}\|_{0,K} &\le Ch_K(\|\tilde{f}\|_{0,K} + \|p_h\|_{0,K} + |p|_{1,K} + |p_h - p|_{1,K}) \\ &\le Ch_K(\|\tilde{f} - f\|_{0,K} + \|f\|_{0,K} + \|p_h\|_{0,K} + |p|_{1,K} + |p_h - p|_{1,K}) \\ &\le Ch_K(h_K|f|_{1,K} + \|f\|_{0,K} + \|p_h\|_{0,K} + |p|_{1,K} + |p_h - p|_{1,K}).\end{aligned}$$

Summing over K, we have

$$\|\mathbf{u}_h - \mathbf{u}\|_{0,\Omega} \le Ch(h|f|_h + \|f\|_{0,\Omega} + \|p_h\|_{0,\Omega} + |p|_{1,\Omega} + |p_h - p|_h).$$

Using the triangle inequality and Theorem 4.2, we get

$$\begin{aligned}\|\mathbf{u}_h - \mathbf{u}\|_{0,\Omega} &\le Ch(h|f|_h + \|f\|_{0,\Omega} + \|p_h - p\|_{0,\Omega} + \|p\|_{0,\Omega} + |p|_{1,\Omega} + |p_h - p|_h)\\ &\le Ch(\|f\|_{0,\Omega} + \|p\|_{0,\Omega} + |p|_{1,\Omega} + |f|_h).\end{aligned}$$

Next we estimate the $H(\text{div})$ norm error:

$$\begin{aligned}\|\nabla\cdot\mathbf{u} - \nabla\cdot\mathbf{u}_h\|_{0,K} &= \|(f - \alpha_K p) - (f_K - \alpha_K p_K)\|_{0,K}\\ &\le \|\tilde{f} - f_K\|_{0,K} + \alpha_K\|p - p_K\|_{0,K}\\ &\le \|\tilde{f} - f_K\|_{0,K} + \alpha_K(\|p - p_h\|_{0,K} + \|p_h - p_K\|_{0,K}).\end{aligned}$$

Hence

$$\|\nabla\cdot\mathbf{u} - \nabla\cdot\mathbf{u}_h\|_{0,K} \le C(h_K|f|_{1,K} + \|p - p_h\|_{0,K} + h_K|p_h|_{1,K}). \tag{4.15}$$

Summing over K, using the triangle inequality and Theorem 4.2, we have

$$\begin{aligned}\|\mathbf{u}_h - \mathbf{u}\|_{H(\text{div}),\Omega} &\le C(h|f|_h + \|p - p_h\|_{0,\Omega} + h|p_h|_1)\\ &\le C(h|f|_h + \|p - p_h\|_{0,\Omega} + h|p|_1 + h|p_h - p|_1)\\ &\le Ch(|f|_h + |p|_1 + h\|f\|_{0,\Omega} + h|f|_h). \qquad \square\end{aligned}$$

Remark 4.4. In the conforming case, we know that $\mathbf{u}_h$ is not in $H(\text{div};\Omega)$. So we define a piecewise $H(\text{div})$ norm by

$$|\mathbf{u} - \mathbf{u}_h|^2_{h,H(\text{div})} := \sum_K \|\nabla\cdot\mathbf{u} - \nabla\cdot\mathbf{u}_h\|^2_{0,K}. \tag{4.16}$$

Then we argue as before and consequently by (4.15) we still have

$$|\mathbf{u} - \mathbf{u}_h|_{h,H(\text{div})} \le Ch(|f|_h + |p|_1 + h\|f\|_{0,\Omega}). \tag{4.17}$$

5. Numerical examples: Nonconforming case. In this section we give numerical results for the case $\tilde{f} = f$ which is the standard $P1$ nonconforming case. The case $\tilde{f} = f_K, \alpha = 0$ was reported in [13].

Notice that the error estimate theorem is valid for both Dirichlet and Neumann problems and we present numerical results for both cases. We partition the unit square $[0,1]\times[0,1]$ into squares evenly in both directions with the diagonals running from the upper-left corner of each triangle to its lower-right corner. The integral of f over element K is computed by the midpoint rule using the three edges of a triangle. Our experiments suggest a superconvergence property in a discrete L^2 norm of the flux $\mathbf{u}_h$. The discrete norms in which the errors are estimated are as follows.

5.1. Choice of discrete norms. In the error estimate section we predicted first order convergence in the L^2 norm for flux $\mathbf{u}$ and second order convergence in the L^2 norm for the pressure p. For computational reasons, we need to choose more convenient discrete norms to measure the error between the true and computed solutions.

Let (x_i, y_j) be the center of square (i,j) with $x_i = (i-1/2)h$, $y_j = (j-1/2)h$, $h = 1/n, i,j = 1,2,\dots,n$. Let p_{ij} be the computed pressure at (x_i, y_j). We define

$$pErr\_L2 := \left[\sum_{i,j=1}^{n} h^2(p(x_i,y_j) - p_{ij})^2\right]^{\frac{1}{2}},$$

i.e., a discrete L^2 norm of the error $p - p_h$. Also we compute the l^2 error of flux $\mathbf{u}$ across each edge in the normal direction.

$$uErr\_nml := \left\{ \sum_{e \in \mathcal{A}} \left[\int_e (\mathbf{u} - \mathbf{u}_h) \cdot \mathbf{n} ds \right]^2 \right\}^{1/2},$$

where the edge integrals are evaluated by the midpoint rule and $\mathbf{n}$ is a prefixed normal direction on corresponding edge. This discrete norm is similar to the ones defined in [12] and is actually the sum of three discrete L^2 norms defined there. This includes horizontal, vertical, and diagonal sweeps. On the other hand, it is also a discrete $H(\mathrm{div})$ norm, which is equivalent to a discrete L^2 norm due to the local conservation property [6].

5.2. Dirichlet problems. We consider the following Dirichlet problem:

$$\begin{cases} -\nabla \cdot \mathcal{K} \nabla p + \alpha p &= f \quad \text{in } \Omega, \\ \qquad\qquad p &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

Let the true pressure be $p = (x^2 - x)(y^2 - y)$ on the unit square in subsection 5.2. Note that the solution is a polynomial.

Below there is a table corresponding to each example. For instance, Table 1 corresponds to Example 1. In each table the corresponding example number is in the upper left corner.

TABLE 1
Dirichlet problem: $\mathcal{K} = \mathrm{diag}(1 + 10x^2 + y^2, 1 + x^2 + 10y^2)$, $\alpha = 0$.

Example 1	$h = 1/16$	$h = 1/32$	$h = 1/64$	$h = 1/128$	Order
pErr_L2	8.7872e-5	2.2256e-5	5.5828e-6	1.3969e-6	≈ 2
uErr_nml	0.0095	0.0024	5.9932e-4	1.5006e-4	≈ 2
Length of P	736	3,008	12,160	48,896	
Length of U	736	3,008	12,160	48,896	

TABLE 2
Dirichlet problem: $\mathcal{K} = \mathrm{diag}(10^4, 1)$, $\alpha = 0$.

Example 2	$h = 1/16$	$h = 1/32$	$h = 1/64$	$h = 1/128$	Order
pErr_L2	3.3867e-5	8.4986e-6	2.1158e-6	5.2599e-7	≈ 2
uErr_nml	4.5246	1.1407	0.2862	0.0716	≈ 2
uErr_rel	0.0032	7.8357e-4	1.9424e-4	4.8339e-5	≈ 2
Length of P	736	3,008	12,160	48,896	
Length of U	736	3,008	12,160	48,896	

Example 1. The mobility tensor $\mathcal{K} = \mathrm{diag}(1 + 10x^2 + y^2, 1 + x^2 + 10y^2)$, $\alpha = 0$. Note that the entries of $\mathcal{K}$ are of moderate size and continuous. The solutions are well behaved.

Example 2. The mobility tensor $\mathcal{K} = \mathrm{diag}(10^4, 1)$, $\alpha = 0$. We also report the relative errors due to the large magnitude of one entry in $\mathcal{K}$.

Example 3. In this example, we study the effect of discontinuous mobility matrix $\mathcal{K}$. Let $\mathcal{K} = \left[\begin{smallmatrix} 10000 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ on the left half unit square, and $\mathcal{K} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right]$ on the right half unit square, $\alpha = 0$. Note that the grid conforms with the interface.

TABLE 3
Dirichlet problem: $\mathcal{K}$ is discontinuous and $\alpha = 0$.

Example 3	$h = 1/16$	$h = 1/32$	$h = 1/64$	$h = 1/128$	Order
pErr_L2	1.0255e-4	2.6039e-5	6.5410e-6	1.6373e-6	≈ 2
pErr_rel	0.0031	7.8117e-4	1.9623e-4	4.9120e-5	≈ 2
uErr_nml	3.1981	0.8062	0.2022	0.0506	≈ 2
uErr_rel	0.0032	7.8319e-4	1.9413e-4	4.8311e-5	≈ 2
Length of P	736	3,008	12,160	48,896	
Length of U	736	3,008	12,160	48,896	

TABLE 4
Dirichlet problem: $\mathcal{K} = full\_tensor$, $\alpha = 0$.

Example 4	$h = 1/16$	$h = 1/32$	$h = 1/64$	$h = 1/128$	Order
pErr_L2	1.6297e-4	4.1601e-5	1.0458e-5	2.6183e-6	≈ 2
uErr_nml	0.0140	0.0036	9.0996e-4	2.3042e-4	≈ 2
Length of P	736	3,008	12,160	48,896	
Length of U	736	3,008	12,160	48,896	

Example 4. Let $\mathcal{K} = \begin{bmatrix} 1+10x^2+y^2 & 1/2+x^2+y^2 \\ 1/2+x^2+y^2 & 1+x^2+10y^2 \end{bmatrix}$, $\alpha = 0$.

Example 5. The mobility tensor $\mathcal{K} = \mathrm{diag}(1 + 10x^2 + y^2, 1 + x^2 + 10y^2)$, $\alpha = 1$.

Example 6. The mobility tensor $\mathcal{K} = \mathrm{diag}(1 + 10x^2 + y^2, 1 + x^2 + 10y^2)$, $\alpha = 1$. $\mathbf{C}_K = 0$ (cf. Remark 3.4.1 and Theorem 3.2).

In the beginning of this paper, we assume that α is nonnegative. In fact, we still can get good approximation when α is a small negative number.

Example 7. The mobility tensor $\mathcal{K} = \mathrm{diag}(1+10x^2+y^2, 1+x^2+10y^2)$, $\alpha = -1$.

5.3. Neumann problems.

$$\begin{cases} -\nabla \cdot \mathcal{K}\nabla p + \alpha p &= f \text{ in } \Omega, \\ \mathcal{K}\nabla p \cdot \mathbf{n} &= 0 \text{ on } \partial\Omega. \end{cases} \tag{5.2}$$

In this subsection, we take the true pressure $p = \cos(2\pi x)\cos(2\pi y)$.

Example 8. $\mathcal{K} = \mathrm{diag}(\cos(2\pi y) + 2, \cos(2\pi x) + 2)$, $\alpha = 1$.

TABLE 5
Dirichlet problem: $\mathcal{K} = \mathrm{diag}(1 + 10x^2 + y^2, 1 + x^2 + 10y^2)$, $\alpha = 1$.

Example 5	$h = 1/16$	$h = 1/32$	$h = 1/64$	$h = 1/128$	Order
pErr_L2	8.7619e-5	2.2195e-5	5.5677e-6	1.3931e-6	≈ 2
uErr_nml	0.0095	0.0024	5.9903e-4	1.4998e-4	≈ 2
Length of P	736	3,008	12,160	48,896	
Length of U	736	3,008	12,160	48,896	

6. Numerical examples: Conforming case. Recall that the $P1$ conforming finite element space is

$$X_{h,0} := \{q \in L^2(\Omega) : q|_K \in P_1(K)\ \forall K \in \mathcal{T}_h; q \text{ is continuous on interior edges and vanishes at the boundary edges}\}.$$

The standard $P1$ conforming FEM discretization is to find $p_h \in X_{h,0}$ such that

$$a_h(p_h, q_h) = (\tilde{f}, q_h) \qquad \forall q_h \in X_{h,0}, \tag{6.1}$$

Table 6
Dirichlet problem: $\mathcal{K} = \text{diag}(1 + 10x^2 + y^2, 1 + x^2 + 10y^2)$, $\alpha = 1$, $\mathbf{C}_K = 0$ *(box method); check Theorem* 3.2.

Example 6	$h = 1/16$	$h = 1/32$	$h = 1/64$	$h = 1/128$	Order
pErr_L2	8.7291e-5	2.2204e-5	5.5757e-6	1.3955e-6	≈ 2
uErr_nml	0.0113	0.0029	7.1905e-4	1.8023e-4	≈ 2
Length of P	736	3,008	12,160	48,896	
Length of U	736	3,008	12,160	48,896	

Table 7
Dirichlet problem: $\mathcal{K} = \text{diag}(1 + 10x^2 + y^2, 1 + x^2 + 10y^2)$, $\alpha = -1$.

Example 7	$h = 1/16$	$h = 1/32$	$h = 1/64$	$h = 1/128$	Order
pErr_L2	8.8132e-5	2.2319e-5	5.5984e-6	1.4008e-6	≈ 2
uErr_nml	0.0095	0.0024	5.9967e-4	1.5016e-4	≈ 2
Length of P	736	3,008	12,160	48,896	
Length of U	736	3,008	12,160	48,896	

where

$$a_h(p_h, q_h) := \sum_K \int_K (\mathcal{K}\nabla p_h) \cdot \nabla q_h d\mathbf{x} + \alpha_K \int_K p_h q_h. \tag{6.2}$$

We can still apply the formula (2.5) to obtain flux approximation after we have computed the pressure. The error estimates in Theorem 4.4 hold for the conforming case too (cf. Remarks 4.2 and 4.4), although we will not see the superconvergence property in the numerical example for flux variable $\mathbf{u}$ in the normal direction $\mathbf{n}$. In fact, unlike in the nonconforming case, the flux across interior edge in the normal direction may not even be continuous. Nevertheless we still have local conservation of mass $-\nabla \cdot \mathbf{u}_h = f_K - \alpha_K p_K$. We will demonstrate only for the Dirichlet case, since the Neumann boundary condition is not valid for the conforming case, as mentioned earlier. Also we run two cases: one in which $\mathbf{C}_K$ is defined by (2.21) and one in which $\mathbf{C}_K = \mathbf{0}$.

Table 8
Neumann problem: $\mathcal{K} = \text{diag}(\cos(2\pi y) + 2, \cos(2\pi x) + 2)$, $\alpha = 1$.

Example 8	$h = 1/16$	$h = 1/32$	$h = 1/64$	$h = 1/128$	Order
pErr_L2	0.0025	6.2201e-4	1.5475e-4	3.8610e-5	≈ 2
uErr_nml	0.0988	0.0250	0.0063	0.0016	≈ 2
Length of P	800	3,136	12,416	49,408	
Length of U	800	3,136	12,416	49,408	

6.1. Numerical examples for conforming method. We first choose proper discrete norms to measure the error between true solution and computed solution.

Let (x_i, y_j) be interior vertexes of unit square with $x_i = ih$, $y_j = jh$, $h = 1/n, i, j = 1, \ldots, n-1$. Let p_{ij} be the computed pressure at (x_i, y_j). We define

$$pErr\_L2 = \|p - p_h\| := \left[\sum_{i,j=1}^{n-1} h^2 (p(x_i, y_j) - p_{ij})^2 \right]^{\frac{1}{2}},$$

i.e., a discrete L^2 norm of the error $p - p_h$. uErr_nml are defined in the same way as the nonconforming case.

TABLE 9

Dirichlet problem: $\mathcal{K} = \mathrm{diag}(1 + 10x^2 + y^2, 1 + x^2 + 10y^2)$, $\alpha = 1$, *conforming method.*

Example 9	$h = 1/16$	$h = 1/32$	$h = 1/64$	$h = 1/128$	Order
pErr_L2	6.5500e-5	1.6505e-5	4.1340e-6	1.0340e-6	≈ 2
uErr_nml	0.1697	0.0877	0.0445	0.0224	≈ 1
Length of P	225	961	4,225	16,129	
Length of U	736	3,008	12,160	48,896	

TABLE 10

Dirichlet problem: $\mathcal{K} = \mathrm{diag}(1 + 10x^2 + y^2, 1 + x^2 + 10y^2)$, $\alpha = 1$, *conforming method,* $C_K = 0$ *for* $K \in \mathcal{T}_h$.

Example 10	$h = 1/16$	$h = 1/32$	$h = 1/64$	$h = 1/128$	Order
pErr_L2	6.5500e-5	1.6505e-5	4.1340e-6	1.0340e-6	≈ 2
uErr_nml	0.2529	0.1287	0.0649	0.0326	≈ 1
Length of P	225	961	4,225	16,129	
Length of U	736	3,008	12,160	48,896	

We take true pressure to be $p = (x^2 - x)(y^2 - y)$ on the unit square.

Example 9. The mobility tensor $\mathcal{K} = \mathrm{diag}(1 + 10x^2 + y^2, 1 + x^2 + 10y^2)$, $\alpha = 1$.

Example 10. The mobility tensor $\mathcal{K} = \mathrm{diag}(1 + 10x^2 + y^2, 1 + x^2 + 10y^2)$, $\alpha = 1$, but $\mathbf{C}_K = 0$ in formula (2.5).

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