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Power Series Solution to a Simple Pendulum with Oscillating Support

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POWER SERIES SOLUTION TO A SIMPLE PENDULUM WITH OSCILLATING SUPPORT *

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Abstract. The problem of determining some of the effects of a small forcing term on a regular perturbation solution to a nonlinear oscillation problem is studied via a simple example. In particular, we investigate the periodic solution of a simple pendulum with an oscillating support. A power series solution is constructed in terms of $\epsilon = (\frac{\omega}{\omega_o})^2 \frac{a}{L}$, where ω_o and ω are the natural and driving frequencies respectively, a is the amplitude of the support oscillation, and L is the length of the pendulum. These solutions are analyzed for three cases: above resonance ($\omega > \omega_o$), below resonance ($\omega < \omega_o$), and at resonance ($\omega = \omega_o$). In each case, the approximate location of the nearest singularities which limit the convergence of the power series are obtained by using Padé approximants. Using this information, a new expansion parameter δ is introduced, where the radius of convergence of the transformed series is greater than the original series. The effects of primary and higher order resonances on the convergence of the series solution is noted and discussed.

Key words. forced nonlinear oscillation, Padé approximants, power series, radius of convergence, regular perturbation expansion, simple pendulum

AMS (MOS) subject classification. 41

1. Introduction. We wish to study the use of a straightforward power series solution of a forced nonlinear oscillation problem involving a small perturbation parameter. The leading terms in many such problems have been calculated and studied by several investigators and their results are discussed, along with relevant references, in various texts, e.g., [3]. However, to date, relatively little has been established about the convergence and hence domain of validity of these series solutions. The purpose of this work is to investigate in some detail the power series solution to a simple forced nonlinear oscillation problem, with the goal of obtaining some insight into the range of validity of the series solution [4]. For the special case of a free nonlinear oscillation, this analysis was carried out successfully in [2], using some of the ideas presented in [1]. For many such cases, as we shall see below, a large number of terms in a power series solution can be calculated using a computer program. From these terms, an estimate of the radius of convergence of the series can be obtained [4] and, in most cases, the series can be recast into a form which converges for much larger values of the perturbation parameter. However, as we shall see, the radius of convergence of the series we shall be considering depends critically upon the value of the forcing fre-

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quency. Thus, unlike the free oscillation case where no such frequency was present, the range of values of the small parameter for which the first few terms in the perturbation solution provide a good approximation to the true solution will vary significantly with the value of the forcing frequency.

The model system we wish to study consists of a simple pendulum (e.g., a rigid rod with a mass attached) with its point of suspension moving horizontally according to a specified function of time (Fig. 1). The coordinates of the pendulum mass are

$$x = f(\tau) + L\sin\theta \quad \text{and} \quad y = -L\cos\theta$$

where $f(\tau)$ is the horizontal coordinate of the support and L is the length of the pendulum. Then the angle $\theta(\tau)$ which the pendulum makes with the vertical coordinate is governed by the equation

$$mL\ddot{\theta} + RL\dot{\theta} + mg\sin\theta = -mf''(\tau)\cos\theta. \tag{1.1}$$

In this equation $\dot{\theta}$ and $\ddot{\theta}$ are the first and second derivatives with respect to time τ, while $-RL\dot{\theta}$ is a damping force, m is the mass attached to the pendulum and g is the accelerate due to gravity. We shall assume that $R > 0$, i.e., that there is some positive damping in our model.

For simplicity, we shall assume that the horizontal oscillation of the point of suspension is described by $f(\tau) = a\sin\omega\tau$, where a is the amplitude and ω is the frequency of the oscillation. We then define dimensionless variables

$$t = \omega\tau, \quad \mu = \frac{\omega}{\omega_o}, \quad \epsilon = \frac{\omega^2}{\omega_o{}^2}\frac{a}{L} = \mu^2\frac{a}{L}, \quad r = \frac{R\omega}{m\omega_o{}^2} = \mu\frac{R}{m\omega_o} \tag{1.2}$$

where $\omega_o = \sqrt{g/L}$ is the natural frequency for small oscillations of the pendulum.

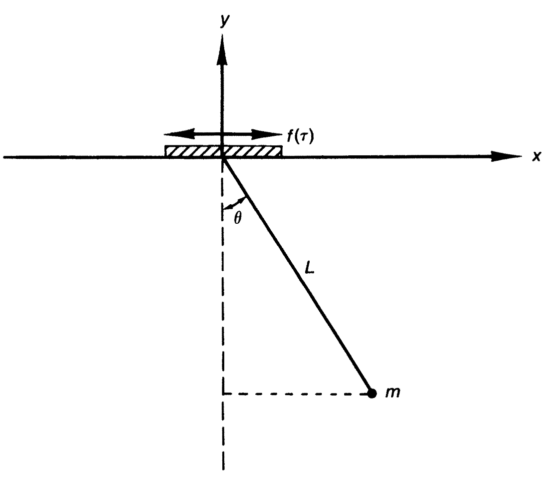

FIG. 1. *A simple pendulum with a support which moves horizontally as a function of time.*

Then, in terms of these variables, (1.1) becomes

$$\mu^2\ddot{\theta} + r\dot{\theta} + \sin\theta = \epsilon \sin t \, \cos\theta. \tag{1.3}$$

This is the equation we wish to study when ϵ is small. In particular, we wish to investigate periodic solutions to (1.3) for various values of the parameters μ, r, and ϵ. (We note that, since we are assuming that $r > 0$, the only periodic solution to (1.3) when $\epsilon = 0$ is $\theta \equiv 0$. Also, we restrict our attention to finding only 2π periodic solutions.)

2. Power series solution for small ϵ**.** The solution to equation (1.3) depends upon the parameters μ, r and ϵ. By using a regular perturbation expansion in terms of the parameter ϵ, a 2π-time-periodic solution of (1.3) can be found in the form

$$\theta = \theta(t,\epsilon) = \sum_{j=1}^{\infty} \theta_j(t)\epsilon^j, \qquad \theta_j(t+2\pi) = \theta_j(t) \tag{2.1}$$

where the coefficients $\theta_j(t)$s will depend on μ and r. Substitution of (2.1) into (1.3) leads to the following sequence of ordinary differential equations for the $\theta_j(t)$:

$$\mu^2\ddot{\theta}_j + r\dot{\theta}_j + \theta_j = f_j, \qquad j = 1, 2, \cdots \tag{2.2}$$

with

$$f_j = \sin t \sum_{k=0}^{j-1} b_k \theta_{k,j-1-k} \; - \; \sum_{k=2}^{j} a_k \theta_{k,j-k}, \qquad j = 1, 2, \cdots \tag{2.3}$$

where

$$a_{2k} = 0 \quad \text{and} \quad a_{2k+1} = \frac{(-1)^k}{(2k+1)!}, \qquad k = 0, 1, \cdots,$$

$$b_{2k+1} = 0 \quad \text{and} \quad b_{2k} = \frac{(-1)^k}{(2k)!}, \qquad k = 0, 1, \cdots. \tag{2.4}$$

From these equations, each θ_j and f_j can be evaluated successively. (Note that each f_j involves only the θ_k's with $k < j$.) The functions $\theta_{k,j}$ appear when (2.1) are used to represent $\sin\theta$ and $\cos\theta$. The recurrence relation for these functions is:

$$\theta_{0,0} = 1, \quad \theta_{0,j} = 0, \qquad j \geq 1, \quad \theta_{1,j} = \theta_{j+1}, \qquad j \geq 0,$$

$$\theta_{k,j} = \sum_{l=1}^{j+1} \theta_{k-1,j+1-l}\,\theta_l, \qquad k \geq 2. \tag{2.5}$$

To obtain explicit exressions for the periodic solutions for θ_j, we represent each θ_j and f_j as a finite linear combination of $\sin nt$ and $\cos nt$, $n = 1, 2, \cdots$. We compute the first few θ_j and f_j explicitly by hand and then use induction to show that

$$\theta_j(t) = C_j + \sum_{n=1}^{j} \alpha_{j,n} \sin nt + \beta_{j,n} \cos nt, \qquad j = 1, 2, \cdots,$$

$$f_j(t) = C'_j + \sum_{n=1}^{j} \gamma_{j,n} \sin nt + \delta_{j,n} \cos nt, \qquad j = 1, 2, \cdots \tag{2.6}$$

where $C_j, C'_j, \alpha_{j,n}, \beta_{j,n}, \gamma_{j,n}$, and $\delta_{j,n}$ are certain constants, with $\alpha_{j,n} = \beta_{j,n} = \gamma_{j,n} = \delta_{j,n} = 0$, when either j or n is an even number. We also note that the coefficients in $\theta_j(t)$ can be expressed in terms of the coefficients in $f_j(t)$ as

$$\alpha_{j,n} = \frac{(1-n^2\mu^2)\gamma_{j,n} + nr\delta_{j,n}}{(1-n^2\mu^2)^2 + (nr)^2}, \quad \beta_{j,n} = \frac{(1-n^2\mu^2)\delta_{j,n} - nr\gamma_{j,n}}{(1-n^2\mu^2)^2 + (nr)^2}, \quad j = 1, 2, \cdots. \tag{2.7}$$

The problem of determining the $\theta_j(t)$ has now been reduced to a purely numerical problem. For this purpose, we have constructed a FORTRAN program to carry out the required numerical calculations in extended precision arithmetic (to reduce the effects of roundoff errors). Due to storage limitations, we could only compute approximately 50 terms in the series (2.1) and hence we must content ourselves with an approximate solution for $\theta(t)$ in the form

$$\theta(t, \epsilon) = \sum_{j=1}^{50} \theta_j(t)\epsilon^j + O(\epsilon^{51}). \tag{2.8}$$

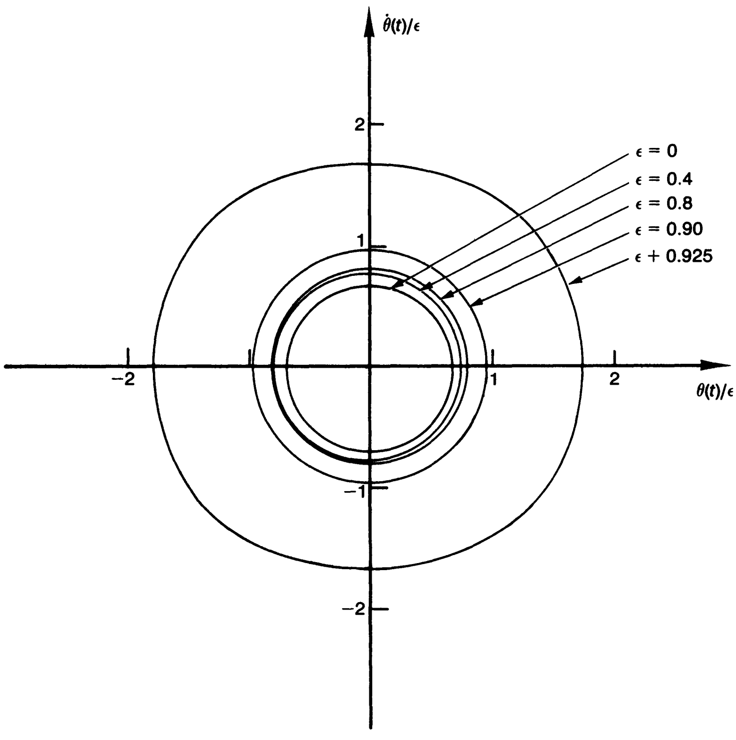

FIG. 2. *Phase plane plots for the limit cycle $\theta(t, \epsilon)$ obtained from (2.8), with $\mu = 1.5$ and $r = 0.075$.*

3. Convergence of the power series. We now use the results of §2 to compute the power series solution of a simple pendulum with oscillating support for some particular values of the parameters μ and r and discuss how well these series can approximate the exact solution. We find it convenient to distinguish three cases, corresponding to the driving frequency ω being greater than, equal to, or less than the natural frequency ω_o. (The case $\omega = \omega_o$ corresponds to the primary resonance case.)

3.1. $\mu = \omega/\omega_o > 1$ (Above the resonance). We have studied this case for several values of μ and r, with the particular values of $\mu = 1.5$ and $r = 0.075$ yielding typical results. To display and analyze our results, we compute $\theta_j(t)$, $j = 1, 2, \cdots, 50$ for $0 \leq t \leq \pi$ at intervals of $\Delta t = 2\pi/100$. It can be shown that the coefficients $\theta_j(t)$ and $\dot{\theta}_j(t)$ satisfy the relations

$$\theta_j(t) = -\theta_j(t-\pi), \qquad \dot{\theta}_j(t) = -\dot{\theta}_j(t-\pi) \tag{3.1}$$

and hence they need to be evaluated only for $0 \leq t \leq \pi$.

In order to study the behavior of the resulting periodic solution and to see for what values of the parameter ϵ the series converges, we first plot the limit cycles in the phase plane for various values of ϵ. A closed phase plane trajectory corresponds to periodic motion. Figure 2 shows the limit cycles for some selected values of ϵ and for $\mu = 1.5$ and $r = 0.075$. From these diagrams, we observe that for $\epsilon < 0.9$, the limit cycles are closed and smooth, but begin to grow very large for $\epsilon \geq 0.9$.

To investigate this behavior more carefully, we have used the ratio and root tests on the coefficients $\theta_j(t)$ for some predetermined values of t. Figure 3 shows some of these results. By means of a linear extrapolation, we estimate from the ratio tests that the radius of convergence R for the series $\theta_j(t)$ is approximately 0.80, while from the root tests R seems to be approximately 0.83. The results of these tests are very consistent for different values of t. In particular, we divided the interval $(0, \pi)$ into 50 equally spaced subintervals and performed these tests on the series $\theta_j(t)$ for a representative value of t within each of these subintervals. We found no significant variation of our estimate of R with t. Thus, it appears that the radius of convergence of these series is independent of the value of t.

Using Padé approximants (see [1], [2] and [4]) on the series $\theta_j(t)$, we have determined the locations of the singularities of $\theta(t, \epsilon)$ near the origin in the complex ϵ-plane. The zeros of the numerator and denominator are located near the imaginary axis. Based on the results of the Padé approximants, it appears that zeros of the numerators and denominators are simulating branch point type singularities. (See [1] and [2] where a similar pattern of zeros and poles was observed. In addition, we have studied the use of Padé approximants with several model functions involving branch point singularities. In each case, we used Padé approximants to locate the branch points from the Taylor series coefficients and found excellent agreement in all cases.) The distance R of the singularity (or singularities, if more than one) closest to the origin determines the radius of convergence of the series [4]. From our results we write the singularity closest to the origin in the form $Re^{\pm i\beta}$ and estimate

$$R \cong 0.81 \quad and \quad \beta \cong \frac{\pi}{2} \tag{3.2}$$

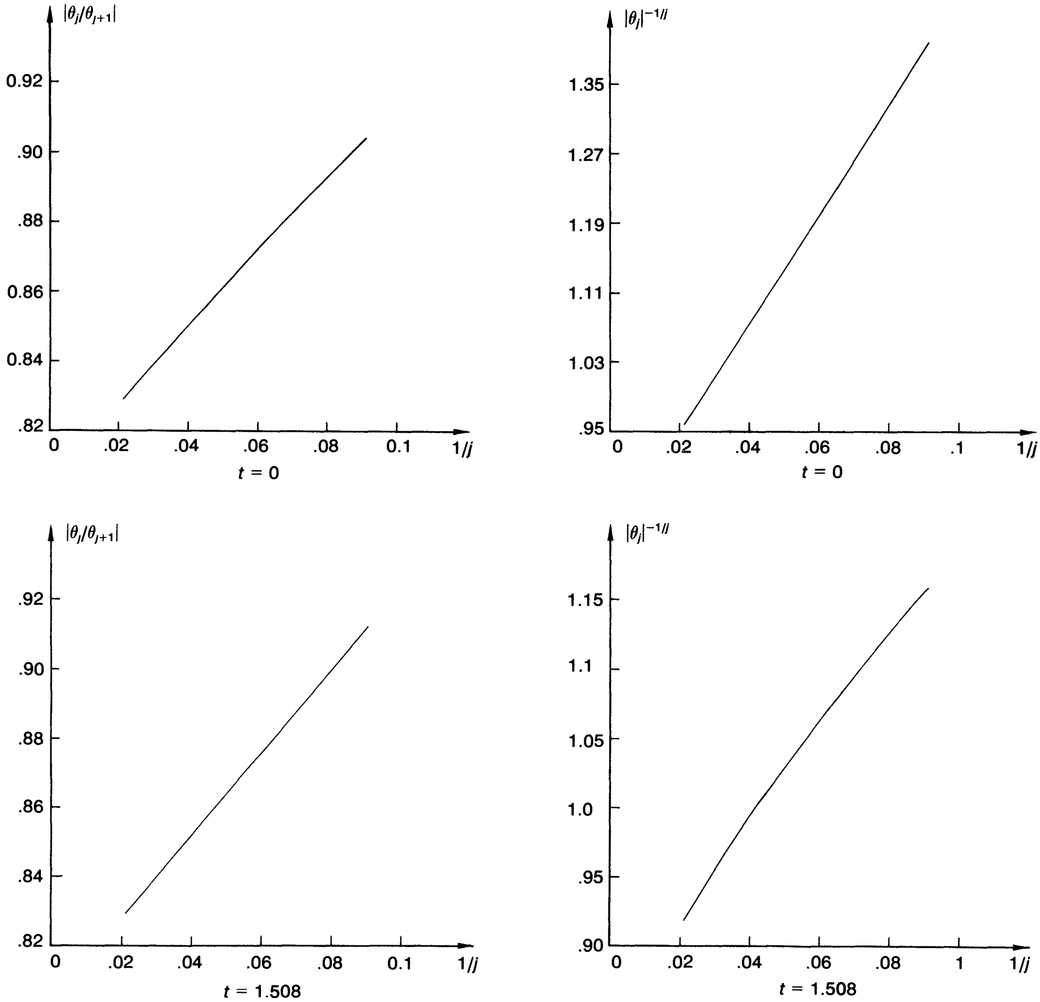

FIG. 3. *The ratio and root tests on the coefficients* $\theta_j(t)$ *of (2.8), with* $\mu = 1.5$ *and* $r = 0.075$.

where R is the distance of the singularity from the origin and β is the argument of the singularity with respect to the positive real axis. We observe that the results from Padé approximants agree with those obtained by using the ratio and root tests. Moreover, these parameters remain fixed for all 50 different values of t $(0 \leq t \leq \pi)$ which we have examined and thus they confirm our previous statement, i.e., parameters R and β appear to be constant as t changes from 0 to π.

3.1.1. Transformation of the series. Using the information about the location of the nearest singularities in the complex ϵ-plane, we find that the two nearest singularities are located on the imaginary axis as a pair of complex conjugate points at

$$\epsilon = Re^{\pm i\beta}. \tag{3.3}$$

The transformation formula should map these points to infinity and should at the same time bring $\epsilon = \infty$ to the unit distance from the origin [4]. Therefore, we have

$$\delta(\epsilon) = \frac{\epsilon}{(R^2 - 2R\epsilon\cos\beta + \epsilon^2)^{1/2}}$$

which, using (3.2), becomes

$$\delta(\epsilon) = \frac{\epsilon}{(R^2 + \epsilon^2)^{1/2}}. \tag{3.4}$$

With this transformation formula, the origin remains fixed and with (3.2) the following relations are satisfied:

$$\delta(\infty) = 1 \quad \text{and} \quad \delta(Re^{i\beta}) = \delta(Re^{-i\beta}) = \infty. \tag{3.5}$$

We note that a similar transformation was used successfully in [1] and [2] when branch point singularities were involved. In particular, we also note that (3.5) maps points close to (3.3) far from the origin and hence outside of the unit circle in the transformed plane.

Using (3.4), we convert the series (2.1) in the complex ϵ-plane to a new series in the δ-plane. In the complex δ-plane, the solution is approximated by

$$\theta(t, \delta) = \sum_{j=1}^{50} \tilde{\theta}_j(t)\delta^j + O(\delta^{51}) \tag{3.6}$$

where $\tilde{\theta}_j(t)$s are new coefficients which can be determined from the $\theta_j(t)$ in a straightforward manner. We expect that this series will converge for $\delta \leq 1$ or its equivalent, $\epsilon < \infty$. Using Padé approximants on the new series, the radius of convergence R_δ in the complex δ-plane is

$$R_\delta \cong 1.00. \tag{3.7}$$

This indicates that our estimates for R and β are reasonably accurate, and that the definition (3.4) is the proper transformation formula to use. Roundoff errors have not been serious in our computation.

The new series (3.6) can be used to approximate the periodic solution for all values of ϵ. Figure 4 shows the limit cycles for different values of ϵ. As ϵ increases, the limit cycles become larger and approach a unique closed curve. We observe that as $\epsilon \rightarrow \infty$, the radius of the limit cycles approaches a fixed value of about 1.38 and thereafter remains constant. This indicates that the periodic motion of the pendulum does not depend upon the parameter ϵ for significantly large values of ϵ.

We have investigated this series solution for other values of μ and r (above resonance). Table 1 shows the results. As μ increases, the radius of convergence of the series increases. This appears to be true for μ up to about 10.0, which is above the first subharmonic resonance of our system. We shall discuss this matter further in §4.

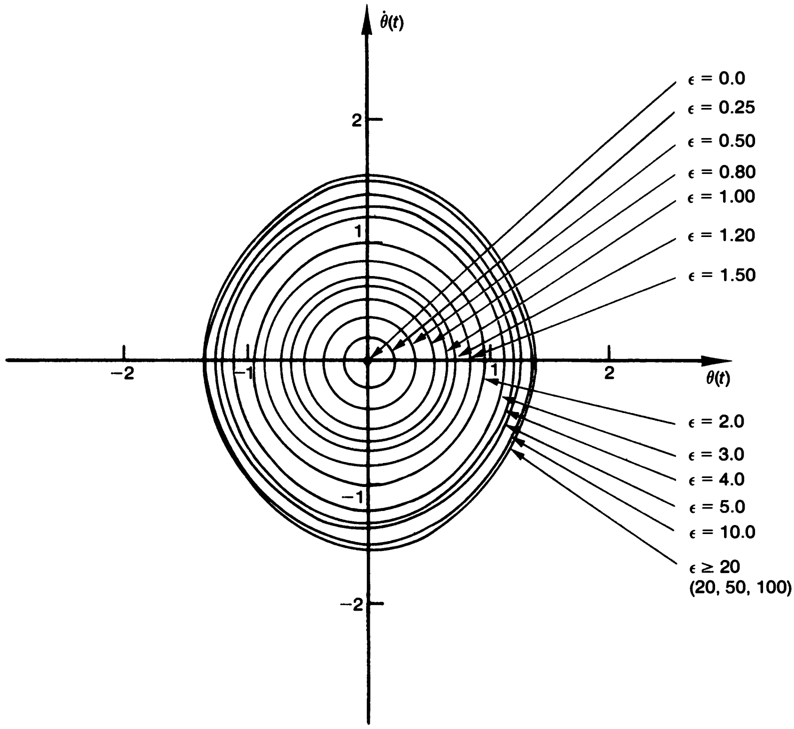

FIG. 4. *The limit cycles of the series solution (3.6) in terms of the parameter expansion* δ, *with* $\mu = 1.5$ *and* $r = 0.075$.

3.2. $\mu < 1$ **(Below the resonance).** We have performed a similar analysis for values of $\mu < 1$, for which the case of $\mu = 0.5$ and $r = 0.025$ is a typical case. The radius of convergence of the power series is about 1. In this case, the transformed series obtained in terms of the variable

$$\delta(\epsilon) = \frac{\epsilon}{\left(R^4 - 2R^2\epsilon^2 \cos 2\beta + \epsilon^4\right)^{1/4}} \tag{3.8}$$

with $R = 1.07$ and $\beta = 1.47$ has a radius of convergence of about 2.9.

Since we do not have accurate estimates for R and β, the transformed series does not approximate the solution for large values of ϵ. The radius of convergence R of the series solution for other values of μ are shown in Table 1 and are illustrated in Fig. 7. The relative minima which appear in the figure for $\mu < 1$ will be discussed in §4.

3.3. $\mu = 1$ **(Resonance).** The case $\mu = \omega/\omega_o = 1$, for which the free and forced oscillations have the same frequencies, corresponds to primary resonance of our system. In this section we investigate the series solution to the pendulum for $\mu = 1$

TABLE 1

The parameters R and β of the nearest singularities in the first quadrant of the complex ε-plane using [12/12] Padé approximants.

μ	r	R	β
0.100	0.0050	0.80	1.48
0.180	0.0090	0.777	1.466
0.190	0.0095	0.619	1.366
0.200	0.0100	0.45	0.80
0.210	0.0105	0.697	0.260
0.220	0.0110	0.983	0.163
0.250	0.0125	0.95	1.46
0.300	0.0150	0.69	1.37
0.310	0.0155	0.607	1.332
0.320	0.0160	0.505	1.248
0.330	0.0165	0.401	0.975
0.333	0.0166	0.391	0.824
0.336	0.0168	0.404	0.671
0.340	0.0170	0.448	0.517
0.350	0.0175	0.593	0.352
0.400	0.0200	1.10	1.46
0.500	0.0250	1.07	1.47
0.600	0.0300	1.07	1.28
0.700	0.0350	0.96	0.98
0.800	0.0400	0.60	0.00
0.900	0.0450	-	- *
1.000	0.0500	0.02	0.04
1.250	0.0625	0.318	$\pi/2$
1.500	0.0750	0.81	$\pi/2$
1.750	0.0875	1.414	$\pi/2$
2.000	0.1000	2.119	$\pi/2$
2.250	0.1125	2.923	$\pi/2$
2.500	0.1250	3.824	$\pi/2$
2.750	0.1375	4.820	$\pi/2$
2.850	0.1425	5.246	$\pi/2$
2.900	0.1450	5.464	$\pi/2$
3.000	0.1500	5.913	$\pi/2$
3.500	0.1750	8.384	$\pi/2$
4.000	0.2000	11.237	$\pi/2$
4.500	0.2250	14.471	$\pi/2$
5.000	0.2500	18.086	$\pi/2$
5.500	0.2750	22.110	$\pi/2$
6.000	0.3000	26.49	$\pi/2$
6.500	0.3250	31.31	$\pi/2$
7.000	0.3500	36.46	$\pi/2$
10.000	0.5000	75.79	$\pi/2$

* For all values of t except $t = 12 * 2\pi/100$, the zeros of numerator and denominator cancel each other.

and illustrate our results for the case when $r = 0.05$. In order to avoid some numerical difficulties, we introduce a new parameter $\epsilon' = \epsilon/r$ and rewrite equation (1.3) as

$$\mu^2\ddot{\theta} + r\dot{\theta} + \sin\theta = \epsilon' \sin t\,(r\cos\theta). \tag{3.9}$$

Using [12/12] Padé approximants, we can estimate the radius of convergence of the series (2.1). The parameters R and β of the singularities in the complex ϵ'-plane nearest the origin remain constant as t changes from 0 to π. We find $R_{\epsilon'} \cong 0.40$, which corresponds to $R_\epsilon \cong 0.02$, and $\beta \cong 0.79$. The limit cycles (Fig. 5) are circles with finite radii for $\epsilon' < 0.45$ and then start to grow indefinitely. As ϵ' increases, the limit cycles deviate from circles and for $\epsilon' \geq 0.50$, the series diverges rapidly.

Using $R_{\epsilon'} = 0.40$ and $\beta = 0.79$ in transformation (3.8), we obtain the revised series in the complex δ-plane. The radius of convergence of the new series from Padé approximants is estimated to be $R_\delta \cong 1.00$. Figure 6 represents some limit cycles for different values of ϵ'. The limit cycles are circles in which their radii vary as a function of ϵ'. As ϵ' increases, these circles grow and, for large values of ϵ' ($\epsilon' \geq 1.0$), approach

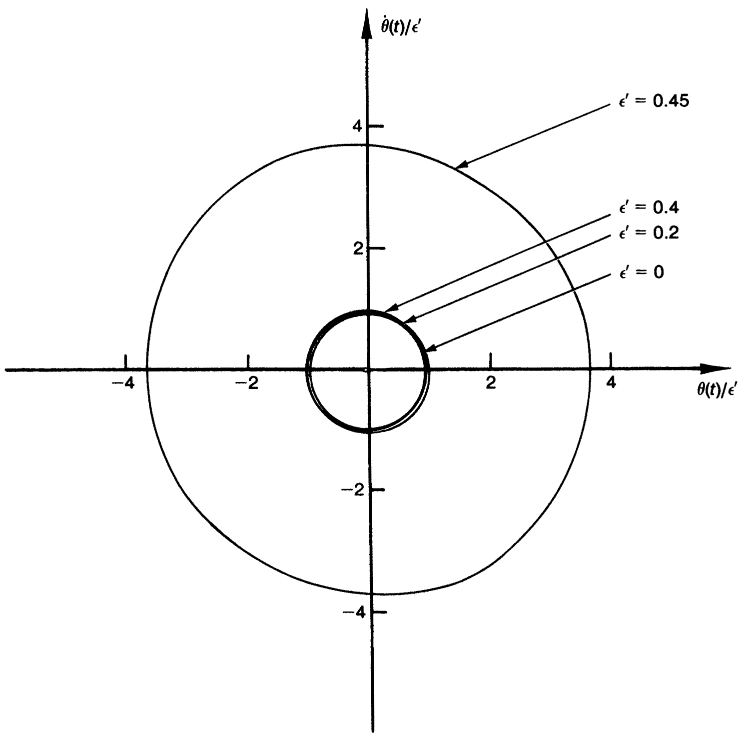

FIG. 5. *Phase plane trajectories for the limit cycle $\theta(t, \epsilon)$ computed from the series (2.8) with $\mu = 1.0$ and $r = 0.05$.*

a unique circle. The circles indicate that the periodic motion of the pendulum is well approximated by simple harmonic motion. Thus, our method can be used to approximate the solution for all values of the parameter ϵ.

4. Effects of secondary resonances and concluding remarks. For a simple pendulum with oscillating support, we have found an approximate periodic solution in the form of a convergent power series in the parameter ϵ. We have done this for three different cases, corresponding to above, below and at the primary resonance.

In the three cases studied, the power series solution to this problem is limited to finite values of ϵ. In order to obtain an approximate solution for a larger range of values of ϵ, we have introduced a new parameter δ as a function of ϵ and have transformed the original series into a new series in terms of the new expansion parameter δ. In most cases, the transformed series can be used to approximate the solution of the pendulum for all real values of ϵ.

Using Padé approximants, we have computed the parameters R and β of the

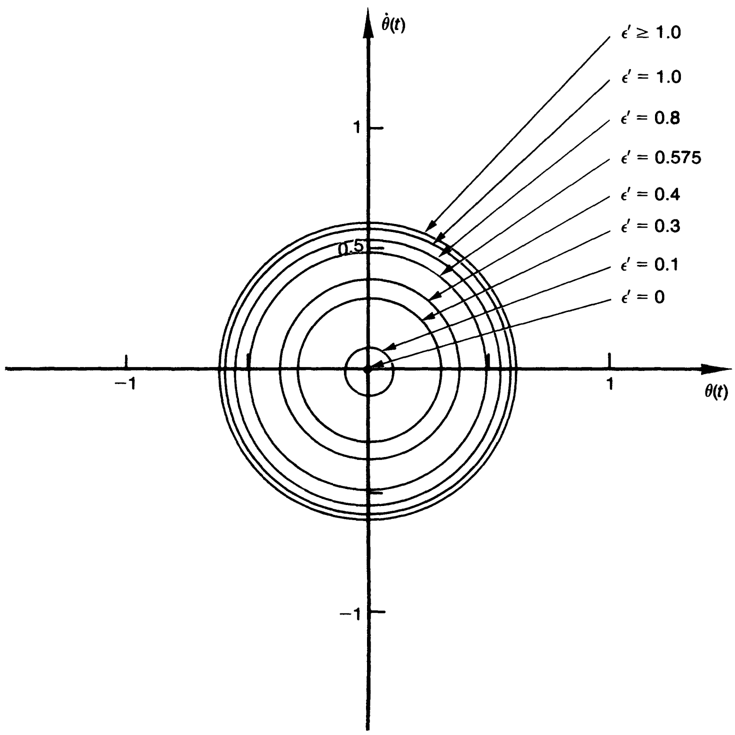

FIG. 6. *Phase plane trajectories for the limit cycle $\theta(t, \epsilon)$ computed from the transformed series, with* $\mu = 1.0$ *and* $r = 0.05$.

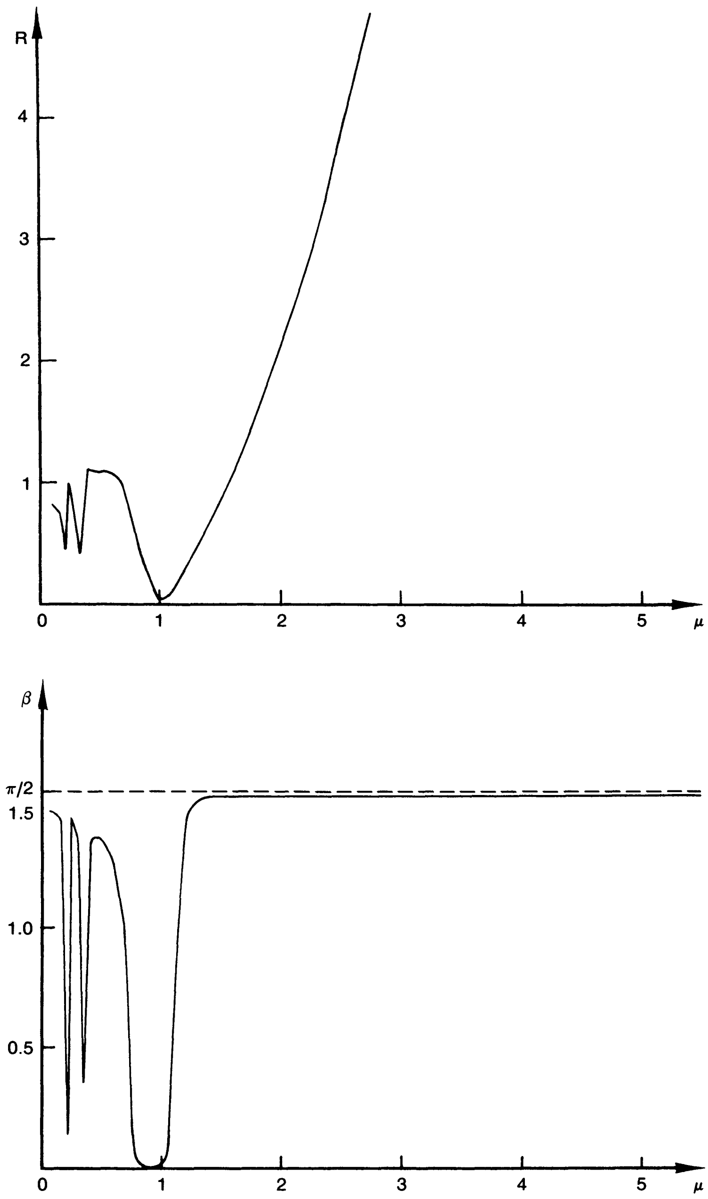

FIG. 7. *The parameters R and β of the nearest singularities in the upper half complex ϵ-plane of the [12/12] Padé approximants to the series (2.8) as a function of $\mu = \frac{\omega}{\omega_o}$.*

singularities in the complex ϵ-plane closest to the origin. The results are shown in Table 1 and Fig. 7. (The results we have shown here are representative of several different combinations of parameter values we have investigated. Qualitatively, the other results are not significantly different from those we have shown.) It appears that in the neighborhood of the primary resonance ($\mu = 1$), the power series solution is valid only for ϵ less than about 0.1. Also, from Fig. 7, it appears that the radius of convergence of our series has a local minimum at the primary and higher order (superharmonic) resonances [3]. In particular, we notice that for each of the superharmonic resonances of $\omega = \frac{1}{3}\omega_o$ ($\mu = \frac{1}{3}$) and $\omega = \frac{1}{5}\omega_o$ ($\mu = \frac{1}{5}$), there exists a local minimum for the radius of convergence of the power series. Presumably, this will also be true for all higher order resonances. In each case, however, the values of R and β appear to be independent of t, which is in contrast to the case of the free van der Pol limit cycle [2]. For this case, it was found that the distance of the singularity closest to the origin varied with t for $0 \leq t \leq 2\pi$ and gave rise to a radius of convergence of the series which varied with t.

For values of $\mu > 1$, the radius of convergence of our series increases monotonically with μ (see Table 1). In particular, the subharmonic resonances (e.g. $\omega = 3\omega_o$ or $\mu = 3$) do not seem to have the same effects on R as the superharmonic resonances.

Finally to investigate the behavior of the maximum angular displacement (am-

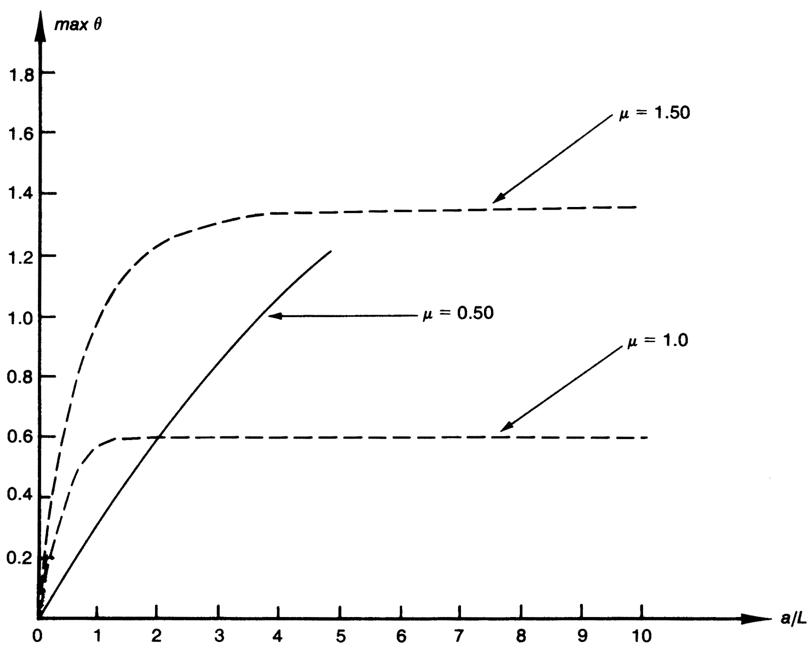

FIG. 8. *The maximum angular displacement as a function of* $\frac{a}{L}$, *for* $\mu = 0.5$, $\mu = 1.0$ *and* $\mu = 1.5$.

plitude of the periodic motion) as a function of ϵ (or more explicitly as a function of a/L), we have plotted in Fig. 8 the maximum amplitude of the motion vs. a/L for three representative cases. For $\mu = 1$ and $\mu = 1.5$, we observe that the angular amplitude rapidly approaches a limit and thereafter remains essentially constant as a/L increases. For $\mu = 0.5$, since the transformed series in the complex δ-plane does not converge for large values of ϵ, we are not able to predict the behavior of the amplitude as a/L increases. Presumably it approaches a finite limit for large values of a/L.

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