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Fixed Point Theorem Of Discontinuity And Weak Compatibility In Non complete Non-Archimedean Menger PM-Spaces

Pooja Sharma (Corresponding author)
Career College, Barkatullah University
M.I.G-9 sector 4B Saket Nager Bhopal, India
Tel: +919301030702 E-mail: poojasharma020283@gmail.com

R.S.Chandel
Govt. Geetanjali College, Barkatullah University
Tulsi Nager, Bhopal, India
Tel: +919425650235 E-mail: rschandel_2009@yahoo.com

Abstract

The aim of this paper is to prove a related common fixed point theorem for six weakly compatible self maps in non complete non-Archimedean menger PM-spaces, without using the condition of continuity and give a set of alternative conditions in place of completeness of the space.

Keywords: key words, Non-Archimedean Menger PM-space, R-weakly commutting maps, fixed points.

1. Introduction

There have been a number of generalizations of metric spaces, one of them is designated as Menger space propounded by Menger in 1972. In 1976, Jungck established common fixed point theorems for commuting maps generalizing the Banach's fixed point theorem. Sessa (1982) defined a generalization of commutativity called weak commutativity. Futher Jungck (1986) introduced more generalized commutativity, which is called compatibility. In 1998, Jungck & Rhodes introduced the notion of weakly compatible maps and showed that compatible maps are weakly compatible but converse need not true. Sharma & Deshpande (2006) improved the results of Sharma & Singh (1982), Cho (1997), Sharma & Deshpande (2006). Chugh and Kumar (2001) proved some interesting results in metric spaces for weakly compatible maps without appeal to continuity. Sharma and deshpande (2006) proved some results in non complete Menger spaces, for weakly compatible maps without appeal to continuity. In this Paper, we prove a common fixed point theorem for six maps has been proved using the concept of weak compatibility without using condition of continuity.

We will improve results of Sharma & Deshpande (2006) and many others.

Preliminary notes

Definition 1.1 Let X be any nonempty set and D be the set of all left continuous distribution functions. An order pair (X, F) is called a non-Archimedean probabilistic metric space, if F is a mapping from $X \times X$

Into D satisfying the following conditions

(i) $F_{x,y}(t) = 1$ for every $t > 0$ if and only if $x = y$,

(ii) $F_{x,y}(0) = 0$ for $x, y \in X$

(iii) $F_{x,y}(t) = F_{y,x}(t)$ for every $x, y \in X$

(iv) If $F_{x,y}(t_1) = 1$ and $F_{y,z}(t_2) = 1$,

Then $F_{x,z}(\max\{t_1, t_2\}) = 1$ for every $x, y, z \in X$,

Definition 1.2 A Non- Archimedean Manger PM-space is an order triple (X, F, Δ), where Δ is a t-norm and (X, F) is a Non-Archimedean PM-space satisfying the following condition.

(v) $F_{x,z}(\max\{t_1, t_2\}) \geq \Delta(F_{x,y}(t_1), F_{y,z}(t_2))$ for x,y,z $\in$ X and $t_1, t_2 \geq 0$.

The concept of neighborhoods in Menger PM-spaces was introduced by Schwizer-Skla (1983). If x$\in$ X, t > 0 and $\lambda \in$ (0, 1), then and $(\in, \lambda)$-neighborhood of x, denoted by $U_x(\varepsilon, \lambda)$ is defined by

$$U_x(\varepsilon, \lambda) = \{y \in X : F_{y,x}(t) > 1 - \lambda\}$$

If the t-norm Δ is continuous and strictly increasing then (X, F, Δ) is a Hausdorff space in the topology induced by the family $\{U_x(t, y) : x \in X, t > 0, \lambda \in (0, 1)\}$ of neighborhoods .

Definition 1.3 A t-norm is a function $\Delta : [0, 1] \times [0, 1] \rightarrow [0, 1]$ which is associative, commutative, non decreasing in each coordinate and $\Delta(a, 1) = a$ for every a$\in$ [0, 1].

Definition 1.4 A PM- space (X, F) is said to be of type $(C)_g$ if there exists a g $\in \Omega$ such that

$$g(F_{x,y}(t)) \leq g(F_{x,z}(t)) + g(F_{z,y}(t))$$

for all x, y, z $\in$ X and t $\geq$ 0, where $\Omega = \{g : g[0,1] \rightarrow [0, \infty)$ is continuous, strictly decreasing, g(1) = 0 and g(0) > ∞ }.

Definition 1.5 A pair of mappings A and S is called weakly compatible pair if they commute at coincidence points.

Definition 1.6 Let A, S: X $\rightarrow$ X be mappings. A and S are said to be compatible if

$\lim_{n\to\infty} g(FASx_n, SAx_n(t)) = 0$

For all t > 0, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some z$\in$ X.

Definition 1.7 A Non-Archimedean Manger PM-space (X, F, Δ) is said to be of type $(D)_g$ if there exists a g $\in \Omega$ such that

$g(\Delta(S, t) \leq g(S) + g(t))$ for all S, t $\in$ [0, 1].

Lemma 1.1 If a function $\phi : [0, +\infty) \rightarrow [0, -\infty)$ satisfying the condition

($\varnothing$), then we have

(1) For all t $\geq$ 0, $\lim_{n\to\infty} \phi^n(t)$ 0, where $\phi^n(t)$ is the n-th iteration of $\phi(t)$.

(2) If $\{t_n\}$ is non-decreasing sequence real numbers and $t_{n+1} \leq \phi(t_n)$, n = 1,2,….., then

$\lim_{n\to\infty} t_n = 0$. In particular, if t $\leq \phi(t)$ for all t $\geq$ 0, then t = 0.

Lemma 1.2 Let $\{y_n\}$ be a sequence in X such that $\lim_{n\to\infty} F_{y_n\, y_{n+1}}(t) = 1$ for all t > 0.

If the sequence $\{y_n\}$ is not a Cauchy sequence in X, then there exit $\varepsilon_0 > 0$, $t_0 > 0$, two sequences

$\{m_i\}$, $\{n_i\}$ of positive integers such that

(1) $m_i > n_i + 1$, $n_i \rightarrow \infty$ as i $\rightarrow \infty$,

(2) $F_{y_{mi}, y_{ni}}(t_0) < 1 - \varepsilon_0$ and $F_{y_{mi-1}, y_{ni}}(t_0) \geq 1 - \varepsilon_0$, i = 1,2,….

Main Results

Theorem 2.1 : Let A, B, S, T, P and Q be a mappings from X into itself such that

(i) $P(X) \subset AB(X)$, $Q(X) \subset ST(X)$

(ii) $g(FPx, Qy(t)) \leq \phi[\max\{g(FABy, STx(t)), g(FPx, STx(t)), g(FQy, ABy(t)), g(FQy, STx(t)), g(FPx, ABy(t))\}]$

(for all x, y$\in$ X and t>0, where a function $\phi:[0,+\infty)\to[0,+\infty)$ satisfies the condition ($\varnothing$).

(iii) A(X) or B(X) is complete subspace of X, then
(a) P and ST have a coincidence point.
(b) Q and AB have a coincidence point.

Further, if

(iv) The pairs (P, ST) and (Q, AB) are R-weakly compatible then A,B,S,T,P and Q have a unique common fixed point.

Proof: Since $P(X) \subset AB(X)$ for any $x_0 \in X$, there exists a point $x_1 \in X$, Such that $Px_0 = ABx_1$. Since $Q(X) \subset ST(X)$ for this point x_1, we can choose a point $x_2 \in X$ such that $Bx_1 = Sx_2$ and so on.. Inductively, we can define a sequence $\{y_n\}$ in X such that $y_{2n} = Px_{2n} = ABx_{2n+1}$ and $y_{2n+1} = Qx_{2n+1} = STx_{2n+2}$, for n=1,2,3……..

Before proving our main theorem we need the following :

Lemma 2.2: Let A,B,S,T,P,Q :X$\to$X be mappings satisfying the condition (i) and (ii).

Then the sequence $\{y_n\}$ define above, such that

$$\lim_{n\to\infty} g(Fy_n, y_{n+1}(t)) = 0, \quad \text{For all } t>0 \text{ is a Cauchy sequence in X.}$$

Proof of Lemma 2.2: Since g$\in\Omega$, it follows that

$$\lim_{n\to\infty} Fy_n, y_{n+1}(t) = 1 \text{ for all } t>0 \text{ if and only if}$$

$\lim_{n\to\infty} g(Fy_n, y_{n+1}(t)) = 0$ for all t>0. By Lemma 1.2, if $\{y_n\}$ is not a Cauchy sequence in X, there exists $\in_0 > 0$, t>0, two sequence $\{m_i\}$,$\{n_i\}$of positive integers such that

(A) $m_i > n_i + 1$, and $n_i \to \infty$ as i $\to \infty$,

(B) $(Fym_i, yn_i(t_0)) > g(1-\in_0)$ and $g(Fym_{i+1}, yn_i(t_0)) \leq g(1-\in_0)$, i=1,2,3……

Thus we have

$$g(1-\in_0) < (Fym_i, yn_i(t_0))$$

$\leq$ g(Fy m_i, $y_{m_{i-1}}(t_0)$) + g(F $y_{m_{i-1}}$,y $n_i(t_0)$)

(v) $\leq$ g(1-$\in_0$) + g(F y_{m_i}, $y_{m_{i-1}}(t_0)$)

Thus i $\longrightarrow \infty$ in (v), we have

(vi) $\lim_{n\to\infty}$ g(F y_{m_i}, y $n_i(t_0)$) = g(1-$\in_0$).

On the other hand, we have

(vii) g(1-$\in_0$) < g(F y_{m_i}, y $n_i(t_0)$)

$\leq$ g(F y_{n_i}, $y_{n_{i+1}}(t_0)$) + g(F $y_{n_{i+1}}$, $y_{m_i}(t_0)$).

Now, consider g(F $y_{n_{i+1}}$, $y_{m_i}(t_0)$) in (vii. Without loss generality, assume that both n_i and m_i are even. Then by (ii), we have

g(F $yn_{i+1,}$ $ym_i(t_0)$) = g(FP x_{mi} , Q $x_{ni+1}(t_0)$)

$\leq$ $\varnothing$[max{g(FST x_{mi} , AB $x_{ni+1}(t_0)$) , g(FST x_{mi} , P $x_{mi}(t_0)$) , g(FAB x_{ni+1} , Q $x_{ni+1}(t_0)$) , g(FST x_{mi}, Q $x_{ni+1}(t_0)$) , g(FAB x_{ni+1}, P $x_{mi}(t_0)$)}]

(viii) = $\varnothing$**[**max{g(Fy m_{i-1} ,y $n_i(t_0)$), g(Fy m_{i-1}, y $m_i(t_0)$), g(Fy n_i ,y $n_{i+1}(t_0)$), g(Fy m_{i-1} ,y $n_{i+1}(t_0)$), g(Fy n_i, y $m_i(t_0)$) }**]**

Using (vi),(vii),(viii) and letting $i \longrightarrow \infty$ n(viii), we have

g(1-$\in_0$) $\leq$ $\varnothing$[max{g(1-$\in_0$), 0, 0, g(1-$\in_0$), g(1-$\in_0$)}]

= $\varnothing$ (g(1-$\in_0$))

< g(1-$\in_0$),

This is a contradiction. Therefore {y $_n$ } is a Cauchy sequence in X.

Proof of the Theorem 2.1: If we prove lim $n \longrightarrow \infty$ g(Fy n, y n_{i+1}(t)) = 0 for all t>0, Then by Lemma(2.2) the sequence {y $_n$ } define above is a Cauchy sequence in X.

Now, we prove lim $n \longrightarrow \infty$ g(Fy n, y $n+1$ (t)) = 0 for all t > 0. In fact by (ii), we have

g(Fy $_{2n}$, y $_{2n+1}$(t)) = g(FP X_{2n}, Q X_{2n+1}(t))

$\leq \varnothing$[max {g(FST X_{2n}, AB X_{2n}, AB X_{2n+1} (t)), g(FST X_{2n},P X_{2n} (t)),

g(FAB X_{2n+1}, Q X_{2n+1} (t)), g(F X_{2n}, Q X_{2n+1} (t)), g(FAB X_{n+1},P X_{2n} (t))}]

$=\varnothing$[max{g(Fy $_{2n-1}$, y $_{2n}$ (t)), g(Fy $_{2n-1}$, y $_{2n}$ (t)), g(Fy $_{2n}$,y $_{2n+1}$(t)),

g(Fy $_{2n-1}$, y $_{2n+1}$ (t))}],

$= \varnothing$[max{g(Fy $_{2n-1}$, y $_{2n}$ (t)), g(Fy $_{2n-1}$, y $_{2n}$ (t)), g(Fy $_{2n}$,y $_{2n+1}$(t)),

g(Fy $_{2n-1}$, y $_{2n}$ (t)) + g(Fy $_{2n}$,y $_{2n+1}$(t)), 0}]

If g(Fy $_{2n-1}$, y $_{2n}$ (t) $\leq$ g(Fy $_{2n}$,y $_{2n+1}$(t) for all t > 0, then we have

g(Fy $_{2n}$,y $_{2n+1}$(t)) $\leq \varnothing$ g(Fy $_{2n}$,y $_{2n}$,y $_{2n+1}$(t)),

Which means that,by Lemma 1.1, g(Fy $_{2n}$,y $_{2n+1}$(t)) = 0 for all t >0.

Similarly, we have g(Fy $_{2n}$,y $_{2n+1}$(t)) = 0 for all t > 0. Thus we have $\lim n \to \infty$ g(Fy n, y $n+1$ (t)) = 0 for all t > 0. On the othr hand, if g(Fy $_{2n-1}$, y $_{2n}$ (t)) $\geq$ g(Fy $_{2n}$,y $_{2n+1}$(t)), then by (ii), we have

g(Fy $_{2n}$,y $_{2n+1}$(t)) $\leq$ g(Fy $_{2n-1}$, y $_{2n}$ (t)), for t > 0.

Similarly, g(Fy $_{2n+1}$,y $_{2n+2}$ (t)) $\leq$ g(Fy $_{2n}$, y $_{2n+1}$(t)) for all t > 0 .

g(Fy n, y $n+1$ (t)) $\leq$ g(Fy $n-1$,y n (t)), for all t >0 and n= 1,2,3,…….
Therefore by Lemma (1.1)
$\lim n \to \infty$ g(Fy n, y $n+1$ (t)) = 0 for all t > 0, which implies that {y n } is a Cauchy sequence in X.
Now suppose that ST(X) is a complete. Note that the subsequence {y $n+1$} is contained in ST(X) and a limit in ST(X) . Call it z. Let p$\in$ (ST) $^{-1}$ z.

We shall use that fact that the subsequence {y $_{2n}$ } also converges to z. By (ii), we have

g(FP p, y $_{2n+1}$ (kt)) = g(FP p,Qx $_{2n+1}$ (kt))

$\leq \varnothing$[max{g(FST p, ABx $_{2n+1}$ (t)), g(FST p, P p (t)), g(FAB x_{2n+1}, Q x_{2n+1} (t)),

g(FST p, Q x_{2n+1} (t)), g(FAB x_{2n+1}, P p (t))}]

$=\varnothing$[max{FST p, y_{2n} (t),g(FST p,P p (t)), g(F y_{2n}, y_{2n+1}(t)), g(F ST p, y_{2n+1}(t)), g(F y_{2n}, P p (t))}]

Taking the limit $n \to \infty$, we obtain

g(FP p , z(kt))≤ $\varnothing$[max{g(Fz, z(t)), g(Fz, P p (t)), g(Fz, z(t), g(Fz, z(t)), g(Fz, P p (t))}]

< $\varnothing$(g(FP p , z(t))),

For all t>0,which means that P p = z and therefore, P p = ST p =z, i.e. p is a coincidence point of P and ST. This proves (i). Since P(X)⊂AB(X) and , P p = z implies that z∈ AB(X).

Let q∈ $(AB)^{-1}$ z. Then q = z.

It can easily be verified by using similar arguments of the previous part of the proof that Qq = z.

If we assume that ST(X) is complete then argument analogous to the previous completeness argument establishes (i) and (ii).

The remaining two cases pertain essentially to the previous cases. Indeed, if B(X) is complete, then by (2.1), z ∈ Q(X) ⊂ ST(X).

Similarly if P(X) ⊂AB(X). Thus (i) and (ii) are completely established.

Since the pair {P, ST} is weakly compatible therefore P and ST commute at their coincidence point i.e. PST p = STP p or Pz = STz. Similarly QABq = ABQq or Qz = ABz.

Now, we prove that Pz = z by (2.2) we have

g(FPz, y_{2n+1}(t)) = g(FPz, Q x_{2n+1}(t))

≤ $\varnothing$[max{g(FSTz, AB x_{2n+1})), g(FSTz, Pz(t)), g(FAB x_{2n+1}, Q x_{2n+1}(t)),

g(FSTz , Q x_{2n+1}(t)), g(FST x_{2n+1}, Pz(t))}].

By letting $n \to \infty$, we have

g(FPz , z(t)) ≤ $\varnothing$[max{g(FPz, z(t)), g(FPz, Pz(t)), g(Fz, z(t)), g(FPz, Pz(t)), g(Fz, Pz(t))}],

Which implies that Pz =z =STz.

This means that z is a common fixed point of A,B, S, T, P, Q. This completes the proof.

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