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FRACTIONAL FACTORIAL DESIGN AND ORTHOGONAL ARRAYS

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Abstract

This study aims to conduct statistical analysis of various types of FRACTIONAL FACTORIAL DESIGN (orthogonal arrays), comparisons between various types of orthogonal arrays with and without replication for the determination of the precision with which factor effects and interactions are estimated

Keywords: FRACTIONAL FACTORIAL DESIGN, orthogonal arrays

1. Introduction

1.1 Basic Of Experimental Design

Various types of experiments are conducted in almost all fields (medical, agricultural, educational, etc). Most of these experiments are carried out either to verify existing theories or to explore new ones. The former are called confirmatory whereas the latter are called exploratory. Precision plays a very important role in confirmatory experiments but it plays a moderate or even a minor role in exploratory experiments.

One of the main objectives of experimentation is to determine and describe the effect(s) of a single or several factors on a particular characteristic (variable) of interest representing the response of the experimental units to the treatment(s) of the experiment. Another objective is to make comparisons among the effects of two or more factors (studied) in the experiment.

Symmetric Sn factorial experiment is a multi-factor experiment involving n factors, each having S levels. This type of experiment creats a total of Sn = SxSx … xS experimental conditions treatments. A more general type of factorial experiments (containing the Sn factorial experiments) is when each of the n factors ($n \geq 1$) is investigated at different number of levels. Such experiments are called asymmetric $S_1 \times S_2 \times \ldots\ldots \times S_n$ factorial experiments, where Si represents the number of levels of the ith factor (i = 1, 2, …., k). They are also called asymmetric $S_1^{n_1} \times S_2^{n_2} \times \ldots . \times S_k^{n_k} = \overset{k}{\underset{i=1}{\pi}} S_i^{ni}$ factorial experiments (n = n1+n2+ ….nk), where each $S_i^{n_i}$ factorial subexperiment is represented n_i times, (i = 1,2,…k).

In experimental design terminology, an experimental condition representing a level of a single-factor is called a treatment whereas an experimental condition representing a combination of levels of a multi-factor experiment is called a treatment combination. Each experiment whether single-factor or multi-factor should be carried out according to a particular design in order to maximize the amount of information about the effect(s) of the factor(s) and their interactions (under the given experimental constraints).

Therefore, some designs are more appropriate for particular type experiments than other designs. One of the basic requirement in experimental design problems is the employment of homogeneous experimental units.

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That is, experimental units for a particular experiment (factorial or single-factor) should be as homogeneous as possible prior to the conduct of the experiment. The number of homogeneous experimental units assigned to each treatment (experimental condition) is called the number of replications of that treatment. This number must be determined before hand, since it has an important impact on the precision of inferences associated with that specific treatment. The larger the replication number is, the more precise inferences about the factors (associated with that treatment) will be. This largeness of replication entails, however, more cost and more experimental effort. Therefore, a compromise attitude is often taken between cost and accuracy.

To ensure unbiasedness and to avoid systematic biases, experimental units should be assigned randomly to the various treatments. That is, random assignment gives equal chances for all units to be treated by any treatment (in the experiment).

Also the order by which units got treated should as well, be done by a random mechanism.

The application of a treatment (treatment combination) to a particular experimental unit is often called an experimental run or just a run (of the experiment).

Once all homogeneous experimental units have received the treatments of an experiment, these units will undergo some changes. These changes form the basis for various comparisons about the treatments and their effects. These comparisons are, in fact, the main part of the statistical analysis in any experimental investigation.

This analysis in one of the two major tasks in any experimental research: the designing task and the statistical analysis task.

It is worth noting that, a design problem arises when there are not enough homogeneous experimental units to carry out all experimental conditions (treatments) of a particular experiment. This problem arises mainly in factorial type experiments, since such experiments often involve a large number of experimental conditions (treatments). In fact, this number of treatments becomes even larger when the number of levels of each factor gets larger and larger. This design problem is resolved by blocking the factorial experiment where blocks of homogeneous experimental units are used, and variation among these blocks is considered as an additional explanatory factor (i.e source of variation) besides the effect of the factors and their interactions.

A second design problem arises when cost of factorial experimentation is extremely important and budgetary constraints don't allow conducting large size (i.e costly) factorial experiments. In these cases, cost of factorial experimentation is reduced by assuming that some factorial effects (mainly high-order interactions) are negligible and have a priori zero effect on the experimental response. Negligibility of higher order interactions parallels that of a Taylor series expansion for a multi-variable function where only terms involving products (i.e. interactions) of at most two or three variables are retained in the expansion while higher order products (i.e. interactions) are assumed negligible (i.e. Zero).

The assumption of negligibility of high order interactions entails that a fraction of the full factorial experiment is to be carried out for the analysis and estimation of the subset of non-negligible factorial effects and their interactions. The fraction size must be at least the size of non-negligible factorial effects. These fractions are often called fractional designs. It is worth mentioning that running a fractional factorial design instead of the complete factorial design for the analysis of the full factorial structure (without the negligibility of any interaction effect) leads to a design problem called aliasing where factorial effects get mixed with each other and it becomes difficult to tell whether the observed experimental differences are due to which factor effect.

The selection of a given fractional factorial design for a particular fractionated factorial experiment is a combinatorial problem where different fractions lead to different patterns of aliasing. It is a general strategy in selecting fractional factorial designs to get lower-order factorial effects aliased with higher order factorial effects. So, assuming that higher-order interaction effects are negligible (i.e. zero effect) and can be eliminated from further investigation leaves the factorial effects aliased with them free and not aliased.

Hence experimental data from fractional factorial designs can be used to get estimates (and conduct tests of significance) for these non-negligible effects.

There are two main types of fractional factorial designs just as there are two types of factorial designs (the asymmetric and the symmetric).

The first type is called symmetric fractional factorial designs and the second type is called the asymmetric fractional factorial designs.

Symmetric fractional factorial designs are subsets of the full Sn factorial design whereas asymmetric fractional factorial designs are subsets of the full asymmetric $\pi_{i=1}^{k} S_i^{n_i}$ factorial design.

Furthermore, symmetric fractional factorial designs are subdivided into two parts: the regular fractional factorial designs and the irregular fractional factorial designs.

Regular fractional factorial designs are often denoted by S^{n-p} where a fraction of $\frac{1}{S^p}$ of the full Sn factorial design is considered ($1 \le p \le n$). The construction of some regular S^{n-p} fractional factorial designs is mainly based on solving simultaneously properly chosen independent linear modular equations. In fact, every regular S^{n-p} fractional factorial designs is also on orthogonal array. (due to RakToe, Hedayat and Federer (1981)). Fractional factorial designs that are not $\frac{1}{S^p}$ fractions of the Sn factorial designs are called irregular fractional factorial designs.

Some irregular fractional factorial designs are orthogonal arrays. (Hedayat, Sloane and Stufken, 1999).

Since fractionating a complete Sn factorial design leads to different aliasing among factorial effects (main effects and interaction) and since major interest in fractional factorial designs is in main factors effects and two-factor interactions, then fractional factorial designs are classified by the resolution concept into three subclasses.

1.2 Resolution III, IV, V Regular Fractional Factorial Design.

Regular S^{n-p} fractional factorial designs (fractions) are classified into three main categories according to the aliasing of main effects. Two-factor interactions:

Resolution III regular fractional designs:

These are designs where no main effect is aliased with any other main effect, but main effects are aliased with two-factor interactions and two-factor interactions may be aliased with each other.

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Resolution IV regular fractional designs.:

These are designs where no main effect is aliased with any other main effect or with any two-factor interaction, but two-factor interactions are aliased with other.

Resolution V Regular fractional designs:

These are designs where no main effect or two-factor interaction is aliased with any other main effect or two-factor interaction, but main effects and two-factor interactions are aliased with three-factor and higher-order interactions.

For illustration of these three resolution types of (regular) fractional factorial designs, we consider the following:

a) Resolution III fractional factorial design; Table (2.1) below represents a full 23 factorial design involving three 2-level factors A, B, C where the first column represents the eight treatment combinations (i.e experimental conditions) upon which this full factorial design is based.

These eight treatment combinations are written in two notations. Notation (1) is well-known for 2-level factorial designs. In this notation, the eight treatment combinations form an abelian group under multiplication modulo 2.

Notation (2) for the 8 treatment combinations is the additive representation of a groups of order 8. The other eight columns (under the heading factorial effects) represent all eight factorial effects: the three main effects A, B and C, the three two-factor interactions AB, AC and BC, and the last column containing the three-factor interaction ABC.

The 8 treatment combinations in table (2.1) are also an orthogonal array OA (8,3,2,3).

Table (2.1): Plus and Minus signs for 23 factorial design:

Treatment Combinations of 23 Design		Factorial Effects			
Notation (1)	Notation (2)	B	C	C	BC
a	100				
b	010				
c	001				
abc	111				
ab	110				
ac	101				
bc	011				
(1)	000				

Table (2.1) represents also the mean response vector EY (a,b,c,ab,ac,bc,abc,(1)) in the first column linearly in terms of all factorial effects (μ ,A,B,C,AB,AC,BC and ABC) according to the linear model E $\underset{\sim}{Y}$ =X $\underset{\sim}{B}$ …………..(2.1)

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Where the 8 x 8 matrix X is the 8 columns of pluses and minuses in table (2.1), where $\underset{\sim}{\beta}$ is 8 x 1 column of all 8 factorial effects μ ,A, B, C, AB, AC, BC, ABC.

As a standard experimental design notation a plus in table (2.1) represents a plus one and a minus represents a minus one.

From fractional factorial point of view and under the assumption that the all interaction effects (i.e AB, AC, BC, ABC) are negligible and have zero effect on the experimental response, only four runs out of the 8 runs in the first columns of table (2.1) are (only) needed for the estimation of the three main effects A, B, C. There will be a total $\binom{8}{4}$ = 70 fractions possible. One of these 70 fractions, the one selected according to the defining contrast I = ABC. This fraction consists of the first four runs of table (2.1), namely runs a, b, c and abc. That is runs a = 100, b = 010, c = 001 and abc =111 are the solutions (modulo 2) of the single linear modular equation:

$$x_1 + x_2 + x_3 \equiv 0 \quad \text{(mod2).}$$

The four runs in this fraction form a subgroup of the full group of 8 runs.

They are also an OA (4, 3, 2, 2). The Alias structure for this four-run 2^{3-1} fractional factorial design is:

I = ABC

A = BC ……………………….(2..2)

B = AC

C = AB

That is, the estimable functions among the 8 factorial effects (I, A, B, C, AB, AC, BC, ABC) when the half fraction (I = ABC) is used are:

μ + ABC

A + BC ………………… (2.3)

B + AC

C + AB

The aliasing among the eight factorial effects occurs since the four data responses: Y(a), Y(b), Y(c) and Y(abc) are not enough to estimate the 8 unknown effects A, B, C, AB, AC, BC and ABC as well as the overall mean I.

A glance at the alias structure in (2.3) shows that factorial effects are aliased together in such a manner that this fractional factorial design (with four runs) is of resolution III. Once the effects on the right hand side of (2.2) are dropped and regarded negligible or have zero effect, this leaves the effects on the left hand side free from aliasing and each main effect becomes estimable.

Moving now to another resolution III example but with higher degree of fractionation of the full factorial design. That is, a much higher fractionated factorial design of resolution III results when not half but

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rather one-quarter of a full 25 factorial design is considered.

For instance, a full factorial experiment investigating all the five two-level factors (A, B, C, D, E) and all their interactions requires a total of 25 = 32 treatment combinations, but under the assumption that all three-factor and all higher order interactions and some of the two–factor interactions are negligible, one quarter fraction will be enough for the estimation of all main effects if this fraction is to be of resolution III: These are $\binom{32}{8}$ = possible quarter fractions, one of them is the fraction given by table (2.2) with the defining contrast I= ABD = ACE= BCDE.

That is, the 8 runs in the second column of table (2.2) are solutions of the simultaneous linear system of two modular equations:

$$x_1 + x_2 + x_4 = 0 \pmod 2$$
$$x_1 + x_3 + x_4 = 0 \pmod 2$$

These eight runs form a subgroup of the full group of 32 runs. These 8 runs are also an OA (8,5,2,2). More about orthogonal arrays provided in chapter three. Table (2.2) gives, in a similar way as table (2.1), the linear modeling of the non negligible factorial effects in term of treatment responses.

Table (2.2): Plus- minus signs for 2^{5-2} fractional factorial design:

Run	Treatment combinations	Factorial effects I	A	B	C	D =AB	E =AC
1	de	+	-	-	-	+	+
2	a	+	+	-	-	-	-
3	be	+	-	+	-	-	+
4	abd	+	+	+	-	+	-
5	cd	+	-	-	+	+	-
6	ace	+	+	-	+	-	+
7	bc	+	-	+	+	-	-
8	abcde	+	+	+	+	+	+

The alias structure for this quarter fraction in table (2.2) is as follows:

$$
\begin{aligned}
I &= ABD = ACE = BCDE \\
A &= BD = CE = ABCDE \\
B &= AD = ABCE = CDE \\
C &= ABCD = AE = BDE \\
D &= AB = ACDE = BCE \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2.4) \\
E &= ABDE = AC = BCD \\
BE &= ADE = ABC = CD
\end{aligned}
$$

That is, the estimable functions among the thirty two parameters representing all five main effects and their interaction of order 2, 3, 4 and 5 (μ, A, B, AB, C, AC, BC, ABC, D, AD, BD, ABD, CD, ACD, BCE, ABCD, E, AE, BE, ABE, CE, ACE, BCE, ABCE, DE, ADE, BDE, ABDE, CDE, ACDE, BCDE, ABCDE) when the quarter fraction (I = ABD = ACE= BCDE) is used are:

$$
\begin{aligned}
&\mu + ABD + ACE + BCDE \\
&A + BD + CE + ABCDE \\
&B + AD + ABCE + CDE \\
&C + ABCD + AE + BDE \\
&D + AB + ACDE + BCE \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.5) \\
&E + ABDE + AC + BCD \\
&BE + ADE + ABC + CD
\end{aligned}
$$

A glance at the alias structure in (2.4) reveals that this fraction is of resolution III but this 2^{5-2} fraction involves a higher degree fractionation than the earlier 2_{III}^{3-1} fraction in table (2.1), where here each factorial effect is aliased with two other effects. That is, the higher the degree of fractionation is the higher the degree of aliasing will be.

Once the three effects on the right hand side of each equation in (2.4) are dropped and regarded negligible, this leaves the effects on the left handside free – from aliasing and all (left hand side effects) become estimable.

This 2^{5-2} fraction in table (2.2) gives 2 degrees of freedom for the experimental error once the two-factor interactions BE and CD are regarded negligible. On the other hand the earlier 2_{III}^{3-1} fraction in table (2.1) is saturated and does not allow any error degrees of freedom.

b) Resolution IV (regular) fractional factorial designs: A full factorial experiment investigating all the four factors A, B, C and D, and all their interactions requires a total 2^4 =16 treatment combinations, but under the assumption that three and four-factor interactions and some of the two-factor interactions are negligible, one-half fraction will be enough.

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There are $\binom{16}{8}$ = 1287 possible fractions; one of them is the fraction given by table (2.3) with the defining contrast I = ABCD (ie. Design generator D = ABC).

That is, the 8 runs in the second column of table (2.3) are solutions to the linear modular equation: $x_1 + x_2 + x_3 + x_4 \equiv 0 \, (\text{mod } 2)$.

These eight runs are a subgroup of the full group of 2^4 = 16 runs. They are also an OA (8, 4, 2, 3).

Table (2.3): Plus and minus signs for 2^{4-1} fractional factorial design:

Runs	Treatment combination	Factorial effects I	A	B	C	D=ABC
1	(1)	+	-	-	-	-
2	ad	+	+	-	-	+
3	bd	+	-	+	-	+
4	ab	+	+	+	-	-
5	cd	+	-	-	+	+
6	ac	+	+	-	+	-
7	bc	+	-	+	+	-
8	abcd	+	+	+	+	+

The Alias structure for this half fraction is:

I = ABCD

A = BCD

B = ACD

C = ABD ………………………… (2.6)

D = ABC

AB = CD

AC = BD

AD = BC

A glance at alias structure (2.6) reveals that this 2^{4-1} fraction is of resolution IV. Once the effects on the right hand side of (2.6) are regarded negligible, this leaves the effects on the left hand side free from aliasing and all become estimable.

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c) Resolution V (regular) fractional factorial design:

A full factorial experiment investigating all the five factors A, B, C, D, E and all their interactions requires a total of 2^5 = 32 treatment combinations but under the assumption that three-factor and higher order interactions are negligible, one half fraction will be enough. There are $\binom{32}{16}$ = 19389690 possible fractions, one of them is the fraction given by second column of table (2.4) with the defining contrast I = ABCDE (i.e. design generator E = ABCD). That is, the 16 runs in second column of table (2.4) are the solution of the linear modular equation

$$x_1 + x_2 + x_3 + x_4 + x_5 \equiv 0 \pmod 2$$

These 16 runs are a subgroup of the 2^5 = 32 runs in the complete 2^5 factorial design. They are also an OA (16, 5, 2, 4).

Table (2.4): Plus – minus signs for 2^{5-1} fractional factorial design:

Runs	Treatment combination	Factorial effect I	A	B	C	D	E= ABCD
1	e	+	-	-	-	-	+
2	a	+	+	-	-	-	-
3	b	+	-	+	-	-	-
4	abe	+	+	+	-	-	+
5	c	+	-	-	+	-	-
6	ace	+	+	-	+	-	+
7	bce	+	-	+	+	-	+
8	abc	+	+	+	+	-	-
9	d	+	-	-	-	+	-
10	ade	+	+	-	-	+	+
11	bde	+	-	+	-	+	+
12	abd	+	+	+	-	+	-
13	cde	+	-	-	+	+	+

14	acd	+	+	-	+	+	-
15	bcd	+	-	+	+	+	-
16	bcde	+	+	+	+	+	+

The Alias structure for this half fraction is:

I = ABCDE

A = BCDE

B = ACDE

C = ABDE

D = ABCE

E = ABCD ……………………….. (2.7)

AB= CDE

AC = BDE

AD = BCE

AE = BCD

BC = ADE

BD = ACE

BE = ACD

CD = ABE

CE = ABD

DE = ABC

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That is, the estimable functions among the thirty- two factorial effects (μ, A, B, AB, C, AC, BC, ABC, D, AD, BD, ABD, CD, ACD, BCD, ABCD, E, AE, BE, ABE, CE, ACE, BCE, ABCE, DE, ADE, BDE, ABDE, CDE, ACDE, BCDE, ABCDE) are the following sixteen linear parametric functions:

μ + ABCDE
A + BCDE
B + ACDE
C + ABDE
D + ABCE
E + ABCD
AB + CDE
AC + BDE
AD + BCE ……………………….. (2.8)
AE + BCD
BC+ ADE
BD + ACE
BE + ACD
CD + ABE
CE + ABD
DE +ABC

A glance at alias structure (2.7) reveals that this 2^{5-1} fraction is of resolution V. Once the effects on the right hand side of (2.7) are dropped and regarded negligible, this leaves the effects on the left hand side free from aliasing and all become estimable.

2.0: Definition of orthogonal arrays:

Orthogonal arrays are fractional factorial designs for the orthogonal investigation of the effect of several factors on an experimental response under assumption that high order interactions are negligible.

Two factors are regarded orthogonal to each other in a factorial design if each level of the first factor occur the same number of times with every level of the second factor. Hence, orthogonal arrays are fractional factorial designs.

Regular Sn-P fractional factorial designs do the same job as that of the orthogonal arrays but the latter are often more economic as they require smaller number of experimental runs, especially for large number of factors. Hedyat, Sloane and Stufken (1999).

The mathematical definition for orthogonal arrays is of combinatorial nature and is stated as follows:

2.1Definition: (symmetrical orthogonal arrays)

An N x k array A with entries from set S = {0, 1, ….., s-1} is said to be an orthogonal array of strength t (for some t: $0 \le t \le k$) and (integer) index λ, if every N x t subarray of array A contains each t-tuples exactly

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λ times as a row, where set S is structured as Galois field. That is, an orthogonal array contains $\binom{k}{t}$ complete St factorial subdesigns for $t \le k$.

It is worth noting that, the N rows of the orthogonal arrays are a subset (i.e. a fraction) of the set of all Sk treatment combinations in the full Sk factorial experiment.

If N = Sn-P then regular Sn-P fractional design (of chapter two) are a subclass of orthogonal arrays. If further index λ of the orthogonal array OA(N,k,s,t) is a power of s, then the orthogonal array is called a hypercube of strength t.

The strength (t) of the orthogonal array is related to the highest degree of non-negligible interaction that need to be investigated and estimated.

Orthogonal arrays of strength two are fractional factorial designs of resolution III. Orthogonal arrays of strength three are fractional factorial designs of resolution IV. and orthogonal arrays of strength four are resolution V fractional factorial design

Orthogonal arrays are often denoted by OA(N,k,s,t). So, orthogonal arrays OA(N,k,s,t) of strength t are fractional factorial designs of resolution (t + 1). It is worth noting that not all resolution R fractional factorial designs are orthogonal arrays (Raktoe, Hedayat and Federer (1981)).

For an example on orthogonal arrays is:

0 0 0 0	
0 0 1 1	
0 1 0 1	
0 1 1 0	…………….. (3.2)
1 0 0 1	
1 0 1 0	
1 1 0 0	
1 1 1 1	

Which is denoted by OA(8,4,2,3).

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This is an orthogonal array based on four two-level factors with strength three, of index unity (λ =1). This array can also be regarded as regular 24-1 fractional factorial designs with defining contrast I = ABCD. For an example of irregular fractional factorial designs is the irregular fractional 24-1 factorial design:

0 0 0 0
0 0 1 1
0 1 0 1
0 1 1 0 ………….. (3.3)
1 0 0 1
1 0 1 0
1 1 0 0
1 1 0 1

Non-regularity of the fraction (3.3) is due to the fact that it is not a subgroup of the complete 24 factorial design and it has no defining contrast.

The full 24 factorial experiment requires all 16 possible treatment combinations of which the arrays in (3.2) and (3.3) are subsets (i.e. fraction). It is worth noting that symmetric orthogonal arrays OA(N,k,s,t) don't exists for any value of the four parameters N,k,s,t in definition (3.1). This is due to the fact that the parameters of the orthogonal array should satisfied the constraint $N = \lambda s^t$. The following inequalities for orthogonal arrays must hold if symmetric orthogonal arrays should exist, (due to RakToe, Hedayat and Federer (1981)):

For $u \geq 0$:

1) $N \geq \sum_{i=0}^{u} \binom{k}{i} (s-1)^i$, if t =2u.

2) $N \geq \sum_{i=0}^{u} \binom{k}{i} (s-1)^i + \binom{k-1}{u} (s-1)^{u+1}$, t = 2u+1 ………….. (3.4a)

with reference to (3.4), the orthogonal array in (3.2) has u = 1 and t = 2(1) + 1 = 3; hence

$$N \geq \binom{4}{0}(2-1)^0 + \binom{4}{1}(2-1)^1 + \binom{3}{1}(2-1)^{1+1}$$

$$N \geq 1+4+3=8$$

A subclass of orthogonal arrays called complete orthogonal arrays are those orthogonal arrays attaining the bound in (3.4a). The orthogonal array in (3.2) is complete.

For OA(N,k,s,t) with index $\lambda = 1$, the bounds in (3.4a) reduce to:

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a) $k \leq s+t-1$ if s is even ……………. (3.4b)

b) $k \leq s+t-2$ if s is odd, $t \geq 3$

Two related problems for the existence of orthogonal arrays are the following two questions:

For given values of N,s,t, what is the largest possible number of factors k that can be studied in an orthogonal array OA(N,k,s,t). This number is denoted by the function f (N,s,t).

For given values of k,s,t, what is the minimum number of runs N in an orthogonal array OA(N,k,s,t). This number is denoted by the function F(k,s,t).

These two numbers (i.e. functions) in (a) and (b) are related as follows:

F(k,s,t) = min {N: f(N,s,t) $\geq k$ }

F(N,s,t) $\leq$ max {k: F(k,s,t) $\leq N$ } ……………………… (3.5)

That is, the values of f(N,s,t) completely determine those of F(k,s,t) but values of F(k,s,t) provide only an upper bound for the values of f(N,s,t). However, the determination of f(N,s,t) is more difficult than the determination of F(k,s,t). Explicit bounds for f(N,s,t) exist in the literature for special cases of parameter values. For instance,

In an OA(λs^2 , k,s,2), the maximum number of factors k(k=f(N,s,t) is such that $k \leq \frac{\lambda s^2-1}{s-1}$. (Hedayat, Sloane and Stufken (1991)).

For example, take the following OA(9,4,3,2):

0 0 0 0

0 1 1 1

0 2 2 2

1 0 2 1

1 1 0 2 ……………... (3.6)

1 2 1 0

2 0 1 2

2 1 2 0

2 2 0 1

Here, k = f(8,3,2) $\leq \frac{9-1}{3-1} = 4$.

In an OA(λs^3,k,s,3), the maximum number of factors k is such that k = f(N,s,t) $\leq \frac{\lambda s^3-1}{s-1}$ + 1, (due to Hedayat, Sloane and Stufken (1999)).

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For example, take the following OA(8,4,2,3):

0 0 0 0
0 0 1 1
0 1 0 1
0 1 1 0 …………… (3.7)
1 0 0 1
1 0 1 0
1 1 0 0
1 1 1 1

Where k = f(8,2,3) $\leq \frac{8-1}{2-1}+1=7+1=8$

In OA(st, k,s,t):

$k \leq t+1$ if $s \leq t$

$k \leq s+t-2$ if $s > t \geq 3$ and s is odd.

$k \leq s+t-1$ in all other cases.

(Due to Hedayat, Sloane and Stufken (1991)).

For example, take the OA(8,4,2,3) in (3.7) where s =2 $\leq$ 3 = t, so k = f(8,2,3) $\leq$ 4.

Definition (3.1) of orthogonal array is restrictive since all k factors are assumed to have the same number of levels namely s.

The following definition generalizes definition (3.1) to allow for factors to have different number of levels.

2.2Definition (3.8): (asymmetrical orthogonal arrays)

A mixed orthogonal array OA(N, $s_1^{k_1} s_2^{k_2} \ldots s_v^{k_v}, t$) is an array of size N x k, where k = k1+k2+…+kv is the total number of factors, in which the first k1 columns have symbols from set {0,1, …., s_1-1}, the next k2 columns have symbols from set {0,1, ….., s_2 -1}, and so on, with the property that in any N x t subarray, every possible t-tuple occurs an equal number of times as a row. Sets {0,1, …., (s_1-1)}, ….. , {0,1, …, (s_v -1)} are often Galois fields where s_1, s_2, …., s_v are primes or prime powers.

It is worth noting that regular $s_1^{k_1-P_1} \times s_2^{k_2-P_2} \times \ldots \times s_v^{k_v-P_v}$ fractional factorial design are a subclass

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of the asymmetric orthogonal arrays.

Unlike symmetric orthogonal arrays where the index λ was fixed single value, the index of asymmetric orthogonal arrays depends on which t factors are chosen. Illustration follows next: we consider the OA(12, $2^4 \times 3^1, 2$) in a transposed form (for economy of space):

001100110011	
010101010101	
001111001001	……………. (3.9)
010110011010	
000011112222	

For the array in (3.9) where strength t = 2, the possible pairs in the last two factors (i.e last two rows of (3.9) are 00, 01, 02, 10, 11,12 and each pair occurs twice, whereas the possible pairs in the first two factors (i.e. first two rows of (3.9) are 00, 01, 10, 11 and each pair occurs three times.

Therefore, the number of runs N in mixed orthogonal arrays must be a multiple of every number $s_1^{i_1} s_2^{i_2} \dots s_v^{i_v}$, where $0 \le i_1 \le k_1$, …., $0 \le i_v \le k_v$ and $i_1 + i_2 + \dots + i_v \le t$ in order that the strength of the array be t. (RakToe, Hedayat and Federer (1981)).

The full asymmetric $2^4 \times 3^1$ factorial experiment requires a total of 48 treatment combinations of which the asymmetric orthogonal array in (3.9) is a subset (i.e a quarter fraction). In fact, the orthogonal array in (3.9) represents a quarter fraction of 12 runs out of the complete $2^4 \times 3^1$ factorial design.

In a parallel way to the bounds in (3.4a), the parameters of the asymmetric array OA(N, $s_1^{k_1} s_2^{k_2} \dots s_v^{k_v}, t$) in definition (3.8) for $s_1 \le s_2 \le \dots \le s_v$ and for $u \ge 0$ satisfy (due to Hedayat, Sloane, Stufken 1999).

$$N \ge \sum_{m=0}^{u} \sum_{\substack{I(v)\\ m}} \binom{k_1}{i_1}\binom{k_2}{i_2}\dots\binom{k_v}{i_v}(s_1-1)^{i_1}(s_2-1)^{i_2}\dots(s_v-1)^{i_v}$$

(1) if t = 2u

(2)

$$N \ge \sum_{m=0}^{u} \sum_{\substack{I(v)\\ m}} \binom{k_1}{i_1}\binom{k_2}{i_2}\dots\binom{k_v}{i_v}(s_1-1)^{i_1}(s_2-1)^{i_2}\dots(s_v-1)^{i_v}$$

……..(3.10)

$$+ \sum_{\substack{I(v)\\ u}} \binom{k_1}{i_1}\binom{k_2}{i_2}\dots\binom{k_{v-1}}{i_{v-1}}\binom{k_v-1}{i_v}(s_1-1)^{i_1}\dots(s_{v-1}-1)^{i_{v-1}}(s_v-1)^{i_v+1}$$

if t = 2u+1

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Where the set $\underset{m}{I}(v)$ ($m \geq 0$ and $v \geq 1$ are integers) is defined as follows:

$$\underset{m}{I}(v) = \{(i_1 i_2 \ldots i_v) : i_1 \geq 0, \ldots, i_v \geq 0, \sum_{L=1}^{v} i_L = m\}.$$

2.3 Properties of orthogonal arrays:

Orthogonal arrays are studied by Raktoe, Hedayat and Federer (1981) and by Hedayat, Sloane and Stufken (1999) as well as by others yet they are continued to be researched.

Symmetric orthogonal arrays have many properties; some of them are:

(1) The parameters of a symmetrical orthogonal array (i.e N,k,s,t, λ) satisfy the equality: N= λs^t.

For illustration, let us take the orthogonal arrays in example (3.2) in which N= 8, S=2 and t = 3, so $\lambda = 1$, hence every 3-tuple occurs once (i.e $\lambda = 1$) as a row and N $= \lambda s^t = (1)(2)^3$.

(2) Any orthogonal array of strength t is also an orthogonal array of strength t', $0 \leq t' < t$ and the index of the array becomes $\lambda s^{t-t'}$, where λ denotes the index of the array. In example (3.2) where t = 3, if we regard this orthogonal array as having strength $t' = 2$, then the index of this strength 2, orthogonal array becomes = $\lambda s^{t-t'} = 1(2^{3-2}) = 2$ where every 2-tuple occurs twice.

(3) If A_i, i = 1, …., r is an OA(N_i,k,s,t_i), then the array A obtained from juxtaposition of these r arrays, A = $\begin{bmatrix} A_1 \\ \vdots \\ A_r \end{bmatrix}$ is an orthogonal arrays OA(N,k,s,t) where N = $N_1 + N_2 + + \ldots + N_r$ and the strength is t for some t $\geq \min\{t_1, \ldots, t_r\}$. For illustration: if we have the two orthogonal arrays:

OA(4,3,2,2): 0 0 0
0 1 1
1 0 1
1 1 0

OA(4,3,2,2): 1 0 0
0 1 0
0 0 1
1 1 1

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Then by juxtapositoining these two arrays, we get the OA(8,3,2,2): 0 0 0

0 1 1
1 0 1
1 1 0
1 0 0
0 1 0
0 0 1
1 1 1

This last array is, in fact, the one in (3.2) and it is the complete 23 factorial design.

(4) A permutation of the runs or factors in an orthogonal array results in an orthogonal array with the same parameter N,k,s,t, λ.

(5) A permutation of the levels of any factor in an orthogonal array results in an orthogonal array with the same parameters: N,k,s,t, λ .

(6) Any N x k' subarray of an OA(N,k,s,t) is an OA(N, k',s, t') where t' = min {k',t}. For illustration, if we have an OA(4,3,2,2):

0 0 0
0 1 1
1 0 1
1 1 0

And by just considering the first two factors (rather than the three), we get the following

OA(4,2,2,2):	0 0
	0 1
	1 0
	1 1

Where t' = 2

(7) Taking the runs in an OA(N,k,s,t) that begin with 0 (or any other symbol from (0, 1, ….(s-1)) and omitting the first column of zeros yields an OA(N/s, k-1, s, t-1).

For illustration: taking the OA(8,4,2,3) in example (3.2) and the subarray corresponding to zeros in the first columns of OA(8,4,2,3), i.e. 0 0 0

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0 1 1
1 0 1
1 1 0

Then these four runs are, in fact, the OA(8, 4-1, 2, 3-1) = OA(4, 3, 2, 2).

(8) If A= $\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$ is an OA(N,k,s,t), where A_1 is an OA(N_1, k,s, t_1), then A_2 is an OA(N- N_1, k,s, t_2) with $t_2 \geq \min\{t, t_1\}$.

For illustration, taking the OA(8,3,2,2) in the preceding third property and letting the first four runs be OA(4,3,2,2), then the last four runs are the OA(8-4, 3, 2,2). That is, complements of regular Sn-P fractional designs (i.e Sn-Sn-P) are also orthogonal arrays.

(9) An orthogonal array OA(N,k,s,t) is simple if all its N k-dimentoinal runs are distinct.

(10) An orthogonal array OA(N,k,s,t) is linear if it is simple and its N k-dimentional runs are a vector space over GF(s). That is, if Ri and Rj are two rows of the array, then $C_1R_i + C_2R_j$ is a row in the array for $C_1, C_2 \in GF(s)$.

Linear orthogonal arrays should have N be integral powers of s.

(11) Orthogonal arrays OA(N,k,s,t) with entries from GF(s) have the property that any t columns of A are linearly independent over GF(s).

(12) Let A be an N x k matrix whose rows are k-dimentional vectors from GF(s) x GF(s)x ….xGF(s). (k-times).

If any t columns of A are linearly independent over GF(s), then A is an orthogonal array OA(N,k,s,t). Thus any N x k matrix over GF(s) array to be an orthogonal array should have its rows linearly independent. So not every N x k array is an orthogonal array, for an example:

Take the 9 x 4 array A:

0 0 0 0
0 0 0 0
0 1 0 0
0 0 1 0
0 0 0 1
0 1 1 1
1 1 0 0
1 0 1 0
1 0 0 1

This is not an orthogonal array.

(13) Non existence of OA(λs^t, k',s,t) implies non-existence of OA(λs^t,k,s,t) for $k' > k$. All four factors in this 9 x 4 array are now at 2 levels, but since N is odd and not powers of two, this array cannot be an orthogonal array.

Having defined orthogonal arrays (symmetric and asymmetric) and having studied properties of symmetric orthogonal arrays, we in the following section move to the some methods that generate different orthogonal arrays.

The statistical analysis of orthogonal arrays will be discussed in chapter IV.

3.0 Construction Methods for symmetrical orthogonal arrays.

There are various construction methods for generating orthogonal arrays: symmetrical orthogonal arrays that are regular Sn-P fractional factorial designs are constructed by solving properly chosen system of independent linear modular equations embodied in their defining contrasts.

Since not all orthogonal arrays are regular Sn-P fractional factorial designs, some other construction methods will be described and studied. We will discuss only four construction method. A separate subsection will be given for each method and it will be illustrated by examples. All these construction methods are studied by Hedayat, Sloane and Stufken (1999) and by Raktoe, Hedayat and Federer (1981).

3. 1. (a) Constructing orthogonal arrays using difference schemes:

Difference schemes are defined as:

Definition (3.11):

An r x c array D with entries from set A = {0, 1, …., (s-1)} is called a difference scheme (c$\leq$ r) based on a group (A, +) if it has the property that for any two columns i and j of array D with $1 \leq i, j \leq c, i \neq j$, the vector difference between the ith and jth columns contains every elements of set A equally often.

Set A is often taken to be a Galois field on {0, 1, 2, …., (s-1)} where s is prime or prime power. The difference scheme in definition (3.11) is denoted by D(r,c,s) and r = λs where λ is the number of times each element of set A = {0, 1, 2, …., (s-1)} occurs in the difference of any two columns of D.

For an illustration: The difference scheme D (9, 9, 3) based on (GF(3), +) is:

000 000 000	
012 012 012	
021 021 021	
000 222 111	………………………. (3.12)
012 201 120	
021 210 102	
000 111 222	
012 120 201	
021 102 201	

This difference scheme in (3.12) has r = 9 = 3 x 3 (s = 3, λ = 3). Like orthogonal arrays, it should be noted that difference schemes don't always exist for any values of c and r; they exist for certain value of r and c. In fact, the difference schemes in (3.12) satisfies the conditions of the following theorem which guarantees the existence of difference schemes in certain special cases.

This theorem is due to Hedayat, Sloane and Stufken (1999).

Theorem (3.13):

A difference scheme D(P^m, P^m, P^n) exists for any prime P and integers $m \geq n \geq 1$.

Over the set A = {0, 1, … ($P^n - 1$)} representing the GF(P^n). The proof is constructive and produces an algorithm that generates the difference schemes D(P^m, P^m, P^n).

We start with this proof as follows:

Proof: Let the elements of Galois field GF(P^m) be represented by polynomials:

$$\beta_0 + \beta_1 x + ... + \beta_{n-1} x^{n-1} + ... + \beta_{m-1} x^{m-1}$$

Where coefficients $\beta_0, \beta_1, ..., \beta_{m-1} \in GF(P)$. (More about Galois fields is in Appendix A). Since $m \geq n \geq 1$, GF(P^n) is an additive subgroup of GF(P^m), (Herstein, (1975)); we identify elements of GF(P^n) with the subset of GF(P^m) consisting of all polynomials of the form:

$$\beta_0 + \beta_1 x + ... + \beta_{n-1} x^{n-1}$$

This identification is described next. Let D* be the P^m x P^m multiplication table of GF(P^m).

(entries in this table are polynomials of degree at most (m-1) from GF(P^m)). Then, we map every entry $\beta_0 + \beta_1 x + ... + \beta_{m-1} x^{m-1}$ in this 2-dimentional table to $\beta_0 + \beta_1 x + ... + \beta_{n-1} x^{n-1}$ (i.e $\phi : \beta_0 + \beta_1 x + ... + \beta_{m-1} x^{m-1} \rightarrow \beta_0 + \beta_1 x + ... + \beta_{n-1} x^{n-1}$). Hence, we get the desired difference scheme D(P^m, P^m, P^n) in the theorem.

Array D is a $P^m x P^m$ array with entries now from GF(P^n) (not from GF(P^m)).

The difference of two columns of the difference scheme D(P^m, P^m, P^n) will have the form

$$\begin{pmatrix} \phi(\beta\alpha_0) \\ \vdots \\ \phi(\beta\alpha_{P^{m-1}}) \end{pmatrix} - \begin{pmatrix} \phi(\gamma\alpha_0) \\ \vdots \\ \phi(\gamma\alpha_{P^{m-1}}) \end{pmatrix}$$

Where $\beta, \gamma \in GF(P^m), \beta \neq \gamma$.

From the definition of the mapping ϕ, it follows that $\phi(\beta\alpha_i) - \phi(\gamma\alpha_i) = \phi(\beta\alpha_i - \gamma\alpha_i)$ and so the above vector difference is equal to

$$\begin{pmatrix} \phi(\beta-\gamma)\alpha_0 \\ \vdots \\ \phi(\beta-\gamma)\alpha_{P^{m-1}} \end{pmatrix}.$$

Since every element of GF(P^m) appears once in every row (column) of the P^m x P^m multiplication table in the elements $(\beta-\gamma)\alpha_i ; 0 \leq i < P^m$ of the vector difference, then every element of GF(P^n) appears P^{m-n} times among the elements of the vector difference $\phi((\beta-\gamma)\alpha_i), 0 \leq i < P^m$. Hence, this completes the proof of theorem (3.13).

For illustration of the construction of difference schemes according to theorem (3.13), we consider the following example: let P =3, m = 2, n = 1. The primitive polynomial f(x) for GF(32) is f(x) = x2 + x + 2 (i.e x2 = 2x + 1 (mod3)).

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Table (3.1): The 9 x 9 multiplicative table for GF(32) is:

	*			2	x +1	x +2	x x	2 x+1	2 x+2	2
				0	0	0	0	0	0	0
				2	x +1	x +2	x x	2 x+1	2 x+2	2
				1 x	2 x+2	2 x+1	2	x +2`	x +1	x
			x	2 x+1	2	1 +1	x +2	x x+2	2	2
+1		+1	x+2	2	1 +2	x x	2	2	x x+1	2
+2		+2	x+1	2 +1	x x	2	2 x+2	2	1	x
x		x		2 +2	x	2 x+2	2 x+1	2 +1	x	1
x+1		x+1	+2	x x+2	2	x	1 +1	x	2 x	2
x+2		x+2	+1	x	2 x+1	2	x	1 x	2 +2	x

The mapping process that generates the difference scheme D(9, 9, 3) in (3.12) is as follows: every entry (i.e. $\beta_0 + \beta_1 X$) in the 9 x 9 multiplicative table of GF(32) is now mapped into β_0 in GF(3). So, we get the difference scheme D(9, 9, 3) in (3.12) by just reducing the linear entries in the table (3.1) to their constant.

Having defined difference schemes and having known when difference schemes exist, we next use difference schemes to construct orthogonal arrays.

3.1. (b) Construction of orthogonal arrays by developing difference schemes:

This development process of difference schemes that leads to orthogonal arrays works as follows:

If D is a difference scheme D(r,c,s) based on set (A, +) where A = { $\sigma_0, \ldots, \sigma_{s-1}$ }(often A is a Galois field), then we get D_i = The r x c array obtained from D by adding σ_i (from Galois field A) to each of its entries. Array D_i remains a difference scheme with the same parameters as those of D. This addition process on difference scheme D has then yielded new S additional difference schemes $D_0, D_1, \ldots, D_{s-1}$; where $D_i = \sigma_i + D$, i = 0, 1, …,(s-1) and $\sigma_i \in GF(s)$. We next juxtapose all s difference schemes D_i's, underneath each other to obtain an orthogonal array of strength two.

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i.e. $A = \begin{bmatrix} D_0 \\ D_1 \\ M \\ D_{s-1} \end{bmatrix}$ where $D_i = D + \sigma_i$; i = 0, 1, …., (s-1) ………………… (3.14)

This orthogonal array has the parameters OA(rs,c,s,2).

Equivalent to the above juxtapositioning in (3.14) is the following kroncker product representation of the array A = $(\sigma_0, \sigma_1, ..., \sigma_{s-1})^T \otimes D$ …………… (3.15).

Now to prove that this array in (3.14) is an orthogonal array we must satisfy definition (3.1), since strength of the generated orthogonal array is two, select two factors (from the k factors) say F_1 and $F_2, F_1 \neq F_2$, and two elements from set A say σ and σ', allowing the possibility that $\sigma = \sigma'$. We must now show that the number of runs with factor F_1 at level σ and factor F_2 at level σ' is equal to rs/$s^2 = \lambda$. If C_1 and C_2 denote the columns of the difference scheme D in (3.14) corresponding to factors F_1 and F_2, respectively, then λ entries in the column difference (C_1 - C_2) are equal to ($\sigma - \sigma'$). For each occurrence of ($\sigma - \sigma'$) in column difference (C_1 - C_2), there is a unique row in a unique the difference scheme D_i in which F_1 is at level σ and F_2 is at level σ'. Since these are the only runs with factor F_1 at level σ and factor F_2 at level σ', we conclude that there are indeed λ such runs in set A. This then complete the proof.

For an illustration on how difference schemes are used to construct orthogonal arrays, we use this development process in (3.14) on the following difference scheme D(3, 3, 3): 0 0 0

0 1 2

0 2 1

To get the following orthogonal array OA(9, 3, 3, 2): $\begin{bmatrix} D_0 \\ D_1 \\ D_2 \end{bmatrix} = \begin{bmatrix} 000 \\ 012 \\ 021 \\ 111 \\ 120 \\ 102 \\ 222 \\ 201 \\ 210 \end{bmatrix}$ …… (3.16).

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This orthogonal array in (3.16) can be regarded as regular 3^{3-1} fractional factorial design with defining contrast I = ABC. The difference scheme D (3, 3, 3) can also be generated by theorem (3.13).

We next move to the resolvability of some orthogonal arrays, where some orthogonal arrays are constructed, so that they can be partitioned into subarrays.

4.0 Statistical analysis

Statistical analysis for an irregular fractional factorial design that is also an orthogonal array.

This orthogonal array OA (12, 11, 2, 2) can be obtained by Hadamard matrix H12, technique II, and then omitting its first column to get table (4.18):

Table (4.18): Orthogonal array OA (12,11, 2, 2) and its responses:

Number of runs	Run label (additive form)	Response
1	11111111111	1.9
2	01011100010	2.3
3	00101110001	3.3
4	10010111000	4.7
5	01001011100	5.9
6	00100101110	6.9
7	00010010111	7.7
8	10001001011	8.8
9	11000100101	9.8
10	11100010010	10.3
11	01110001001	11.6
12	10111000100	12.2

This orthogonal array in table (4.18) is not regular fraction from the complete 212 factorial design since N=12 which is not a power of 2 yet this irregular fraction yields orthogonal estimation for all twelve main effects. This is unlike the irregular 24-1 fraction in subsection (4.4.1) whose number of runs is a power of 2 (namely 8) yet it produces correlated estimates for factor main effects. Linear modeling of the orthogonal array in table (4.18) is:

$$Y = \mu + \sum_{i=1}^{11} A_i x_i + \in \qquad (4.34)$$

and unbiased least squares estimates of the twelve factorial effects in (4.34) (according to (4.5)) are:

$\hat{A}_1 =$ 0.833 $\quad \hat{A}_5 = -1.383 \quad \hat{A}_9 = 0.283$

$\hat{A}_2 = -0.150 \quad \hat{A}_6 = -2.300 \quad \hat{A}_{10} = -0.800$

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$\hat{A}_3 = 0.583$ $\hat{A}_7 = -1.483$ $\hat{A}_{11} = 0.0667$

$\hat{A}_4 = -0.383$ $\hat{A}_8 = -0.483$

with Var $\hat{A}_1 =$ Var $\hat{A}_2 =$ Var $\hat{A}_3 =$ Var $\hat{A}_4 =$ Var $\hat{A}_5 =$ Var $\hat{A}_6 =$ Var $\hat{A}_7$ = Var $\hat{A}_8 =$ Var $\hat{A}_9$ = Var $\hat{A}_{10}$ = Var $\hat{A}_{11} = \frac{1}{3}\sigma^2$. The 12 ×12 design matrix X for the orthogonal array in table (4.18) is diagonal and is equal to 12 I12, meaning that this orthogonal array leads to or thogonal estimates .

From ANOVA (4.19) below, it is clear that we can't make tests of significance since error has degree of freedom equal zero.

ANOVA (4.19): Analysis of variance of an orthogonal Array OA (12, 11, 2, 2)

Source of variation	Degree of freedom	Sum of squares	Mean squares	F-value	P-value
A1	1	8.333	8.333	-	-
A2	1	0.270	0.270	-	-
A3	1	4.083	4.083	-	-
A4	1	1.763	1.763	-	-
A5	1	22.963	22.963	-	-
A6	1	63.480	63.480	-	-
A7	1	26.403	26.403	-	-
A8	1	2.803	2.803	-	-
A9	1	0.963	0.963	-	-
A10	1	7.680	7.680	-	-
A11	1	0.053	0.053	-	-
Error	0	-	-	-	-
Total	11	-	-	-	-

To solve this problem, we may replicate the fractional design in (4.18) at least twice as in table (4.20) although this may increase the cost of experimentation.

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Table (4.20): Double replicate of the orthogonal array OA (12, 11, 2, 2)

Number o f runs	Run label additive form	Response	
		Replicate (1)	Replica te (2)
1	11111111111	1.9	2.9
2	01011100010	2.3	3.3
3	00101110001	3.3	4.3
4	10010111000	4.7	5.7
5	01001011100	5.9	6.9
6	00100101110	6.9	7.9
7	00010010111	7.7	8.7
8	10001001011	8.8	9.8
9	11000100101	9.8	10.8
10	11100010010	10.3	11.3
11	01110001001	11.6	12.6
12	10111000100	12.2	13.2

Least squares estimates of effects in linear (4.5) modeling (4.34) according to (4.5) and from the replicated fraction in table (4.20) are

$\hat{\mu} = 7.617$

$\hat{A}_1 = 1.666$ $\hat{A}_4 = -0.766$ $\hat{A}_8 = -0.966$

$\hat{A}_2 = -0.300$ $\hat{A}_5 = -4.600$ $\hat{A}_9 = 0.566$

$\hat{A}_3 = 1.166$ $\hat{A}_6 = -2.966$ $\hat{A}_{10} = -1.600$

$\hat{A}_7 = -2.966$ $\hat{A}_{11} = 0.134$

with Var $\hat{A}_1 =$ Var $\hat{A}_2 =$ Var $\hat{A}_3 =$ Var $\hat{A}_4 =$ Var $\hat{A}_5 =$ Var $\hat{A}_6 =$ Var $\hat{A}_7 =$ Var $\hat{A}_8 =$ Var $\hat{A}_9 =$ Var $\hat{A}_{10} =$ Var $\hat{A}_{11} = \frac{1}{6}\sigma^2$

That is, the 12 × 12 design matrix is diagonal where XtX = 6 I12. Analysis of variance for this replicated orthogonal array is summarized in the table (4.21) where here the analysis under type I is the same as analysis under type III due to orthogonality of this replicated orthogonal array.

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Table (4.21): Analysis of variance of replicated OA (24,11, 2, 2)

Source of variation	Degree of freedom	Sum of squares	Mean squares	F-value	P-value
A1	1	16.667	16.667	33.33	< 0.0001
A2	1	0.540	0.540	1.08	0.3192
A3	1	8.167	8.167	16.33	0.0016
A4	1	3.527	3.527	7.05	0.0210
A5	1	45.92	45.9	91.85	< 0.0001
A6	1	126.960	126.960	253.92	< 0.0001
A7	1	52.807	52.807	105.61	< 0.0001
A8	1	5.607	5.607	11.21	0.0058
A9	1	1.927	1.927	3.85	0.0732
A10	1	15.36	15.36	30.72	0.0001
A11	1	0.107	0.107	0.21	0.6524
Error	12	6.000	0.500		
Total	23	283.593			

Replicating the entire orthogonal array twice increases the cost of experimentation but allows for possibility to conduct tests of significance.

To achieve further economy in cost of experimentation, we use a different replication strategy where we replicate only one run of the orthogonal array in order to get an estimate of the experimental error.

4.1Conclusion

we have considered statistical analysis of various types of FRACTIONAL FACTORIAL DESIGN (orthogonal arrays). This conducted comparisons between various types of orthogonal arrays with and without replication for the determination of the precision with which factor effects and interactions are estimated. Replication has increased precision but also has increased experimentation cost.

The recommendation is that cost can be reduced by assuming high order interactions negligible. This assumption eliminates the need for replication and allows for the possibility of conducting tests of significance on various factor effects.

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