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Chaotic Business Cycles and the Stabilization Policy in a Dynamic Macroeconomic Model

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1 Introduction

Time series of many macroeconomic variables have the appearance of irregular fluctuations[1]. The traditional explanation of this is that an essentially stationary economy is subject to random shocks. Important examples of this approach can be found in the works of Lucas (1975) and Sargent (1979). They argue that when the economy is subject to a sequence of random shocks, it will behave in a way which resembles real-life business cycles .

An alternative way in handling irregular bussiness cycles would be to use a model involving non-linear difference (or differential) equations such as the ones illustrated by Schinasi (1981,1982), Torre (1977). If a model of this sort would exhibit chaotic or cyclic behavior, then it could provide an explanation for the irregular business cycles[2]. Most recent works of this approach are done by Day and Shafer (1985), who show how chaotic behavior can emerge in the standard fixed price macroeconomic model when induced investment is strong enough. Their dynamic model is reduced to a single first-order difference equation on GDP. The chaotic behavior that emerges in their model is caused by non-linearities of demand for money and investment commodities. This paper shows that endogenous business cycles (including periodic and chaotic behavior) would emerge from a standard macroeconomic model with a negatively sloped IS curve and a positively LM curve in the interest rate-output space, respectively. Economic dynamics of our model is described by two-dimensional first order deference equations on GDP and interest rate. Therefore our model is an extension of Day-Shafer model. Next we demonstate that a discretionary monetary policy, which reduces the deviation between an actual interest rate and its target level, can stabilize any endogenous business cycles. Moreover, we illustrate that the stabilization policy is able to stabilize the irregular endogenous business cycles to which is added random shocks.

The paper is organized as follows. First, we specify the structure of a standard macroeconomic model in Sec.2. Then we demonstrate a necessary and sufficient condition for local stability of the economic equilibrium and also illustrate global dynamics which would emerge from the model in Sec.3. We shows a stabilization policy, that is, a discretionary monetary policy, that can stabilize any endogenous business cycle in Sec.4. A few concluding remarks are given in Sec.5.

2 Model

The model is a dynamic intermediate-run IS-LM model

$$Y_{t+1} = Y_t + \alpha F(Y_t, R_t) + \epsilon_t, \quad (0 < \alpha < 1). \tag{1}$$

$$R_{t+1} = R_t + \beta G(Y_t, R_t) + v_t, \quad (0 < \beta < 1). \tag{2}$$

Equation (1) and (2) represent traditional macroeconomic disequilibrium adjustment process for the commodities and money markets respectively. F represents excess demand for commodities and services and G represents excess demand for money. Both are continuous functions of the interest rate (R), national product (Y). α and β are the adjustment speeds of commodity market and money market respectively. ϵ_t and v_t are the stochastic error terms that follow the normal distribution.

In fixed price regimes, each equation will also be parametrized by the fixed price level $(P = 1)$.

We define F and G as follows :

$$F(Y_t, R_t) = C(Y_t) + I(Y_t) - Y_t. \tag{3}$$

$$G(Y_t, R_t) = L(Y_t, R_t) - M/P. \tag{4}$$

Here C$(\cdot)$, I$(\cdot)$, and L$(\cdot)$ are the consumption function, the investment demand function, the money demand function respectively with M_t

the nominal money stock. For IS-LM model, a good discussion is illustrated by Branson (1972). We assume that M_t is constant at M until the section 3 that we discuss about a stabilization policy. We further specify these functions as follows :

$$C(Y_t) = c_0 + c_1(Y_t), \quad (c_0 > 0, 0 < c_1 < 1). \tag{5}$$

$$I(Y_t, R_t) = a_0 + a_1 Y_t - a_2 R_t, \quad (a_0 > 0, a_1 > 0, a_2 > 0). \tag{6}$$

$$L(Y_t, r_t) = b_0 \exp(b_1(R^c - R_t))Y_t, \quad (b_0 > 0, b_1 > 0, R^c > 0). \tag{7}$$

Before analyzing the dynamic model it will be useful and illuminating to consider the geometric properties of the model in (Y, R) space (the IS-LM analysis). We get the IS relation and LM relation as follows :

$$\text{IS}: \quad R_t = \frac{1}{a_2}[(a_0 + c_0) + (a_1 + c_1 - 1)Y_t]. \tag{8}$$

$$\text{LM}: \quad R_t = \frac{1}{b_1}[\ln b_0 + b_1 R^c + \ln Y_t - \ln M]. \tag{9}$$

The IS relation has negative first derivative, and the LM relation has positive first derivative in (Y, R) space. Therefore there exists the unique equilibrium of the model (Y^*, R^*) where $Y^* > 0$ and $R^* > 0$. Figure 1 illustrates IS-LM curves under $c_0 = 1$, $c_1 = 0.6$, $a_0 = 1$, $a_1 = 0.2$, $a_2 = 0.1$, $b_0 = 0.01$, $b_1 = 1$, $R^c = 7$ and $M = 4$.

3 Dynamics

First of all we demonstrate the condition for locally stability of the equilibrium (Y^*, R^*).

The Jacobian of the model (1) and (2) evaluated at the equilibrium (Y^*, R^*) is

$$J = \begin{vmatrix} 1 + \alpha F_Y & \alpha F_R \\ \beta G_Y & 1 + \beta G_R \end{vmatrix} \tag{10}$$

where $F_Y = a_1 + c_1 - 1,$, $F_R = -a_2$, $G_Y = b_0 \exp(b_1(R^c - R^*))$ and $G_R = -b_0 b_1 \exp(b_1(R^c - R^*))Y^*$.
Then the characteristic equation of J is

$$\lambda^2-(2+\alpha F_Y+\beta G_R)\lambda+1+\alpha F_Y+\beta G_R+\alpha\beta(F_Y G_R-F_R G_Y)=0. \quad (11)$$

The characteristic roots are

$$\lambda_{1,2}=(A\pm\sqrt{A^2-4B})/2. \quad (12)$$

where $A=2+\alpha F_Y+\beta G_R$ and $B=1+\alpha F_Y+\beta G_R+\alpha\beta(F_Y G_R-F_R G_Y)$.

The necessary and sufficient condition for local stability of the equilibrium (Y^*, R^*) is that the absolute values of $\lambda_{1,2}$ are less than 1. If the following conditions are satisfied, then the equilibrium (Y^*, R^*) is locally stable.

The stability conditions for the equilibrium

1. $\alpha F_Y+\beta G_R+\alpha\beta(F_Y G_R-F_R G_Y)<0$
2. $(F_Y G_R-F_R G_Y)>0$
3. $4+2\alpha F_Y+2\beta G_R+\alpha\beta(F_Y G_R-F_R G_Y)>0$.

Note that the changes of α and *beta* does not influence the equilibrium (Y^*, R^*). We specify the parameters as follows :

$c_0=1$, $c_1=0.6$, $a_0=1$, $a_1=0.2$, $a_2=0.1$,
$b_0=0.01$, $b_1=1$, $R^c=7$, $M=4$ and $\alpha=0.5$.

We will use this set of the parameters in all figures below. With the parameters the equilibrium (Y^*, R^*) is $(8.429843188, 3.140313623)$, and the above local stability conditions are

$$0<\beta<0.5015657$$

When $\beta=0.2$ the characteristic roots are

$$\lambda_1=-0.206845584, \quad \text{and} \quad \lambda_2=-0.893154416.$$

Therefore in this case the equilibrium (Y^*, R^*) is locally stable. Figure 2 (a), (b) show that the time series of GDP Y_t and interest rate R_t where $\beta=0.4$, $\epsilon_t\sim N(0,0.005)$ and $v_t\sim N(0,0.05)$. Both of time series are appearently irregular. Since under this set of parameters the economic equilibrium is stable, the irregularity is caused by only the random shocks ϵ_t and v_t. On the other hand, under $\beta = 0.7$ the characteristic roots are

$$\lambda_1 = 1.793834936, \quad \text{and} \quad \lambda_2 = -0.893834936.$$

In this case the equilibrium (Y^*, R^*) is locally unstable.

Figure 3 (a), (b) illustrate the time series of Y_t and R_t where $\beta = 0.7$, $\epsilon_t \sim N(0, 0.005)$ and $v_t \sim N(0, 0.05)$. Both of the time series are apparently irregular. The time series with no stochastic noise, that is, $\epsilon_t = 0$ and $v_t = 0$, are deterministically chaotic as we see below. Hence the irregularity is caused by the endogenous power of the economic system derived by the nonlinearlity of the money demand function (7) and the random shocks ϵ_t and v_t.

Comparing Figure 2 with Figure 3, the range of fluctuations of interest rate in Figure 2 (b) is apparently smaller than the one in Figure 3 (b). While the time series of GDP in Figure 2 (a) fluctuates around the equilibrium level Y^*, that in Figure 3 (a) fluctuates below the equilibrium level Y^*. This suggests that stabilizing endogenous business cycles to the neighborhood of Y^* raises the average value of GDP.

Although it is difficult to analyze mathematically the global properties of the two-dimensional difference equations (1) and (2), the global properties can be illustrated by performing the computer simulations. In order to analyze deterministic dynamics of the model we assume $\epsilon_t = 0$ and $v_t = 0$. Figure 4 is the bifurcation diagrams with respect to β that varies smoothly from 0 to 1. The bifurcation diagram shows chaos occurring through a sequence of period doublings which is one route to chaos as β increases. We present a more global view of the dynamics of IS-LM model (1) and (2) through a bifurcation diagram with respect to the pair of (β, M), (Period-Chaos Plot). Figure 5 is Period-Chaos Plot for the dynamic IS-LM model. The parameter set consists of all pairs of (α, β) for the bifurcation parameter β between 0.39 and 1, and M between 0.15 and 5. Figure 5 shows that except when the money supply M is extremely small, chaos occurs through a sequence of period doublings as the adjustment speed of money market β becomes faster.

4 Stabilization Policy

The problem faced by the authorities in this model is that irregular endogenous business cycles will reduce the economic welfare. Figure 3 (a) shows that the average GDP is lower than the equilibrium GDP Y^*. Suppose that the authorities wish to use money supply to achieve the equilibrium interest rate R^*, and also wish to ensure that the monetary policy stabilize the economy. We consider the following simple discretionary monetary policy rule.

$$M_t = \bar{M} + \delta(R_t - R^*). \tag{13}$$

where $\bar{M}$ is exogenous money stock, and δ is the monetary policy parameter.

The logic behind this policy is simple to increase money supply wherever interest rate is above its target level R^*, and to decrease it when it is below its target, provided that optimal δ is positive. Considering the feedback control rule of money supply (12) in conjunction with the IS-LM model (1) and (2), we obtain the new economic system. The Jacobian matrix of the model at (Y^*, R^*) is

$$\begin{vmatrix} 1+\alpha F_Y & \alpha F_R \\ \beta G_Y & 1+\beta G_R - \delta \end{vmatrix} = 0. \tag{14}$$

The characteristic equation is

$$\begin{aligned} \lambda^2 &- (2+\alpha F_Y + \beta G_R + \delta)\lambda + 1 + \alpha F_Y + \beta G_R & (15)\\ &+ \alpha\beta(F_Y G_R - F_R G_Y) - (1+\alpha F_Y)\delta = 0. & (16) \end{aligned}$$

The conditions for the stability of (Y^*, R^*) in the dynamic model are all of the characteristic roots of the Jacobian matrix are less than 1. Thus the conditions for stabilizing endogenous business cycles by the discretionary monetary policy (12) follow :

1. $\delta > (\alpha F_Y + \beta G_R + \alpha\beta(F_Y G_R - F_R G_Y))/(1+\alpha F_Y)$
2. $\delta < \beta(F_Y G_R - F_R G_Y)/F_Y$
3. $\delta < (4 + 2\alpha F_Y + 2\beta G_R + \alpha\beta(F_Y G_R - F_R G_Y))/(2+\alpha F_Y)$.

under $\alpha = 0.5$, $\beta = 0.7$, $c_0 = 1$, $c_1 = 0.6$, $a_0 = 1$, $a_1 = 0.2$, $a_2 = 0.1$, $b_0 = 0.01$, $b_1 = 1$, $R^c = 7$, and $M = 4$, the conditions for stabilization are as follows :

$$-2.8926582 < \delta < -2.1186262.$$

Figure 6 (a),(b) and (c) illustrate the stabilizing effect of the discretionary monetary policy (12) with random noises $\epsilon_t \sim N(0, 0.005)$ and $v_t \sim N(0, 0.05)$. The time series of GDP and interest rate of the area A are the economic dynamics (1), (2) with $\beta = 0.7$ and no discretionary monetary policy $\delta = 0$. These fluctuate irregularly. The authority begins to implement the discretionary monetary policy at time a. The time series of the area B are the economic dynamics after the implementation of the monetary policy with $\delta = -2.7$. In Figure 6 (b) the fluctuations of interest rate after the implementation of the discretionary monetary policy becomes dramatically smaller than that of interest rate before the implementation of the discretionary monetary policy. Furthermore the average value of GDP after the implementation of the discretionary monetary policy becomes higher than the average value of GDP before the implementation of the discretionary monetary policy. Therefore we can say that the discretionary monetary policy raises the economic welfare.

5 Concluding Remarks

We demonstrate that (1) endogenous business cycles occurs form a dynamic IS-LM model with the standard assumptions when the adjustment speed of money market is fast, and that (2) the discretionary monetary policy can stabilize the endogenous business cycles and can increase the average GDP level.

It remains to be seen how these findings can be extended to more general and perusable frameworks including the labor market and the international trade.

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[1]For instance the existence of significant non-linearities have been reported for time series of U.S. unemployment, employment, and industrial production by Brock and Sayers (1988), Scheinkman and LeBaron (1989).

[2]Useful surveys of the non-linear economic dynamics literature can be found in Gabisch and Lorenz (1987), Kelsey (1988), Baumol and Benhabib (1989), Boldrin and Woodford (1990).

Figure 1 : IS-LM Curves

$c_0 = 1$, $c_1 = 0.6$, $a_0 = 1$, $a_1 = 0.2$, $a_2 = 0.1$, $b_0 = 0.01$, $b_1 = 1$, $R^c = 7$ and $M = 4$.

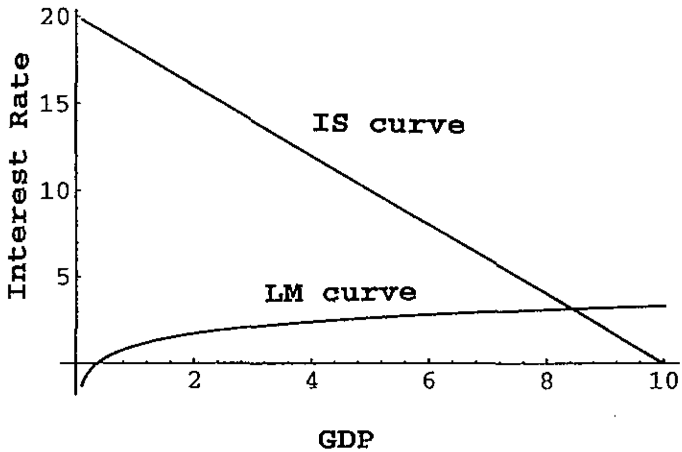


Figure 2 (a) : Random Walk of GDP Y_t

$\alpha = 0.5$, $\beta = 0.4$, $c_0 = 1$, $c_1 = 0.6$, $a_0 = 1$, $a_1 = 0.2$, $a_2 = 0.1$, $b_0 = 0.01$, $b_1 = 1$, $R^c = 7$ and $M = 4$, $\epsilon_t \sim N(0, 0.005)$ and $v_t \sim N(0, 0.05)$.

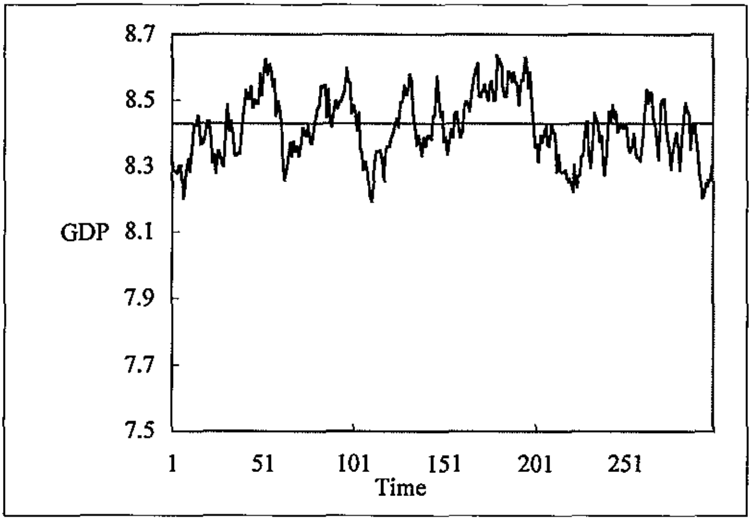


Figure 2 (b) : Random Walk of the interest rate R_t

$\alpha = 0.5$, $\beta = 0.4$, $c_0 = 1$, $c_1 = 0.6$, $a_0 = 1$, $a_1 = 0.2$, $a_2 = 0.1$, $b_0 = 0.01$, $b_1 = 1$, $R^c = 7$ and $M = 4$, $\epsilon_t \sim N(0, 0.005)$ and $v_t \sim N(0, 0.05)$.

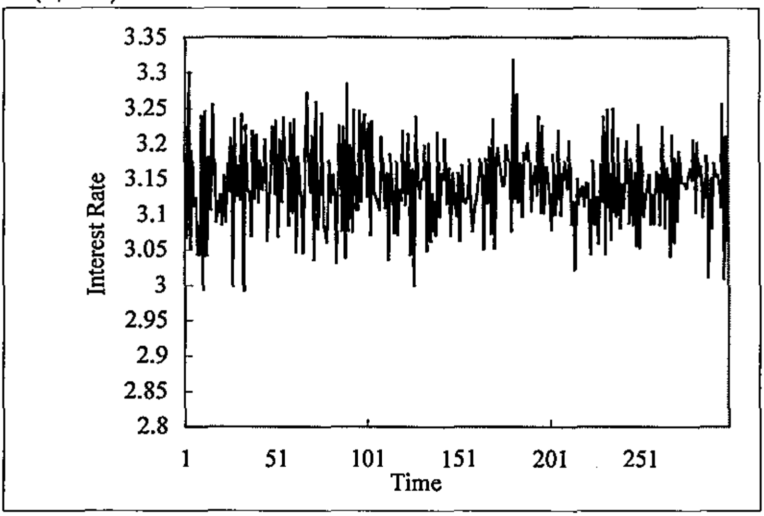


Figure 3 (a) : Noisy Chaos of GDP Y_t
$\alpha = 0.5$, $\beta = 0.7$, $c_0 = 1$, $c_1 = 0.6$, $a_0 = 1$, $a_1 = 0.2$, $a_2 = 0.1$, $b_0 = 0.01$, $b_1 = 1$, $R^c = 7$ and $M = 4$, $\epsilon_t \sim N(0, 0.005)$ and $\upsilon_t \sim N(0, 0.05)$.

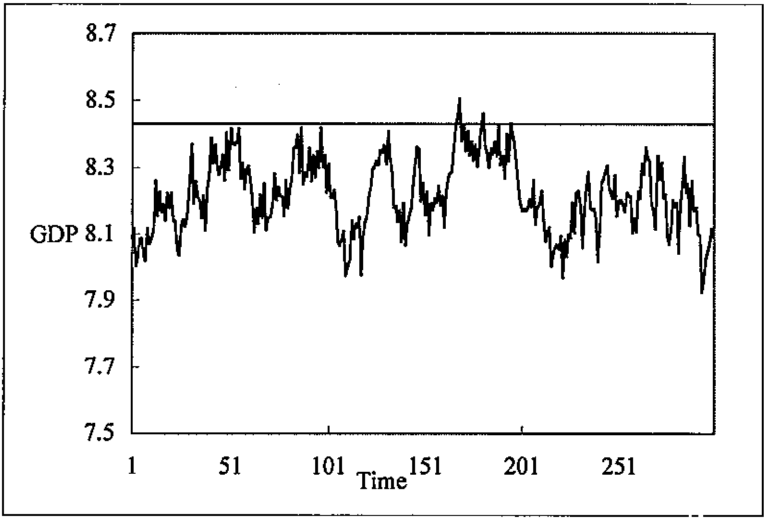


Figure 3 (b) : Noisy Chaos of the interest rate R_t
$\alpha = 0.5$, $\beta = 0.7$, $c_0 = 1$, $c_1 = 0.6$, $a_0 = 1$, $a_1 = 0.2$, $a_2 = 0.1$, $b_0 = 0.01$, $b_1 = 1$, $R^c = 7$ and $M = 4$, $\epsilon_t \sim N(0, 0.005)$ and $\upsilon_t \sim N(0, 0.05)$.

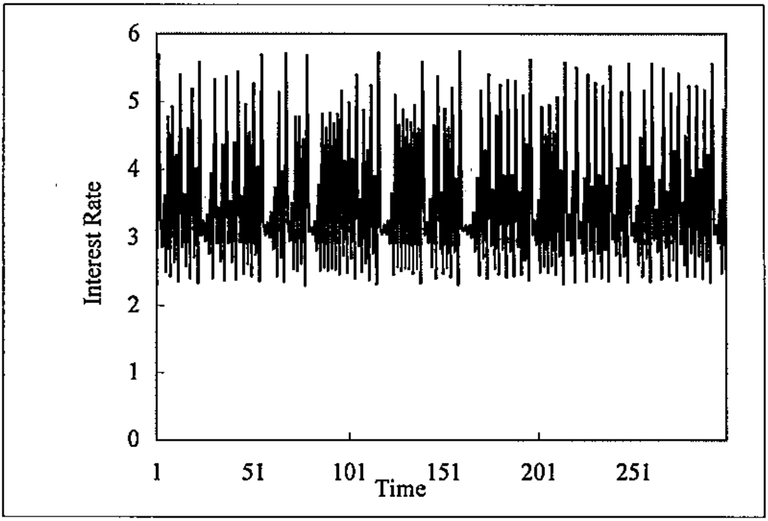


Figure 4 : Bifurcation Diagrams with respect to β

The bifurcation diagram shows chaos occurring through a sequence of period doublings which is one route to chaos as β increases under $\alpha = 0.5$, $c_0 = 1$, $c_1 = 0.6$, $a_0 = 1$, $a_1 = 0.2$, $a_2 = 0.1$, $b_0 = 0.01$, $b_1 = 1$, $R^c = 7$ and $M = 4$, $\epsilon_t = 0$ and $v_t = 0$.

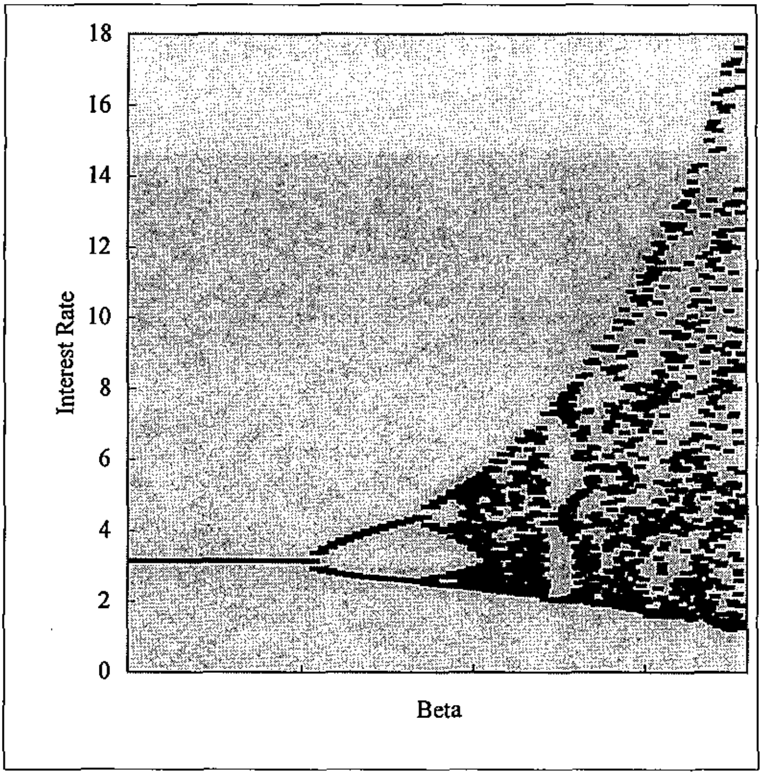


Figure 5 : Period-Chaos Plot for the dynamic IS-LM model

The bifurcation diagram shows chaos occurring through a sequence of period doublings which is one route to chaos as β increases under $\alpha = 0.5$, $c_0 = 1$, $c_1 = 0.6$, $a_0 = 1$, $a_1 = 0.2$, $a_2 = 0.1$, $b_0 = 0.01$, $b_1 = 1$, $R^c = 7$ and $M = 4$, $\epsilon_t = 0$ and $v_t = 0$.

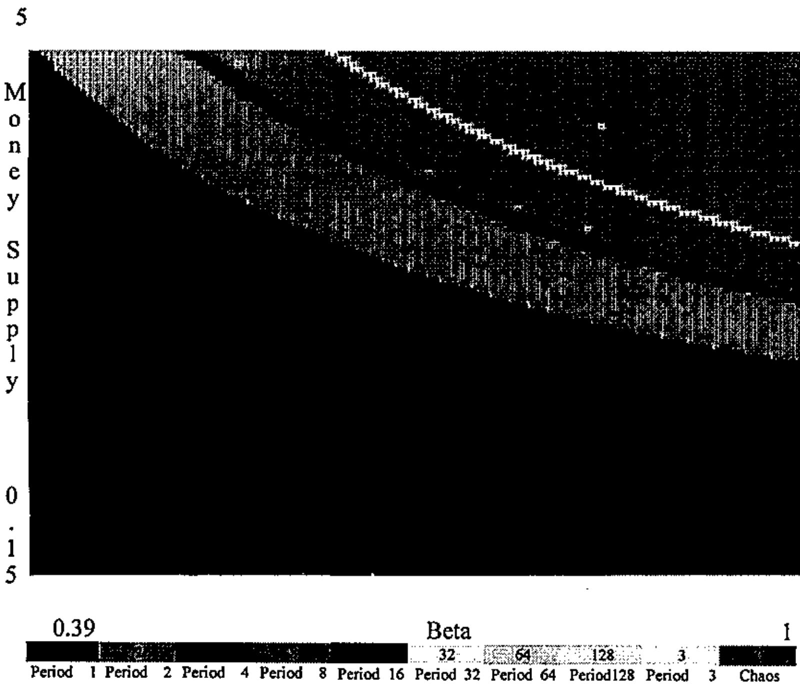
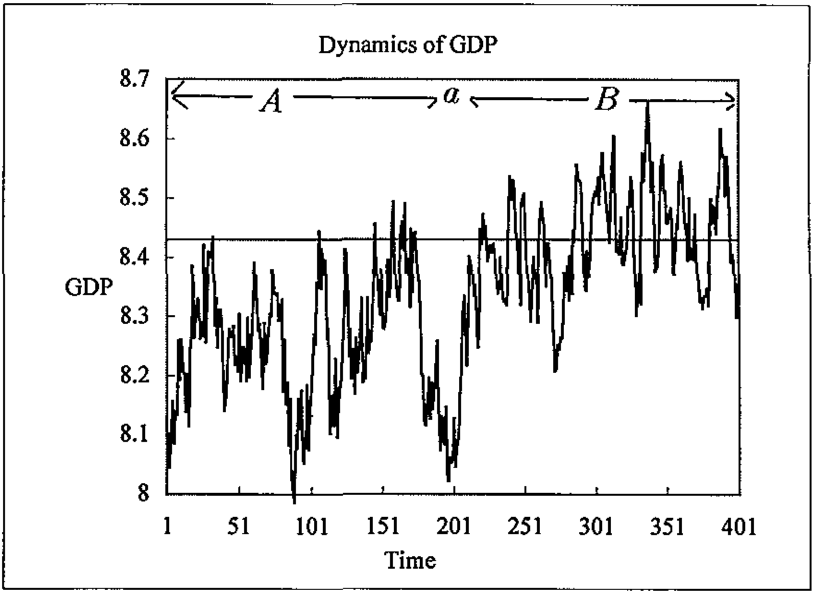
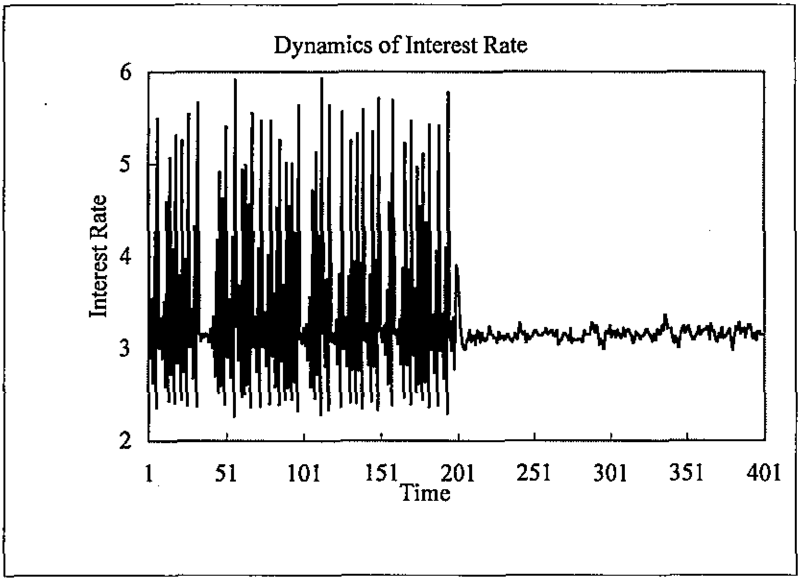


Figure 6 : The Stabilizing Effect of the Monetary Policy for Stochastic Endogenous Business Cycles

Figure 6 (a),(b) and (c) illustrate the stabilizing effect of the discretionary monetary policy (12) with random noises $\epsilon \sim N(0, 0.005)$ and $v \sim N(0, 0.05)$. The time series of GDP and interest rate of the area A are the economic dynamics (1), (2) with $\beta = 0.7$ and no discretionary monetary policy $\delta = 0$. These fluctuates irregularly. The authority begins to implement the discretionary monetary policy at time a. the time series of the area B are the economic dynamics after the implementation of the monetary policy with $\delta = -2.7$. In Figure 6 (b) the fluctuations of interest rate after the implementation of the discretionary monetary policy becomes dramatically smaller than that of interest rate before the implementation of the discretionary monetary policy. Furthermore the average value of GDP after the implementation of the discretionary monetary policy becomes higher than the average value of GDP before the implementation of the discretionary monetary policy. Therefore we can say that the discretionary monetary policy raises the economic welfare.

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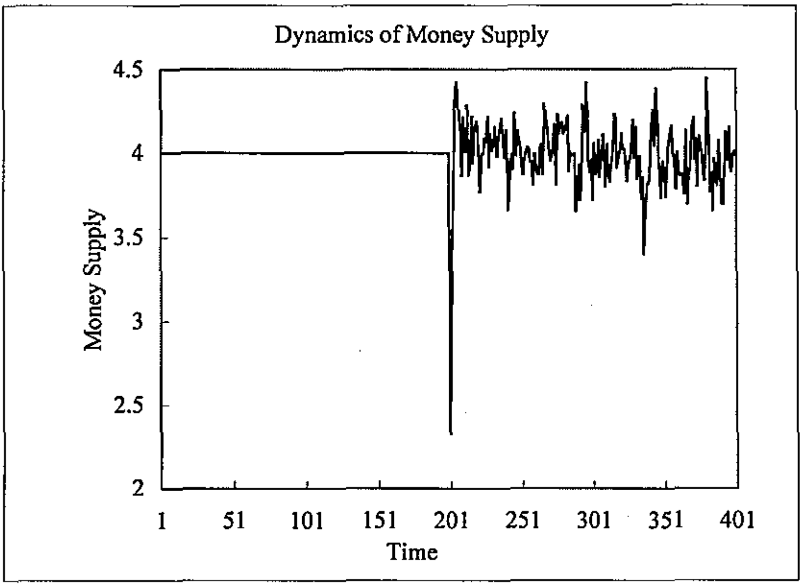
Dynamics of Money Supply
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動学的マクロ経済モデルにおける 不規則な景気循環とその安定化政策

<要　約>

海蔵寺　大成，張　喬森

国内総生産を始めとする多くの経済時系列データ（ただし，タイムトレンドや季節変動を除去した系列）はランダムに近い不規則な動きをしていることが，多くの実証研究によって確かめられている。

本稿の目的はこのような不規則変動が何故発生するのかを標準的なマクロ経済モデルであるIS-LM Modelを用いて理論的に調べ，その不規則変動を政策的に安定化する方法を提案することである。

消費関数，投資関数，および貨幣需要関数が通常の仮定を満たすとき，周知のように右下がりのIS曲線と右上がりのLM曲線が得られる。我々はこの様な自然な経済状況においても，不規則な経済変動（カオス）が発生し得ることを示す。特筆すべき点は，このような不規則変動は経済システムの内生的な力によって創り出されるのであって，多くの経済学者がしばしば論じる外生的なランダムショックに起因するものではないことである。次に，中央銀行による裁量的金融政策を取り上げ，金融政策によってこのような不規則変動を安定化する方法を示すとともに，コンピューターシュミレーションによってその有効性を確かめる。結果として金融政策による経済変動の安定化は国内総生産を引き上げることがわかる。最後に，この安定化政策（金融政策）は，経済モデルに外生的ランダムショックを付加した場合にも有効であることが示される。