

On Some Fourier Asymptotes of Fractal Measures

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Abstract
We give a condition for a quasi-regular set to satisfy certain density, if μ is absolutely continuous with respect to $\mu_{\alpha/E}$ and an inequality was hold. We investigate a Fourier asymptotic of fractal measures with a sharp bound. For a continuous measure with a monotone discrete sequence a best estimate was proved.
Keywords: Maximal functions, Wiener's Measures, Fractal Measures, quasi-regular

1.Introduction

So much of harmonic analysis being with maximal functions, and maximal functions are understood via covering lemmas. One of the most powerful covering lemmas is the following, due to Besicovitch (a short proof found in de Guzman [5]). Here $B_r(x)$ denotes the open ball of radius r centered at x. But really not necessary that we deal with balls – for example, cubes would do as well, but not general rectangles – but it is essential that the set be centered at x.

Proposition1.1: There exists a constant c_n depending only on the dimension, such that if $A \subset R^n$. is measurable and a collection $\{B_{r(x)}(x)\}_{x\in A}$ of balls centered at each point of A is given with the radii $r(x)$ arbitrary but uniformly bounded, then there exists a finite or countable sub-collection $\{B_k\}$ which covers A with no more than c_n overlaps; i.e.

$$x_A \le \sum x_{B_k} \le c_n \qquad on \qquad R^2. \tag{1}$$

Let μ be any locally finite measure on R^n . (Actually we could do with the following hypothesis: for $\mu-$almost every x there exists $r>0$ such that $0<\mu(B_r(x))<\infty$). We define the centered maximal function

$$M_\mu f(x) = \sup_{r\succ 0} \mu(B_r(x))^{-1} \int_{B_r(x)} |f|\, d\mu \tag{2}$$

For any locally integrable f ,where we take $0/0=0 if\, \mu(B_r(x))=0$. It is easy to see that $M_\mu f$ is measurable.

Theorem1.1: The operator M_μ satisfies the weak-L^1 estimate

$$\mu:\{x: M_\mu f(x)>s\} \le c_n s^{-1}\|f\|_1 \tag{3}$$

For all $f\in L^1(d\mu)$, and the L^p estimate

$$\|M_p f\|_p \le c_p \|f\|_p \tag{4}$$

For all $f\in L^p(d\mu), 1<p\le\infty$, where all L^p norms are with respect to μ .

Proof.

Let $E_s=\{x: M_\mu f(x)>s\}$. For every $x\in E_s$ there exists r such that

$$\int_{B_r(x)} |f| \, d\mu \geq s\mu(B_r(x))$$

Assume first that E_s is bounded, so that we may apply the Besicovitch covering lemma to obtain $\{B_k\}$, and then

$$c_n \|f\|_1 \geq \sum \int_{B_k} |f| \, d\mu \geq \sum s_\mu(B_k) \geq s_\mu(E_s)$$

By (1), which is (3). In the general case we partition R^n into a countable union of bounded sets, run the above on each bounded set, and then sum. Then (4) follows by the Marcinkiewiecz interpolation theorem in[3] using the trivial $p = \infty$ case.

This result is also proved in [3]. The next result is proved by different method by [2] - [7], but also using his covering lemma.

Corollary1.1: For any $f \in L^1(d\mu)$,

$$\lim_{r \to 0} \mu(B_r(x))^{-1} \int_{B_r(x)} f d\mu = f(x) \qquad (5)$$

for $\mu-$almost every x and in fact also

$$\lim_{r \to \sigma} \mu(B_r(x))^{-1} \int_{B_r(x)} |f(y) - f(x)| \, d\mu(y) = 0 \qquad (6)$$

proof. Continuous functions are dense in $L^1(d\mu)$ because μ is $\sigma-$ finite hence regular. Since (5) and (6) are obviously true for this dense subclass, the result follows for all $L^1(d\mu)$ by general functional analysis principles and the estimate (3).

Corollary 1.2: For any $f \in L^p(d\mu), 1 < p < \infty$,

$$\lim_{r \to 0} \mu(B_r(x))^{-1} \int_{B_r(x)} f d\mu = f(x) \qquad in \qquad L^p(d\mu) \qquad (7)$$

Proof:

Convergence almost everywhere follows the previous Corollary (localized), and then L^p convergence follows from (4) by the dominated convergence theorem.

Now fix a real value α satisfying $0 < \alpha \leq n$, and define the $\alpha-$dimensional centered maximal function by

$$M_x f(x) = \sup_{r \succ o} r^{-x} \int_{B_{r(x)}} |f| \, d\mu. \qquad (8)$$

Similarly we define the local $\alpha-$dimensional centered maximal function by

$$m_x f(x) = \sup_{0 \prec r \leq 1} r^{-x} \int_{B_{r(x)}} |f| \, d\mu$$

Observe that these maximal functions depend on the measure μ, but this dependence is suppressed in the notation.

We will say that the measure μ is uniformly $\alpha-$dimensional if there exists a constant c such that

$$\mu(B_r(x)) \leq cr^x \quad \text{for all } x \quad and \quad r > 0 \qquad (9)$$

Similarly, we say that μ is locally uniformly $\alpha-$dimensional if (9) holds for $0 < r \leq 1$. It is easy to see that a locally uniformly $\alpha-$dimensional measure must be absolutely continuous with respect to $\alpha-$dimensional Hausdorff measure μ_x, but such a measure need not exhibit any actual "fractal" behavior. Thus, for example, Lebesgue is locally uniformly $\alpha-$dimensional for any $\alpha \leq n$. We can allow $\alpha = 0$ in these definitions, in which case a measure is uniformly 0-dimensional if and only if it is finite, and locally uniformly 0-dimensional if and only if $\mu(B_1(x))$ is uniformly bounded in x.

2. Maximal functions and Wiener's Measures

Corollary2.1: If x is uniformly $\alpha-$dimensional then M_x is bounded on $L^p(d\mu) for 1 < p \leq \infty$ and satisfies a weak-L^1 estimate, similarly for m_x if μ is locally uniformly $\alpha-$dimensional.

Proof:
$M_x f \le cM_\mu f$ in the first case, and $m_x f \le cM_\mu f$ in the second case. It is also interesting to ask if these results remain true if we drop the requirement that the balls be centered at x, and only require that they contain x. Journe [4] shows that this is the case when the dimension n = 1, but not when $n \ge 2$.
If the measure μ satisfies a doubling condition, then all these results are known . However, most fractal measures do not satisfy a doubling condition.

Let μ be a positive measure with no infinite atoms, not necessarily $\sigma-$ finite, and let ν be a $\sigma-$ finite positive measure which is absolutely continuous with respect to μ ,$\nu << \mu$, in the usual sense $\left(\mu(E)=0\ implies \nu\ (E)=0\right)$. The Radon-Nikodym theorem does not apply in this situation but there is a simple substitute result. We will say that a measure ν is null with respect to μ, written $\nu << \mu$,if $\mu(E)<\infty$ implies $\nu(E)=0$. Clearly this is a stronger condition than absolute continuity, and it implies that $\nu(E)=0$ if E is an $\sigma-$ finite set for μ. In particular, if μ were $\sigma-$ finite, then only the zero measure could be null with respect to μ. But for non-$\sigma-$ finite measure μ, such as counting measure on R , it is easy to give examples of non-trivial measures which are null with respect to μ. But again, if $d\nu = fd\mu$ for a measurable non-negative function f, then we cannot have ν null with respect to μ unless ν is the zero measure. Thus null measures and the Radon-Nikodym measures with respect to μ form mutually exclusive classes.

Theorem 2.1: Let μ be a measure with no infinite atoms, and let ν be $\sigma-$ finite and absolutely continuous with respect to μ ,$\nu << \mu$. Then there exists a unique decomposition $\nu=\nu_1+\nu_2$ such that $d\nu_1 = fd\mu$ for a non-negative measurable function f, and ν_2 is null with respect to μ ,$\nu_2 << \mu$.

Proof:
The uniqueness has already been noted. For existence it suffices to consider the case where ν is a finite measure. Then let sl denote the set of measurable sets A such that $\nu(A)>0$ and μ restricted to A is $\sigma-$ finite. Let a denote the sup of $\nu(A)$ for $A \in sl$, and choose a sequence of sets $A_j \in sl$ such that $\lim_{j\to\infty} \nu(A_j)=a$, and set $B=\bigcup_{j=1}^{\infty} A_j$. We claim $\nu_1 = \nu|_B$ and $\nu_2 = \nu|_{cB}$ is the desired decomposition.
Indeed $d\nu_1 = fd\mu$ by the Radon-Nikodym theorem since $\mu|_B$ is $\sigma-$ finite. To show $\nu_2 << \mu$ Assume $\mu(E)<\infty$. Then $\nu_2(E)=0$ for if not we would have $\nu(B\cup E)>a$ and $B\cup E \in sl$, a contradiction
Note that. If μ is counting measure, then the decomposition $\nu=\nu_1+\nu_2$ is just the familiar decomposition of a measure into discrete and continuous parts.
Now we specialize to the case $\mu=\mu_x$, the Hausdorff measure of dimension α on R^n. The definition of the $\alpha-$upper density(see[4])

$$\overline{D}_x(\nu,x)=\limsup_{r\to 0}(2r)^{-x}\nu(B_r(x))$$

Of a measure ν. Similarly the $\alpha-$lower density $\underline{D}_x(\nu,x)$ is defined with the liminf in place of limsup.

Theorem 2.2: If ν is a locally finite measure on R^n that is null with respect to μ_x ,$\nu << \mu_x$, then $\overline{D}_x(\nu,x)=0$ for μ_x-almost every x.

Proof: Let E_k denote the set of $x \in R^n$ such that for all $\varepsilon>0$ there exists $r \le \varepsilon$ with $(2r)^{-x}\nu(B_r(x)) \ge 1/k$. It is easy to see that the union of the sets E_k is exactly the set of points where $\overline{D}_x(\nu,x)>0$, so it suffices to show $\mu_x(E_k)=0$ for every k. we do this first for the case when ν is a finite measure.

Now we apply the Besicovitch covering lemma to the balls whose existence define E_k, and obtain a

cover $\{B_r(x_j)\}$ of E_k such that $\sum x_{B_{rj}}(x) \le c_n$ everywhere. However, each ball has radius $r_j \le \varepsilon$, so $B_{rj}(x_j) \subseteq E_{k,\varepsilon}$ where $E_{k,\varepsilon}$ denotes the set of points of distance $\le \varepsilon$ from E_k. Thus $\sum x_{B_{rj}}(x_j) \le c_n x_{E_{k,\varepsilon}}$ hence $\sum \nu(B_{rj}(x_j)) \le c_n \nu(E_{k,\varepsilon})$. But since we also have $(2r_j)^x \le k\nu(B_{rj}(x_j))$ we have $\sum (2r_j)^x \le c\nu(E_{k,\varepsilon})$, and letting $\varepsilon \to 0$ this shows $\mu_x(E_k) \le c\nu(E_k)$ by the definition of μ_x and the fact that $E_k = \bigcap_\varepsilon E_{k,\varepsilon}$ and is finite this means $\mu_x(E_k) < \infty$ hence $\nu(E_k) = 0$ hence $\mu_x(E_k) = 0$.
Finally, if ν is only a locally finite measure, we can apply the same argument to the restriction of ν to any fixed ball B to show $\mu_x(E_k \cap B) = 0$ hence $\mu_x(E_k) = 0$.
Using the same method of proof, we can give some refinements of Corollaries (1.1), (1.2), and (2.1). We assume now that μ is locally uniformly $\alpha-$ dimensional. It is easy to see that this implies $\mu << \mu_x$, Let $\mu = \mu_1 + \mu_2$ be the decomposition of Theorem (2.1), and let E be a set that supports μ_1. (The fact that μ_x contains no infinite atoms follows from a deep theorem of Besicovitch, see below.)

Theorem 2.3: For any $f \in L^1(d\mu)$,

$$\lim_{r\to 0} r^{-x} \int_{B_r(x)} f d\mu = 0 \qquad (10)$$

For μ_x-almost every x in the complement of E.

Proof:
We may assume $f \ge 0$ and μ is finite, without loss of generality.
For each k let

$A_k = \left\{ x \notin E : for\ all \quad \varepsilon > 0 there \quad exists \quad r \le \varepsilon such \quad that\ r^{-x} \int_{B_r(y)} f d\mu \ge 1/k \right\}$. It suffices to show $\mu_x(A_k) = 0$ for each , since $\bigcup A_k$ is the subset of the complement of E where (10) fails to hold.
Assume first that E supports μ, so $\int_{A_k} f d\mu = 0$. We apply the Besicovitch covering lemma to obtain a covering of A_k by balls $\{B_{rj}(x_j)\}$ such that $\sum x_{B_{rj}}(x_j) \le c_n x_{Ak,\varepsilon}$. Since $r_j^x \le k \int_{B_{rj}(xj)} f d\mu$ we have $\sum r_j^x \le kc_n \int_{Ak,\varepsilon} fdu$ which shows $\mu_x(A_k) \le c\int_{Ak} fdu = 0$.

Now in the general case E supports μ_1, so let E_2 be disjoint from E and support μ_2. The above argument shows (10) holds μ_x-almost everywhere on the complement of $E \cup E_2$, so it suffices to show (10) holds μ_x- almost everywhere on E_2. But the above argument also shows $\lim_{r\to 0} r^{-x} \int_{B_r(x)} f d\mu_1 = 0$ μ_x-almost everywhere on E_2, so it remains to show $\lim_{r\to 0} r^{-x} \int_{B_r(x)} f d\mu_2 = 0$ for μ_x-almost every $x \in E_2$. But this is Theorem (2.2) for $\nu = f d\mu_2$.

We can combine this result with Corollary (1.1) to obtain precise estimates for $\limsup_{r\to 0} r^{-x} \int_{B_r(x)} fdu$ in case μ is the restriction of μ_x to a set E. We say that a set E is locally uniformly $\alpha-$dimensional if the restriction of μ_x to E is locally uniformly $\alpha-$dimensional. A powerful theorem of Besicovitch [4] shows that every Borel set of infinite μ_x measure contains subsets of arbitrary finite μ_x measure that are locally uniformly $\alpha-$dimensional. (Besicovitch only proved the result for $F_{\sigma\delta\sigma}$ sets; the extension to Borel sets is due to Davies [6].

Corollary 2.2: Let E be locally uniformly $\alpha-$dimensional, let μ denote the restriction of μ_x to E, let $f \in L^1(d\mu)$ be non-negative, and set $f(x) = 0$ for $x \notin E$. Then

$$2^{-x} f(x) \le \limsup_{r\to 0} (2r)^{-x} \int_{B_r(x)} f d\mu \le f(x) \qquad (11)$$

for μ_x − almost every x.

Proof:

For $x \notin E$ this is just (10), For μ − almost every $x \in E$ we have (5) by Corollary (1.1), hence

$$\limsup_{r\to 0} (2r)^{-x} \int_{B_r(x)} f d\mu = \overline{D}_x(\mu, x) f(x)$$

The result follows since it is known that $2^{-x} \le \overline{D}_x(\mu, x) \le 1$ for μ-almost every $x \in E$ (this result is also due to Besicovitch).

Note that. In fact it is easy to show that every Borel set E of finite, positive μ_x measure contains locally uniformly α − dimensional subsets E_ε with $\mu_x(E_\varepsilon) \ge \mu_x(E) - \varepsilon$ for every $\varepsilon > 0$. Indeed, let

$$F_k = \left\{ x \in E : \sup_{0 < r \le 1} r^{-x} \mu_x\left(B_r(x) \cap E\right) \le k \right\}$$

It is easy to see that F_k is measurable and increasing with k, and each F_k is locally uniformly α − dimensional. But μ-almost every $x \in E$ belongs to $\bigcup_k F_k$ since $\overline{D}_x(\mu, x) \le 1$ for μ_x-almost every $x \in E$, so $\lim_{k\to\infty} \mu_x(F_k) = \mu_x(E)$. Of course, the constant of local uniform α − dimensionality tends to infinity with k. Nevertheless, the result is interesting because sometimes we obtain estimates that are independent of this constant.

These results give us control of $m_x f(x)$ for x outside the support of μ. Indeed if $\limsup_{r\to 0} r^{-r} \int_{B_r(x)} |f| d\mu$ is finite then so is $m_x f(x)$, since

$$\sup_{\varepsilon \le r \le 1} r^{-x} \int_{B_x(x)} |f| d\mu \le \varepsilon^{-x} \int_{B_r(x)} |f| d\mu.$$

Thus if E is as in Corollary (1.1) then $m_x f(x)$ is finite for μ_x-almost every x. More generally, if μ is any locally uniformly α − dimensional measure supported on a set E, then $m_x f(x)$ is finite μ_x-almost everywhere on the complement of E. To see this, assume on the contrary that there exists a set E_1 disjoint from E with $\mu_x(E_1) > 0$ and $m_x f(x) = +\infty$ on E_1. By the above remarks there exists a locally uniformly α − dimensional subset $E_2 \subseteq E_1$ with $0 < \mu_x(E_2) < \infty$. Let $\nu = \mu_{x/E_2}$ and consider the measure $\mu + \nu$ and the function f which is extended to be zero on E_2. Clearly $\mu + \nu$ is locally uniformly α − dimensional, and the maximal function $m_x f$ formed with respect to $\mu + \nu$ is the same as the one formed with respect to μ. But then Corollary (2.1) applied to $\mu + \nu$ shows $m_x f$ is finite almost everywhere with respect to ν, a contradiction.

We begin with a simple measure theoretic lemma valid for any σ-finite measure μ on a measure space for which points are measurable and with atoms of bounded size. Write $\mu = \mu_1 + \mu_2$, where μ_2 is continuous and $\mu_1 = \sum c_j \delta(x - a_j)$, is discrete.

Lemma 2.1: Let μ be as above with $c_j \le M$ for all j. Then for any $f \in L^2(d\mu)$ we have

$$\iint x(x = y) f(x) \overline{f(y)} d\mu(x) d\mu(y) = \sum \left| f(a_j) \right|^2 c_j^2, \qquad (12)$$

Where $x(x = y)$ denotes the characteristic function of the diagonal.

Proof:

By Fubini's theorem it suffices to verify the result for one iterated integral. Since $f(x) = f(y)$ whenever

$x(x=y)$ is different from zero we can write the integral as $\int\left(f\left|f(x)\right|^2 x(x=y)\,d\mu(y)\right)d\mu(x)$. Doing the y integration first we obtain

$$\int\left|f(x)\right|^2 \mu(\{x\})\,d\mu(x)$$

Which equals $\sum\left|f(a_j)\right|^2 c_j^2$, and this is finite because $c_j \le M$ and $f \in L^2$.

Now let μ be a measure on R^n which is locally uniformly zero dimensional, meaning

$$\left|\mu(B)\right| \le M \qquad (13)$$

For any ball B of radius one. Clearly this implies that μ is σ-finite and satisfies the hypothesis of Lemma (2.1) It is also easy to verify that if $f \in L^2(d\mu)$ then $fd\mu$ is a tempered distribution and so $(fd\mu)^\wedge$ is well-defined as a tempered distribution.

Lemma 2.2: Under the above hypotheses, $(fd\mu)^\wedge \in L^2_{loc}$ in fact

$$\int_{\mathbb{R}^n}\left|(fd\mu)^\wedge(\xi)\right|^2 e^{-1|\xi|^2}d\xi \prec \infty \; for \quad all\, t \succ 0.$$

Proof:
By definition

$$\left\langle (fd\mu)^\wedge, \varphi\right\rangle = \int \varphi(x) f(x)\,d\mu(x)$$

for any $\varphi \in \Psi$. Thus to show $(fd\mu)^\wedge \in L^2\left(e^{-1|\xi|^2}d\xi\right)$ it suffices to establish the estimate

$$\left|\left\langle (fd\mu)^\wedge, \varphi(\xi)e^{-1|\xi|^2}\right\rangle\right| \le c_1\left(\int\left|\psi(\xi)^2 e^{-1|\xi|^2}d\xi\right|\right)^{1/2} \qquad (14)$$

for all $\psi \in \Psi$. To do this we set $\varphi(\xi) = \psi(\xi)e^{-(1/2)t|\xi|^2}$, so that (14) becomes

$$\left|\left\langle (fd\mu)^\wedge, \varphi(\xi)e^{-(1/2)t|\xi|^2}\right\rangle\right| \le c_1\left\|\varphi\right\|_2.$$

But we know that

$$\left(\varphi(\xi)e^{-(1/2)t|\xi|^2}\right)^\wedge(x) = c_t\int\varphi(x-y)e^{-(1/2)t|y|^2}dy$$

So that we need only show

$$\left|\iint\varphi(x-y)e^{-t|y|^2}dyf(x)\,d\mu(x)\right| \le c_t\left\|\varphi\right\|_2 \qquad (15)$$

after some trivial changes in notation. We can restate (15) as follows: the operator T defined by

$$T\varphi = e^{-t|x|^2} * \varphi$$

Is a bounded operator from $L^2(dx) to L^2(d\mu)$.

But now by the Riesz interpolation theorem it suffices to show that T is bounded from $L^1(dx) to L^1(d\mu)$ and from $L^\infty(dx) to L^\infty(d\mu)$. The second statement is trivial, since T maps $L^\infty(dx)$ to continuous bounded functions. For the first, we observe that (13) implies

$$\int e^{-t|x-y|^2}d\mu(x) \le c_t M, \qquad (16)$$

And so

$$\int\left|T\varphi(x)\right|d\mu(x) \le \iint\left|\varphi(y)\right|e^{-t|x-y|^2}dyd\mu(x) \le c_t M\int\left|\varphi(y)\right|dy$$

Theorem 2.4: Under the same hypotheses as Lemma (2.2), we have

$$\lim_{t\to 0} t^{n/2}\int\left|(fd\mu)^{\wedge}(\xi)\right|^2 e^{-t|\xi|^2}d\xi = \pi^{n/2}\sum\left|f(a_j)\right|^2 c_j^2 \qquad (17)$$

Proof:
A formal calculation shows

$$t^{n/2}\int\left|(fd\mu)^{\wedge}(\xi)\right|^2 e^{-t|\xi|^2}d\xi$$
$$= t^{n/2}\iiint f(x)\overline{f(y)}e^{i(x-y),\xi}e^{-t|\xi|^2}d\mu(y)d\xi$$
$$= \pi^{n/2}\iint e^{-|x-y|^{2/4t}} f(x)\overline{f(y)}d\mu(x)d\mu(y) \qquad (18)$$

And as $t\to 0$ the integrand tends to $x(x=y)f(x)\overline{f(y)}$, so that (17) would follow from Lemma (2.1), provided we could justify the interchange of limit and integral and the formal computation.
Therefore we begin by looking at

$$\iint e^{-|x-y|^{2/4t}} f(x)\overline{f(y)}d\mu(x)d\mu(y).$$

For $t\le 1/4$ the integrand is dominated by $e^{-|x-y|^2}\left|f(x)f(y)\right|$, and we will show this belongs to $L^1(\mu\times\mu)$. This clearly follows if we can show that the operator S defined by $Sf(x)=\int e^{-|x-y|^2} f(y)d\mu(y)$ is bounded on $L^2(d\mu)$. But both statements are easy consequences of (16).
Thus we know that the integral in (18) is absolutely convergent, and the dominated convergence theorem applies to establish

$$\lim_{t\to 0}(4\pi)^{-n/2}\iint e^{-|x-y|^{2/4t}} f(x)\overline{f(y)}d\mu(x)d\mu(y)$$
$$=(4\pi)^{-n/2}\sum\left|f(a_j)\right|^2 c_j^2.$$

Finally to justify (18) we note first that if we assume $f\in L^1\cap L^2(d\mu)$ then all the integrals in (18) are absolutely integrable, so (18) is valid by Fubini's theorem in (18) are general $f\in L^2(d\mu)$, we consider the sequence $f_k(x)=f(x)x(|x|\le k) in L^1\cap L^2(d\mu)$ which converges to f in $L^2(d\mu)$. Then

$$\lim_{k\to\infty}(4\pi)^{-n/2}\iint e^{-|x-y|^{2/4t}} f_k(x)\overline{f_k(y)}d\mu(x)d\mu(y)$$
$$=(4\pi)^{-n/2}\iint e^{-|x-y|^{2/4t}} f(x)\overline{f(y)}d\mu(x)d\mu(y)$$

by the argument above and the dominated convergence theorem, while

$$\lim_{k\to\infty} t^{n/2}\int\left|(f_k d\mu)^{\wedge}(\xi)\right|^2 e^{-t|\xi|^2}d\xi$$
$$= t^{n/2}\int\left|(f\,d\mu)^{\wedge}(\xi)\right|^2 e^{-t|\xi|^2}d\xi$$

by the proof of Lemma (2.2).
Note that. The proof also shows

$$\sup_{0\le t\le 1} t^{n/2}\int\left|(f\,d\mu)^{\wedge}(\xi)\right|^2 e^{-t|\xi|^2}d\xi \le c\int\left|f(x)\right|^2 d\mu(x).$$

Theorem 2.5: Let ν be any complex measure on R^n satisfying

$$\sum_{k\in\square^n}\left(|\nu|(Q(k))\right)^2<\infty \qquad (19)$$

Where $Q(k)$ denotes the cube of side length 1 centered at k, and write

$$\nu=\sum c_j\delta(x-a_j)+\nu_2,$$

Where ν_2 is continuous. Then $\nu\in y'(R^n), \hat{\nu}\in L^2_{loc}(R^n)$ and

$$\lim_{r\to\infty} \frac{1}{\Omega r^n} \int_{|\xi|\le r} \left|\hat{\nu}(\xi)\right|^2 d\xi = \sum \left|c_j\right|^2, \tag{20}$$

Where Ω denotes the volume of the unit ball. Furthermore we have

$$\sup_{r\ge 1} \frac{1}{r^n} \int_{|\xi|\le r} \left|\hat{\nu}(\xi)\right|^2 d\xi \le c \sum_{k\in\mathbb{Z}^n} \left(|\nu|(Q(k))\right)^2 \tag{21}$$

Proof.

Define a positive measure μ by $\mu(A) = |\nu|(A)/|\nu|(Q(k))$ for $A \subseteq Q(k)$, so clearly (13) is satisfied. Furthermore we have $d\nu = f d\mu$ where $|f(x)| = |\nu|(Q(k))$ for $x \in Q(k)$, so $f \in L^2(d\mu)$ by (19). Therefore Theorem (2.4) applies to $d\nu = f d\mu$, so

$$\lim_{t\to 0} t^{n/2} \int \left|\hat{\nu}(\xi)\right|^2 e^{-t|\xi|^2} d\xi = \pi^{n/2} \sum \left|c_j\right|^2$$

and (20) follows by a familiar Tauberian theorem. Finally (21) follows from the note following the proof of Theorem (2.4).

We may consider

$$\hat{\nu}(\xi) = \sum c_j e^{ia.\xi} + \nu_2(\xi)$$

as a sum of an almost periodic function and some "noise" $\hat{\nu}_2(\xi)$, so that Wiener's theorem says that Bohr mean of $|\hat{\nu}|^2$ picks out the total energy of the almost periodic component. In Wiener's version, where ν is a finite measure, we have $\sum |c_j| < \infty$ so the almost periodic component is uniformly almost periodic, and in fact has an absolutely convergent Fourier series. In our version, the restriction on the almost periodic component is that

$$\sum_{k\in\mathbb{Z}^n} \left(\sum_{a_j \in Q(k)} |c_j| \right)^2 < \infty \tag{22}$$

Which is considerably weaker, but not as weak as Besicovitch's B^2 class of almost periodic functions [1] of which we only need

$$\sum \left|c_j\right|^2 < \infty \tag{23}$$

However, there are uniformly almost periodic functions which do not satisfy (22), essentially because the left side of (22) fails to be dilation invariant.

It would appear that the B^2 class of almost periodic functions is the natural class to consider for a generalization of Wiener's theorem of the form: Bohr mean $\left(|f + noise|^2\right) = \sum |c_j|^2$ since Besicovitch shows Bohr mean $\left(|f|^2\right) = \sum |c_j|^2$ for

$$f(\xi) \to \sum c_j e^{ia_j\xi} \tag{24}$$

Under the assumption (23) alone.

We close this section with a brief discussion of the analogue of Wiener's theorem for Hermite and related expansions.We restrict ourselves to the simplest cases; there are clearly many generalizations possible in the spirit of the other results.

On R^1 we consider the normalized Hermite functions $h_k(x) = \left(2^k k!\sqrt{\pi}\right)^{-1/2} e^{-x^2/2} H_k(x)$ where $H_k(x) = (-1)^k e^{x^2} (d/dx)^k e^{-x^2}$ is the kth Hermite polynomial. Then $\|h_k\|_2 = 1$ with respect to Lebesgue measure, and

$$\left(-\frac{d^2}{dx^2} + x^2\right) h_k(x) = (2k+1) h_k(x)$$

In fact the system $\{h_k\}_{k=0}^{\infty}$ is the complete eigenfunction system associated with the self-adjoint operator $\left(-\left(d^2/dx^2\right)+x^2\right)$ on $L^2\left(R^t,dx\right)$.

For any finite measure μ on R, let $\hat{\mu}(k)=\int h_k(x)\,d\mu(x)$ so $\sum_0^\infty \hat{\mu}(k)h_k(x)$ is the Hermite expansion for μ.

Theorem 2.6: Let $\mu=\sum c_j\delta\left(x-a_j\right)+\mu'$, where μ' is a continuous measure. Then

$$\lim_{t\to 1^-}\left(1-t^2\right)^{1/2}\sum_0^\infty\left|\hat{\mu}(k)\right|^2 t^k=\pi^{-1/2}\sum\left|c_j\right|^2 \tag{25}$$

and

$$\lim_{N\to\infty} N^{-1/2}\sum_{k=0}^{N-1}\left|\hat{\mu}(k)\right|^2=\sqrt{2}\pi^{-1}\sum\left|c_j\right|^2 \tag{26}$$

Proof.

The basic generating function identify for Hermite polynomials is

$$\sum_0^\infty H_k(x)H_k(y)\frac{t^k}{2^k\,k\,!}=\left(1-t^2\right)^{-1/2}\exp\left(\frac{2xyt-\left(x^2+y^2\right)t^2}{1-t^2}\right) \tag{27}$$

for $0<t<1$. Therefore

$$\left(1-t^2\right)^{1/2}\sum_0^\infty\left|\hat{\mu}(k)\right|^2 t^k$$

$$=\int\left(1-t^2\right)^{1/2}\sum_0^\infty t^k h_k(y)\,d\mu(x)\overline{d\mu(y)}$$

$$=\pi^{-1/2}\int\exp\left(-\left(\frac{x^2+y^2}{2}\right)+\frac{2xyt-\left(x^2+y^2\right)t^2}{1-t^2}\right)d\mu(x)\overline{d\mu(y)}$$

Now (25) follows by the dominated convergence and Lemma (2.1) since the function

$$G_t(x,y)=\exp\left(-\left(\frac{x^2+y^2}{2}\right)+\frac{2xyt-\left(x^2+y^2\right)t^2}{1-t^2}\right)$$

$$=\exp\left(-\left(\frac{1}{1-t^2}\right)(x-y)^2-\frac{1}{2}\left(\frac{1-t}{1+t}\right)\left(x^2+y^2\right)\right)$$

is uniformly bounded by one and

$$\lim_{t\to 1^-}G_t(x,y)=\begin{cases}0 & if \quad x\neq y\\ 1 & if \quad x=y.\end{cases}$$

If we set $\varepsilon=1-t$ we can rewrite this as

$$\lim_{\varepsilon\to 0^+}\varepsilon^{1/2}\sum\left|\hat{\mu}(k)\right|^2 e^{-\varepsilon k}=(2\pi)^{-1/2}\sum\left|c_j\right|^2$$

Then (26) follows by a Tauberian theorem with $n=1$, $t=1/2$.

The surprising feature of (26) is the power of N that occurs. A similar result holds in R^n.

More generally, we consider the self-adjoint operator $-A+A|x|^\beta$ on R^n, where A denotes the Laplacian and $\beta>1$. Let $\{\varphi_k\}$ denote a complete set of eigenfunctions with eigenvalues $\lambda_1\le\lambda_2\le\cdots$ arranged in non-decreasing order. It is known that

$$\lambda_k\to(ak)^{1/\gamma}\; as \quad k\to\infty, \tag{28}$$

where

$$\gamma=\frac{n}{2}+\frac{n}{\beta}$$

and

$$a=2^{n} A^{n/\beta}\frac{\Gamma(n/2+1)\Gamma(\gamma+1)}{\Gamma(n/\beta+1)}$$

(Actually one has the same result for $R-A+V$ if $V=|x|^{\beta}+V_1$ and V_1 is suitably small.)

For μ a finite measure one R^n write $\hat{\mu}(k)=\int\varphi_k(x)d\mu(x)$.

Theorem 2.7:

Let $\mu=\sum c_j\delta(x-a_j)+\mu'$ where μ' is a continuous measure. Then

$$\lim_{N\to\infty} N^{-\beta/(\beta+2)}\sum_{1}^{N}|\hat{\mu}(k)|^2=b\sum|c_j|^2, \qquad (29)$$

where

$$b=\left(\frac{2^2\Gamma(\gamma)}{\Gamma(n/\beta+1)}\right)^{\beta/(\beta+2)}\frac{A^{n/(\beta+2)}}{(4\pi)^{n/2}\Gamma(n/2+1)^{2/(\beta+2)}}$$

Proof.

Let $K(t_1x,y)=\sum e^{-t\lambda k}\varphi_k(x)\varphi_k(y)$ denote the heat kernel for the operator $-A+A|x|^{\beta}$. It is known that the behavior as $t\to 0$ the same as the Euclidean heat kernel $(4\pi t)^{-n/2}e^{-|x-y|2/4t}$, hence

$$\lim_{t\to 0} t^{n/2}\sum e^{-t\lambda k}|\hat{\mu}(k)|^2=(4\pi)^{-n/2}\sum|c_j|^2 \qquad (30)$$

By Lemma (2.1). But then (28) and (29) imply (30) by a Tauberian theorem.

3.Main Results

Theorem 3.1: For a constant c and $f\in L^p(d\mu)$ where $\mu<<\mu_\alpha$, and $\mu=\mu_{\alpha/E}+\mu'$, E is regular while $\mu_\alpha(\cdot)<\infty$,such that

$$\lim_{r\to\infty}\frac{1}{r^{n-\alpha}}\int_{B_r(y)}|F(x)|^2dx=c\int_E|f|^2d\mu_a$$

Proof. with the upper density $\bar{D}_a(\mu,x)=1$ and regularity of E gives

$$\limsup_{r\to\infty}\int_{B_r(y)}|F(x)|^2dx\le c\int_E|f|^2d\mu_a$$

and similarly if E is a C^1 manifold with lower density $\underline{D}_\alpha(\mu_{a/E},x)\ge c>\frac{n}{n+1}$ for any $n\ge 1$ we have

$$\liminf_{r\to\infty}\frac{1}{r^{n-\alpha}}\int_{B_r(y)}|F(x)|^2dx\le c\int_E|f|^2d\mu_a$$

and if $\mu<<\mu_{\alpha/E}$ then

$$\lim_{r\to\infty}\frac{1}{r^{n-\alpha}}\int_{B_r(y)}|F(x)|^2dx=c\int_E|f|^2d\mu_a$$

Corollary 3.1: Show that

$$(f\ d\mu)^{\wedge}\in L^2\left(e^{-t|\beta|^2}d\beta\right)$$

Proof. From the definition

$$\langle(fd\mu)^{\wedge},f\rangle=\int|f(x)|^2d\mu(x)$$

we can establish the estimate

$$\left|\left\langle (fd\mu)^{\wedge}, f(\beta)e^{-\frac{1}{2}t|\beta|}\right\rangle\right| \le c_t\left(\int\left|\varphi(\beta)^2 e^{-\frac{1}{2}t|\beta|^2} d\beta\right|\right)^{1/2}$$

Now set $f(\beta)=\varphi(\beta)e^{-\frac{1}{2}t|\beta|^2}$ so

$$\left|\left\langle (fd\mu)^{\wedge}, f(\beta)e^{-\frac{1}{2}t|\beta|}\right\rangle\right| \le c_t\left(\int|f(\beta)|^2 d\beta\right)^{1/2}$$

$$= c_t\|f\|_2$$

But $\left(f(\beta)e^{-\frac{1}{2}t|\beta|^2}\right)^{\wedge}(x)=c_t\int f(x-y)e^{-\frac{1}{2}t|y|^2}dy$ we show

$$\left|\iint f(x-y)e^{-t|y|^2}dy\, f(x)d\mu(x)\right|$$

$$\le \iint\left|f(x-y)e^{-t|y|^2}dy\right| f(x)d\mu(x)$$

$$=\int\left(f(\beta)e^{-\frac{1}{2}t|\beta|^2}\right)^{\wedge}|f(x)|d\mu(x)$$

$$\le c_t\|f\|_2$$

Corollary 3.2:

Let $\mu=\mu_1+\mu_2$, where μ_2 is a continuous measure and $\mu_1=\sum_{j=0}^{\infty}c_j\delta(x\text{-}a_j)$ is discrete with $\{c_j\}$ is monotone such that $|c_j|\le M$, for $j\ge 1$. Then

$$\lim_{t\to 1^-}\sum_{k=0}^{\infty}\frac{|\hat{\mu}(k)|^2 t^k}{(1-t)^{1/2}}\le\frac{M^2}{\sqrt{\pi}}$$

Proof. For any $f\in L^p(d\mu)$, Theorem (2.1) implies that $\iint x\,|f(x)|^2|d\mu(x)|^2=\sum_{j=0}^{\infty}|f(a_j)|^2c_j^2$ where x is the characteristic function of the diagonal. Set $|\hat{\mu}(k)|^2=\frac{k_k^2(x)}{2^k k!}$, then the Hermite polynomial is given by

$$\sum_{k=0}^{\infty}\frac{k_k^2(x)\,t^k}{2^k k!}=\left(1-t^2\right)^{-\frac{1}{2}}e^{\frac{2x^2t}{1+t}}\quad,\ 0<t<1$$

we have

$$\lim_{t\to 1^-}\sum_{k=0}^{\infty}\frac{|\hat{\mu}(k)|^2 t^k}{(1-t)^{1/2}}=\lim_{t\to 1^-}\left|\int\left(1-t^2\right)^{\frac{1}{2}}\sum_{k=0}^{\infty}t^k k_k(x)|d\mu|^2\right|$$

$$\le\lim_{t\to 1^-}\pi^{-\frac{1}{2}}\int e^{\left(-x^2+\frac{2x^2t}{1+t}\right)}|d\mu|^2$$

$$=\lim_{t\to 1^-}\pi^{-\frac{1}{2}}\int e^{\frac{x^2(t-1)}{t+1}}|d\mu|^2$$

set $\left|f\ (a_j)\right|^2 = e^{\frac{a_j^2 (t-1)}{t+1}}$ and using the Lebesgue deminated convergence Theorem and for any $f \in L^p(d\mu)$ we have

$$\int \left|f\ (x)\right|^2 \left|d\mu\right|^2 = \lim_{t \to 1^-} \sum_{j=0}^{\infty} \left|f\ (a_j)\right|^2 c_j^2$$

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