

# Extension of Some Common Fixed Point Theorems of Integral Type Mappings in Hilbert Space

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## ABSTRACT

The object of this paper is to obtain common fixed point theorems for two continuous mappings of a Hilbert space for integral type mapping.

**Key words:** common fixed point, rational expression, Hilbert space .uniqueness

**AMS Subject Classification:** 47H10, 54H25

## 2. Introduction and Preliminaries

Impact of fixed point theory in different branches of mathematics and its applications is immense. The first result on fixed points for Contractive type mapping was the much celebrated Banach's contraction principle by S. Banach [13] in 1922. In the general setting of complete metric space, this theorem runs as the follows

**Theorem 2.1** (Banach's contraction principle) Let  $(X,d)$  be a complete metric space,  $c \in (0,1)$  and  $f: X \rightarrow X$  be a mapping such that for each  $x, y \in X$ ,

$d(fx, fy) \leq cd(x,y)$  Then  $f$  has a unique fixed point  $a \in X$ , such that for each

$$x \in X, \lim_{n \rightarrow \infty} f^n(x) = a.$$

After the classical result, Kannan [11] gave a subsequently new contractive mapping to prove the fixed point theorem. Since then a number of mathematicians have been worked on fixed point theory dealing with mappings satisfying various type of contractive conditions.

In 2002, A. Branciari [1] analysed the existence of fixed point for mapping  $f$  defined on a complete metric space  $(X,d)$  satisfying a general contractive condition of integral type.

**Theorem 2.2 (Branciari)** Let  $(X,d)$  be a complete metric space,  $c \in (0,1)$  and let  $f: X \rightarrow X$  be a mapping such that for each  $x, y \in X$ ,

$$\int_0^{d(fx,fy)} \phi(t) dt \leq c \int_0^{d(x,y)} \phi(t) dt \text{ where } \phi: [0, +\infty) \rightarrow [0, +\infty) \text{ is a Lebesgue integrable}$$

mapping which is summable on each compact subset of  $[0, +\infty)$ , non negative, and such that for each

$$\varepsilon > 0, \int_0^\varepsilon \phi(t) dt > 0, \text{ then } f \text{ has a unique fixed point } a \in X, \text{ such that for each } x \in X,$$

$$\lim_{n \rightarrow \infty} f^n(x) = a.$$

After the paper of Branciari, a lot of research works have been carried out on generalizing contractive condition of integral type for different contractive mappings satisfying various known properties. A fine work has been done by Rhoades [2] extending the result of Branciari by replacing the condition [1.2] by the following

$$\int_0^{d(fx,fy)} \phi(t) dt \leq \int_0^{\max\{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy)+d(y,fx)}{2}\}} \phi(t) dt.$$

Koparde and Waghmode [9] extended the result of Jungck[6] and Fisher[3] to common fixed points. Here we proceed to extend and sharpen the main result of

Koparde and Waghmode [9] by modifying their procedure.

Jungck [6] obtained the following

**Theorem 2.3:** Let  $f$  be a continuous mapping of a complete metric space  $(X,d)$  into itself. Then  $f$  has a fixed point in  $X$  if there exists  $\alpha \in (0,1)$  and a mapping  $g: X \rightarrow X$  which commutes with  $f$  and satisfies

$$g(X) \subset f(X) \text{ and } d(g(x), g(y)) \leq \alpha d(f(x), f(y)) \quad \forall x, y \in X$$

Indeed,  $f$  and  $g$  have a unique common fixed point.

Fisher [3] extended and modified the above result in the form of

**Theorem 2.4:** Let  $S$  and  $T$  be continuous mapping of complete metric space  $(X,d)$  into itself. Then  $S$  and  $T$  have a common fixed point in  $X$  iff there exists a continuous mapping  $A$  of  $X$  into  $SX \cap TX$ , which commute with  $S$  and  $T$  and satisfies the inequality

$$d(Ax, Ay) \leq \alpha d(Sx, Sy) \quad \forall x, y \in X$$

Where  $0 < \alpha < 1$ . Indeed  $S$ ,  $T$ , and  $A$  then have a unique common fixed point.

Extending further, Koparde and Waghmode [9] obtained the following in setting of Hilbert space.

**Theorem 2.5:** Let  $S$  and  $T$  be continuous mapping of Hilbert space  $(X,d)$  into itself. Then  $S$  and  $T$  have a common fixed point in  $X$  iff there exists a continuous mapping  $A$  of  $X$  into  $SX \cap TX$ , which commute with  $S$  and  $T$  and satisfies the inequality

$$\|Ax - Ay\| \leq \alpha \|Ax - Sx\| + \beta \|Ay - Ty\| + \gamma \|Sx - Ty\|$$

For all  $x, y$  in  $X$ , where  $\alpha, \beta, \gamma$  are non-negative reals with  $0 < \alpha + \beta + \gamma < 1$ . Indeed,  $S, T$  and  $A$  then have a unique common fixed point.

The aim of this paper is to generalize and modified the above result in integral type mappings.

### 3. MAIN RESULTS

**Theorem 3.1:**

Let  $S$  and  $T$  be continuous mapping of Hilbert space  $(X,d)$  into itself. Then  $S$  and  $T$  have a common fixed point in  $X$  iff there exists a continuous mapping  $A$  of  $X$  into  $SX \cap TX$ , which commute with  $S$  and  $T$  and satisfies the inequality

$$\begin{aligned} \int_0^{\|Ax - Ay\|^2} \phi(t) dt &\leq \alpha \int_0^{\|Ax - Sx\|^2} \phi(t) dt + \beta \int_0^{\|Ay - Ty\|^2} \phi(t) dt \\ &+ \gamma \int_0^{\|Ax - Sy\|^2} \phi(t) dt + \delta \int_0^{\|Ax - Ty\|^2} \phi(t) dt \\ &+ \eta \int_0^{\max\{\|Ay - Sy\|^2, \|Ay - Tx\|^2\}} \phi(t) dt \end{aligned}$$

For all  $x, y$  in  $X$ ; where  $\alpha, \beta, \gamma, \delta, \eta$  are non-negative reals with  $0 < \alpha + \beta + 2\eta + \gamma + \delta < 1$ . Also  $\phi: [0, +\infty) \rightarrow [0, +\infty)$  is a Lebesgue integrable mapping which is summable on each compact subset of  $[0, +\infty)$ , non-negative, and such that for each  $\varepsilon > 0$ ,  $\int_0^\varepsilon \phi(t) dt < \varepsilon$ , then  $S, T$  and  $A$  then have a unique common fixed point.

**Proof:** First of all we prove that the existence of such a mapping  $A$  is necessary. For this suppose,  $Sz = z = Tz$  for some  $z$  in  $X$ .

Let us define a mapping  $A$  of  $X$  into  $X$  by  $Ax = z$  for all  $x$  in  $x$ . Then clearly,  $A$  is continuous mapping of  $X$  into  $SX \cap TX$ . Since,  $Sx, Tx \in X$ , for all

$x \in X$  and  $Ax = z$  for all  $x \in X$ , we get

$$ASx = z, SAx = Sz = z, ATx = z, TAX = Tz = z.$$

Hence,  $A$  commutes with  $S$  and  $T$ . Now for any  $\alpha, \beta, \gamma, \delta, \eta$  with

$$0 < \alpha + \beta + 2\eta + \gamma + \delta < 1, \text{ it is observe that}$$

$$\int_0^{\infty} \|Ax - Ay\|^2 \phi(t) dt \leq \alpha \int_0^{\infty} \|Ax - Sx\|^2 \phi(t) dt + \beta \int_0^{\infty} \|Ay - Ty\|^2 \phi(t) dt$$

$$+ \gamma \int_0^{\infty} \|Ax - Sy\|^2 \phi(t) dt + \delta \int_0^{\infty} \|Ax - Ty\|^2 \phi(t) dt$$

$$+ \eta \int_0^{\infty} \max\{\|Ay - Sy\|^2, \|Ay - Tx\|^2\} \phi(t) dt$$

Which gives

$$0 \leq \alpha \int_0^{\infty} \|z - Sx\|^2 \phi(t) dt + \beta \int_0^{\infty} \|z - Ty\|^2 \phi(t) dt$$

$$+ \gamma \int_0^{\infty} \|z - Sy\|^2 \phi(t) dt + \delta \int_0^{\infty} \|z - Ty\|^2 \phi(t) dt$$

$$+ \eta \int_0^{\infty} \max\{\|z - Sy\|^2, \|z - Tx\|^2\} \phi(t) dt$$

For all  $x, y$  in  $X$ . This proves the existence of such a mapping  $A$  is necessary.

To prove sufficient part, sequence  $\{x_n\}$  is constructed as follows. Let  $x_0 \in X$  be an arbitrary point.

Since  $AX \subset SX$ , we choose a point  $x_1$  in  $X$  such that  $Sx_1 = Ax_0$ .

Also  $AX \subset TX$  and hence we can choose  $x_2 \in X$  such that  $Tx_2 = Ax_1$ .

Continuing in this way, we get a sequence  $\{x_n\}$  as follows :

$$Sx_{2n-1} = Ax_{2n-2}, Tx_{2n} = Ax_{2n-1} \quad n = 1, 2, 3, \dots$$

We proceed to show that  $\{Ax_n\}$  is a Cauchy sequence.

For this we have the inequality,

$$\int_0^{\infty} \|Ax_{2n} - Ax_{2n-1}\|^2 \phi(t) dt = \int_0^{\infty} \|Ax_{2n-1} - Ax_{2n}\|^2 \phi(t) dt$$

$$\leq \alpha \int_0^{\infty} \|Ax_{2n-1} - Sx_{2n-1}\|^2 \phi(t) dt + \beta \int_0^{\infty} \|Ax_{2n} - Tx_{2n}\|^2 \phi(t) dt$$

$$+ \gamma \int_0^{\infty} \|Ax_{2n-1} - Sx_{2n}\|^2 \phi(t) dt + \delta \int_0^{\infty} \|Ax_{2n-1} - Tx_{2n}\|^2 \phi(t) dt$$

$$+ \eta \int_0^{\infty} \max\{\|Ax_{2n} - Sx_{2n}\|^2, \|Ax_{2n} - Tx_{2n-1}\|^2\} \phi(t) dt$$

$$\leq \alpha \int_0^{\infty} \|Ax_{2n-1} - Ax_{2n-2}\|^2 \phi(t) dt + \beta \int_0^{\infty} \|Ax_{2n} - Ax_{2n-1}\|^2 \phi(t) dt$$

$$+ \gamma \int_0^{\infty} \|Ax_{2n-1} - Ax_{2n-1}\|^2 \phi(t) dt + \delta \int_0^{\infty} \|Ax_{2n-1} - Ax_{2n-1}\|^2 \phi(t) dt$$

$$+ \eta \int_0^{\infty} \max\{\|Ax_{2n} - Ax_{2n-1}\|^2, \|Ax_{2n} - Ax_{2n-2}\|^2\} \phi(t) dt \dots \dots \dots (3.1.1)$$

**Case: 1** If

$$\max\{\|Ax_{2n} - Ax_{2n-1}\|^2, \|Ax_{2n} - Ax_{2n-2}\|^2\} = \|Ax_{2n} - Ax_{2n-1}\|^2$$

From (3.1.1)

$$\int_0^{\infty} \|Ax_{2n} - Ax_{2n-1}\|^2 \phi(t) dt \leq \alpha \int_0^{\infty} \|Ax_{2n-1} - Ax_{2n-2}\|^2 \phi(t) dt +$$

$$(\beta + \eta) \int_0^{\infty} \|Ax_{2n} - Ax_{2n-1}\|^2 \phi(t) dt$$

$$\int_0^{\infty} \|Ax_{2n} - Ax_{2n-1}\|^2 \phi(t) dt \leq \frac{\alpha}{1-\beta-\eta} \int_0^{\infty} \|Ax_{2n-1} - Ax_{2n-2}\|^2 \phi(t) dt \quad \text{--- (3.1.2)}$$

Further, it is seen that

$$\int_0^{\infty} \|Ax_{2n+1} - Ax_{2n}\|^2 \phi(t) dt = \int_0^{\infty} \|Ax_{2n} - Ax_{2n+1}\|^2 \phi(t) dt$$

$$\leq \alpha \int_0^{\infty} \|Ax_{2n} - Sx_{2n}\|^2 \phi(t) dt + \beta \int_0^{\infty} \|Ax_{2n+1} - Tx_{2n+1}\|^2 \phi(t) dt$$

$$+ \gamma \int_0^{\infty} \|Ax_{2n} - Sx_{2n+1}\|^2 \phi(t) dt + \delta \int_0^{\infty} \|Ax_{2n} - Tx_{2n+1}\|^2 \phi(t) dt$$

$$+ \eta \int_0^{\infty} \|Ax_{2n+1} - Sx_{2n+1}\|^2 \phi(t) dt$$

$$= \alpha \int_0^{\infty} \|Ax_{2n} - Ax_{2n-1}\|^2 \phi(t) dt + \beta \int_0^{\infty} \|Ax_{2n+1} - Ax_{2n}\|^2 \phi(t) dt + \eta$$

$$\int_0^{\infty} \|Ax_{2n+1} - Ax_{2n}\|^2 \phi(t) dt$$

This implies that

$$\int_0^{\infty} \|Ax_{2n+1} - Ax_{2n}\|^2 \phi(t) dt \leq \frac{\alpha}{1-\beta-\eta} \int_0^{\infty} \|Ax_{2n} - Ax_{2n-1}\|^2 \phi(t) dt$$

$$= \lambda_1 \int_0^{\infty} \|Ax_{2n} - Ax_{2n-1}\|^2 \phi(t) dt \quad \text{--- (3.1.3)}$$

Where  $\lambda_1 = \frac{\alpha}{1-\beta-\eta}$

**Case: 2** If

$$\max\{\|Ax_{2n} - Ax_{2n-1}\|^2, \|Ax_{2n} - Ax_{2n-2}\|^2\} = \|Ax_{2n} - Ax_{2n-2}\|^2$$

From (3.1.1)

$$\int_0^{\infty} \|Ax_{2n} - Ax_{2n-1}\|^2 \phi(t) dt \leq \alpha \int_0^{\infty} \|Ax_{2n-1} - Ax_{2n-2}\|^2 \phi(t) dt +$$

$$\beta \int_0^{\infty} \|Ax_{2n} - Ax_{2n-1}\|^2 \phi(t) dt + \eta \int_0^{\infty} \|Ax_{2n} - Ax_{2n-2}\|^2 \phi(t) dt$$

$$\leq \alpha \int_0^{\infty} \|Ax_{2n-1} - Ax_{2n-2}\|^2 \phi(t) dt + \beta \int_0^{\infty} \|Ax_{2n} - Ax_{2n-1}\|^2 \phi(t) dt$$

$$+ \eta \left[ \int_0^{\infty} \|Ax_{2n} - Ax_{2n-1}\|^2 \phi(t) dt + \int_0^{\infty} \|Ax_{2n-1} - Ax_{2n-2}\|^2 \phi(t) dt \right]$$

$$\int_0^{\infty} \|Ax_{2n} - Ax_{2n-1}\|^2 \phi(t) dt \leq \frac{\alpha+\eta}{1-\beta-\eta} \int_0^{\infty} \|Ax_{2n-1} - Ax_{2n-2}\|^2 \phi(t) dt \quad \text{--- (3.1.4)}$$

Further, it is seen that

$$\int_0^{\infty} \|Ax_{2n+1} - Ax_{2n}\|^2 \phi(t) dt = \int_0^{\infty} \|Ax_{2n} - Ax_{2n+1}\|^2 \phi(t) dt$$

$$\begin{aligned}
 &\leq \alpha \int_0^{\infty} \|Ax_{2n} - Sx_{2n}\|^2 \phi(t) dt + \beta \int_0^{\infty} \|Ax_{2n+1} - Tx_{2n+1}\|^2 \phi(t) dt \\
 &+ \gamma \int_0^{\infty} \|Ax_{2n} - Sx_{2n+1}\|^2 \phi(t) dt + \delta \int_0^{\infty} \|Ax_{2n} - Tx_{2n+1}\|^2 \phi(t) dt \\
 &+ \eta \int_0^{\infty} \|Ax_{2n+1} - Tx_{2n}\|^2 \phi(t) dt \\
 &= \alpha \int_0^{\infty} \|Ax_{2n} - Ax_{2n-1}\|^2 \phi(t) dt + \beta \int_0^{\infty} \|Ax_{2n+1} - Ax_{2n}\|^2 \phi(t) dt \\
 &+ \eta \int_0^{\infty} \|Ax_{2n+1} - Ax_{2n-1}\|^2 \phi(t) dt \\
 &\leq \alpha \int_0^{\infty} \|Ax_{2n} - Ax_{2n-1}\|^2 \phi(t) dt + \beta \int_0^{\infty} \|Ax_{2n+1} - Ax_{2n}\|^2 \phi(t) dt \\
 &+ \eta \left[ \int_0^{\infty} \|Ax_{2n+1} - Ax_{2n}\|^2 \phi(t) dt + \int_0^{\infty} \|Ax_{2n} - Ax_{2n-1}\|^2 \phi(t) dt \right]
 \end{aligned}$$

This implies that

$$\begin{aligned}
 \int_0^{\infty} \|Ax_{2n+1} - Ax_{2n}\|^2 \phi(t) dt &\leq \frac{\alpha + \eta}{1 - \beta - \eta} \int_0^{\infty} \|Ax_{2n} - Ax_{2n-1}\|^2 \phi(t) dt \\
 &= \lambda_2 \int_0^{\infty} \|Ax_{2n} - Ax_{2n-1}\|^2 \phi(t) dt \quad \text{--- (3.1.5)}
 \end{aligned}$$

Where  $\lambda_2 = \frac{\alpha + \eta}{1 - \beta - \eta}$

From both cases

Taking  $\lambda = \max \{ \lambda_1, \lambda_2 \}$

$0 < \lambda < 1$  as we have

$$0 < \alpha + \beta + 2\eta < 1$$

Now, from the both cases

$$\begin{aligned}
 \int_0^{\infty} \|Ax_{n+1} - Ax_n\|^2 \phi(t) dt &\leq \lambda \int_0^{\infty} \|Ax_n - Ax_{n-1}\|^2 \phi(t) dt \\
 &\leq \lambda^2 \int_0^{\infty} \|Ax_{n-1} - Ax_{n-2}\|^2 \phi(t) dt \\
 &\dots\dots\dots \\
 &\leq \lambda^n \int_0^{\infty} \|Ax_1 - Ax_0\|^2 \phi(t) dt
 \end{aligned}$$

For large n. Now, it can be seen that  $\{Ax_n\}$  is a Cauchy sequence and so it has a limit x in X. Since sequences  $\{Sx_{2n+1}\}$  and  $\{Tx_{2n}\}$  are subsequences of  $\{Ax_n\}$ , they have the same limit z. As S and A are commuting mapping, we can have

$$\begin{aligned}
 Sz &= \lim_{n \rightarrow \infty} S Ax_{2n+1} \\
 &= \lim_{n \rightarrow \infty} A S x_{2n+1} \\
 &= Az
 \end{aligned}$$

Similarly, we get  $Tz = Az$

This gives us,

$$Tz = Az = Sz$$

It comes out that

$$\begin{aligned} \int_0^{\infty} \|Az - AAz\|^2 \phi(t) dt &\leq \alpha \int_0^{\infty} \|Az - Sz\|^2 \phi(t) dt + \beta \int_0^{\infty} \|AAz - TAz\|^2 \phi(t) dt \\ &+ \gamma \int_0^{\infty} \|Az - SAz\|^2 \phi(t) dt + \delta \int_0^{\infty} \|Az - TAz\|^2 \phi(t) dt \\ &+ \eta \int_0^{\infty} \max\{\|AAz - SAz\|^2, \|AAz - Tz\|^2\} \phi(t) dt \end{aligned}$$

Commutativity of A with S and T gives the following by

$$\|AAz - TAz\| = \|AAz - ATz\| = \|AAz - AAz\| = 0$$

And

$$\|Az - SAz\| = \|Az - AAz\| \text{ and } \|Az - TAz\| = \|Az - AAz\|$$

Also ,

$$\|AAz - SAz\| = \|AAz - ASz\| = \|AAz - AAz\| = 0$$

Resorting to these, we arrive at

$$\int_0^{\infty} \|Az - AAz\|^2 \phi(t) dt \leq (\gamma + \delta + \eta) \int_0^{\infty} \|Az - AAz\|^2 \phi(t) dt$$

Since  $\alpha + \beta + 2\eta + \gamma + \delta < 1$  , we have that  $Az = AAz$  .

Finally, putting  $Az = Z_1$  , we have  $AZ_1 = Az = AAz = Z_1$

Similarly,  $TZ_1 = TAz = ATz = AAz = Az = Z_1$  ,

$SZ_1 = Z_1$  .

So ,  $Z_1$  is a fixed point of S ,T and A.

Next , to show uniqueness of this Common fixed point , let us suppose that  $Z_2$  is also a common fixed point of S ,T and A other than  $Z_1$  .

Then  $SZ_2 = Z_2$  ,  $TZ_2 = Z_2$  ,  $AZ_2 = Z_2$  , and also  $\|Z_1 - Z_2\| \neq 0$  .

Hence, it follows that

$$\begin{aligned} \int_0^{\infty} \|z_1 - z_2\|^2 \phi(t) dt &= \int_0^{\infty} \|Az_1 - Az_2\|^2 \phi(t) dt \\ &\leq \alpha \int_0^{\infty} \|Az_1 - Sz_1\|^2 \phi(t) dt + \beta \int_0^{\infty} \|Az_2 - Tz_2\|^2 \phi(t) dt \\ &+ \gamma \int_0^{\infty} \|Az_1 - Sz_2\|^2 \phi(t) dt + \delta \int_0^{\infty} \|Az_1 - Tz_2\|^2 \phi(t) dt \\ &+ \eta \int_0^{\infty} \max\{\|Az_2 - Sz_2\|^2, \|Az_2 - Tz_1\|^2\} \phi(t) dt \end{aligned}$$

Implies

$$\int_0^{\infty} \|z_1 - z_2\|^2 \phi(t) dt \leq (\gamma + \delta + \eta) \int_0^{\infty} \|z_1 - z_2\|^{p/q} \phi(t) dt$$

Which is a contradiction as  $\alpha + \beta + 2\eta + \gamma + \delta < 1$ .

Thus occurs the uniqueness.

**Theorem 3.2:**

Let S and T be continuous mapping of Hilbert space (X,d) into itself. Then S and T have a common fixed point in X iff there exists a continuous mapping A of X into  $SX \cap TX$ , which commute with S and T and satisfies the inequality

$$\int_0^{\infty} \|Ax-Ay\|^2 \phi(t) dt \leq \alpha \int_0^{\infty} \|Ax-Sx\| \|Ay-Sy\| \phi(t) dt + \beta \int_0^{\infty} \|Ax-Tx\| \|Ay-Ty\| \phi(t) dt$$

$$+ \gamma \int_0^{\infty} \|x-y\|^2 \phi(t) dt + \delta \int_0^{\infty} \|Sx-Ty\|^2 \phi(t) dt$$

$$+ \eta \int_0^{\infty} \|Sx-Tx\|^2 \phi(t) dt$$

For all  $x,y$  in X ; where  $\alpha, \beta, \gamma, \delta, \eta$  are non-negative reals with  $0 < \alpha + \beta + 2\eta + \gamma + \delta < 1$ . Also  $\phi: [0, +\infty) \rightarrow [0, +\infty)$  is a Lebesgue integrable mapping which is summable on each compact subset of  $[0, +\infty)$ , non-negative, and such that for each  $\varepsilon > 0$ ,  $\int_0^{\varepsilon} \phi(t) dt > 0$ , then S,T and A then have a unique common fixed point.

Proof: First of all we prove that the existence of such a mapping A is necessary. For this suppose,  $Sz=z=Tz$  for some  $z$  in X.

Let us define a mapping A of X into X by  $Ax=z$  for all  $x$  in X. Then clearly, A is continuous mapping of X into  $SX \cap TX$ . Since,  $Sx, Tx \in X$ , for all

$x \in X$  and  $Ax=z$  for all  $x \in X$ , we get

$$ASx = z, SAx = Sz = z, ATx = z, TAx = Tz = z.$$

Hence, A commutes with S and T. Now for any  $\alpha, \beta, \gamma, \delta, \eta$  with

$$0 < \alpha + \beta + 2\eta + \gamma + \delta < 1, \text{ it is observe that}$$

$$\int_0^{\infty} \|Ax-Ay\|^2 \phi(t) dt \leq \alpha \int_0^{\infty} \|Ax-Sx\| \|Ay-Sy\| \phi(t) dt + \beta \int_0^{\infty} \|Ax-Tx\| \|Ay-Ty\| \phi(t) dt$$

$$+ \gamma \int_0^{\infty} \|x-y\|^2 \phi(t) dt + \delta \int_0^{\infty} \|Sx-Ty\|^2 \phi(t) dt$$

$$+ \eta \int_0^{\infty} \|Sx-Tx\|^2 \phi(t) dt$$

This gives,

$$0 \leq \alpha \int_0^{\infty} \|Ax-Sx\| \|Ay-Sy\| \phi(t) dt + \beta \int_0^{\infty} \|Ax-Tx\| \|Ay-Ty\| \phi(t) dt$$

$$+ \gamma \int_0^{\infty} \|x-y\|^2 \phi(t) dt + \delta \int_0^{\infty} \|Sx-Ty\|^2 \phi(t) dt$$

$$+ \eta \int_0^{\infty} \|Sx-Tx\|^2 \phi(t) dt$$

For all  $x,y$  in X. This proves the existence of such a mapping A is necessary.

To prove sufficient part, sequence  $\{x_n\}$  is constructed as follows. Let  $x_0 \in X$  be an arbitrary point.

Since  $AX \subset SX$ , we choose a point  $x_1$  in X such that  $Sx_1 = Ax_0$ .

Also  $AX \subset TX$  and hence we can choose  $x_2 \in X$  such that  $Tx_2 = Ax_1$ .

Continuing in this way, we get a sequence  $\{x_n\}$  as follows :

$$Sx_{2n-1} = Ax_{2n-2}, Tx_{2n} = Ax_{2n-1} \quad n = 1, 2, 3, \dots$$

We proceed to show that  $\{Ax_n\}$  is a Cauchy sequence .

For this we have the inequality,

$$\begin{aligned} \int_0^{\infty} \|Ax_{2n+1} - Ax_{2n}\|^2 \phi(t) dt &\leq \alpha \int_0^{\infty} \|Ax_{2n+1} - Sx_{2n+1}\| \|Ax_{2n} - Sx_{2n}\| \phi(t) dt \\ &\quad + \beta \int_0^{\infty} \|Ax_{2n+1} - Tx_{2n+1}\| \|Ax_{2n} - Tx_{2n}\| \phi(t) dt \\ &\quad + \gamma \int_0^{\infty} \|x_{2n+1} - x_{2n}\|^2 \phi(t) dt + \delta \int_0^{\infty} \|Sx_{2n+1} - Tx_{2n}\|^2 \phi(t) dt \\ &\quad + \eta \int_0^{\infty} \|Sx_{2n+1} - Tx_{2n+1}\|^2 \phi(t) dt \\ &\leq \alpha \int_0^{\infty} \|Ax_{2n+1} - Ax_{2n}\| \|Ax_{2n} - Ax_{2n-1}\| \phi(t) dt \\ &\quad + \beta \int_0^{\infty} \|Ax_{2n+1} - Ax_{2n}\| \|Ax_{2n} - Ax_{2n-1}\| \phi(t) dt \\ &\quad + \gamma \int_0^{\infty} \|x_{2n+1} - x_{2n}\|^2 \phi(t) dt + \delta \int_0^{\infty} \|Ax_{2n} - Ax_{2n-1}\|^2 \phi(t) dt \\ &\quad + \eta \int_0^{\infty} \|Ax_{2n} - Ax_{2n}\|^2 \phi(t) dt \\ &\leq (\alpha + \beta) \int_0^{\infty} \left[ \frac{\|Ax_{2n+1} - Ax_{2n}\|^2}{2} + \frac{\|Ax_{2n} - Ax_{2n-1}\|^2}{2} \right] \phi(t) dt \\ &\quad + \gamma \int_0^{\infty} \|x_{2n+1} - x_{2n}\|^2 \phi(t) dt + \delta \int_0^{\infty} \|Ax_{2n} - Ax_{2n-1}\|^2 \phi(t) dt \\ &\quad + \eta \int_0^{\infty} \|Ax_{2n} - Ax_{2n}\|^2 \phi(t) dt \end{aligned}$$

This gives

$$\begin{aligned} 2 \int_0^{\infty} \|Ax_{2n+1} - Ax_{2n}\|^2 \phi(t) dt &\leq (\alpha + \beta) \int_0^{\infty} [\|Ax_{2n+1} - Ax_{2n}\|^2 + \|Ax_{2n} - Ax_{2n-1}\|^2] \phi(t) dt \\ &\quad + 2\gamma \int_0^{\infty} \|x_{2n+1} - x_{2n}\|^2 \phi(t) dt + 2\delta \int_0^{\infty} \|Ax_{2n} - Ax_{2n-1}\|^2 \phi(t) dt \\ &\quad + 2\eta \int_0^{\infty} \|Ax_{2n} - Ax_{2n}\|^2 \phi(t) dt \end{aligned}$$

Resulting in

$$\int_0^{\infty} \|Ax_{2n+1} - Ax_{2n}\|^2 \phi(t) dt \leq \left( \frac{\alpha + \beta + 2\delta}{2 - \alpha - \beta} \right) \int_0^{\infty} \|Ax_{2n} - Ax_{2n-1}\|^2 \phi(t) dt$$



$$+ \left( \frac{2\gamma}{2-\alpha-\beta} \right) \int_0^{\|x_{2n+1}-x_{2n}\|^2} \phi(t) dt \quad \text{----- (3.2.1)}$$

Further, observe that

$$\begin{aligned} \int_0^{\|Ax_{2n}-Ax_{2n-1}\|^2} \phi(t) dt &\leq \alpha \int_0^{\|Ax_{2n}-Sx_{2n}\| \|Ax_{2n-1}-Sx_{2n-1}\|} \phi(t) dt \\ &+ \beta \int_0^{\|Ax_{2n}-Tx_{2n}\| \|Ax_{2n-1}-Tx_{2n-1}\|} \phi(t) dt \\ &+ \gamma \int_0^{\|x_{2n}-x_{2n-1}\|^2} \phi(t) dt + \delta \int_0^{\|Sx_{2n}-Tx_{2n-1}\|^2} \phi(t) dt \\ &+ \eta \int_0^{\|Sx_{2n}-Tx_{2n}\|^2} \phi(t) dt \end{aligned}$$

Simplifying this using Young's inequality, we get

$$\begin{aligned} \int_0^{\|Ax_{2n}-Ax_{2n-1}\|^2} \phi(t) dt &\leq \left( \frac{\alpha+\beta+2\delta}{2-\alpha-\beta} \right) \int_0^{\|Ax_{2n-1}-Ax_{2n-2}\|^2} \phi(t) dt \\ &+ \left( \frac{2\gamma}{2-\alpha-\beta} \right) \int_0^{\|x_{2n}-x_{2n-1}\|^2} \phi(t) dt \quad \text{----- (3.2.2)} \end{aligned}$$

Since  $\alpha + \beta + 2\eta + \gamma + \delta < 1$ . We find that

$$\lambda_1 = \frac{\alpha+\beta+2\delta}{2-\alpha-\beta}, \lambda_2 = \frac{2\gamma}{2-\alpha-\beta} \in (0,1)$$

Suppose, now

$$\lambda = \max\{\lambda_1, \lambda_2\}$$

then  $0 < \lambda < 1$ .

From (3.2.1) and (3.2.2), we conclude that

$$\begin{aligned} \int_0^{\|Ax_{n+1}-Ax_n\|^2} \phi(t) dt &\leq \lambda \int_0^{[\|Ax_n-Ax_{n-1}\|^2 + \|Ax_{n+1}-Ax_n\|^2]} \phi(t) dt \\ &\leq \lambda^2 \int_0^{[\|Ax_{n-1}-Ax_{n-2}\|^2 + \|Ax_n-Ax_{n-1}\|^2]} \phi(t) dt \\ &+ \lambda \int_0^{[\|Ax_{n+1}-Ax_n\|^2]} \phi(t) dt \\ &\text{-----} \\ &\leq \lambda^n \int_0^{[\|Ax_1-Ax_0\|^2 + \|Ax_2-Ax_1\|^2]} \phi(t) dt + \varepsilon, \end{aligned}$$

For large n.

. Now, it can be seen that  $\{Ax_n\}$  is a Cauchy sequence and so it has a limit x in X. Since sequences  $\{Sx_{2n+1}\}$  and  $\{Tx_{2n}\}$  are sub-sequences of  $\{Ax_n\}$ , they have the same limit z. As S and A are commuting mapping, we can have

$$Sz = \lim_{n \rightarrow \infty} S Ax_{2n+1}$$

$$= \lim_{n \rightarrow \infty} A S x_{2n+1}$$

$$= Az$$

Similarly, we get  $Tz = Az$

This gives us ,

$$Tz = Az = Sz$$

It comes out that

$$\int_0^{\|Az-AAz\|^2} \phi(t) dt \leq \alpha \int_0^{\|Az-Sz\| \|AAz-SAz\|} \phi(t) dt$$

$$+ \beta \int_0^{\|Az-Tz\| \|AAz-TAz\|} \phi(t) dt$$

$$+ \gamma \int_0^{\|z-Az\|^2} \phi(t) dt + \delta \int_0^{\|Sz-TAz\|^2} \phi(t) dt$$

$$+ \eta \int_0^{\|Sz-Tz\|^2} \phi(t) dt$$

Commutativity of A with S and T gives the following by

$$\|AAz - TAz\| = \|AAz - ATz\| = \|AAz - AAz\| = 0$$

And

$$\|Az - SAz\| = \|Az - AAz\| \text{ and } \|Az - TAz\| = \|Az - AAz\|$$

Also,

$$\|AAz - SAz\| = \|AAz - ASz\| = \|AAz - AAz\| = 0$$

Resorting to these, we arrive at

$$\int_0^{\|Az-AAz\|^2} \phi(t) dt \leq \gamma \int_0^{\|z-Az\|^2} \phi(t) dt + \delta \int_0^{\|Az-AAz\|^2} \phi(t) dt$$

Since  $\gamma + \delta < 1$ , we must have  $Az = AAz$

Finally, putting  $Az = Z_1$ , we have  $AZ_1 = Az = AAz = Z_1$

Similarly,  $TZ_1 = TAz = ATz = AAz = Az = Z_1$ ,

$$SZ_1 = Z_1.$$

So,  $Z_1$  is a fixed point of S, T and A.

Next, to show uniqueness of this Common fixed point, let us suppose that  $Z_2$  is also a common fixed point of S, T and A other than  $Z_1$ .

Then  $SZ_2 = Z_2$ ,  $TZ_2 = Z_2$ ,  $AZ_2 = Z_2$ , and also  $\|z_1 - z_2\| \neq 0$ .

Hence, it follows that

$$\int_0^{\|z_1-z_2\|^2} \phi(t) dt = \int_0^{\|Az_1-Az_2\|^2} \phi(t) dt$$

$$\leq \alpha \int_0^{\|Az_1-Sz_1\| \|Az_2-Sz_2\|} \phi(t) dt$$

$$+ \beta \int_0^{\|Az_1-Tz_1\| \|Az_2-Tz_2\|} \phi(t) dt$$

$$+ \gamma \int_0^{\|z_1-z_2\|^2} \phi(t) dt + \delta \int_0^{\|Sz_1-Tz_2\|^2} \phi(t) dt$$

$$+ \eta \int_0^{\|Sz_1-Tz_1\|^2} \phi(t) dt$$

Implies

$$\int_0^{\|z_1 - z_2\|^2} \phi(t) dt \leq (\gamma + \delta) \int_0^{\|z_1 - z_2\|^2} \phi(t) dt$$

Which is a contradiction as  $\alpha + \beta + 2\eta + \gamma + \delta < 1$ .

Thus occurs the uniqueness.

**Theorem 3.3:**

Let S and T be continuous mapping of Hilbert space (X,d) into itself. Then S and T have a common fixed point in X iff there exists a continuous mapping A of X into  $SX \cap TX$ , which commute with S and T and satisfies the inequality

$$\begin{aligned} \int_0^{\|Ax - Ay\|^{p/q}} \phi(t) dt &\leq \alpha \int_0^{\|Ax - Sx\|^{p/q}} \phi(t) dt + \beta \int_0^{\|Ay - Ty\|^{p/q}} \phi(t) dt \\ &+ \gamma \int_0^{\|Ax - Sy\|^{p/q}} \phi(t) dt + \delta \int_0^{\|Ax - Ty\|^{p/q}} \phi(t) dt \\ &+ \eta \int_0^{\max\{\|Ax - Tx\|^{p/q}, \|Ay - Sy\|^{p/q}\}} \phi(t) dt \end{aligned}$$

For all x,y in X ; where  $\alpha, \beta, \gamma, \delta, \eta$  are non-negative reals with  $0 < \alpha +$

$\beta + \eta + 2(\gamma + \delta)2^{p/q} < 1$  where p,q are positive real numbers with

$p/q < 1$  Also  $\phi: [0, +\infty) \rightarrow [0, +\infty)$  is a Lebesgue integrable mapping which is summable on each

compact subset of  $[0, +\infty)$ , non negative, and such that for each  $\varepsilon > 0, \int_0^\varepsilon \phi(t) dt$ , then S,T and A

then have a unique common fixed point.

Proof : First of all we prove that the existence of such a mapping A is necessary. For this suppose,  $Sz = z = Tz$  for some z in X.

Let us define a mapping A of X into X by  $Ax = z$  for all x in x. Then clearly, A is continuous mapping of X into  $SX \cap TX$ . Since,  $Sx, Tx \in X$ , for all

$x \in X$  and  $Ax = z$  for all  $x \in X$ , we get

$$ASx = z, SAx = Sz = z, ATx = z, TAx = Tz = z.$$

Hence, A commutes with S and T. Now for any  $\alpha, \beta, \gamma, \delta, \eta$  with

$0 < \alpha + \beta + \eta + 2(\gamma + \delta)2^{p/q} < 1$ , it is observe that

$$\begin{aligned} \int_0^{\|Ax - Ay\|^{p/q}} \phi(t) dt &\leq \alpha \int_0^{\|Ax - Sx\|^{p/q}} \phi(t) dt + \beta \int_0^{\|Ay - Ty\|^{p/q}} \phi(t) dt \\ &+ \gamma \int_0^{\|Ax - Sy\|^{p/q}} \phi(t) dt + \delta \int_0^{\|Ax - Ty\|^{p/q}} \phi(t) dt \\ &+ \eta \int_0^{\max\{\|Ax - Tx\|^{p/q}, \|Ay - Sy\|^{p/q}\}} \phi(t) dt \end{aligned}$$

$$\text{Which gives } 0 \leq \alpha \int_0^{\|Ax - Sx\|^{p/q}} \phi(t) dt + \beta \int_0^{\|Ay - Ty\|^{p/q}} \phi(t) dt$$

$$\begin{aligned}
 & + \gamma \int_0^{\infty} \|Ax - Sy\|^{p/q} \phi(t) dt + \delta \int_0^{\infty} \|Ax - Ty\|^{p/q} \phi(t) dt \\
 & + \eta \int_0^{\infty} \max\{\|Ax - Tx\|^{p/q}, \|Ay - Sy\|^{p/q}\} \phi(t) dt
 \end{aligned}$$

For all  $x, y$  in  $X$ . This proves the existence of such a mapping  $A$  is necessary.

To prove sufficient part, sequence  $\{x_n\}$  is constructed as follows. Let  $x_0 \in X$  be an arbitrary point.

Since  $AX \subset SX$ , we choose a point  $x_1$  in  $X$  such that  $Sx_1 = Ax_0$ .

Also  $AX \subset TX$  and hence we can choose  $x_2 \in X$  such that  $Tx_2 = Ax_1$ .

Continuing in this way, we get a sequence  $\{x_n\}$  as follows :

$$Sx_{2n-1} = Ax_{2n-2}, Tx_{2n} = Ax_{2n-1} \quad n = 1, 2, 3, \dots$$

We proceed to show that  $\{Ax_n\}$  is a Cauchy sequence.

For this we have the inequality,

$$\begin{aligned}
 & \int_0^{\infty} \|Ax_{2n+1} - Ax_{2n}\|^{p/q} \phi(t) dt \leq \alpha \int_0^{\infty} \|Ax_{2n+1} - Sx_{2n+1}\|^{p/q} \phi(t) dt \\
 & \quad + \beta \int_0^{\infty} \|Ax_{2n} - Tx_{2n}\|^{p/q} \phi(t) dt \\
 & \quad + \gamma \int_0^{\infty} \|Ax_{2n+1} - Sx_{2n}\|^{p/q} \phi(t) dt + \delta \int_0^{\infty} \|Ax_{2n+1} - Tx_{2n}\|^{p/q} \phi(t) dt \\
 & \quad + \eta \int_0^{\infty} \max\{\|Ax_{2n+1} - Tx_{2n+1}\|^{p/q}, \|Ax_{2n} - Sx_{2n}\|^{p/q}\} \phi(t) dt \\
 & = \alpha \int_0^{\infty} \|Ax_{2n+1} - Ax_{2n}\|^{p/q} \phi(t) dt + \beta \int_0^{\infty} \|Ax_{2n} - Ax_{2n-1}\|^{p/q} \phi(t) dt \\
 & \quad + \gamma \int_0^{\infty} \|Ax_{2n+1} - Ax_{2n-1}\|^{p/q} \phi(t) dt + \delta \int_0^{\infty} \|Ax_{2n+1} - Ax_{2n-1}\|^{p/q} \phi(t) dt \\
 & \quad + \eta \int_0^{\infty} \max\{\|Ax_{2n+1} - Ax_{2n}\|^{p/q}, \|Ax_{2n} - Ax_{2n-1}\|^{p/q}\} \phi(t) dt \\
 & = \alpha \int_0^{\infty} \|Ax_{2n+1} - Ax_{2n}\|^{p/q} \phi(t) dt + \beta \int_0^{\infty} \|Ax_{2n} - Ax_{2n-1}\|^{p/q} \phi(t) dt \\
 & \quad + (\gamma + \delta) \int_0^{\infty} \|Ax_{2n+1} - Ax_{2n} + Ax_{2n} - Ax_{2n-1}\|^{p/q} \phi(t) dt \\
 & \quad + \eta \int_0^{\infty} \max\{\|Ax_{2n+1} - Ax_{2n}\|^{p/q}, \|Ax_{2n} - Ax_{2n-1}\|^{p/q}\} \phi(t) dt
 \end{aligned}$$

$$\begin{aligned} &\leq \alpha \int_0^{\|Ax_{2n+1}-Ax_{2n}\|^{p/q}} \phi(t)dt + \beta \int_0^{\|Ax_{2n}-Ax_{2n-1}\|^{p/q}} \phi(t)dt \\ &+ (\gamma + \delta)2^{p/q} \left[ \int_0^{\|Ax_{2n+1}-Ax_{2n}\|^{p/q}} \phi(t)dt + \int_0^{\|Ax_{2n}-Ax_{2n-1}\|^{p/q}} \phi(t)dt \right] \\ &+ \eta \int_0^{\max\{\|Ax_{2n+1}-Ax_{2n}\|^{p/q}, \|Ax_{2n}-Ax_{2n-1}\|^{p/q}\}} \phi(t)dt \end{aligned}$$

Because  $\|a + b\|^p \leq 2^p(\|a\|^p + \|a\|^p)$  for  $0 < p < 1$

-----3.3.1

**Case : 1** If

$$\max\{\|Ax_{2n+1} - Ax_{2n}\|^{p/q}, \|Ax_{2n} - Ax_{2n-1}\|^{p/q}\} = \|Ax_{2n+1} - Ax_{2n}\|^{p/q}$$

From (1.5.1)

$$\begin{aligned} \int_0^{\|Ax_{2n+1}-Ax_{2n}\|^{p/q}} \phi(t)dt &\leq \alpha \int_0^{\|Ax_{2n+1}-Ax_{2n}\|^{p/q}} \phi(t)dt \\ &+ \beta \int_0^{\|Ax_{2n}-Ax_{2n-1}\|^{p/q}} \phi(t)dt \\ &+ (\gamma + \delta)2^{p/q} \left[ \int_0^{\|Ax_{2n+1}-Ax_{2n}\|^{p/q}} \phi(t)dt + \int_0^{\|Ax_{2n}-Ax_{2n-1}\|^{p/q}} \phi(t)dt \right] \\ &+ \eta \int_0^{\|Ax_{2n+1}-Ax_{2n}\|^{p/q}} \phi(t)dt \\ &= (\alpha + \eta + (\gamma + \delta)2^{p/q}) \int_0^{\|Ax_{2n+1}-Ax_{2n}\|^{p/q}} \phi(t)dt \\ &+ (\beta + (\gamma + \delta)2^{p/q}) \int_0^{\|Ax_{2n}-Ax_{2n-1}\|^{p/q}} \phi(t)dt \end{aligned}$$

Which leads to

$$\begin{aligned} (1 - \alpha - \eta - (\gamma + \delta)2^{p/q}) \int_0^{\|Ax_{2n+1}-Ax_{2n}\|^{p/q}} \phi(t)dt &\leq \\ (\beta + (\gamma + \delta)2^{p/q}) \int_0^{\|Ax_{2n}-Ax_{2n-1}\|^{p/q}} \phi(t)dt & \end{aligned}$$

Which yields

$$\int_0^{\|Ax_{2n+1}-Ax_{2n}\|^{p/q}} \phi(t)dt \leq \left( \frac{\beta + (\gamma + \delta)2^{p/q}}{1 - \alpha - \eta - (\gamma + \delta)2^{p/q}} \right) \int_0^{\|Ax_{2n}-Ax_{2n-1}\|^{p/q}} \phi(t)dt$$

(3.3.2)

Similarly we have

$$\int_0^{\infty} \|Ax_{2n} - Ax_{2n-1}\|^{p/q} \phi(t) dt \leq \left( \frac{\beta + (\gamma + \delta) 2^{p/q}}{1 - \alpha - \eta - (\gamma + \delta) 2^{p/q}} \right) \int_0^{\infty} \|Ax_{2n-1} - Ax_{2n-2}\|^{p/q} \phi(t) dt$$

(3.3.3)

Hence, one concludes that

$$\int_0^{\infty} \|Ax_{n+1} - Ax_n\|^{p/q} \phi(t) dt \leq \left( \frac{\beta + (\gamma + \delta) 2^{p/q}}{1 - \alpha - \eta - (\gamma + \delta) 2^{p/q}} \right) \int_0^{\infty} \|Ax_n - Ax_{n-1}\|^{p/q} \phi(t) dt$$

Which implies that

$$\begin{aligned} \int_0^{\infty} \|Ax_{n+1} - Ax_n\| \phi(t) dt &\leq \left( \frac{\beta + (\gamma + \delta) 2^{p/q}}{1 - \alpha - \eta - (\gamma + \delta) 2^{p/q}} \right)^{q/p} \int_0^{\infty} \|Ax_n - Ax_{n-1}\| \phi(t) dt \\ &\leq \lambda_1 \int_0^{\infty} \|Ax_n - Ax_{n-1}\| \phi(t) dt \end{aligned}$$

Where  $\lambda_1 = \left( \frac{\beta + (\gamma + \delta) 2^{p/q}}{1 - \alpha - \eta - (\gamma + \delta) 2^{p/q}} \right)^{q/p}$

**Case : 2** If

$$\max \left\{ \|Ax_{2n+1} - Ax_{2n}\|^{p/q}, \|Ax_{2n} - Ax_{2n-1}\|^{p/q} \right\} = \|Ax_{2n} - Ax_{2n-1}\|^{p/q}$$

From (3.3.1)

$$\begin{aligned} \int_0^{\infty} \|Ax_{2n+1} - Ax_{2n}\|^{p/q} \phi(t) dt &\leq \alpha \int_0^{\infty} \|Ax_{2n+1} - Ax_{2n}\|^{p/q} \phi(t) dt \\ &\quad + \beta \int_0^{\infty} \|Ax_{2n} - Ax_{2n-1}\|^{p/q} \phi(t) dt \\ &\quad + (\gamma + \delta) 2^{p/q} \left[ \int_0^{\infty} \|Ax_{2n+1} - Ax_{2n}\|^{p/q} \phi(t) dt + \int_0^{\infty} \|Ax_{2n} - Ax_{2n-1}\|^{p/q} \phi(t) dt \right] \\ &\quad + \eta \int_0^{\infty} \|Ax_{2n} - Ax_{2n-1}\|^{p/q} \phi(t) dt \\ &= (\alpha + (\gamma + \delta) 2^{p/q}) \int_0^{\infty} \|Ax_{2n+1} - Ax_{2n}\|^{p/q} \phi(t) dt \\ &\quad + (\beta + \eta + (\gamma + \delta) 2^{p/q}) \int_0^{\infty} \|Ax_{2n} - Ax_{2n-1}\|^{p/q} \phi(t) dt \end{aligned}$$

This leads to

$$\left( 1 - \alpha - (\gamma + \delta) 2^{p/q} \right) \int_0^{\infty} \|Ax_{2n+1} - Ax_{2n}\|^{p/q} \phi(t) dt \leq$$

$$(\beta + \eta + (\gamma + \delta)2^{p/q}) \int_0^{\|Ax_{2n} - Ax_{2n-1}\|^{p/q}} \phi(t) dt$$

Which yields

$$\int_0^{\|Ax_{2n+1} - Ax_{2n}\|^{p/q}} \phi(t) dt \leq \left( \frac{\beta + \eta + (\gamma + \delta)2^{p/q}}{1 - \alpha - (\gamma + \delta)2^{p/q}} \right) \int_0^{\|Ax_{2n} - Ax_{2n-1}\|^{p/q}} \phi(t) dt$$

(3.3.4)

Similarly we have

$$\int_0^{\|Ax_{2n} - Ax_{2n-1}\|^{p/q}} \phi(t) dt \leq \left( \frac{\beta + \eta + (\gamma + \delta)2^{p/q}}{1 - \alpha - (\gamma + \delta)2^{p/q}} \right) \int_0^{\|Ax_{2n-1} - Ax_{2n-2}\|^{p/q}} \phi(t) dt \quad (3.3.5)$$

Hence , one concludes that

$$\int_0^{\|Ax_{n+1} - Ax_n\|^{p/q}} \phi(t) dt \leq \left( \frac{\beta + \eta + (\gamma + \delta)2^{p/q}}{1 - \alpha - (\gamma + \delta)2^{p/q}} \right) \int_0^{\|Ax_n - Ax_{n-1}\|^{p/q}} \phi(t) dt$$

Which implies that

$$\begin{aligned} \int_0^{\|Ax_{n+1} - Ax_n\|} \phi(t) dt &\leq \left( \frac{\beta + \eta + (\gamma + \delta)2^{p/q}}{1 - \alpha - (\gamma + \delta)2^{p/q}} \right)^{q/p} \int_0^{\|Ax_n - Ax_{n-1}\|} \phi(t) dt \\ &\leq \lambda_2 \int_0^{\|Ax_n - Ax_{n-1}\|} \phi(t) dt \end{aligned}$$

Where  $\lambda_2 = \left( \frac{\beta + \eta + (\gamma + \delta)2^{p/q}}{1 - \alpha - (\gamma + \delta)2^{p/q}} \right)^{q/p}$

From both cases

Taking  $\lambda = \max \{ \lambda_1, \lambda_2 \}$

$0 < \lambda < 1$  as we have

$$0 < \alpha + \beta + \eta + 2(\gamma + \delta)2^{p/q} < 1$$

Now ,from the both cases

$$\begin{aligned} \int_0^{\|Ax_{n+1} - Ax_n\|} \phi(t) dt &\leq \lambda \int_0^{\|Ax_n - Ax_{n-1}\|} \phi(t) dt \\ &\leq \lambda^2 \int_0^{\|Ax_{n-1} - Ax_{n-2}\|} \phi(t) dt \end{aligned}$$

$$\dots\dots\dots$$

$$\leq \lambda^n \int_0^{\|Ax_1 - Ax_0\|} \phi(t) dt$$

For large n. Now, it can be seen that  $\{Ax_n\}$  is a Cauchy sequence and so it has a limit x in X. Since sequences  $\{Sx_{2n+1}\}$  and  $\{Tx_{2n}\}$  are subsequences of  $\{Ax_n\}$ , they have the same limit z. As S and A are commuting mapping, we can have

$$\begin{aligned} Sz &= \lim_{n \rightarrow \infty} S Ax_{2n+1} \\ &= \lim_{n \rightarrow \infty} A S x_{2n+1} \\ &= Az \end{aligned}$$

Similarly, we get  $Tz = Az$

This gives us,

$$Tz = Az = Sz$$

It comes out that

$$\begin{aligned} \int_0^{\|Az - AAz\|^{p/q}} \phi(t) dt &\leq \alpha \int_0^{\|Az - Sz\|^{p/q}} \phi(t) dt + \beta \int_0^{\|AAz - TAz\|^{p/q}} \phi(t) dt \\ &+ \gamma \int_0^{\|Az - SAz\|^{p/q}} \phi(t) dt + \delta \int_0^{\|Az - TAz\|^{p/q}} \phi(t) dt \\ &+ \eta \int_0^{\max\{\|Az - Tz\|^{p/q}, \|AAz - SAz\|^{p/q}\}} \phi(t) dt \end{aligned}$$

Commutativity of A with S and T gives the following by

$$\|AAz - TAz\| = \|AAz - ATz\| = \|AAz - AAz\| = 0$$

And

$$\|Az - SAz\| = \|Az - AAz\| \text{ and } \|Az - TAz\| = \|Az - AAz\|$$

Also,

$$\|AAz - SAz\| = \|AAz - ASz\| = \|AAz - AAz\| = 0$$

Resorting to these, we arrive at

$$\int_0^{\|Az - AAz\|^{p/q}} \phi(t) dt \leq (\gamma + \delta) \int_0^{\|Az - AAz\|^{p/q}} \phi(t) dt$$

Since  $\alpha + \beta + \eta + 2(\gamma + \delta)2^{p/q} < 1$ , we must have  $Az = AAz$

Finally, putting  $Az = Z_1$ , we have  $AZ_1 = Az = AAz = Z_1$

Similarly,  $TZ_1 = TAz = ATz = AAz = Az = Z_1$ ,

$SZ_1 = Z_1$ .

So,  $Z_1$  is a fixed point of S, T and A.

Next, to show uniqueness of this Common fixed point, let us suppose that  $Z_2$  is also a common fixed point of S, T and A other than  $Z_1$ .

Then  $SZ_2 = Z_2$ ,  $TZ_2 = Z_2$ ,  $AZ_2 = Z_2$ , and also  $\|Z_1 - Z_2\| \neq 0$ .

Hence, it follows that

$$\begin{aligned} \int_0^{\|z_1 - z_2\|^{p/q}} \phi(t) dt &= \int_0^{\|Az_1 - Az_2\|^{p/q}} \phi(t) dt \\ &\leq \alpha \int_0^{\|Az_1 - Sz_1\|^{p/q}} \phi(t) dt + \beta \int_0^{\|Az_2 - Tz_2\|^{p/q}} \phi(t) dt \end{aligned}$$



$$\begin{aligned}
 & + \gamma \int_0^{\infty} \|Az_1 - Sz_2\|^{p/q} \phi(t) dt + \delta \int_0^{\infty} \|Az_1 - Tz_2\|^{p/q} \phi(t) dt \\
 & + \eta \int_0^{\infty} \max\{\|Az_1 - Tz_1\|^{p/q}, \|Az_2 - Sz_2\|^{p/q}\} \phi(t) dt
 \end{aligned}$$

Implies

$$\int_0^{\infty} \|z_1 - z_2\|^{p/q} \phi(t) dt \leq (\gamma + \delta) \int_0^{\infty} \|z_1 - z_2\|^{p/q} \phi(t) dt$$

Which is a contradiction as  $\alpha + \beta + \eta + 2(\gamma + \delta)2^{p/q} < 1$ .

Thus occurs the uniqueness.

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