THE ANALYSIS OF DECICIONS ON FINANCIAL LEVERAGE AND INFORMATION*

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1. Introduction

Our primary objectives are to evaluate the value of financial leverage of a firm in view of various kinds of utility functions and to study financial leverage in terms of information.

Teble 1 provides definitions of symbols that are used in this paper. A tilde over a symbol is used to indicate a random variable. A bar indicates the expected value of a random variable. There are two fundamental assumptions throughout this paper:

1. The rate of return on capital r is a random variable having a finite mean and variance where $\sigma_r \neq 0$. Its probability distribution is independent of financial leverage.
2. That the firm could borrow unlimited amount at the rate of interest i equal to the lending rate.[1]

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Symbols (Table 1)

$e\,(e_j)$	rate of return on equity (of j)
E	equity
FLL	Financial Leverage Line
i	rate of interest
Ir	quantity of prior information
Is	quantity of sample information
K, K'	constant
L	Liability
r	rate of return on total capital
u	utility function
$\tilde{v}$	standard normal random variable; N (0,1)
α, λ	coefficients of risk aversion
β	ratio of cost of information over $(\bar{r}-i)$
η	financial leverage $\frac{L}{E}$
ξ	slope of FLL
σ, σ_j	Standard deviation (of j)

The rate of return on equity after interest and before tax is defined to be

$$e=\frac{r(E+L)-iL}{E} \tag{1}$$

$$=r+(r-i)\frac{L}{E} \tag{2}$$

$$=r+(r-i)\eta \tag{3}$$

$$=(1+\eta)r-i\eta \tag{4}$$

Since r is assumed to be a random variable, e becomes a random variable, too.

$$\tilde{e}=\tilde{r}+(\tilde{r}-i)\eta \tag{5}$$

$$=(1+\eta)\tilde{r}-i\eta \tag{6}$$

Expected value of $\tilde{e}$ is

$$\bar{e}=\bar{r}+(\bar{r}-i)\eta \tag{7}$$

Standard deviation of $\tilde{e}$ is

$$\sigma_e = \sigma_r (1+\eta) \qquad (8)$$

Variance of $\tilde{e}$ is

$$\sigma_e^2 = \sigma_r^2 (1+\eta)^2 \qquad (9)$$

Initially, we must consider that the behavior of point $(\bar{e}, \sigma e)$ corresponds to the change of η.

From Eq. (7)

$$\eta = \frac{\bar{e}-\bar{r}}{\bar{r}-i} \qquad (10)$$

We substitute Eq. (10) into Eq. (8)

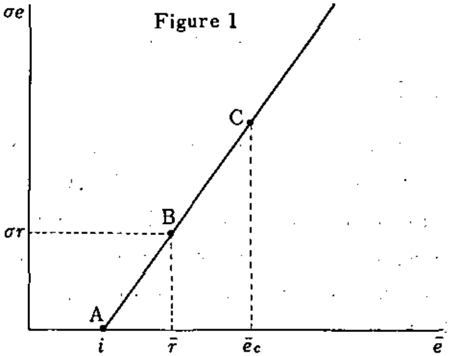
$$\sigma_e = \frac{\bar{e}-i}{\bar{r}-i}\,\sigma_r \qquad (11)$$

Eq. (11) is depicted as straight line A B C in the $\bar{e}-\sigma_e$ plane in Fig. 1.

Point A is at $\eta = -1$. This means that all equity is lent at the rate of interest i. It is risk-free.

Point B is at $\eta = 0$. This means that there is no liability in the capital structure, that is, all the capital consists only of the quity whose unlevered firm is operating with the rate of retun on capital being $\bar{r}$.

Point C is at $\eta > 0$. From Eq. (10), $\eta = \frac{\bar{e}_c - \bar{r}}{\bar{r}-i} = \frac{\overline{BC}}{\overline{AB}}$, where $\overline{AB}$ and BC are line segments. The ray ABC is called "the financial leverage line", (FLL).[2] The slope of FLL ξ is: $\xi = \frac{\sigma_r}{(\bar{r}-i)}$



Five types of utility function in the financial decision situation will be discussed.

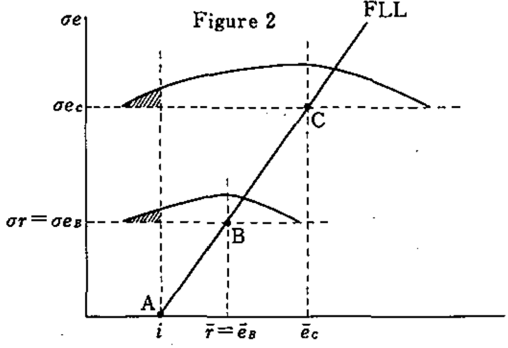
2. Survival Model (S Model)

In the survival model, the rate of interest i is a critical value. Naturally, if the rate of return on equity declines to less than i, the probability of bankruptcy of the firm is much greater.

Even in the case of the unlevered firm B, this is because such an inefficient firm would fade away from the capital market. So the probability of rate of retun on equity being less than i is defined as the probability of bankruptcy.

One should investigate the probability of bankruptcy of any firm on the financial leverage line.

Figure 2



Following Roy (4), it can easily be shown that

$$P(\tilde{e}_B \leqq i)=P\left(\tilde{v} \leqq \frac{\bar{e}_B-i}{\sigma r}\right)=P\left(\tilde{v} \leqq \frac{\bar{r}-i}{\sigma r}\right)=P\left(\tilde{v} \leqq \frac{1}{\xi}\right) \tag{12}$$

$$P(\tilde{e}_c \leqq i)=P\left(\tilde{v}=\frac{\bar{e}_c-i}{\sigma e_c}\right)=P\left(\tilde{v} \leqq \frac{1}{\xi}\right) \tag{13}$$

where $\tilde{v}=\frac{\tilde{e}_j-i}{\sigma_j} \qquad j=B,\ C$ (14)

So the probability distribution of $\tilde{v}$ is assumed to be N (0,1).

From Eq. (12) and (13), it is evident that whatever financial leverage of any firm on the financial leverage line may be, it is indifferent for survival.

Investors who are separated from managing a firm could be indifferent to its capital structure, but the corporate management has to take the raising of capital into account.

3. Stochastic Dominance Model (SD Model)

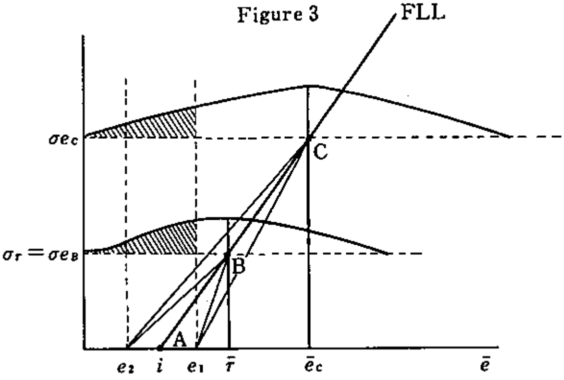
An important issue of financial study concerns the conflict between the Stochastic Dominance (SD) and the Expected Value-Variance (EV) Model in choosing optimal portfolio of risky assets, as is pointed out by Burrporter (5). He stands for SD model.

According to Hadar and Russel (6), Stochastic Dominance is the fact that the value of the cumulative distribution of the preferred prospect never exceeds that of the inferior prospect.

At present, this Stochastic Dominance is called the first-degree stochastic dominance (FSD). Additionally, we have the second-degree stochastic dominance (TSD) by Hadar and Russell (7), and the third-degree stochastic dominance (TSD) by Whitmore (8).

It has been verified that FSD implies SSD and TSD. Our discussion will be confined to FSD.

Figure 3



In the previous section, we considered the indifference between B and C for survival, in the case of P($\tilde{e}_B \leqq i$) and P($\tilde{e}_c \leqq i$).

In this section, we will consider the two cases $e_1 > i$ and $e_2 < i$.

Case 1: $e_1 > i$

Referring to Fig. 3, we can easily reason as follow:

$$P(\tilde{e}_B \leqq e_1) = P\left(\tilde{v} \leqq \frac{e_1 - \bar{e}_B}{\sigma e_B}\right) = P\left(\tilde{v} \leqq \frac{r - \bar{r}}{\sigma_r}\right) \tag{15}$$

$$P(\tilde{e}_C \leqq e_1) = P\left(\tilde{v} \leqq \frac{e_1 - \bar{e}_C}{\sigma e_C}\right) \tag{16}$$

$$\frac{e_1 - \bar{e}_B}{\sigma e_B} \geqq \frac{e_1 - \bar{e}_C}{\sigma e_C} \tag{17}$$

$$\therefore\ P(\tilde{e}_B \leqq e_1) \geqq P(\tilde{e}_C \leqq e_1) \tag{18}$$

Case 2: $e_2 < i$

We can similarly show that

$$P(\tilde{e}_B \leqq e_2) \leqq P(\tilde{e}_C \leqq e_2) \tag{19}$$

If P($\tilde{r} \leqq i$) is negligible, P($\tilde{e} \leqq i$) is also negligible. In this case, C is said to dominate B by FSD. Generally, any firm on the upper part of FLL stochastically dominates firms on the lower part of FLL.[3]

If P($\tilde{r} \geqq i$) is negligible, B will stochastically dominate C.

If $\tilde{r}$ is at times less than i, and another times more than i, there is no stochastic dominance between B and C.

4. Expected-Value-Standard Deviation Model (ESD Model)

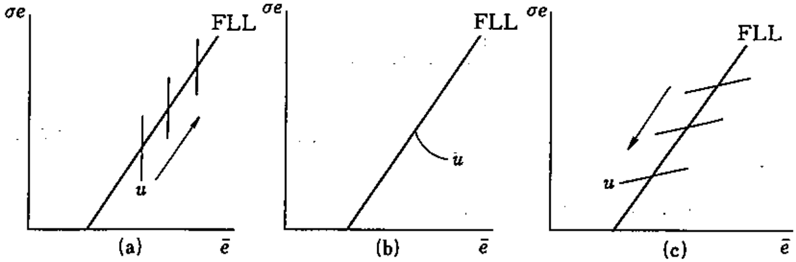
The utility function u of this model is the following function.

$$u = \bar{e} - \lambda\sigma_e \tag{20}$$

$$= (\bar{r} - \lambda\sigma_e) + (\bar{r} - i - \lambda\sigma_r)\eta \tag{21}$$

where λ is the coefficient of risk aversion.

Figure 4



(a) (b) (c)

$$\bar{r} - i - \lambda\sigma_r \gtreqqless 0 \Longleftrightarrow \lambda \lesseqqgtr \frac{\bar{r} - i}{\sigma_r} = \frac{1}{\xi} \tag{22}$$

where $\sigma r \neq 0$

If $\lambda < \frac{1}{\xi}$, u is an increasing function with regard to η. Therefore, the optimal value of η is infinite to maximize u. (Fig. 4(a))

If $\lambda = \frac{1}{\xi}$, any financial leverage on FLL is indifferent. (Fig. 4(b))

If $\lambda > \frac{1}{\xi}$, u is a decreasing function with regard to η. Then, the optimal value of η is 0, that is, unlevered, provided that lending is not permitted. If lending is feasible, the optimal value is $\eta = -1$.

The optimal behaviors of financial leverage at $\lambda = \frac{1}{\xi}$ and $\lambda \gtrless \frac{1}{\xi}$ in the ESD model are equivalent to those of the S model and SD model, respectively.[4]

5. Expected Value-Variance Model (EV Model)

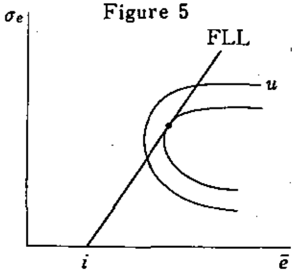
The utility function in EV model is as follos:

$$u = \bar{e} - \frac{1}{2}\alpha\sigma_e^2 \tag{23}$$

$$= \bar{r} + (\bar{r} - i)\eta - \frac{\alpha}{2}\sigma_r^2 (1+\eta)^2 \tag{24}$$

where α is the coefficient of risk aversion.

Figure 5



The necessary condition to maximize u with regard to η is

$$\frac{du}{d\eta} = (\bar{r} - i) - \alpha\sigma_r^2 (1+\eta) = 0 \tag{25}$$

Therefore, the optimal value of η is

$$\eta = \frac{(\bar{r} - i) - \alpha\sigma_r^2}{\alpha\sigma_r^2} \tag{26}$$

We can not get uniquely any optimal finite value of η to maximize the utility functions in S, SD, and ESD models, other than the extreme points $\eta=-1$ or $\eta=0$.

On the other hand, the optimal value of η is finite using Eq. (25) in E-V model.

One must recognize the difference between Expected Value-Standard Deviation model and Expected Value-Variance model.

The quadratic utility function like E-V model has been criticized for several years.

Pratt (II) said that a quadratic utility could not be a decreasing risk-averse on any interval and that this severely limited the usefullness of quadratic utility, however nice it would be to have expected utility depend only on the mean and variance of the probability distribution.

Arrow (12) also discussed the same results.

Linter (1) criticized normality and derived "market opportunity line", skillfully using Roy's survival model.

In the following section, we will construct a model, mainly following Pratt.

6. Decreasing Risk Aversion Model (DRA Model)

In his paper, the function $r(x)=u''(x)/u'(x)$ is defined as a measure of local risk aversion, and considered a measure of the concavity of u at the point x where x is the amount of holding assets.

"A man's utility system is the result of his social situation, and of society around him. But his social situation depends in turn on economic organization", said Marris (13).

Referring to his ideas, it seems to me that a man is decreasing a degree of risk aversion against a given risk as he reaches the empire of power.
So x is defined as a measure of holding not only assets, but also other managerial powers of the firm.

Expected utility is as follows

$$E\{u(x+\tilde{e})\}=E\{u(x)+\tilde{e}u'(x)+\frac{1}{2}\tilde{e}^2u''(x)+0(\tilde{e}^3)\} \tag{27}$$

$$=u(x)+\bar{e}u'(x)+\frac{1}{2}(\sigma_e^2+\bar{e}^2)u''(x) \tag{28}$$

$$=u(x)+\{\bar{r}+(\bar{r}-i)\eta\}u'(x)+\frac{1}{2}\{(1+\eta)^2\sigma_r^2 + [\bar{r}+(\bar{r}-i)\eta]^2\}u''(x) \tag{29}$$

The first derivative of Eq. (29) with respect to η, is the following

$$\frac{dE\{u(x+\tilde{e})\}}{d\eta}=(\bar{r}-i)u'(x)+\{\sigma r^2+(\bar{r}-i)\bar{r} + [\sigma r^2+(\bar{r}-i)^2]\eta\}u''(x) \tag{30}$$

The necessary condition to maximize E $[u(x+\tilde{e})]$ with respect to η is

$$\frac{dE}{d\eta}=0$$

Therefore $$\eta=\frac{(\bar{r}-i)-[\sigma r^2+\bar{r}(\bar{r}-i)]r(x)}{[\sigma r+(r-i)^2]r(x)} \tag{31}$$

where $$r(x)=-\frac{u''(x)}{u'(x)} \tag{32}$$

This η in the DRA model is correspondent to that of that in the E-V model. $r(x)$ is to α in Eq. (26). α is a constant but $r(x)$ is a decreasing function of x. So that η is an increasing function of x. In other words, financial leverage will increase as the assets and other resources of a firm increase.

We can not recognize the behavior of financial leverage in the dynamic setting without using $r(x)$. So Eq. (31) is very helpful to study the dynamic financial leverage. But, since we can keep $r(x)$ constant to study financial leverage in the static state, the η of Eq. (26) is useful instead of the of Eq. (31).

In the next section, we would like to analyse financial leverage further, using E-V model, mainly because it is much easier to manipulate the of Eq. (26) than that of Eq. (31).

7. Financial leverage, Risk aversion and Information

In this section we will consider the next two relations using Eq. (26).

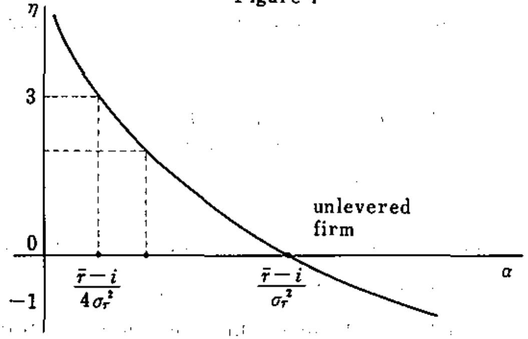
Case a: between financial leverage and risk aversion

Case b: between financial leverage and information

Case a: between financial leverage and risk aversion

The function $\eta(\alpha)$ is depicted at Fig. 7 where given $\bar{r}>i$ and $\sigma_r \neq 0$. η is needed to be less than or equal to 3 by the rule of tuumb. Using Eq. (25), the value of α is at $\eta=3$,

Figure 7



$$\alpha=\frac{1}{4}\left[\frac{\bar{r}-i}{\sigma_r^2}\right] \tag{33}$$

At $\eta=0$, the value of α is

$$\alpha=\frac{\bar{r}-i}{\sigma_r^2} \tag{34}$$

From Eq. (33) and (34),

$$\frac{\alpha(\eta=3)}{\alpha(\eta=0)}=\frac{1}{4} \tag{35}$$

This rule of thumb says that, ceteris paribus, the decision-maker should not hve α less than a fourth of the unlevered coefficient of risk aversion.

Many Japanese companies have $\eta>3$. For instance, the η of the Mitsubishi Trading Company is 30.44 and that of the Mitsui Bussan Company is 28.85, in 1973.

At $\eta=29$, the value of α is,

$$\alpha = \frac{1}{30}\left[\frac{\bar{r}-i}{\sigma_r^2}\right] \tag{36}$$

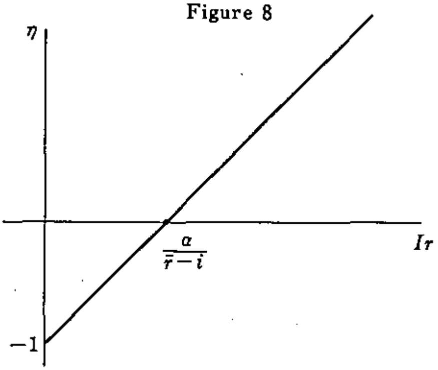
This α is a thirtieth of the unlevered coefficient of risk aversion.

Case b: between financial leverage and information

$\frac{1}{\sigma r^2} = Ir$, is called the quantity of information by Raiffa and Schlaifer (14). Substituting Ir into Eq. (26).

$$\eta = \frac{1}{\alpha}(\bar{r}-i)\,Ir - 1 \tag{37}$$

Figure 8



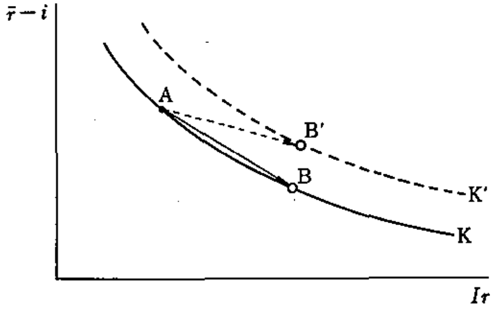
The optimal leverage of a firm is a linear increasing function of the quantity of information which the firm has in the data bank.

Given η and α,

$$(1+\eta)\,\alpha = (\bar{r}-i)\,Ir = K \text{ (Const.)} \tag{38}$$

Let the quantity of additional sample information of r be Is and its cost be β% of $(\bar{r}-i)$. Assume that the sample mean is the same as $\bar{r}$.

Figure 9



At Fig. 9, point A stands for the state of having prior information. The condition that point B remains on the same trade-off curve k as point A does will be derived.

In order to get the condition, we have the following equation, using Bayes' Theorem.

$$(\bar{r}-i)\, Ir=(\bar{r}-i)\,(1-\beta)\,(Ir+Is) \tag{39}$$

$$\therefore\ \beta=\frac{Is}{Ir+Is} \tag{40}$$

If the cost of the additional information is equal to $\frac{Is}{Ir+Is}(\bar{r}-i)$, we can reach point B.

If the cost is less than $\frac{Is}{Ir+Is}(\bar{r}-i)$, the optimal η becomes greater shifting into curve K', given α.

Conclusion

Among our models, S Model, SD Model and ESD Model have no finite optimal financial leverage. In order to get finite optimal financial leverage, EV Model or DRA Model has to be used.

EV Model is criticized in terms of DRA Model. But it is easy to manipulate EV Model. So that we considered the relations between financial leverage and risk aversion, and between financial leverage and information in terms of EV Model with caution paid to its criticism.

It is interesting to say that financial leverage is much connected with information, given risk aversion.

(November 3, 1974)

Notes

1) Assumption 2 is the same as Lintner (1) did. (p. 1)
2) The close relationship between Fisher's "Market Opportunity Line" (2) or Sharpe's "Capital Market Line" (3) in portfolio theory, and our financial leverage line should be noted. (p. 3)
3) The fact that an efficient portfolio with high mean - high standard deviation is preferred according to SD as Burrporter (5) did has something to do with the above mentioned characteristic of SD. (p. 6)
4) Baumol (9) considered dominant portfolio in the ESD model using his "lower confidence limit, L"

$$L=E-k\sigma$$

In his (E. L) model,

$$\sigma=f(E)$$
$$\sigma'=f'(E)>0$$
$$\sigma''=f''(E)>0$$

But in our model, σe is a linear increasing function of . Taking into account this difference, our result from the ESD model is consistent with his results. (p. 7)

Selected Bibliography

(1) Lintner, John, "The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolio and Capital Budgets" The Review of Economics and Statistics, Vol. XLVII No. 1 (Feb., 1965), pp. 13-37.
(2) Fisher, Irving, The Theory of Interest (New York, 1930).
(3) Sharpe, William F., "Capital Asset Price: A Theory of Market Equilibrium under Conditions of Risk" The Journal of Finance, Vol. XIX, No. 3 (Sept., 1964), pp. 425-42.
(4) Roy, A.P., "Safety First and the Holding of Assets" Econometrica, Vol. XX (Jul., 1952) pp. 431-49.
(5) Burrporter, R. "An Empirical Comparison of Stochastic Dominance and Mean-Variance Portfolio Choice Criterion "The Journal of Financial and Quantitative Analysis (Sept., 1973)
(6) Hadar, J and Russell, W. R., "Rules for Ordering Uncertain Prospects" American Economic Review (Mar. 1969) Vol. 49, pp. 25-34.
(7) Ibid.
(8) Whitmore, G.A., "Third-Degree Stochastic Dominance" American Economic Review Vol. 60 (Jun., 1970) pp. 457-59.
(9) Baumol, William J., "An Expected Gain – Confidence Limit Criterion for Portfolio Selection", Management Science, Vol. X (Oct., 1963) pp. 174-82.
(10) Markowitz, Harry, Portfolio Selection (New York, 1959).
(11) Pratt, John W., "Risk Aversion in the Small and in the Large" Econometrica, Vol. 32, No. 1-2 (Jan-Apr, 1964), pp. 122-136.
(12) Arrow, K., "Comment on the Portfolio Approach to the Demand for Money and Other Assets", The Review of Economics and Statistics, Supplement, XLV (Feb., 1963), pp. 24-27.
(13) Marris, R. The Economic Theory of Managerial Capitalism, The Free Press of Glencoe, 1964 p. 5.
(14) Raiffa, H. and Schlaifer, R. Applied Statistical Decision Theory, Harvard University, 1961 pp. 62-66.

財務梃子率と情報の決定分析

〈要　　約〉

藤　田　　忠

財務的意思決定者が生存モデル，確率的支配モデル，期待値ー標準偏差モデル，期待値ー分散モデルおよび逓減的危険回避モデルの効用関数をとったとき，財務梃子率がどのような態様を示すかを研究した。その結果，生存，確率的支配，期待値ー標準偏差モデルでは有限な（ただし，財務梃子率＝－１あるいは０以外の）最適な財務梃子率がないかあるいはどのような梃子率をとっても無差別である場合以外ないことが明らかにされた。

期待値ー分散モデルはPrattあるいはArrowによる逓減的危険回避モデルの観点から批判されている。期待値ー分散モデルも局所的には利用可能である点を考慮して，ＥＶモデルによって，さらに次の２点を検討した。

ケースａ：　財務梃子率と危険回避

ケースｂ：　財務梃子率と情報

ケースａにおいて財務梃子率と危険回避係数との関係を考察した。これによって，企業の財務行動が効用理論に立つ意思決定モデルに関連がつけられた。

ケースｂによって，危険回避係数が所与ならば，財務梃子率は情報システムと関連を持っていることが指摘される。ベイズ決定理論を用いて，経済的な情報が利用可能ならば，最適財務梃子率が増加することが示された。