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ORIENTED MANIFOLDS WITH COMPACT SUPPORT AND COHOMOLOGY ALGEBRA

Md. Shafiul Alam
Department of Mathematics, University of Barishal
Barishal-8200, Bangladesh
Tel: +8801819357123 E-mail: shafiulmt@gmail.com

Abstract

The cohomology of M with compact supports is the graded algebra $\Omega(G_c(M),\delta)$ and is given by $\Omega_c(M)=\sum_{k=0}^{n}\Omega_c^k(M)$. The bilinear map $\Omega(M)\times\Omega_c(M)\to\Omega_c(M)$ is induced by $G(M)\times G_c(M)\to\Omega_c(M)$ and makes $\Omega_c(M)$ into a left graded $\Omega(M)$-module. $\Omega(S^n)$, which is the cohomology of S^n, is determined by $\Omega^0(S^n)\cong\Omega^n(S^n)\cong\mathbb{R}$ and $\Omega^k(S^n)=0$ for $n\geq 1$. Also, we determine the cohomology of $\mathbb{R}^n$ with compact supports. Finally, it is shown that the map $i_M:\Omega(M)\to\Omega_c(M)^*$ is a linear isomorphism.

***Keywords*:** Compact manifold, cohomology, graded algebra, isomorphism, bilinear map.

1. Introduction

Let M be an n-manifold, then the graded algebra of differential forms on M is defined as $G(M)=\sum_{k=0}^{n}G^k(M)$ and $G(M)$ is converted into a graded differential algebra by the exterior derivative (Greub et al., 1972). The differential forms Φ satisfying the condition $\delta\Phi=0$ construct cocycles in this differential algebra and this differential form is closed. The closed forms are graded subalgebra $Z(M)$ of $G(M)$ as δ is an antiderivation (Bott and Tu, 1982). The subset $H(M)=\delta G(M)$ is a graded ideal in $Z(M)$. The differential forms in $G(M)$ are called coboundaries and the corresponding cohomology algebra is defined by $\Omega(M)=Z(M)/H(M)$ and this cohomology algebra is called the de Rham cohomology algebra of M (Iversen, 1986).

The cohomology of M with compact supports is the graded algebra $\Omega(G_c(M),\delta)$ (Grivaux, 2010). It is denoted by $\Omega_c(M)$ and is defined by

$$\Omega_c(M)=\sum_{k=0}^{n}\Omega_c^k(M),\quad n=\dim M.$$

Multiplication in $G(M)$ is restricted to a real bilinear map as $G_c(M)$ is an ideal (Kobayashi and Nomizu, 1963). $G_c(M)$ is confined into a left graded $G(M)$-module by this multiplication which is given by

$$G(M)\times G_c(M)\to\Omega_c(M).$$

The bilinear map $\Omega(M)\times\Omega_c(M)\to\Omega_c(M)$ is induced by the above map and makes $\Omega_c(M)$ into a left graded $\Omega(M)$-module (Sternberg, 1964). This map can be written as

$$(\lambda,\mu)\mapsto\alpha*\beta,\lambda\in\Omega(M),\mu\in\Omega_c(M).$$

In the same way, $\Omega_c(M)$ can be converted into a right graded $\Omega(M)$-module and we can write $\mu*\lambda,\mu\in$

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$\Omega_c(M), \lambda \in \Omega(M)$. Also, the algebra homomorphism

$$(\tau_M)_{\#}: \Omega_c(M) \to \Omega(M)$$

is induced by the inclusion map $\tau_M: G_c(M) \to G(M)$. The above module structures can be converted to ordinary multiplication by this homomorphism (Haller and Rybicki, 1999).

2. Preliminaries and Auxiliary Results

Let $\Omega: \mathbb{R} \times M \to N$ be a smooth map. Two smooth maps $f, g: M \to N$ are said to be homotopic (Eilenberg and Maclane, 1950) if $\Omega(0, x) = f(x)$ and $\Omega(1, x) = g(x)$. We can define a linear map $h: G(N) \to G(M)$ homogeneous of degree -1 for such a homotopy Ω by

$$h = I_0^1 \circ i(T) \circ \Omega^*.$$

Consider the spaces $\Omega^k(M)$ having finite dimension, then the kth Betti number of M is defined by $b_k = \dim \Omega^k(M)$ and the Poincaré polynomial of M is defined by

$$p_M(t) = \sum_{k=0}^{n} b_k\, t^k.$$

If M consists of a single point, then $\Omega^k(M) = 0$ $(k \geq 1)$ and $\Omega^0(M) = \mathbb{R}$.

The Euler-Poincaré characteristic of M is defined by the alternating sum $\zeta_M = \sum_{k=0}^{n} (-1)^k b_k = p_M(-1)$.

Now, we discuss the axioms for de Rham cohomology. The axioms for de Rham cohomology are given below:

(a) $\Omega(\text{point}) = \mathbb{R}$

(b) If M is the disjoint union of open submanifolds M_α, then

$$\Omega(M) \cong \prod_\alpha \Omega(M_\alpha) \text{ (disjoint union)}$$

(c) If $f \sim g: M \to N$, then $f^{\#} = g^{\#}$ (homotopy axiom)

(d) If $M = U \cup V$ (U, V are open), there is an exact triangle (Mayer-Vietoris)

$$\begin{tikzcd}
\Omega(M) \arrow[rr] & & \Omega(U) \oplus \Omega(V) \arrow[dl] \\
& \Omega(U \cap V) \arrow[ul] &
\end{tikzcd}$$

Consider a manifold M which is the disjoint union $M = \bigcup_\nu M_\nu$ of open submanifolds M_ν. A

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homomorphism $h_\nu^*: G(M) \to G(M_\nu)$ is induced by the inclusion map $h_\nu^*: M_\nu \to M$. We obtain a homomorphism $h^*: G(M) \to \prod_\nu G(M_\nu)$ given by $(h^*\Phi)_\nu = h_\nu^*$, where $\Phi \in G(M)$ and $\prod_\nu G(M_\nu)$ is the direct product of the algebras $G(M_\nu)$.

If δ_ν denotes the exterior derivative in $\Omega(M_\nu)$, then $\prod_\nu \Omega(M_\nu)$ is given by the differential operator $\prod_\nu \Omega(M_\nu)$. As a result, h^* is an isomorphism of graded differential algebras $\Omega(M_\nu)$ and h^* induces the following isomorphism

$$h^*: \Omega(M) \xrightarrow{\cong} \prod_\nu \Omega(M_\nu)$$

given by

$$(h^*\gamma)_\nu = h_\nu^*(\gamma),\ \gamma \in \Omega(M).$$

Consider a manifold M and two open subsets X_1, X_2 such that $X_1 \cup X_2 = M$. Let us consider the following inclusion maps

$$u_1: X_1 \cap X_2 \to X_1,\ \ u_2: X_1 \cap X_2 \to X_2$$
$$v_1: X_1 \to M,\ \ v_2: X_2 \to M.$$

which induce a sequence of linear mappings

$$0 \longrightarrow \Omega(M) \xrightarrow{\lambda} \Omega(X_1) \oplus \Omega(X_2) \xrightarrow{\mu} \Omega(X_1 \cap X_2) \longrightarrow 0$$

given by

$$\lambda\Phi = (v_1^*\Phi, v_\lambda^*\Phi),\ \ \Phi \in \Omega(M)$$

and

$$\mu(\Phi_1, \Phi_2) = u_1^*\Phi_1 - u_2^*\Phi_2,\ \ \Phi_i \in \Omega(U_i),\ \ i = 1, 2.$$

Let $\delta_1, \delta_2, \delta_{12}$ and δ be the exterior derivatives in $\Omega(X_1), \Omega(X_2), \Omega(X_1 \cap X_2)$ and $\Omega(M)$ respectively, then we have

$$\lambda \circ \delta = (\delta_1 \oplus \delta_2) \circ \alpha \ \text{ and } \ \mu \circ (\delta_1 \oplus \delta_2) = \delta_{12} \circ \mu.$$

Consequently, the following linear maps are induced by λ and μ:

$$\lambda_\#: \Omega(M) \to \Omega(X_1) \oplus \Omega(X_2),\ \ \mu_\#: \Omega(X_1) \oplus \Omega(X_2) \to \Omega(X_1 \cap X_2).$$

Lemma 1. The following sequence of linear mappings is exact

$$0 \longrightarrow \Omega(M) \xrightarrow{\lambda} \Omega(X_1) \oplus \Omega(X_2) \xrightarrow{\mu} \Omega(X_1 \cap X_2) \longrightarrow 0.$$

Proof. We have to consider the following three cases:

(a) $\ker \mu = \operatorname{Im} \lambda$

(b) λ is injective

(c) μ is surjective

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(a) Since it is obvious $\mu \circ \lambda = 0$, so $\mathrm{Im}\,\lambda \subset \ker \mu$. We need only to show that $\ker \mu \subset \mathrm{Im}\,\lambda$.

Let $(\Phi_1, \Phi_2) \in \ker \mu$. If $x \in X_1 \cap X_2$, then $\Phi_1(x) = \Phi_2(x)$. Consequently, we can find a differential form $\Phi \in \Omega(M)$ which is given by

$$\Phi(\mathrm{x}) = \begin{cases} \Phi_1(x), & x \in X_1 \\ \Phi_2(x), & x \in X_2 \end{cases}.$$

Since $\lambda\Phi = (\Phi_1, \Phi_2)$, so $\ker \mu \subset \mathrm{Im}\,\lambda$. Therefore, $\ker \mu = \mathrm{Im}\,\lambda$.

(b) Let $x \in X_1 \cup X_2 = M$. If $\lambda\Phi = 0$, then $\Phi(x) = 0$ for $x \in X_1 \cup X_2 = M$.

(c) Consider the covering $X_1, X_2$of M. Let x_1, x_2 be subordinate to the covering X_1, X_2. Thus, $\{x_1, x_2\}$ is a partition of unity for M. Then, $carr\ v_1^* x_2,\ carr\ v_2^* x_1 \subset X_1 \cup X_2$.

For $\Phi \in \Omega(X_1 \cap X_2)$, we define

$$\Phi_1 = v_1^* x_2 \cdot \Phi \in \Omega(X_1),\ \Phi_2 = v_2^* x_1 \cdot \Phi \in \Omega(X_2).$$

Consequently, we have $\Phi = \mu(\Phi_1, -\Phi_2)$. □

Consider a compact oriented n-manifold M. Then, we have

$$\Omega_c(M) = \Omega(M) \text{ and } i_M : \Omega(M) \xrightarrow{\cong} \Omega(M)^*.$$

Therefore, the bilinear map $\mathcal{P}_M^k :\ \Omega^k(M) \times \Omega^{n-k}(M) \to \mathbb{R}$ represents the Poincaré scalar product.

Theorem 1. If M is any compact manifold, then the dimension of $\Omega(M)$ is finite.

Proof. First we assume that the compact manifold M is orientable. Then the Poincaré scalar product is given by the bilinear map $\mathcal{P}_M^k : \Omega^k(M) \times \Omega^{n-k}(M) \to \mathbb{R}$ and $\mathcal{P}_M^k$ induces the following two linear isomorphisms

$$\Omega^k(M) \xrightarrow{\cong} \Omega^{n-k}(M)^*$$

and

$$\Omega^{n-k}(M) \xrightarrow{\cong} \Omega^k(M)^*.$$

Now, from the related results of elementary linear algebra we can observe that each $\Omega^k(M)$ has finite dimension; hence the theorem is proved in this case.

Again, we assume that the compact manifold M is nonorientable. In this case, the double cover $\widetilde{M}$ is orientable and compact. Consequently, we have

$$\dim \Omega(M) = \dim \Omega_+(\widetilde{M}) \le \dim \Omega(\widetilde{M}) < \infty.$$

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Thus the dimension of $\Omega(M)$ is finite. $\square$

Lemma 2. $\int_M^{\#} : \Omega_c^n(M) \to \mathbb{R}$ is a linear isomorphism if M is a connected oriented n-manifold.

Proof. Let $\Omega(M)$ be the cohomology of an oriented manifold M and $\Omega_c(M)$ be the cohomology of M with compact support. Then the map

$$i_M : \Omega(M) \to \Omega_c(M)^*$$

is a linear isomorphism. Also, we have

$$\dim \Omega_c^n(M) = \dim \Omega^0(M) = 1.$$

Moreover, $\int_M^{\#}$ is surjective. Therefore, $\int_M^{\#} : \Omega_c^n(M) \to \mathbb{R}$ is a linear isomorphism if M is a connected oriented n-manifold. $\square$

Consider an oriented n-manifold M. The linear map $\int_M : G_c^n(M) \to \mathbb{R}$ satisfies $\int_M \circ \delta = 0$ and it is surjective map. The linear map $\int_M^{\#} : \Omega_c^n(M) \to \mathbb{R}$ is induced by $\int_M : G_c^n(M) \to \mathbb{R}$ and this map is also surjective. Let $\lambda \in \Omega^k(M)$ and $\mu \in \Omega_c^{n-k}(M)$. The Pioncaré scalar product

$$\mathcal{P}_M^k : \Omega^k(M) \times \Omega_c^{n-k}(M) \to \mathbb{R}$$

can be expressed as the following bilinear map $\mathcal{P}_M^k(\lambda, \mu) = \int_M^{\#} \lambda * \mu$.

Lemma 3. Let M, N be two manifolds, then the following diagram commutes.

$$\begin{tikzcd}
\Omega(M) \arrow[d, "i_M"'] & \Omega(N) \arrow[l, "\psi^{\#}"'] \arrow[d, "i_N"] \\
\Omega_c(M)^* & \Omega_c(N)^* \arrow[l, "(\psi_c)_{\#}^*"]
\end{tikzcd}$$

Proof. If $\lambda \in \Omega^k(N)$, $\mu \in \Omega_c^{n-k}$, $\zeta \in G^k(N)$, $\xi \in G_c^{n-k}(M)$, then λ and μ are represented by ζ and ξ respectively. Consequently, $(\psi_c)_{\#}\mu \in \Omega_c^{n-k}(N)$ is represented by

$$(\psi_c)_*\xi \quad \text{and} \quad \psi^*(\zeta \wedge ((\psi_c)_*\xi) = \psi^*\zeta \wedge \xi.$$

Since the Pioncaré scalar product $\mathcal{P}_M^k : \Omega^k(M) \times \Omega_c^{n-k}(M) \to \mathbb{R}$ is the bilinear map given by

$$\mathcal{P}_M^k(\lambda, \mu) = \int_M^{\#} \lambda * \mu,$$

thus, $\mathcal{P}_M^k(\psi^\#\lambda,\mu)=\int_M^\#(\psi^\#\lambda)*\mu$ and $\mathcal{P}_N^k(\lambda,(\psi_c)_\#\mu)=\int_M^\#\lambda*(\psi_c)_\#\mu$. Hence we have

$$\mathcal{P}_M^k(\psi^\#\lambda,\mu)=\int_M^\#(\psi^\#\lambda)*\ \mu=\int_M\psi^*\zeta\wedge\ \xi\ =\ \int_N\zeta\wedge\ (\psi_c)_*\xi=\int_M^\#\lambda*(\psi_c)_\#\mu=\ \mathcal{P}_N^k(\lambda,(\psi_c)_\#\mu)$$

Since $\mathcal{P}_M^k(\psi^\#\lambda,\mu)=\mathcal{P}_N^k(\lambda,(\psi_c)_\#\mu)$, we can conclude that the diagram commutes. Hence the proposition is proved. □

3. Main Results

Theorem 2. For $n\geq1$, $\Omega(S^n)$ is determined by $\Omega^0(S^n)\cong\Omega^n(S^n)\cong\mathbb{R}$ and $\Omega^k(S^n)=0$ $(1\leq k\leq n-1)$.

Proof. First we consider an $(n+1)$-dimensional Euclidean space E^{n+1}. Suppose S^n is embedded in E^{n+1}. We know that S^n is connected, thus $\Omega^0(S^n)=\mathbb{R}$. Now let $s\in S^n$ and $\xi\in(0,1)$ where ξ is fixed. Again, we consider open sets $X_1,X_2\subset S^n$ defined by

$$X_1=\{x\in S^n:\langle x,s\rangle>-\xi\},\ \ X_2=\{x\in S^n:\langle x,s\rangle<\xi\}.$$

As a result, $S^n=X_1\cup X_2$ and we have the following exact Mayer-Vietoris sequence

$$\cdots\ \to\ \Omega^k(S^n)\to\Omega^k(X_1)\oplus\Omega^k(X_2)\to\Omega^k(X_1\cap X_2)\to\Omega^{k+1}(S^n)\to\ \cdots.$$

It is clear that S^{n-1} is contained in $X_1\cap X_2$. We observe that X_1 and X_2 are contractible. Consequently, the following exact sequence can be considered as the Mayer-Vietoris sequence

$$\cdots\ \to\ \Omega^k(S^n)\ \to\ \Omega^k(\text{point})\oplus\Omega^k(\text{point})\to\ \Omega^k(S^{n-1})\ \to\ \Omega^{k+1}(S^n)\ \to\ \cdots.$$

The above sequence can be split into the following two sequences

$$0\ \to\ \Omega^0(S^n)\ \to\ \Omega^0(\text{point})\oplus\ \Omega^0(\text{point})\ \to\ \Omega^0(S^{n-1})\ \to\ \Omega^1(S^n)\ \to\ 0$$

and

$$0\longrightarrow\Omega^k(S^{n-1})\xrightarrow{\ \cong\ }\Omega^{k+1}(S^n)\longrightarrow0,\qquad k\geq1.$$

These sequences are exact and from the first sequence we have

$$0=\dim\Omega^1(S^n)-\dim\Omega^0(S^{n-1})+\ 2\dim\Omega^0(\text{point})-\dim\Omega^0(S^n).$$

For $n\geq2$, we observe that S^{n-1} is connected and S^0 consists of two points. Thus we can conclude from the above equation

$$\Omega^1(S^n)\cong\begin{cases}\mathbb{R}, & n=1\\ 0, & n>1\end{cases}.$$

Since $0\longrightarrow\Omega^k(S^{n-1})\xrightarrow{\ \cong\ }\Omega^{k+1}(S^n)\longrightarrow0$ for $k\geq1$, we have

$$\Omega^k(S^n)\cong\Omega^1(S^{n-k+1})\ (1\leq k\leq n).$$

Therefore, $\Omega^0(S^n)\cong\Omega^n(S^n)\cong$ and $\Omega^k(S^n)=0$. Hence, the proposition is proved. □

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Corollary 1. Consider a connected n-manifold M. Then $\Omega^n(M) \cong \mathbb{R}$ when M is compact and orientable. Otherwise, $\Omega^n(M) = 0$.

Proof. First we assume that M is compact. Then, there are two cases:

(i) M is orientable

(ii) M is nonorientable.

If we consider M to be orientable, then from the consequence of Lemma 2 we can deduce that $\Omega^n(M) \cong \mathbb{R}$. If M is nonorientable, then $\Omega^n(M) = 0$.

Next we assume that M is not compact. Then, there are again two cases:

(i) M is orientable

(ii) M is nonorientable.

If the manifold M is orientable, then we have $\Omega^n(M) \cong \Omega_c^0(M)^* = 0$.

If the manifold M is nonorientable, then the double cover $\widetilde{M}$ must be orientable, connected and noncompact. Consequently, we have

$$\Omega^n(M) \cong \Omega_+^n(\widetilde{M}) \subset \Omega^n(\widetilde{M}) = 0.$$

Thus, $\Omega^n(M) \cong \mathbb{R}$ when M is compact and orientable, otherwise, $\Omega^n(M) = 0$. $\square$

Corollary 2. $\Omega_c^k(\mathbb{R}^n) = \begin{cases} 0 \text{ when } k < n \\ \mathbb{R} \text{ when } k = n \end{cases}$ gives the cohomology of $\mathbb{R}^n$ with compact supports.

Proof. The case $n = 0$ is trivial. Assume that S^n is the one-point compactification of $\mathbb{R}^n$ for $n > 0$. Let $s \in S^n$ be the compactifying point, thus we can write $\mathbb{R}^n = S^n - \{s\}$.

The differential forms on S^n are zero in a neighbourhood of s and the ideal of differential forms on S^n is denoted by τ_s. It is clear that $\tau_s = G_c(\mathbb{R}^n)$. Consequently, the following sequence is exact

$$0 \to \tau_s \to G(S^n) \to G_s(S^n) \to 0.$$

In cohomology, we can derive a long exact sequence from the above short exact sequence. As $\Omega\big(G_b(S^n)\big) = \Omega(\text{point})$, we can split this long sequence into the following two exact sequences

$$0 \to \Omega_c^0(\mathbb{R}^n) \to \Omega^0(S^n) \to \mathbb{R} \to \Omega_c^1(\mathbb{R}^n) \to \Omega^1(S^n) \to 0$$

and

$$0 \longrightarrow \Omega_c^k(\mathbb{R}^n) \xrightarrow{\cong} \Omega^k(S^n) \longrightarrow 0, \; k \geq 2.$$

As $\Omega^0(S^n) = \mathbb{R}$ and $\Omega_c^0(\mathbb{R}^n) = 0$, thus the following exact sequence can be derived from the first sequence

$$0 \longrightarrow \Omega_c^1(\mathbb{R}^n) \xrightarrow{\cong} \Omega^1(S^n) \longrightarrow 0.$$

Hence $\Omega_c^k(\mathbb{R}^n) = \begin{cases} 0 \text{ when } k < n \\ \mathbb{R} \text{ when } k = n \end{cases}$ gives the cohomology of $\mathbb{R}^n$with compact supports. □

Theorem 3. Let $\Omega(M)$ be the cohomology of an oriented manifold M and $\Omega_c(M)$ be the cohomology of M with compact support. Then the map $i_M : \Omega(M) \to \Omega_c(M)^*$ is a linear isomorphism.

Proof. To prove the theorem, we have to consider the following three cases:

(i) $M = \mathbb{R}^n$

(ii) M is an open subset of $\mathbb{R}^n$

(iii) M is arbitrary

(i) We have to show that the map $i: \Omega^0(\mathbb{R}^n) \to \Omega_c^k(\mathbb{R}^n)^*$is a linear isomorphism to prove $M = \mathbb{R}^n$ since $\Omega^k(\mathbb{R}^n)$ and $\Omega_c^k(\mathbb{R}^n)$ are given by

$$\Omega^k(\mathbb{R}^n) = \begin{cases} \mathbb{R}, & k = 0 \\ 0, & k \neq 0 \end{cases} \quad \text{and} \quad \Omega_c^k(\mathbb{R}^n) = \begin{cases} \mathbb{R}, & k = n \\ 0, & k \neq n \end{cases}$$

Also, in this case it is sufficient to show that $i \neq 0$ as we have

$$\dim \Omega^0(\mathbb{R}^n) \to \dim \Omega_c^k(\mathbb{R}^n)^*.$$

Assume that $\varphi \in S(\mathbb{R}^n)$ is a nonnegative function and φ is not identically zero. Consider a positive determinant function Δ in $\mathbb{R}^n$.
Now, $\int_{\mathbb{R}^n} \varphi \cdot \Delta = \int_{\mathbb{R}^n} \varphi(x)\, dx^1 \cdots dx^n > 0$ for a suitable basis of $\mathbb{R}^n$.

Consequently, if μ is a non-zero element in $\Omega_c^p(\mathbb{R}^n)$, μ is represented by $f \cdot \Delta$.

From the definitions we have$\langle i(1), \mu \rangle = \int_{\mathbb{R}^n} 1 \wedge (\varphi \cdot \Delta) == \int_{\mathbb{R}^n} \varphi \cdot \Delta \neq 0$.

Therefore, $\langle i(1), \mu \rangle \neq 0$ implies that $i(1) \neq 0$ and so $i \neq 0$.

(ii) Assume that $\{b_1, \cdots, b_n\}$ is a basis of $\mathbb{R}^n$. Then, for $v \in \mathbb{R}^n$, we have $v = \sum_{k=1}^n v^k b_k$.

Then an i-basis for the topology of $\mathbb{R}^n$ can be represented by the open subsets of the form

$$B = \{x \in \mathbb{R}^n : a^k < x^k < b^k, \ k = 1, \cdots, n\}.$$

By the definition of diffeomorphism, B is diffeomorphic to $\mathbb{R}^n$. Therefore, with the help of Case (i) and the result of Lemma 3 we conclude that i_B is an isomorphism for each such B. As a result, for every open subset M of $\mathbb{R}^n$ we have $i_M : \Omega(M) \to \Omega_c(M)^*$ which is an isomorphism.

(iii) Let us assume that every open subset of M is diffeomorphic to open subset of $\mathbb{R}^n$ and $\mathcal{B}$ is the collection of such open subsets of M. Consequently, it is obvious that for the topology of M this collection of open subsets $\mathcal{B}$ is an i-basis. With the help of the results derived in Case (ii) and Lemma 3, we can conclude that i_B is an isomorphism for every $B \in \mathcal{B}$. Therefore, for

□

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every open subset $X \subset M$ we can find an i_X which is an isomorphism. Thus, the map $i_M : \Omega(M) \xrightarrow{\cong} \Omega_c(M)^*$ is a linear isomorphism.

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