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Analysis of an $M^{[X]}/G/1$ Feedback Retrial Queue with Two Phase Service, Bernoulli Vacation, Delaying Repair and Orbit Search

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Abstract
In this paper, we considered a batch arrival feedback retrial queue with two phase of service under Bernoulli vacation schedule and orbit search. At the arrival epoch of a batch, if the server is busy, under repair or on vacation then the whole batch joins the orbit. Where as if the server is free, then one of the arriving customers starts his service immediately and the rest join the orbit. At the completion epoch of each service, the server either goes for a vacation or may wait for serving the next customer. While the server is working with any phase of service, it may breakdown at any instant and the service channel will fail for a short interval of time. The repair process does not start immediately after a breakdown and there is a delay time for repair to start. After vacation completion, the server searches for the customers in the orbit (i.e. customer in the orbit, if any taken for service immediately) or remains idle. The probability generating function of the number of customers in the system and orbit are found using the supplementary variable technique. The mean numbers of customers in the system/orbit and special cases are analyzed. The effects of various parameters on the performance measure are illustrated numerically.
Keywords: Feedback, retrial queue, Bernoulli vacation, delaying repair, orbit search

1. Introduction

Queueing system is a powerful tool for modeling communication and transportation networks, production lines, operating systems, etc. Retrial queues (or queues with repeated attempts) are characterized by the phenomenon that an arriving customer who finds the server busy upon arrival is obliged to leave the service area and repeat his demand after some time. Between trials, a blocked customer who remains in a retrial group is said to be in orbit. Queues in which customers are allowed to conduct retrials have wide applications in telephone switching systems, telecommunication networks and computers to gain service from a central processing unit. Choudhury (2009) investigated a single server retrial queue with an additional phase of second service and general retrial times. There is an extensive literature on the retrial queues. We refer the works by Falin and Templeton (1997) and Artalejo (2010) as a few.

The feedback phenomenon is one of the important tools for communication systems. When the service of a customer is unsatisfied, the service can be retried again until the service is completed successfully. For example, in multiple accesses telecommunication systems, where messages turned out as errors are sent again can be modeled as retrial queues with feedback. Choudhury and Paul (2005) inspected the M/G/1 system with two phases of heterogeneous service and Bernoulli feedback. In this system a tagged customer may get an unsuccessful service and then it retries to take the service until a successful service.

In a vacation queueing system, the server may not be available for a period of time due to many reasons like, being checked for maintenance, working at other queues, scanning for new work (a typical aspect of many communication systems) or simply taking break. This period of time, when the server is unavailable for primary customers is referred as a vacation. Krishnakumar and Arivudainambi (2002) have investigated a single server retrial queue with Bernoulli schedule and general retrial times. These models arise naturally in call centers with multi-task employees, customized manufacturing, telecommunication and computer networks, maintenance activities, production and quality control problem, etc. Some of the authors like, Choudhury and Madhan (2004), Rajadurai *et al.* (2015) have developed queueing models with the concept of general retrial times along with Bernoulli vacation schedule.

The service interruptions are unavoidable phenomenon in many real life situations. In most of the studies, it is assumed that the server is available in the service station on a permanent basis and service station never fails. In practice we often meet the case where service stations may fail and can be repaired. Applications of these models found in the area of computer communication networks and flexible manufacturing system etc. Ke and Choudhury (2012) discussed about the batch arrival retrial queueing system with two phases of service under the concept of breakdown and delaying repair. While the server is working with any phase of service, it may breakdown at any instant and the service channel will fail for a short interval of time. The repair process does not start immediately after a breakdown and there is a delay time for repair to start. Choudhury and Deka (2008) considered a single server queue with two phases of service and the server is subject to breakdown while providing service to the customers. Further, Choudhury and Deka (2012) developed a model with vacation. Authors like Wang and Li (2009) and Rajadurai *et al.* (2014) discussed about the retrial queueing systems with the concept of breakdown and repair. In the retrial setup, after completion of each service the server will remain idle in the system until the arrival of the next primary or retrial customer. Server's idle time is reduced by the introduction of search of orbital customers immediately after a service completion. Search for orbital customers was introduced by Neuts *et al.* (1984) where the authors examined classical queue with search for customers immediately on termination of a service. Orbital search after service have been investigated by Krishnamoorthy *et al.* (2005), Deepak *et al*. (2013), Gao and Wang (2014). In this model, we discussed after vacation completion, the server searches for the customers in the orbit or remain idle. However, no work has been done in the feedback retrial queueing model taking into account, the two phase service under Bernoulli vacation and breakdowns. Hence to fill this gap, we introduced a concept in this paper, a batch arrival feedback retrial queueing system with two phases of service under Bernoulli vacation, orbit search and delaying repair.

The rest of this paper is organized as follows. In section 2, the detailed description of the mathematical model is given. In section 3, we consider the governing equations of our model and also obtain the steady-state solutions. Some performance measures and important special cases are derived in Section 4. In section 5, the effects of various parameters on the system performance are analyzed numerically. Conclusion and application of the work are presented in section 6.

2. Description of the model

In this section, the detailed description of the model is given as follows:

Arrival process: Customers arrive in batches according to a compound Poisson process with rate λ. Let X_k denote the number of customers belonging to the k^{th} arrival batch, where X_k, $k = 1,2,3,\ldots$are with a common distribution $Pr[X_k = n]=\chi_n$, $n = 1,2,3\ldots$ and $X(z)$ denotes the probability generating function of X.

Retrial process: We assume that there is no waiting space and therefore if arriving customers find the server being busy or on vacation, all these customers leave the service area and join a pool of blocked customers called an orbit. Later the customers in the orbit try to get their service. Inter-retrial times have an arbitrary distribution $R(x)$ with corresponding Laplace–Stieltjes transform (LST) $R^*(x)$.

Service process: Service is provided one by one FCFS basis. Every customer has to undergo two stages of heterogeneous service. The service times of both phases follow different general (arbitrary) distributions, the first phase service (FPS) followed by the second phase service (SPS). It is assumed that the i^{th} (i=1,2) phase service follows general random variable S_i with distribution function $S_i(t)$and LST $S_i^*(x)$.

Feedback process: After completion of two stages of services if the customer is unsatisfied with his service then he can immediately join the orbit as feedback customer for receiving another service with probability r or he may depart from the system with probability $1-r$.

Vacation process: After completion two phases of service of each customer, the server may take a vacation with probability p, and with probability 1-p it waits for serving the next customer. The vacation time of the server is of random length V with distribution function $V(t)$ and LST $V^*(x)$.

Orbit search rule: At the end of a vacation, the server searches for the customers in the orbit with probability θ (i.e. customer in the orbit, if any taken for service immediately) or remains idle with probability (1-θ).

Breakdown process: While the server is working with any phase of service, it may breakdown at any time and the service channel will fail for a short interval of time i.e. server is down for a short interval of time. The breakdowns i.e. server's life times are generated by exogenous Poisson processes with rates α_1 for FPS and α_2 for SPS, which we may call some sort of disaster during FPS and SPS periods respectively.

Repair process: As soon as breakdown occurs the server is sent for repair, during that time it stops providing service to the arriving batch of and waits for repair to start, which we may refer to as waiting period of the server. We define the waiting time as delay time. The delay time D_i of the server for i^{th} phase of service follows with d.f. $D_i(t)$ and LST $D_i^*(y)$. The customer who was just being served before server breakdown waits for the remaining service to complete. The repair time (denoted by G_1 for FPS and G_2 for SPS) distributions of the server for both the phases of service are assumed to be arbitrarily distributed with d.f. $G_i(t)$ and LST $G_i^*(y)$.

In addition, let $R^0(t)$, $S_i^0(t)$, $V^0(t)$, $D_i^0(t)$ and $G_i^0(t)$ be the elapsed retrial time, service time, vacation time, delay times and repair time respectively at time t. In the steady state, we assume that $R(0)$=0, $R(\infty)$=1, $S_i(0)$ = 0, $S_i(\infty)$=1, $V(0)$=0, $V(\infty)$=1 are continuous at x = 0 and $D_i(0)$ = 0, $D_i(\infty)$ = 1, $G_i(0)$ = 0, $G_i(\infty)$ =1 are continuous at y = 0.

The state of system at time t can be described by the bivariate Markov process $\{(C(t), X(t), \zeta(t)), t \ge 0\}$ where $C(t)$denotes the server state (0,1,2,3,4) depending if the server is free, busy on FPS or SPS, vacation, delaying repair on FPS or SPS and repair on FPS or SPS respectively. $X(t)$ corresponds to the number of customers in orbit at time t and $\zeta(t)$ represents the elapsed time for server states.

So that the function $a(x)$, $\mu_i(x)$, $\gamma(x)$,$\eta_i(y)$ *and* $\xi_i(y)$ are the conditional completion rates for repeated attempts, for service, vacation, delay and repair times respectively (for i=1,2). Conditional completion rates for repeated attempts, service on both phases, vacation, delay in both phases and under repair on both phases respectively,

$a(x)dx = \frac{dR(x)}{1-R(x)}$ $\mu_i(x)dx = \frac{dS_i(x)}{1-S_i(x)}$ $\gamma(x)dy = \frac{dV(x)}{1-V(x)}$ $\eta_i(y)dy = \frac{dD_i(y)}{1-D_i(y)}$ $\xi_i(y)dy = \frac{dG_i(y)}{1-G_i(y)}$

Now we analyze the stability of this model. At first, we analyze the embedded Markov chain at departure completion epochs. Let $\{t_n ; n = 1,2,\ldots\}$ be the sequence of epochs at which either a service period completion occurs or a vacation time ends. The sequence of random vectors $Z_n = \{C(t_n+), X(t_n+)\}$ form a Markov chain, which is embedded Markov chain for our retrial queueing system, then we have the embedded Markov chain $\{Z_n; n \in N\}$ is ergodic if and only if $\rho < 1$,

where $\rho = \frac{1}{1-r}\left[E(X)[(1-R^*(\lambda)(1-\theta p)] + \lambda E(X)\left[E(S_1)[1+\alpha_1(E(D_1)+E(G_1))] + E(S_2)[1+\alpha_2(E(D_2)+E(G_2))] + pE(V)\right]\right]$

3. Steady state analysis

In this section, assume that the ergodic condition $\rho < 1$ is fulfilled. Then, we shall derive the steady state probability distribution of our queueing model. For the process $\{X(t), t \geq 0\}$, we define the probability $P_0(t) = P\{C(t)=0, X(t)=0\}$ and the probability densities for $t \geq 0$, $x > 0$ and $n \geq 1$

$P_n(x,t)dx = P\left\{C(t)=0, X(t)=n, x \leq R^0(t) < x+dx\right\}$, $\Pi_{i,n}(x,t)dx = P\left\{C(t)=1, X(t)=n, x \leq S_i^0(t) < x+dx\right\}$,

$\Omega_n(x,t)dx = P\left\{C(t)=2, X(t)=n, x \leq V^0(t) < x+dx\right\}$,

$Q_{i,n}(x,y,t)dy = P\left\{C(t)=3, X(t)=n, y \leq D_i^0(t) < y+dy \big/ S_i^0(t)=x\right\}$

$R_{i,n}(x,y,t)dy = P\left\{C(t)=4, X(t)=n, y \leq G_i^0(t) < y+dy \big/ S_i^0(t)=x\right\}$

We assume that the stability condition is fulfilled in the sequence and so that we can set $P_0 = \lim_{t\to\infty} P_0(t)$, $P_n(x) = \lim_{t\to\infty} P_n(x,t)$, $\Pi_{i,n}(x) = \lim_{t\to\infty} \Pi_{i,n}(x,t)$, $\Omega_n(x) = \lim_{t\to\infty} \Omega_n(x,t)$, $Q_{i,n}(x,y) = \lim_{t\to\infty} Q_{i,n}(x,y,t)$, $R_{i,n}(x,y) = \lim_{t\to\infty} R_{i,n}(x,y,t)$. By the method of supplementary variable technique, we obtain the following system of equations that govern the dynamics of the system behavior.

$$\lambda P_0 = (1-\theta)\int_0^\infty \Omega_0(x)\gamma(x)dx + (1-r)q\int_0^\infty \Pi_{2,0}(x)\mu_2(x)dx \tag{3.1}$$

$$\frac{dP_n(x)}{dx} + [\lambda + a(x)]P_n(x) = 0, \; n \geq 1 \tag{3.2}$$

$$\frac{d\Pi_{i,0}(x)}{dx} + [\lambda + \alpha_i + \mu_i(x)]\Pi_{i,0}(x) = \int_0^\infty \xi_i(y)R_{i,0}(x,y)dy, n=0, \; \text{for}(i=1,2) \tag{3.3}$$

$$\frac{d\Pi_{i,n}(x)}{dx} + [\lambda + \alpha_i + \mu_i(x)]\Pi_{i,n}(x) = \lambda\sum_{k=1}^{n}\chi_k\Pi_{i,n-k}(x) + \int_0^\infty \xi_i(y)R_{i,n}(x,y)dy, n \geq 1, \; \text{for}(i=1,2) \tag{3.4}$$

$$\frac{d\Omega_n(x)}{dx} + [\lambda + \gamma(x)]\Omega_n(x) = \lambda\sum_{k=1}^{n}\chi_k\Omega_{n-k}(x), \; n \geq 0 \tag{3.5}$$

$$\frac{dQ_{i,0}(x,y)}{dy} + [\lambda + \eta_i(y)]Q_{i,0}(x,y) = 0, \; n=0, \; \text{for}(i=1,2) \tag{3.6}$$

$$\frac{dQ_{i,n}(x,y)}{dy} + [\lambda + \eta_i(y)]Q_{i,n}(x,y) = \lambda\sum_{k=1}^{n}\chi_k Q_{i,n-k}(x,y), \; n \geq 1, \; \text{for}(i=1,2) \tag{3.7}$$

$$\frac{dR_{i,0}(x,y)}{dy} + [\lambda + \xi_i(y)]R_{i,0}(x,y) = 0, \; n=0, \; \text{for}(i=1,2) \tag{3.8}$$

$$\frac{dR_{i,n}(x,y)}{dy} + [\lambda + \xi_i(y)]R_{i,n}(x,y) = \lambda\sum_{k=1}^{n}\chi_k R_{i,n-k}(x,y), \; n \geq 1, \; \text{for}(i=1,2) \tag{3.9}$$

The steady state boundary conditions are

$$P_n(0)=(1-\theta)\int_0^\infty \Omega_n(x)\gamma(x)dx+(1-r)q\int_0^\infty \Pi_{2,n}(x)\mu_2(x)dx+rq\int_0^\infty \Pi_{2,n-1}(x)\mu_2(x)dx,\ n\geq 1 \tag{3.10}$$

$$\Pi_{1,n}(0)=\int_0^\infty P_{n+1}(x)a(x)dx+\lambda\sum_{k=1}^{n}\chi_k\int_0^\infty P_{n-k+1}(x)dx+\theta\int_0^\infty \Omega_{n+1}(x)\gamma(x)dx+\lambda\chi_{n+1}P_0,\quad n\geq 1 \tag{3.11}$$

$$\Pi_{2,n}(0)=\int_0^\infty \Pi_{1,n}(x)\mu_1(x)dx,\ n\geq 1 \tag{3.12}$$

$$\Omega_n(0)=(1-r)p\int_0^\infty \Pi_{2,n}(x)\mu_2(x)dx+rp\int_0^\infty \Pi_{2,n-1}(x)\mu_2(x)dx,\ n\geq 0 \tag{3.13}$$

$$Q_{i,n}(x,0)=\alpha_i\Pi_{i,n}(x),\ n\geq 1,\ \text{for}(i=1,2) \tag{3.14}$$

$$R_{i,n}(x,0)=\int_0^\infty Q_{i,n}(x,y)\eta_i(y)dy,\ n\geq 1 \tag{3.15}$$

The normalizing condition is

$$P_0+\sum_{n=1}^{\infty}\int_0^\infty P_n(x)dx+\sum_{n=0}^{\infty}\left\{\int_0^\infty \Omega_n(x)dx+\sum_{i=1}^{2}\left[\int_0^\infty \Pi_{i,n}(x)dx+\int_0^\infty\int_0^\infty Q_{i,n}(x,y)dxdy+\int_0^\infty\int_0^\infty R_{i,n}(x,y)dxdy\right]\right\}=1 \tag{3.16}$$

We use the method of probability generating function to solve the above equations (3.1)-(3.15), so we define the probability generating functions, where $|z|\leq 1$, $i=1,2$

$$P(x,z)=\sum_{n=1}^{\infty}P_n(x)z^n\ ;\ P(0,z)=\sum_{n=1}^{\infty}P_n(0)z^n\ ;\ \Pi_i(x,z)=\sum_{n=0}^{\infty}\Pi_{i,n}(x)z^n\ ;\ \Pi_i(0,z)=\sum_{n=0}^{\infty}\Pi_{i,n}(0)z^n$$

$$\Omega(x,z)=\sum_{n=0}^{\infty}\Omega_n(x)z^n\ ;\Omega(0,z)=\sum_{n=0}^{\infty}\Omega_n(0)z^n\ ;Q_i(x,y,z)=\sum_{n=0}^{\infty}Q_{i,n}(x,y)z^n\ ;Q_i(x,0,z)=\sum_{n=0}^{\infty}Q_{i,n}(x,0)z^n$$

$$R_i(x,y,z)=\sum_{n=0}^{\infty}R_{i,n}(x,y)z^n\ ;\ R_i(x,0,z)=\sum_{n=0}^{\infty}R_{i,n}(x,0)z^n\ \text{and}\ X(z)=\sum_{n=1}^{\infty}\chi_n z^n$$

Multiplying the steady state equations and steady state boundary conditions (3.1) - (3.15) by z^n and summing over n,

$$\frac{\partial P(x,z)}{\partial x}+[\lambda+a(x)]P(x,z)=0, \tag{3.17}$$

$$\frac{\partial \Pi_i(x,z)}{\partial x}+[\lambda-\lambda X(z)+\alpha_i+\mu_i(x)]\Pi_i(x,z)=\int_0^\infty \xi_i(y)R_i(x,y,z)dy \tag{3.18}$$

$$\frac{\partial \Omega(x,z)}{\partial x}+[\lambda-\lambda X(z)+\gamma(x)]\Omega(x,z)=0 \tag{3.19}$$

$$\frac{\partial Q_i(x,y,z)}{\partial y}+[\lambda-\lambda X(z)+\eta_i(y)]Q_i(x,y,z)=0 \tag{3.20}$$

$$\frac{\partial R_i(x,y,z)}{\partial y}+[\lambda-\lambda X(z)+\xi_i(y)]R_i(x,y,z)=0 \tag{3.21}$$

$$P(0,z)=(1-\theta)\int_0^\infty \Omega(x,z)\gamma(x)dx+(1-r)q\int_0^\infty \Pi_2(x,z)\mu_2(x)dx+rqz\int_0^\infty \Pi_2(x,z)\mu_2(x)dx-\lambda P_0 \tag{3.22}$$

$$\Pi_1(0,z)=\frac{1}{z}\int_0^\infty P(x,z)a(x)dx+\frac{\lambda X(z)}{z}\left[\int_0^\infty P(x,z)dx+P_0\right]+\frac{\theta}{z}\int_0^\infty \Omega(x,z)\gamma(x)dx \tag{3.23}$$

$$\Pi_2(0,z) = \int_0^\infty \Pi_1(x,z)\mu_1(x)dx \tag{3.24}$$

$$\Omega(0,z) = (1-r+rz)p\int_0^\infty \Pi_2(x,z)\mu_2(x)dx \tag{3.25}$$

$$Q_i(x,0,z) = \alpha_i \Pi_i(x,z) \text{ for(i=1,2)} \tag{3.26}$$

$$R_i(x,0,z) = \int_0^\infty Q_i(x,y,z)\eta_i(y)dy \text{ for(i=1,2)} \tag{3.27}$$

Solving the partial differential equations (3.17)-(3.21), it follows that

$$P(x,z) = P(0,z)[1-R(x)]e^{-\lambda x} \tag{3.28}$$

$$\Pi_i(x,z) = \Pi_i(0,z)[1-S_i(x)]e^{-A_i(z)x}, \tag{3.29}$$

$$\Omega(x,z) = \Omega(0,z)[1-V(x)]e^{-\lambda_0(z)x} \tag{3.30}$$

$$Q_i(x,y,z) = Q_i(x,0,z)[1-D_i(y)]e^{-\lambda_0(z)y}, \tag{3.31}$$

$$R_i(x,y,z) = R_i(x,0,z)[1-G_i(y)]e^{-\lambda_0(z)y}, \tag{3.32}$$

$$A_i(z) = \lambda_0(z) + \alpha_i[1-D_i^*(\lambda_0(z))G_i^*(\lambda_0(z))] \quad \text{and} \quad \lambda_0(z) = \lambda\left(1-X(z)\right)$$

Using (3.30) and (3.29) in (3.22), finally we get

$$P(0,z) = \left(1-r+rz\right)\Pi_1(0,z)\left\{\left(q+p(1-\theta)V^*(\lambda_0(z))\right)B_1^*\left[A_1(z)\right]B_2^*\left[A_2(z)\right]\right\} - \lambda P_0 \tag{3.33}$$

Inserting (3.23) and (3.28) in (3.33), we get

$$P(0,z) = \frac{Nr(z)}{Dr(z)} \tag{3.34}$$

$$Nr(z) = \lambda P_0\left\{X(z)(1-r+rz)\left\{\left(q+(1-\theta)pV^*\left[\lambda_0(z)\right]\right)S_1^*\left[A_1(z)\right]S_2^*\left[A_2(z)\right]\right\} - \left[z-\theta p(1-r+rz)S_1^*\left[A_1(z)\right]S_2^*\left[A_2(z)\right]V^*\left[\lambda_0(z)\right]\right]\right\}$$

$$Dr(z) = \left\{\begin{array}{l} z-\theta p(1-r+rz)S_1^*\left[A_1(z)\right]S_2^*\left[A_2(z)\right]V^*\left[\lambda_0(z)\right] \\ -\left(R^*(\lambda)+X(z)\left(1-R^*(\lambda)\right)\right)(1-r+rz)\left\{\left(q+(1-\theta)pV^*\left[\lambda_0(z)\right]\right)S_1^*\left[A_1(z)\right]S_2^*\left[A_2(z)\right]\right\} \end{array}\right\}$$

Using (3.22)-(3.25) and (3.28)-(3.30) and make some calculation, we get

$$\Pi_1(0,z) = \left\{\lambda P_0 R^*(\lambda)\left[X(z)-1\right]\right\}\Big/ Dr(z) \tag{3.35}$$

Using (3.35) in (3.24), we get

$$\Pi_2(0,z) = \left\{\lambda P_0 R^*(\lambda)\left[X(z)-1\right]S_1^*\left[A_1(z)\right]\right\}\Big/ Dr(z) \tag{3.36}$$

Using (3.36) in (3.25), we get

$$\Omega(z) = pP_0\left\{R^*(\lambda)(1-r+rz)S_1^*\left[A_1(z)\right]S_2^*\left[A_2(z)\right]V^*\left[\lambda_0(z)\right]\right\}\Big/ Dr(z) \tag{3.37}$$

Using (3.31)-(3.32) in (3.26)-(3.27), we get

$$Q_1(x,0,z) = \alpha_1\lambda P_0\left\{R^*(\lambda)\left[X(z)-1\right]\left[1-S_1(x)\right]e^{-A_1(z)x}\right\}\Big/ Dr(z) \tag{3.38}$$

$$Q_2(x,0,z) = \alpha_2\lambda P_0\left\{R^*(\lambda)\left[X(z)-1\right]S_1^*\left[A_1(z)\right]\left[1-S_2(x)\right]e^{-A_2(z)x}\right\}\Big/ Dr(z) \tag{3.39}$$

$$R_1(x,0,z) = \alpha_1\lambda P_0\left\{R^*(\lambda)D_1^*\left[\lambda_0(z)\right]\left[X(z)-1\right]\left[1-S_1(x)\right]e^{-A_1(z)x}\right\}\Big/ Dr(z) \tag{3.40}$$

$$R_2(x,0,z) = \alpha_2\lambda P_0\left\{R^*(\lambda)\left[X(z)-1\right]S_1^*\left[A_1(z)\right]D_2^*\left[\lambda_0(z)\right]\left[1-S_2(x)\right]\exp^{-A_2(z)x}\right\}\Big/ Dr(z) \tag{3.41}$$

For the limiting probability generating functions $P(x,z), \Pi_i(x,z), \Omega(x,z), Q_i(x,y,z)$ and $R_i(x,y,z)$ we define the partial probability generating functions as, for $(i = 1,2)$

$$P(z) = \Pi_i(z) = \int_0^\infty \Pi_i(x,z)dx,\ \Omega(z) = \int_0^\infty \Omega(x,z)dx,\ Q_i(x,z) = \int_0^\infty Q_i(x,y,z)dy,\ Q_i(z) = \int_0^\infty Q_i(x,z)dx,\ R_i(x,z) = \int_0^\infty R_i(x,y,z)dy,\ R_i(z) = \int_0^\infty R_i(x,z)dx,$$

Note that, $P(z)\left(\Pi_1(z), \Pi_2(z), \Omega(z), Q_1(z), Q_2(z), R_1(z), R_2(z)\right)$ is the probability generating function of orbit size when the server is idle (busy on FPS or SPS, on vacation, under delaying repair on FPS or SPS and repair on FPS or SPS respectively).

Let $P(z)=\dfrac{Nr(z)}{Dr(z)}$ (3.42)

$$Nr(z)=P_0\left[1-R^*(\lambda)\right]\left\{X(z)(1-r+rz)\left\{\left(q+(1-\theta)pV^*\left[\lambda_0(z)\right]\right)S_1^*\left[A_1(z)\right]S_2^*\left[A_2(z)\right]\right\}-\left[z-\theta p(1-r+rz)S_1^*\left[A_1(z)\right]S_2^*\left[A_2(z)\right]V^*\left[\lambda_0(z)\right]\right]\right\}$$

$$Dr(z)=\left\{\begin{array}{l} z-\theta p(1-r+rz)S_1^*\left[A_1(z)\right]S_2^*\left[A_2(z)\right]V^*\left[\lambda_0(z)\right] \\ -\left(R^*(\lambda)+X(z)\left(1-R^*(\lambda)\right)\right)(1-r+rz)\left\{\left(q+(1-\theta)pV^*\left[\lambda_0(z)\right]\right)S_1^*\left[A_1(z)\right]S_2^*\left[A_2(z)\right]\right\}\end{array}\right\}$$

$$\Pi_1(z)=P_0\left\{R^*(\lambda)\lambda_0(z)\left[S_1^*\left(A_1(z)\right)-1\right]\right\}\Big/A_1(z)Dr(z) \tag{3.43}$$

$$\Pi_2(z)=P_0\left\{R^*(\lambda)\lambda_0(z)S_1^*\left(A_1(z)\right)\left[S_2^*\left(A_2(z)\right)-1\right]\right\}\Big/A_2(z)Dr(z) \tag{3.44}$$

$$\Omega(z)=pP_0\left\{R^*(\lambda)[(1-r)+r]S_1^*\left[A_1(z)\right]S_2^*\left[A_2(z)\right]V^*\left[\lambda_0(z)\right]\right\}\Big/Dr(z) \tag{3.45}$$

$$Q_1(z)=\alpha_1P_0\left\{R^*(\lambda)\left[1-S_1^*\left(A_1(z)\right)\right]\left[D_1^*\left(\lambda_0(z)\right)-1\right]\right\}\Big/A_1(z)Dr(z) \tag{3.46}$$

$$Q_2(z)=\alpha_2P_0\left\{R^*(\lambda)S_1^*\left[A_1(z)\right]\left[1-S_2^*\left(A_1(z)\right)\right]\left[D_2^*\left(\lambda_0(z)\right)-1\right]\right\}\Big/A_2(z)Dr(z) \tag{3.47}$$

$$R_1(z)=\alpha_1P_0\left\{R^*(\lambda)D_1^*\left[\lambda_0(z)\right]\left[1-S_1^*\left(A_1(z)\right)\right]\left[G_1^*\left(\lambda_0(z)\right)-1\right]\right\}\Big/A_1(z)Dr(z) \tag{3.48}$$

$$R_2(z)=\alpha_1P_0\left\{R^*(\lambda)S_1^*\left[A_1(z)\right]D_2^*\left[\lambda_0(z)\right]\left[1-S_2^*\left(A_2(z)\right)\right]\left[G_2^*\left(\lambda_0(z)\right)-1\right]\right\}\Big/A_2(z)Dr(z) \tag{3.49}$$

Since, P_0 is the probability that the server is idle when no customer in the orbit and it can be determined using the normalizing condition. $P_0+P(1)+\Omega(1)+\Pi_1(1)+\Pi_2(1)+Q_1(1)+Q_2(1)+R_1(1)+R_2(1)=1$, Thus, by setting $z=1$ in (3.42) – (3.49) and applying L-Hospital's rule whenever necessary and we get

$$P_0=\left\{\frac{(1-r)-E(X)\left[(1-\theta p)(1-R^*(\lambda))\right]-\lambda E(X)\left[E(S_1)\left[1+\alpha_1\left(E(D_1)+E(G_1)\right)\right]+E(S_2)\left[1+\alpha_2\left(E(D_2)+E(G_2)\right)\right]+pE(V)\right]}{(1-r)R^*(\lambda)}\right\} \tag{3.50}$$

Then we define the probability generating function of the number of customer in the system is

$$K(z)=P_0+P(z)+\Omega(z)+z\left[\Pi_1(z)+\Pi_2(z)+Q_1(z)+Q_2(z)+R_1(z)+R_2(z)\right]$$

$$K(z)=\left\{P_0R^*(\lambda)\left[z-1\right](1-r)S_1^*\left[A_1(z)\right]S_2^*\left[A_2(z)\right]\right\}\Big/Dr(z) \tag{3.51}$$

The probability generating function of the number of customer in the orbit is

$$H(z)=P_0+P(z)+\Omega(z)+\left[\Pi_1(z)+\Pi_2(z)+Q_1(z)+Q_2(z)+R_1(z)+R_2(z)\right]$$

$$H(z)=\left\{P_0R^*(\lambda)\left[z-1\right]\left[1-r\,S_1^*\left[A_1(z)\right]S_2^*\left[A_2(z)\right]\right]\right\}\Big/Dr(z) \tag{3.52}$$

4. Performance measures

Now we derive the system performance measures of our model.

The mean number of customers in the system L_s under steady state condition is obtained by differentiating (3.51) with respect to z and evaluating at z=1

$$L_s=\lim_{z\to1}K'(z)=P_0R^*(\lambda)\left[\frac{Dr'(1)Nr''(1)-Nr'(1)Dr''(1)}{2\left[Dr'(1)\right]^2}\right]$$

$$Nr'(1)=(1-r)$$

$$Nr''(1)=2(1-r)\left[\lambda E(X)\right]\left[E(S_1)\left(1+\alpha_1\left[E(D_1)+E(G_1)\right]\right)+E(S_2)\left(1+\alpha_2\left[E(D_2)+E(G_2)\right]\right)+pE(V)\right]$$

$$Dr'(1)=(1-r)-E(X)\left[\left(1-\theta p\right)\left(1-R^*(\lambda)\right)\right]-\lambda E(X)\left[E(S_1)\left(1+\alpha_1\left[E(D_1)+E(G_1)\right]\right)+E(S_2)\left(1+\alpha_2\left[E(D_2)+E(G_2)\right]\right)+pE(V)\right]$$

$$
\begin{aligned}
Dr''(1) = -\Big\{ & \big[\lambda E(X)\big]^2 \Big[E(S_1^2)\big(1+\alpha_1\big[E(D_1)+E(G_1)\big]\big)^2 + E(S_2^2)\big(1+\alpha_2\big[E(D_2)+E(G_2)\big]\big)^2 + pE(V^2) \Big] \\
& + \big[\lambda E(X)\big]^2 \Big[\alpha_1 E(S_1)\big[E(D_1^2)+E(G_1^2)+2E(D_1)E(G_1)\big] + \alpha_2 E(S_2)\big[E(D_2^2)+E(G_2^2)+2E(D_2)E(G_2)\big] \Big] \\
& + E(X(X-1))\Big[\lambda\Big(E(S_1^2)\big(1+\alpha_1\big[E(D_1)+E(G_1)\big]\big)^2 + E(S_2^2)\big(1+\alpha_2\big[E(D_2)+E(G_2)\big]\big)^2 + pE(V) \Big) \Big] \\
& + 2\big[\lambda E(X)\big]^2 \Big[E(S_1)E(S_2)\big(1+\alpha_1\big[E(D_1)+E(G_1)\big]\big)\big(1+\alpha_2\big[E(D_2)+E(G_2)\big]\big) \Big] - E(X(X-1))\Big[(1-\theta p)\big(1-R^*(\lambda)\big) \Big] \\
& + 2p\big[\lambda E(X)\big]^2 E(V)\Big[E(S_1)\big(1+\alpha_1\big[E(D_1)+E(G_1)\big]\big) + E(S_2)\big(1+\alpha_2\big[E(D_2)+E(G_2)\big]\big) \Big] \\
& + 2r\lambda E(X)\Big[E(S_1)\big(1+\alpha_1\big[E(D_1)+E(G_1)\big]\big) + E(S_2)\big(1+\alpha_2\big[E(D_2)+E(G_2)\big]\big) + pE(V) \Big] \\
& + 2E(X)\big(1-R^*(\lambda)\big)\left\{ \begin{array}{l} \Big[\lambda E(X)\Big[E(S_1)\big(1+\alpha_1\big[E(D_1)+E(G_1)\big]\big) + E(S_2)\big(1+\alpha_2\big[E(D_2)+E(G_2)\big]\big) + pE(V) \Big] + r \Big] \\ -\theta p\Big[\lambda E(X)\Big[E(S_1)\big(1+\alpha_1\big[E(D_1)+E(G_1)\big]\big) + E(S_2)\big(1+\alpha_2\big[E(D_2)+E(G_2)\big]\big) + E(V) \Big] + r \Big] \end{array} \right\} \Big\}
\end{aligned}
$$

The mean number of customers in the system L_q under steady state condition is obtained by differentiating (3.52) with respect to z and evaluating at $z = 1$,

$$L_q = \lim_{z \to 1} H'(z) = P_0 R^*(\lambda) \left[\frac{Dr'(1)Nr''(1) - Nr'(1)Dr''(1)}{2\big[Dr'(1)\big]^2} \right]$$

$$Nr''(1) = -2r\big[\lambda E(X)\big]\Big[E(S_1)\big(1+\alpha_1\big[E(D_1)+E(G_1)\big]\big) + E(S_2)\big(1+\alpha_2\big[E(D_2)+E(G_2)\big]\big) \Big]$$

The average time a customer spends in the system (W_s) and in the orbit (W_q) under steady-state condition due to Little's formula, we obtain $W_s = L_s / \lambda E(X)$ and $W_q = L_q / \lambda E(X)$

4.1. Special cases

Case (i): *Single Poisson arrival, No Feedback and No orbit search*

Let $r = 0$, $\theta = 0$ and $\alpha_2 = 0$; then if we set $Pr[S_2 = 0] = 1$, our model can be reduced to A Batch arrival retrial queue with general retrial time under Bernoulli vacations for unreliable server and delaying repair. The following expression and the equations coincide with results of Ke and Choudhury (2012).

$$H(z) = \frac{\Big\{ \Big(1 - E(X)\big[1 - R^*(\lambda)\big] - \lambda E(X)\Big[E(S_1)\big[1 + \alpha_1\big(E(D_1)+E(G_1)\big)\big] + pE(V) \Big]\Big)\big[z-1\big] \Big\}}{\Big\{ z - \Big(R^*(\lambda) + X(z)\big(1 - R^*(\lambda)\big)\Big) \Big\}\Big\{ \Big(q + pV^*\big[\lambda_0(z)\big]\Big) S_1^*\big[A_1(z)\big] \Big\}}$$

Case (ii): *Single Poisson arrival, No vacation, No orbit search, No breakdown and No Retrial*

Let $p = 0$, $\theta = 0$ and $Pr[X = 1] = 1$; $\alpha_1 = \alpha_2 = 0$; $R^*(\lambda) \to 1$ then if we get M/G/1 queueing system with two phases of service and Bernoulli feedback. The following expression coincides with results of Choudhury and Paul (2005).

$$K(z) = \frac{\Big\{ \big[(1-r) - \lambda\big[E(S_1)+E(S_2)\big]\big]\big[z-1\big](1-r)S_1^*\big[\lambda - \lambda z\big]S_2^*\big[\lambda - \lambda z\big] \Big\}}{\Big\{ z - [(1-r) + rz]S_1^*\big[\lambda - \lambda z\big]S_2^*\big[\lambda - \lambda z\big] \Big\}}$$

5. Numerical illustration

In this section, we present some numerical examples to study the effect of various parameters in the system performance measures of our system where all retrial times, service times, vacation times, delay times and repair times are exponentially, Erlangianly and hyper-Exponentially distributed. We further assume that customers are arriving one by one, so $E(X)$=1, $E(X(X-1)) = 0$. We assume arbitrary values to the parameters such that the steady state condition is satisfied. The following tables give the computed values of various characteristics of our model like, probability that the server is idle P_0, the utilization factor ρ, the mean orbit size L_q, probability that server is retrial $P(1)$, where exponential distribution is $f(x) = ve^{-vx}, x > 0$, Erlang-2stage distribution is $f(x) = v^2 x e^{-vx}, x > 0$ and hyper-exponential distribution is $f(x) = cve^{-vx} + (1-c)v^2 e^{-v^2 x}, x > 0$.

Table 1 shows that when retrial rate (a) increases, then the probability that server is idle P_0 increases, the mean orbit size L_q decreasing and probability that server is idle during retrial time $P(1)$ also decreasing. Table 2 shows that when feedback (r) increases, then the probability that server is idle P_0 decreases, the mean orbit size L_q and all other characteristics are increasing. For the effect of the parameters a, r, θ, γ and η_1 on the system performance measures are given in following figures. Figure1 shows that the idle probability P_0 increasing for the increasing the value of the orbit search with probability θ. In Figure2 the surface displays downward trend as expected for increasing the value of the vacation rate γ and retrial rate a against the mean orbit size L_q

6. Conclusion

In this paper, we introduced a batch arrival feedback queueing system with general repeated attempts, two phases of service, Bernoulli vacation and orbit search, where the server is subject to breakdown and delaying repair. The probability generating function of the number of customers in the system and orbit are found using the supplementary variable technique. Various performance measures like the mean number of customers in the system/orbit, average waiting time customer spends in the system/orbit and special cases are analyzed. The effects of various parameters on the performance measure are illustrated numerically. Finally, the general decomposition law is shown to hold good for this model. Application of this paper results are useful to network design engineers and software design engineers to design various computer communication systems, packet switched networks, production lines and mail systems.

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Table 1: P_0 and Lq for different Retrial rate (a) for the values of $\lambda = 0.1$, $r = 0.2$; $\theta = 0.5$; $\mu_1 = 5$; $\mu_2 = 3$; $p = 0.5$; $\alpha_1 = 0.2$; $\alpha_2 = 0.1$; $\eta_1 = 3$; $\eta_2 = 2$; $\xi_1 = 2$; $\xi_2 = 1$; $\gamma = 5$; $c = 0.7$;

Retrial time	Exponential			Erlang – 2 stage			Hyper – Exponential		
a	P_0	L_q	$P(1)$	P_0	L_q	$P(1)$	P_0	L_q	$P(1)$
3.00	0.9095	0.0100	0.0009	0.7907	0.0557	0.0093	0.9306	0.0057	0.0002
4.00	0.9097	0.0090	0.0007	0.7930	0.0511	0.0070	0.9307	0.0050	0.0001
5.00	0.9099	0.0085	0.0005	0.7944	0.0484	0.0056	0.9307	0.0047	0.0001
6.00	0.9100	0.0081	0.0005	0.7954	0.0466	0.0046	0.9307	0.0044	0.0001

Table 2: P_0 and Lq for different feedback probabilities (r) for the values of $\lambda = 0.1$, $a = 2$; $\theta = 0.5$; $\mu_1 = 5$; $\mu_2 = 3$; $p = 0.5$; $\alpha_1 = 0.2$; $\alpha_2 = 0.1$; $\eta_1 = 3$; $\eta_2 = 2$; $\xi_1 = 2$; $\xi_2 = 1$; $\gamma = 5$; $c = 0.5$;

Feedback	Exponential			Erlang – 2 stage			Hyper – Exponential		
r	ρ	P_0	L_q	P	P_0	L_q	P	P_0	L_q
0.50	0.2148	0.8245	0.0166	0.4595	0.5959	0.1984	0.1434	0.8886	0.0031
0.55	0.2386	0.7994	0.0220	0.5105	0.5397	0.2757	0.1594	0.8720	0.0045
0.60	0.2685	0.7681	0.0310	0.5743	0.4693	0.4109	0.1793	0.8514	0.0073
0.65	0.3068	0.7279	0.0466	0.6564	0.3789	0.6827	0.2049	0.8248	0.0123
0.70	0.3579	0.6742	0.0754	0.7658	0.2582	1.4007	0.2390	0.7894	0.0217

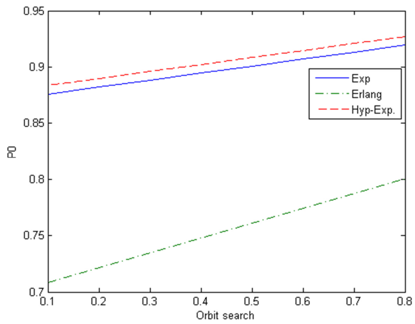

Figure 1. P_0 versus θ

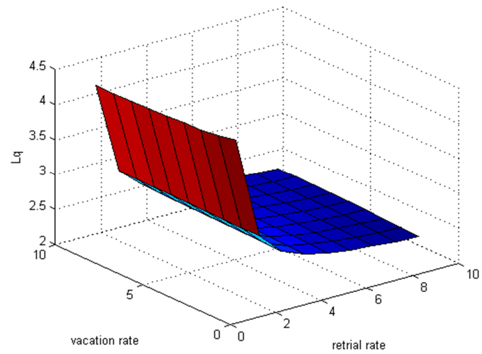

Figure 2. L_q versus a and γ