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Solutions of Relativistic Klein-Gordon Equation with Equal Scalar and Vector Shifted Hulthen plus Angle Dependent Potential

Akaninyene D. Antia*, Sunday E. Etuk and Adebayo O. Adeniran
Department of Physics, University of Uyo, Nigeria.

Abstract
We have solved approximately the relativistic Klein-Gordon equation under the special case of equal scalar and vector shifted Hulthen plus angle dependent potential using the parametric form of NIkiforov-Uvarov method. The energy eigenvalues and the corresponding wave functions expressed in terms of a Jacobi polynomial are obtained. The effect of the angle dependent part on the radial solution is also discussed. We have also discussed few special cases of this potential.
Keywords: relativistic Klein-Gordon equation, shifted Hulthen, angle dependent potential, parametric Nikiforov-Uvarov method, centrifugal term.
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1.0 Introduction

The study of exactly solvable potentials has attracted much attention since the early development of quantum mechanics[1-3] and solving the nonrelativistic and relativistic equations for some potentials of interest is still an interesting work in the existing literature[4-13]. In nuclear and high energy physics, one of the interesting problems is to obtain exact solution of the Klein-Gordon, Duffin-Kemmer-Petiau and Dirac equations. When a particle is in a strong potential field, the relativistic effect must be considered, which gives the correction for nonrelativistic quantum mechanics[14-15].

In nonrelativistic quantum mechanics, it is well known that the exact solutions of Schrödinger equation are possible only for a few set of quantum systems. However, when arbitrary angular momentum quantum number l is present, one can only solve the Schrödinger equation approximately using suitable approximation schemes [16]. Some of such approximations include conventional approximation scheme proposed by Greene and Aldrich [17], improved approximation scheme by Jia *et al.*[18], elegant approximation scheme [19] etc. These approximations are used to deal with the centrifugal term or potential barrier arising from the problem.

The subject of the noncentral potentials has generated a lot of interest [20-23] and the study of noncentral potentials has been carried out in various fields of nuclear physics and quantum chemistry which could be used to discuss the interactions between pair of nuclei and ring-shaped molecules such as benzene [24-26]. Bound states solutions of Schrödinger, Klein-Gordon and Dirac equations for some noncentral potentials have received so much attention and interest from researchers [27-32]. Recently, Hamzavi and Rajabi solved Dirac equation for the Coulomb plus Novel angle dependent potential [33].

In solving nonrelativistic or relativistic wave equation whether for central or no central potential, various methods are used. These methods include asymptotic iteration method (AIM) [34], super symmetric quantum mechanics (SUSYQM) [35], shifted $\frac{1}{N}$ expression [36], factorization method [37, 38], Nikiforov-Uvarov (NU) [39] and others [40, 41].

In the relativistic quantum mechanics, one can apply the Klein-Gordon equation to the treatment of a zero-spin particle. In recent years, many studies have been carried out to explore the relativistic energy eigenvalues and corresponding wave functions of the Klein-Gordon and Dirac equations [14, 15, 42].

The aim of this paper is to apply the parametric generalization of Nikiforov-Uvarov (NU) method to obtain the approximate analytical solutions of the relativistic Klein-Gordon equation under equal scalar and vector shifted Hulthen plus angle dependent (SHAD) potential defined as [43, 44]

$$V(r,\theta) = -\frac{\left(V_0 - \frac{1}{2b^2}\right)}{\left(e^{\frac{r}{b}} - 1\right)} + \frac{\hbar^2}{2\mu}\left(\frac{\gamma + \beta\cos^2\theta + \Im\cos^4\theta}{r^2\cos^2\theta\sin^2\theta}\right), \qquad 1$$

where $\left(V_0 - \frac{1}{2b^2} > 0\right)$, b is the range of the potential and V_0 is the potential depth , μ is the reduced mass and $\hbar$ is the reduced plank's constant , $\gamma,\ \beta\ and\ \Im$ are arbitrary constants.

The potential in Eq. (1) can be expressed as

$$V(r,\theta) = -V_r(r) + \frac{\hbar^2}{2\mu}\frac{V_\theta(\theta)}{r^2}, \qquad 2$$

where $$V_r(r) = \frac{\left(V_0 - \frac{1}{2b^2}\right)}{\left(e^{\frac{r}{b}} - 1\right)} \qquad 3$$

$$V_\theta(\theta) = \frac{\gamma + \beta\cos^2\theta + \Im\cos^4\theta}{\cos^2\theta\sin^2\theta}. \qquad 4$$

This potential could be used to describe nucleon-nucleon interactions, meson-meson interaction and also in various branches of nuclear physics and quantum chemistry which may be used for interactions between the deformed pair of nuclei and ring shaped molecules like benzene. The plot of the behavior of the radial potential with range of the potential b is presented in Fig.1.

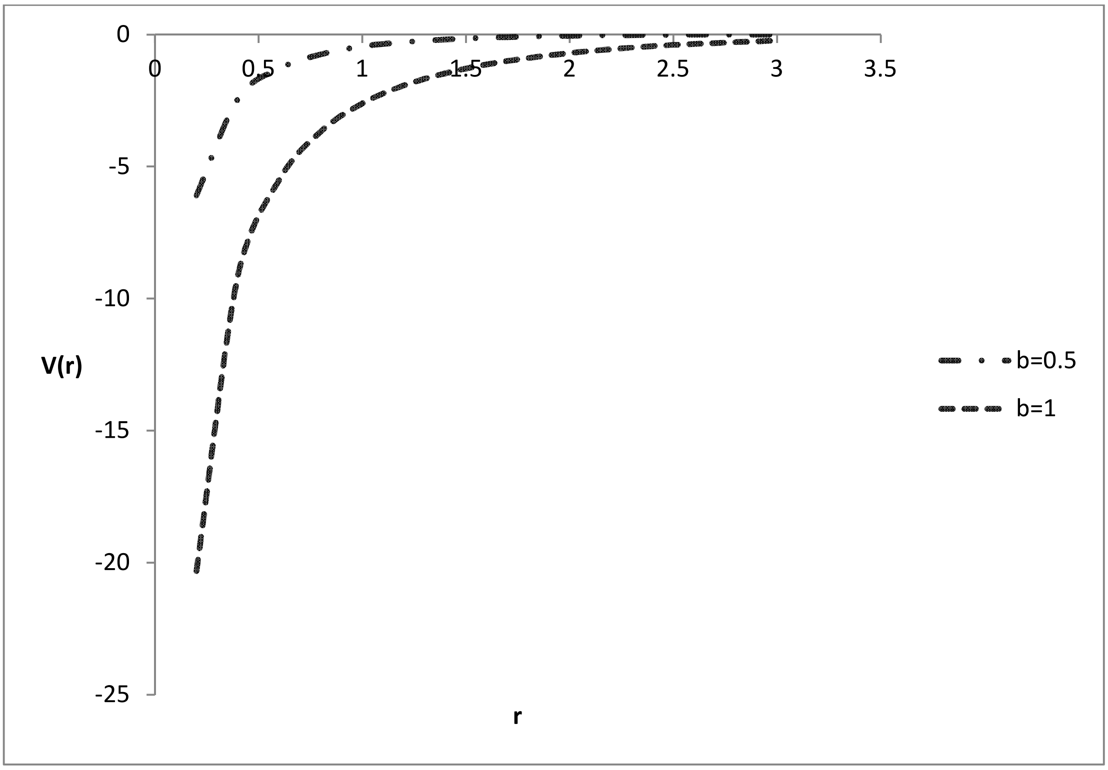

Fig.1: Variation of potential with dept for $b = 0.5, 1.$ with V_0=5

2.0 The Generalized Parametric Nikiforov-Uvarov (NU) Method

The NU method was presented by Nikiforov and Uvarov [39] and has been employed to solve second order differential equations such as the Schrödinger wave equation (SWE), Klein-Gordon equation (KGE), Dirac equation (DE) etc. The SWE

$$\psi''(r)+[E-V(r)]\psi(r)=0 \qquad 5$$

can be solved by transforming it into a hypergeometric type equation through using the transformation, $s=s(x)$ and its resulting equation is expressed as

$$\psi''(s)+\frac{\bar{\tau}(s)}{\sigma(s)}\psi'(s)+\frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi(s)=0, \qquad 6$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ must be polynomials of at most second degree and $\bar{\tau}(s)$ is a polynomial with at most first degree and $\psi(s)$ is a function of the hypergeometric type.

The parametric generalization of the NU method is given by the generalized hypergeometirc-type equation as [45]

$$\psi''(s)+\frac{(c_1-c_2s)}{s(1-c_3s)}\psi'(s)+\frac{1}{s^2(1-c_3s)^2}\left[-\xi_1s^2+\xi_2s-\xi_3\right]\psi(s)=0. \qquad 7$$

Equation (7) is solved by comparing it with Eq. (6) and the following polynomials are obtained:

$$\tilde{\tau}(s)=(c_1-c_2s),\ \sigma(s)=s(1-c_3s), \tilde{\sigma}(s)=-\xi_1s^2+\xi_2s-\xi_3. \qquad 8$$

According to the NU method, the energy eigenvalues equation and eigen functions, respectively, satisfy the following sets of equation

$$c_2n-(2n+1)c_5+(2n+1)\left(\sqrt{c_9}+c_3\sqrt{c_8}\right)+n(n-1)c_3+c_7+2c_3c_8+2\sqrt{c_8c_9}=0, \qquad 9$$

$$\psi(s)=N_{nl}s^{c_{12}}(1-c_3s)^{-c_{12}-(c_{13}/c_3)}P_n^{\left(c_{10}-1,\frac{c_{11}}{c_3}-c_{10}-1\right)}(1-2c_3s), \qquad 10$$

where

$$\begin{aligned}&c_4=\frac{1}{2}(1-c_1),\ c_5=\frac{1}{2}(c_2-2c_3),\ c_6=c_5^2+\xi_1,\ c_7=2c_4c_5-\xi_2,\ c_8=c_4^2+\xi_3,\\ &c_9=c_3c_7+c_3^2c_8+c_6,\ c_{10}=c_1+2c_4+2\sqrt{c_8},\ c_{11}=c_2-2c_5+2\left(\sqrt{c_9}+c_3\sqrt{c_8}\right),\\ &c_{12}=c_4+\sqrt{c_8},\ c_{13}=c_5-\left(\sqrt{c_9}+c_3\sqrt{c_8}\right)\end{aligned} \qquad 11$$

and P_n is the orthogonal Jacobi polynomial.

3.0 Factorization Method

The three dimensional the relativistic Klein-Gordon equation with mixed vector and scalar noncentral potentials is written as

$$\left[\nabla^2+(V(r,\theta)-E)^2-(S(r,\theta)+M)^2\right]\psi(r,\theta,\varphi)=0 \qquad 12$$

Where M is the rest mass, E is the relativistic energy, and $S(r,\theta)$ and $V(r,\theta)$ are the scalar and vector potentials respectively and ∇^2 is the Laplace operator. In spherical coordinate, the Klein-Gordon equation for a particle in the present of shifted Hulthen plus angle dependent potential $V(r,\theta)$ becomes

$$\left[\begin{array}{l}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}-2\left(EV(r,\theta)+MS(r,\theta)\right)\\+V^2(r,\theta)-S^2(r,\theta)+E^2-M^2\end{array}\right]\psi(r,\theta,\varphi)=0. \qquad 13$$

The total wave function in Eq. (13) can be defined as

$$\psi(r,\theta,\varphi)=\frac{R(r)}{r}Y(\theta,\varphi) \qquad 14$$

and by decomposing the spherical wave function in Eq. (13) using Eq. (14) and the potential $V(r,\theta)$ in Eq. (2)) for special case of equal scalar and vector potential (i.e $V_r(r)=S_r(r)$ and $V_\theta(\theta)=S_\theta(\theta)$) , we obtain the following equations:

$$\frac{d^2R(r)}{dr^2}+\left[E^2-M^2+2(E+M)V_r(r)-\frac{\lambda}{r^2}\right]R(r)=0, \qquad 15$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}Y(\theta,\varphi)\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}Y(\theta,\varphi)+[\lambda-2(E+M)V_\theta(\theta)]Y(\theta,\varphi)=0 \qquad 16$$

Substituting $Y(\theta,\varphi)=\Theta(\theta)\Phi(\varphi)$ into Eq. (16), we have:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta(\theta)}{d\theta}\right)+\left(\lambda-\frac{m^2}{\sin^2\theta}-2(E+M)V_\theta(\theta)\right)\Theta(\theta)=0, \qquad 17$$

$$\frac{d^2\Phi(\varphi)}{d\varphi^2}+m^2\Phi(\varphi)=0, \qquad 18$$

where $\lambda=l(l+1)$ and m^2 are the separation constants. The solution of Eq. (18) is well known [46]. Equations (15) and (17) are the radial and angular parts of Klein-Gordon equation respectively which are subject for discussion in the preceding section.

4.0 Solutions of the Radial Klein-Gordon Equation

For eigenvalues and corresponding eigen functions of the radial part of the Klein-Gordon equation, we substitute Eq. (3) into (15) to obtain

$$\frac{d^2R(r)}{dr^2}+\left[E^2-M^2+2(E+M)\frac{\left(V_0-\frac{1}{2b^2}\right)}{\left(e^{\frac{r}{b}}-1\right)}-\frac{\lambda}{r^2}\right]R(r)=0. \qquad 19$$

Equation (19) has no analytical or exact solution for $l \neq 0$ due to the centrifugal term, but can be solved approximately. Here we make use a newly improved approximation scheme [10] as

$$\frac{1}{r^2}\approx\frac{1}{b^2}\left[\frac{1}{12}+\frac{e^{-\frac{r}{b}}}{1-e^{-\frac{r}{b}}}+\frac{e^{-\frac{2r}{b}}}{\left(1-e^{-\frac{r}{b}}\right)^2}\right], \qquad 20$$

Substituting Eq.(20) into Eq.(19) gives

$$\frac{d^2R(r)}{dr^2}+\left[E^2-M^2+2(E+M)\frac{V_0}{\left(e^{\frac{r}{b}}-1\right)}-\frac{(E+M)}{b^2\left(e^{\frac{r}{b}}-1\right)^2}-\frac{\lambda}{b^2}\left[\frac{1}{12}+\frac{e^{-\frac{r}{b}}}{\left(1-e^{-\frac{r}{b}}\right)}+\frac{e^{-\frac{2r}{b}}}{\left(1-e^{-\frac{r}{b}}\right)^2}\right]\right]R(r)=0. \qquad 21$$

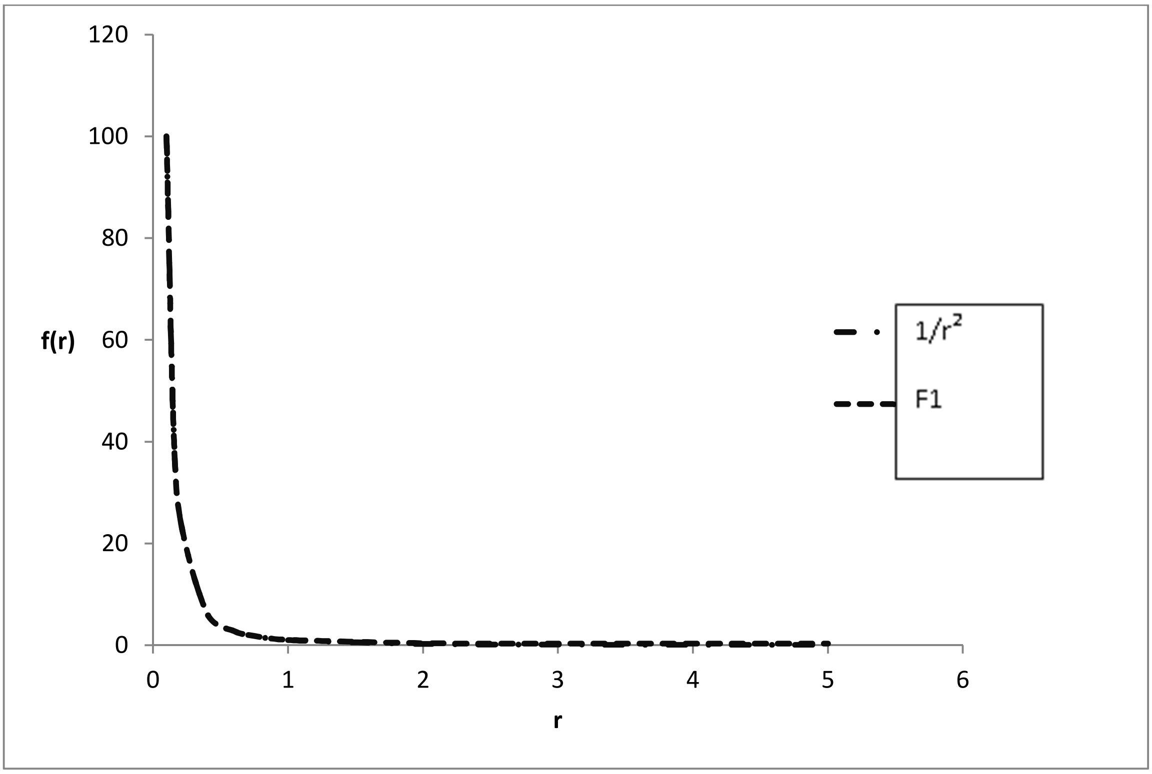

Fig.2: The centrifugal term $1/r^2$ and its approximation for $b = 0.5$.

The comparison of the approximation of Eq.(20) for $b = 0.5$ denoted as F1 with the centrifugal term $\frac{1}{r^2}$ is presented in Fig.2. This shows that the approximation is in good agreement with the centrifugal term.

Taking the transformation, $s = e^{-\frac{r}{b}}$, Eq. (21) reduces to

$$\frac{d^2R(s)}{ds^2} + \frac{(1-s)}{s(1-s)}\frac{dR(s)}{ds} + \frac{1}{s^2(1-s)^2}\left[-Q_1 s^2 + Q_2 s - Q_3\right]R(s) = 0, \qquad 22$$

where the following dimensionless quantities have been defined as

$$-\varepsilon^2 = b^2\left(E^2 - M^2\right), \qquad 23$$

$$Q_1 = \varepsilon^2 + 2b^2(E+M)V_0 - (E+M) + \frac{\lambda}{12}, \qquad 24$$

$$Q_2 = 2\varepsilon^2 + 2b^2(E+M)V_0 - (E+M) - \frac{5\lambda}{6} \qquad 25$$

$$Q_3 = \varepsilon^2 + \frac{\lambda}{12} \qquad 26$$

Comparing Eq. (22) with Eq. (7) and making use of Eq. (11), we obtain the following parameters:

$$c_1 = c_2 = c_3 = 1,$$
$$\xi_1 = Q_1 = \varepsilon^2 + 2b^2(E+M)V_0 - (E+M) + \frac{\lambda}{12}$$
$$\xi_2 = Q_2 = 2\varepsilon^2 + 2b^2(E+M)V_0 - (E+M) - \frac{5\lambda}{6}$$
$$\xi = Q_3 = \varepsilon^2 + \frac{\lambda}{12}$$
$$c_4 = 0,\ c_5 = -\frac{1}{2},\ c_6 = \frac{1}{4} + \varepsilon^2 + 2b^2(E+M)V_0 - (E+M) + \frac{\lambda}{12}$$
$$c_7 = -2\varepsilon^2 - 2b^2(E+M)V_0 + (E+M) + \frac{5\lambda}{6},$$
$$c_8 = \varepsilon^2 + \frac{\lambda}{12},\ c_9 = \lambda + \frac{1}{4},\ c_{10} = 1 + 2\sqrt{\varepsilon^2 + \frac{\lambda}{12}}$$
$$c_{11} = 2 + 2\left(\sqrt{\lambda + \frac{1}{4}} + \sqrt{\varepsilon^2 + \frac{\lambda}{12}}\right),\ c_{12} = \sqrt{\varepsilon^2 + \frac{\lambda}{12}},\ c_{13} = -\frac{1}{2} - \left(\sqrt{\lambda + \frac{1}{4}} + \sqrt{\varepsilon^2 + \frac{\lambda}{12}}\right). \qquad 27$$

Substituting Eqs.(23)-(27) into Eq. (9),we obtain the energy eigenvalues equation for SHAD potential as

$$n^2 + \frac{1}{2}(2n+1) + (2n+1)\left(\sqrt{\lambda + \frac{1}{4}} + \sqrt{\varepsilon^2 + \frac{\lambda}{12}}\right) - 2\varepsilon^2 - 2b^2(E+M)V_0 + (E+M) + \frac{5\lambda}{6}$$
$$+ 2\left(\varepsilon^2 + \frac{\lambda}{12}\right) + 2\sqrt{\left(\varepsilon^2 + \frac{\lambda}{12}\right)\left(\lambda + \frac{1}{4}\right)} = 0. \qquad 28$$

Solving Eq.(28) explicitly, we obtain the energy eigenvalues for the radial part of the Klein-Gordon equation for equal scalar and vector SHAD potential as

$$E^2 - M^2 = -\frac{1}{4b^2}\left[\frac{-2b^2(E+M)V_0 + (E+M) + \frac{15}{64} + \frac{15\lambda}{16} + \left(n + \frac{1}{2} + \frac{1}{4}\left(l + \frac{1}{2}\right)\right)^2}{(n+l+1)}\right]^2 + \frac{\lambda}{12b^2} \qquad 29$$

Using Eqs.(10) and (27), corresponding wave function of the radial part is obtained as

$$R(s) = N_{nl} s^{\mu}(1-s)^{\frac{1}{2}+\nu} P_n^{(2\mu,2\nu)}(1-2s), \qquad 30$$

where $\mu = \sqrt{\varepsilon^2 + \frac{\lambda}{12}}$, $\nu = \sqrt{\lambda + \frac{1}{4}}$.

Using the transformation, $s = e^{-\frac{r}{b}}$, Eq. (26) can also be written as

$$R(r) = N_{nl} e^{-\frac{\mu r}{b}}\left(1 - e^{-\frac{r}{b}}\right)^{\frac{1}{2}+\nu} P_n^{(2\mu,2\nu)}\left(1 - 2e^{-\frac{r}{b}}\right). \qquad 31$$

where N_{nl} is a normalization constant.

5.0 Solutions of the Polar (angular) Part

The eigenvalues and the eigen functions of the polar part of the Klein-Gordon equation in this case can be obtained by making use of Eqs. (4) and (17). Substituting Eq. (4) into Eq. (17), we have:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\theta\right) + \left[\lambda - \frac{m^2}{\sin^2\theta} - 2(E+M)\frac{\gamma + \beta\cos^2\theta + \Im\cos^4\theta}{\cos^2\theta\sin^2\theta}\right]\Theta(\theta) = 0. \qquad 30$$

Using the transformation, $q = \cos^2\theta$, Eq. (28) reduces to

$$\frac{d^2\Theta(q)}{dq^2} + \frac{(1-3q)}{2q(1-q)}\frac{d\Theta(q)}{dq} + \frac{1}{4q^2(1-q)^2}$$

$$\times\begin{bmatrix}-(\lambda+2(E+M)\Im)q^2\\+(\lambda-m^2-2(E+M)\beta)q-2(E+M)\gamma\end{bmatrix}\Theta(q)=0. \qquad 31$$

Comparing Eq. (33) with Eq. (7) we obtain the following parameters:

$$c_1=\frac{1}{2},\ c_2=\frac{3}{2},\ c_3=1,$$
$$\xi_1=\frac{1}{4}(\lambda+2(E+M)\Im),\ \xi_2=\frac{1}{4}(\lambda-m^2-2(E+M)\beta),\ \ \xi_3=\frac{1}{2}(E+M)\gamma,$$
$$c_4=\frac{1}{4},\ c_5=-\frac{1}{4},\ c_6=\frac{1}{16}+\frac{1}{4}(\lambda+2(E+M)\Im)$$
$$c_7=-\frac{1}{8}-\frac{1}{4}(\lambda-m^2-2(E+M)\beta),\ c_8=\frac{1}{16}+\frac{1}{2}(E+M)\gamma,$$
$$c_9=\frac{1}{4}(m^2+2(E+M)\gamma+2(E+M)\beta+2(E+M)\Im),$$
$$c_{10}=1+2\sqrt{\frac{1}{2}(E+M)\gamma+\frac{1}{16}},$$
$$c_{11}=2+2\left(\frac{1}{2}\sqrt{m^2+2(E+M)\gamma+2(E+M)\beta+2(E+M)\Im}+\sqrt{\frac{1}{2}(E+M)\gamma+\frac{1}{16}}\right),$$
$$c_{12}=\frac{1}{4}+\sqrt{\frac{1}{2}(E+M)\gamma+\frac{1}{16}},$$
$$c_{13}=-\frac{1}{4}-\left(\frac{1}{2}\sqrt{m^2+2(E+M)\gamma+2(E+M)\beta+2(E+M)\Im}+\sqrt{\frac{1}{2}(E+M)\gamma+\frac{1}{16}}\right). \qquad 34$$

Substituting Eq. (34) into Eq. (9), we obtain the relation for λ as

$$\lambda=4\left(n+\frac{1}{2}\right)^2+2(2n+1)\left(\sqrt{m^2+2(E+M)\gamma+2(E+M)\beta+2(E+M)\Im}+\sqrt{2(E+M)\gamma+\frac{1}{4}}\right)$$
$$+2\sqrt{(m^2+2(E+M)\gamma+2(E+M)\beta+2(E+M)\Im)\left(2(E+M)\gamma+\frac{1}{4}\right)}$$
$$+m^2+4(E+M)\gamma+2(E+M)\beta, \qquad 35$$

For the corresponding wave function of the angle dependent part, we obtain by substituting Eq. (34) into Eq. (10) as

$$\Theta(q) = N_m s^{\frac{1}{4}+\frac{1}{2}\sqrt{(E+M)\gamma+\frac{1}{4}}}(1-q)^{\frac{1}{2}\sqrt{m^2+2(E+M)\gamma+2(E+M)\beta+2(E+M)\Im}} \times P_n^{\left(\sqrt{2(E+M)\gamma+\frac{1}{4}},\sqrt{m^2+2(E+M)\gamma+2(E+M)\beta+2(E+M)\Im}\right)}(1-2q). \quad 36$$

Equation (36) can further be written as

$$\Theta(\theta) = N_m(\cos\theta)^{\frac{1}{2}+\sqrt{2(E+M)\gamma+\frac{1}{4}}}(\sin\theta)^{\sqrt{m^2+2(E+M)\gamma+2(E+M)\beta+2(E+M)\Im}} \times P_n^{\left(\sqrt{2(E+M)\gamma+\frac{1}{4}},\sqrt{m^2+2(E+M)\gamma+2(E+M)\beta+2(E+M)\Im}\right)}(-\cos 2\theta), \quad 37$$

where N_m is a normalization constant.

For the effect of the angle dependent part on the radial solutions of the Klein-Gordon equation, we substitute Eq.(35) into Eq.(29) and obtain

$$E^2 - M^2 = -\frac{1}{4b^2}\left[\frac{-2b^2(E+M)V_0 + (E+M) + \frac{15}{64} + \frac{15\left(n+\frac{1}{2}\right)^2}{4} + \delta + \left(n+\frac{1}{2}+\frac{1}{4}\left(l+\frac{1}{2}\right)\right)^2}{(n+l+1)}\right]^2 + \frac{(2n+1)\sqrt{2(E+M)\gamma+\frac{1}{4}}}{6b^2} + \frac{\{m^2+2(E+M)\beta+4(E+M)\gamma\}}{12b^2} + \sigma, \quad 38$$

where

$$\delta = \frac{15}{8}(2n+1)\left(\sqrt{m^2+2(E+M)\gamma+2(E+M)\beta+2(E+M)\Im} + \sqrt{2(E+M)\gamma+\frac{1}{4}}\right) + \frac{15}{8}\sqrt{\left(m^2+2(E+M)\gamma+2(E+M)\beta+2(E+M)\Im\right)\left(2(E+M)\gamma+\frac{1}{4}\right)} + \frac{15}{16}\left[m^2+4(E+M)\gamma+2(E+M)\beta\right], \quad 39$$

and

$$\sigma = \frac{1}{12b^2}\left[2(2n+1)\left(\sqrt{m^2+2(E+M)\gamma+2(E+M)\beta+2(E+M)\Im}+\sqrt{2(E+M)\gamma+\frac{1}{4}}\right)\right.$$
$$\left.+\,2\sqrt{\left(m^2+2(E+M)\gamma+2(E+M)\beta+2(E+M)\Im\right)\left(2(E+M)\gamma+\frac{1}{4}\right)}+4\left(n+\frac{1}{2}\right)^2\right], \qquad 40$$

Finally, we can write the total wave function for the system as

$$\psi(r,\theta,\varphi)=\frac{N_{nm}}{\sqrt{2\pi}}\frac{1}{r}e^{-\frac{\mu r}{b}+im\varphi}\left(1-e^{-\frac{r}{b}}\right)^{\frac{1}{2}+\nu}P_n^{(2\mu,2\nu)}\left(1-2e^{-\frac{r}{b}}\right)$$
$$\times(\cos\theta)^{\frac{1}{2}+\sqrt{2(E+M)\gamma+\frac{1}{4}}}(\sin\theta)^{\sqrt{m^2+2(E+M)\gamma+2(E+M)\beta+2(E+M)\Im}}$$
$$\times P_n^{\left(\sqrt{2(E+M)\gamma+\frac{1}{4}},\sqrt{m^2+2(E+M)\gamma+2(E+M)\beta+2(E+M)\Im}\right)}(-\cos 2\theta), \qquad 41$$

where N_{nm} is the normalization constant.

If we set $V_0=\gamma=\Im=0$ and $b=0$, it is found that our potential in Eq. (1) reduces to a potential of the form

$$V(r,\theta)=\frac{\hbar^2}{2\mu}\frac{\beta}{r^2\sin^2\theta}. \qquad 42$$

Substituting these parameters into Eqs.(38) and Eq.(41), we obtain the corresponding energy spectrum and the wave function of this potential respectively. Also, setting $\gamma=\beta=\Im=0,$ and mapping $\frac{1}{2b^2}\rightarrow 0$ reduces Eq. (1) into Hulthen potential of the form [48]

$$V(r)=-\frac{V_0}{\left(e^{\frac{r}{b}}-1\right)}. \qquad 43$$

Substituting these parameters in the energy spectrum of Eq.(38) and wave function of Eq.(41), we obtain the desired energy spectrum and the wave function of the Hulthen potential respectively. Finally, setting $\gamma=\beta=\Im=0,\ b=0$ and mapping $\frac{1}{2b^2}\rightarrow 0$

reduces Eq.(1) to constant potential.

6.0 Conclusion

In this paper, we have obtained the approximate bound state solutions of the relativistic Klein-Gordon equation in the case of equal scalar and vector shifted Hulthen plus angle dependent potential using parametric form of Nikiforov-Uvarov method with the help of approximation scheme in ref.[10] to evaluate the centrifugal term. The bound states energy eigenvalues and the corresponding wave functions in terms of Jacobi polynomial are obtained. Our results could be used to study the interactions and binding energies of the noncentral potential for diatomic molecules in the relativistic framework. The results will also have many applications in chemical and molecular physics and the recently reported result of neutron-proton pairs in heavy nuclei using perturbation theory [49]. Also, this problem under investigation will have great applications in the nonrelativistic quantum mechanics in the limiting cases as reported in recent works [50-52]. By appropriate choice of potential parameters our potential in Eq. (1) reduces to few well known potentials in the literature.

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