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# linear derivative operator with differential subordination of meromorphic $\varepsilon$ -valent functions

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**Abstract:** The present paper is to investigate some inclusion relations between the linear derivative operator and differential subordination with other interesting properties for meromorphic $\varepsilon$ -valent Functions in the puncture unit disk $©^* = \{z \in \mathbb{C}: 0 < |z| < 1\}$.

**Keywords:** Meromorphic functions , differential subordination, the linear derivative operator.

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## Introduction.

Let $\mu_\varepsilon$ be the class of analytic and $\varepsilon$ -valent meromorphic functions defined on

$©^* = \{z \in \mathbb{C}: 0 < |z| < 1\}$.

$$f(z) = z^{-\varepsilon} + \sum_{\mathcal{S}=1}^{\infty} a_{\mathcal{S}-\varepsilon} z^{\mathcal{S}-\varepsilon}, (\varepsilon \in \mathbb{N} = \{1,2,\dots\}) \quad (1)$$

For function $f \in \mu_\varepsilon$ given by ( 1 ) and $q \in \mu_\varepsilon$ defined by

$$q(z) = z^{-\varepsilon} + \sum_{\mathcal{S}=1}^{\infty} b_{\mathcal{S}-\varepsilon} z^{\mathcal{S}-\varepsilon}, (\varepsilon \in \mathbb{N} = \{1,2,\dots\}) \quad (2)$$

the hadamard product of $f$ and $q$ defined by

$$(f * q)(z) = z^{-\varepsilon} + \sum_{\mathcal{S}=1}^{\infty} a_{\mathcal{S}-\varepsilon} b_{\mathcal{S}-\varepsilon} z^{\mathcal{S}-\varepsilon} \quad (3)$$

Let $\mathfrak{D}_*^{t,\varepsilon} f$ denote the linear derivative operator of Ruschwey typ [9][6],

$f \in \mu_\varepsilon$ defined by:

$$\mathfrak{D}_*^{t,\varepsilon} f(z) = \frac{z^{-\varepsilon}}{(1-z)^{t+\varepsilon}} * f(z), \qquad t > -\varepsilon, (z \in ©^*) \quad (4)$$

The (4) can be written by binomial coefficients

$$\mathfrak{D}_*^{t,\varepsilon}\mathfrak{f}(z) = z^{-\varepsilon} + \sum_{\mathcal{S}=1}^{\infty}\binom{t+\mathcal{S}}{\mathcal{S}}a_{\mathcal{S}-\varepsilon}z^{\mathcal{S}-\varepsilon}, \quad t > -\varepsilon. \tag{5}$$

The class of functions $\mathbb{h}$ with $\mathbb{h}(0) = 1$, is, which are convex univalent and analytic in © ={ z ∈ $\mathbb{C}$: $|z| < 1$}.

Recently some authors studied differential subordination of meromorphic functions of different subclasses [1],[2],[3],[4] and [5]

**Definition (1):** If satisfies the subordination condition the function $\mathfrak{f} \in \mu_\varepsilon$ is said to be in the class $\mu_\varepsilon(t, \mathcal{S}: \mathbb{h})$:

$$\frac{(1+t)}{\varepsilon(\varepsilon+1)(\varepsilon+2)}z^{\varepsilon+3}\left(\mathfrak{D}_*^{t,\varepsilon}\mathfrak{f}(z)\right)''' + \frac{t}{\varepsilon(\varepsilon+1)(\varepsilon+2)(\varepsilon+3)}z^{p+4}\left(\mathfrak{D}_*^{t,\varepsilon}\mathfrak{f}(z)\right)'''' \prec \mathbb{h}(z). \tag{6}$$

Where $t \in \mathbb{C}, \mathbb{h} \in \Psi$.

It is necessary to put the restrictions on the operator $\mathfrak{D}_*^{t,\varepsilon}$ such that

$$\mathfrak{D}_*^{t,\varepsilon}(\mathfrak{f}_1 * \mathfrak{f}_2) = (\mathfrak{D}_*^{t,\varepsilon}\mathfrak{f}_1) * \mathfrak{f}_2 = \mathfrak{f}_1 * (\mathfrak{D}_*^{t,\varepsilon}\mathfrak{f}_2), \tag{7}$$

if $\mathfrak{f}_1, \mathfrak{f}_2 \in \mu_\varepsilon(t, \mathcal{S}: \mathbb{h})$, we get the convolution results of the class of multivalent analytic functions $\mu_\varepsilon(t, \mathcal{S}: \mathbb{h})$.

**Lemma 1[8]:** let $q$ be analytic and convex univalent in © and Let $\mathbb{h}$ be analytic in © with $q(0) = \mathbb{h}(0)$. If

$$q(z) + \frac{1}{\mathfrak{M}}z'(z) \prec \mathbb{h}(z), \tag{8}$$

Where $Re\,\mathfrak{M} \geq 0$ and $\mathfrak{M} \neq 0$, then

$$q(z) \prec \mathbb{h}^{\neg}(z) = \mathfrak{M}z^{-\mathfrak{M}}\int_0^z L^{\mathfrak{M}-1}\,\mathbb{h}(L)dL \prec \mathbb{h}(z).$$

And $\mathbb{h}^{\neg}(z)$ is the best dominant of (7).

**Lemma (2)[10]:** let $\mathfrak{f}(z) \prec \emptyset(z)(z \in ©)$ and $q(z) \prec \mathfrak{D}(z)$ $(z \in ©)$ if the function $\emptyset(z)$ and $\mathfrak{D}(z)$ are convex in ©. Then $(\mathfrak{f} * q)(z) \prec (\emptyset * \mathfrak{D})(z)$ $(z \in ©)$.

**Theorem (1):** If the function $\mathfrak{f} \in \mu_\varepsilon(t, \mathcal{S}: \mathbb{h})$, then

$$q(z) = \frac{z^{\varepsilon+3}\left(\mathfrak{D}_*^{t,\varepsilon}f(z)\right)'''}{\varepsilon(\varepsilon+1)(\varepsilon+2)} \prec \mathbb{h}(z), \tag{9}$$

and if $t > 0$, then $q(z) \prec \mathbb{h}^{\neg}(z)$, where

$$\mathbb{h}^{\neg}(z) = \frac{(\varepsilon+3)}{t}z^{\frac{-(\varepsilon+3)}{t}}\int_0^z L^{\frac{(\varepsilon+3)}{t}-1}\,\mathbb{h}(L)dL \prec \mathbb{h}(z) \qquad (z \in ©),$$

$\mathbb{h}^{\neg}(z)$ is the best dominant of subordination $q(z) \prec \mathbb{h}^{\neg}(z)$ $(z \in ©)$

and $\mathbb{h}^{\neg}(z)$ is convex univalent in ©

**Proof.** When t=0, trivial.

If $t > 0$, let $f \in \mu_\varepsilon(\lambda, \mathcal{S}: \mathbb{h})$, then

$$\frac{(1+t)}{\varepsilon(\varepsilon+1)(\varepsilon+2)} z^{\varepsilon+3}\left(\mathfrak{D}_*^{t,\varepsilon} f(z)\right)''' + \frac{t}{\varepsilon(\varepsilon+1)(\varepsilon+2)(\varepsilon+3)} z^{\varepsilon+4}\left(\mathfrak{D}_*^{t,\varepsilon} f(z)\right)'''' \prec \mathbb{h}(z).$$

By (6) and (9)

$$q(z) + \frac{t}{(\varepsilon+3)} z q'(z) \prec \mathbb{h}(z) \ (z \in \mathrm{U}). \tag{10}$$

During Lemma (1) in (10) with $m = \frac{(\varepsilon+3)}{t}$ and $t > 0$, we give

$$q(z) \prec \mathbb{h}^{\neg}(z) = \frac{(\varepsilon+3)}{t} z^{\frac{-(\varepsilon+3)}{t}} \int_0^z L^{\frac{(\varepsilon+3)}{t}-1} \mathbb{h}(L) dL \prec \mathbb{h}(z)$$

Where $q$ is given by (9).

**Theorem (2):** $\mu_\varepsilon(t_1, \mathcal{S}: \mathbb{h}), \subset \mu_\varepsilon(t_2, \mathcal{S}: \mathbb{h})$ if $0 \leq t_2 < t_1$.

**Proof.** Let $f \in \mu_\varepsilon(t_1, \mathcal{S}: \mathbb{h})$.

$$\frac{(1+t_2)}{\varepsilon(\varepsilon+1)(\varepsilon+2)} z^{\varepsilon+3}\left(\mathfrak{D}_*^{t,\varepsilon} f(z)\right)''' + \frac{t_2}{\varepsilon(\varepsilon+1)(\varepsilon+2)(\varepsilon+3)} z^{\varepsilon+4}\left(\mathfrak{D}_*^{t,\varepsilon} f(z)\right)''''$$

$$= \left[1 - \frac{t_2}{t_1}\right] \frac{z^{\varepsilon+3}\left(\mathfrak{D}_*^{t,\varepsilon} f(z)\right)'''}{\varepsilon(\varepsilon+1)(\varepsilon+2)} +$$

$$\frac{t_2}{t_1}\left[\frac{(1+t_2)}{\varepsilon(\varepsilon+1)(\varepsilon+2)} z^{\varepsilon+3}\left(\mathfrak{D}_*^{t,\varepsilon} f(z)\right)''' + \frac{t_2}{\varepsilon(\varepsilon+1)(\varepsilon+2)(\varepsilon+3)} z^{\varepsilon+4}\left(\mathfrak{D}_*^{t,\varepsilon} f(z)\right)''''\right] \tag{11}$$

since $h$ is a convex set and $0 \leq \frac{t_2}{t_1} < 1$. (11) can write as follows:

$$\left[\frac{(1+t_2)}{\varepsilon(\varepsilon+1)(\varepsilon+2)} z^{\varepsilon+3}\left(\mathfrak{D}_*^{t,\varepsilon} f(z)\right)''' + \frac{t_2}{\varepsilon(\varepsilon+1)(\varepsilon+2)(\varepsilon+3)} z^{\varepsilon+4}\left(\mathfrak{D}_*^{t,\varepsilon} f(z)\right)''''\right]$$

$$= \left[1 - \frac{t_2}{t_1}\right] q_1(z) + \frac{t_2}{t_1} q_2(z) = \emptyset(z),$$

Where $q_1(z), q_2(z) \prec \mathbb{h}(z)$, by using definition of convex set and by Theorem (1), since $f \in \mu_\varepsilon(t_1, \mathcal{S}: \mathbb{h})$, we get $\emptyset(z) \prec \mathbb{h}(z)$, then $f \in \mu_\varepsilon(t_2, \mathcal{S}: \mathbb{h})$.

**Theorem (3):** Let $\phi$ defined by

$$\phi(z) = \frac{(\sigma-\varepsilon)}{z^\sigma} \int_0^z L^{\sigma-1} f(L) dL \qquad (Re\{\sigma\} > -\varepsilon),$$

and let the function $f \in \mu_\varepsilon$

If

$$\left[1-\frac{\gamma}{\varepsilon}\right]\frac{z^{\varepsilon+3}\left(\mathfrak{D}_*^{t,\varepsilon}\phi(z)\right)'''}{\varepsilon(\varepsilon+1)(\varepsilon+2)}+\gamma\frac{z^{\varepsilon+4}\left(\mathfrak{D}_*^{t,\varepsilon}\mathfrak{f}(z)\right)}{\varepsilon(\varepsilon+1)(\varepsilon+2)(\varepsilon+3)}\prec \mathbb{h}(z). \qquad (12)$$

Then the function $\phi \in \mu_\varepsilon(0,\mathcal{S}:\mathbb{h}^\neg)$ Where

$$\mathbb{h}^\neg(z)=\frac{(\sigma\varepsilon-\varepsilon)}{\gamma}z^{\frac{(\sigma\varepsilon-\varepsilon)(\varepsilon+3)}{\gamma}}\int_0^z L^{\frac{(\sigma\varepsilon-\varepsilon)(\varepsilon+3)}{\gamma}}\mathbb{h}(L)dL\prec \mathbb{h}(z). \qquad (13)$$

**Proof**. Define

$$\mathfrak{I}(z)=\frac{z^{\varepsilon+3}\left(\mathfrak{D}_*^{t,\varepsilon}\phi(z)\right)'''}{\varepsilon(\varepsilon+1)(\varepsilon+2)}, \qquad (14)$$

then $\mathfrak{I}$ is analytic in ©, $\mathfrak{I}(0)=1$ and

$$\frac{z\mathfrak{I}'(z)}{(\varepsilon+3)}=\mathfrak{I}(z)+\frac{z^{\varepsilon+4}\left(\mathfrak{D}_*^{t,\varepsilon}\phi(z)\right)''''}{\varepsilon(\varepsilon+1)(\varepsilon+2)(\varepsilon+3)}. \qquad (15)$$

Making use of (12), (14) and (15) and by

$$(\sigma\varepsilon-\varepsilon)\mathfrak{f}(z)=\sigma(\varepsilon+3)\phi'''(z)+z\phi''''(z),$$

then

$$\left[1-\frac{\gamma}{\varepsilon}\right]\frac{z^{\varepsilon+3}\left(\mathfrak{D}_*^{t,\varepsilon}\phi(z)\right)'''}{\varepsilon(\varepsilon+1)(\varepsilon+2)}+\frac{\gamma z^{\varepsilon+4}\left(\mathfrak{D}_*^{t,\varepsilon}\mathfrak{f}(z)\right)}{\varepsilon(\varepsilon+1)(\varepsilon+2)(\varepsilon+3)}$$

$$=\left[1-\frac{\gamma}{\varepsilon}\right]\frac{z^{p+3}\left(\mathfrak{D}_*^{t,\varepsilon}\phi(z)\right)'''}{\varepsilon(\varepsilon+1)(\varepsilon+2)}+\frac{\gamma}{(\sigma\varepsilon-\varepsilon)}\left[\frac{\sigma z^{\varepsilon+3}\left(\mathfrak{D}_*^{t,\varepsilon}\phi(z)\right)'''}{\varepsilon(\varepsilon+1)(\varepsilon+2)}+\frac{z^{\varepsilon+4}\left(\mathfrak{D}_*^{t,\varepsilon}\phi(z)\right)''''}{\varepsilon(\varepsilon+1)(\varepsilon+2)(\varepsilon+3)}\right]$$

$$=\mathfrak{I}(z)+\frac{\gamma}{(\sigma\varepsilon-\varepsilon)(\varepsilon+3)}z\mathfrak{I}'(z)$$

$$=\mathfrak{I}(z)+\frac{\gamma}{(\sigma\varepsilon-\varepsilon)(\varepsilon+3)}z\mathfrak{I}'(z)\prec \mathbb{h}(z).$$

where $\mathbb{h}^\neg(z)$ is given by (13) then $\mathfrak{I}(z)\prec \mathbb{h}^\neg(z)$, and $\phi\in\mu_\varepsilon\ (0,\mathcal{S}:\mathbb{h}^\neg)$.

**Theorem (4):** Let $\mathfrak{D}_*^{t,\varepsilon}$ satisfy the condition (7) in definition (1) If

$$\mathfrak{f}_i\in\mu_\varepsilon\left(t,\mathcal{S}:\frac{1+\alpha_i z}{1+\beta_i z}\right)(i=1,2).$$

Then the inclusion relationship are hold:

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$$\mathfrak{I}(z) = \frac{(1+t)}{\varepsilon^2(\varepsilon+1)^2(\varepsilon+2)^2} z^3\left(\mathfrak{D}_*^{t,\varepsilon}(\mathfrak{f}_1{}''' * \mathfrak{f}_2{}''')(z)\right) + \frac{t}{\varepsilon^2(\varepsilon+1)^2(\varepsilon+2)^2(\varepsilon+3)^2} z^4\left(\mathfrak{D}_*^{t,\varepsilon}(\mathfrak{f}_1{}'''' * \mathfrak{f}_2{}'''')(z)\right)$$

$$\in \mu_\varepsilon\left(t, k; \left(\frac{1+\alpha_1 z}{1+\beta_1 z}\right) * \left(\frac{1+\alpha_2 z}{1+\beta_2 z}\right)\right), \quad (16)$$

$$h(z) = \frac{z^3\left(\mathfrak{D}_*^{t,\varepsilon}(\mathfrak{f}_1{}''' * \mathfrak{f}_1{}''')(z)\right)}{\varepsilon^2(\varepsilon+1)^2(\varepsilon+2)^2} \in M_\varepsilon\left(t, \mathcal{S}; \left(\frac{1+\alpha_1 z}{1+\beta_1 z}\right) * \left(\frac{1+\alpha_2 z}{1+\beta_2 z}\right)\right), \quad (17)$$

and

$$\frac{z^{\varepsilon+3}\left(\mathfrak{D}_*^{t,\varepsilon}\mathfrak{f}(z)\right)'''}{\varepsilon(\varepsilon+1)(\varepsilon+2)} \prec \left(\left(\frac{1+\alpha_1 z}{1+\beta_1 z}\right) * \left(\frac{1+\alpha_2 z}{1+\beta_2 z}\right)\right). \quad (18)$$

**Proof.** Since

$$\mathfrak{f}_1 \in \mu_\varepsilon\left(t, \mathcal{S}: \frac{1+\alpha_1 z}{1+\beta_1 z}\right) \; and \; \mathfrak{f}_2 \in \mu_\varepsilon\left(t, \mathcal{S}: \frac{1+\alpha_2 z}{1+\beta_2 z}\right).$$

Then

$$\frac{(1+t)}{\varepsilon(\varepsilon+1)(\varepsilon+2)} z^{\varepsilon+3}\left(\mathfrak{D}_*^{t,\varepsilon}\mathfrak{f}_1(z)\right)''' + \frac{t}{\varepsilon(\varepsilon+1)(\varepsilon+2)(\varepsilon+3)} z^{\varepsilon+4}\left(\mathfrak{D}_*^{t,\varepsilon}\mathfrak{f}_1(z)\right)'''' \prec \left(\frac{1+\alpha_1 z}{1+\beta_1 z}\right), \quad (19)$$

and

$$\frac{(1+t)}{\varepsilon(\varepsilon+1)(\varepsilon+2)} z^{\varepsilon+3}\left(\mathfrak{D}_*^{t,\varepsilon}\mathfrak{f}_2(z)\right)''' + \frac{t}{\varepsilon(\varepsilon+1)(\varepsilon+2)(\varepsilon+3)} z^{\varepsilon+4}\left(\mathfrak{D}_*^{t,\varepsilon}\mathfrak{f}_2(z)\right)'''' \prec \left(\frac{1+\alpha_2 z}{1+\beta_2 z}\right). \quad (20)$$

By Theorem (1), (19) and (20), we give

$$\frac{z^{\varepsilon+3}\left(\mathfrak{D}_*^{t,\varepsilon}\mathfrak{f}_1(z)\right)'''}{\varepsilon(\varepsilon+1)(\varepsilon+2)} \prec \left(\frac{1+\alpha_1 z}{1+\beta_1 z}\right),$$

and

$$\frac{z^{\varepsilon+4}\left(\mathfrak{D}_*^{t,\varepsilon}\mathfrak{f}_2(z)\right)'''}{\varepsilon(\varepsilon+1)(\varepsilon+2)(\varepsilon+3)} \prec \left(\frac{1+\alpha_2 z}{1+\beta_2 z}\right).$$

By (7), (19) and (20) and Lemma (2), we have

$$\frac{(1+t)}{\varepsilon(\varepsilon+1)(\varepsilon+2)}z^{\varepsilon+3}\left(\mathfrak{D}_*^{t,\varepsilon}\mathfrak{y}(z)\right)''' + \frac{t}{\varepsilon(\varepsilon+1)(\varepsilon+2)(\varepsilon+3)}z^{\varepsilon+4}\left(\mathfrak{D}_*^{t,\varepsilon}\mathfrak{y}(z)\right)''''$$

$$= \frac{(1+t)}{\varepsilon(\varepsilon+1)(\varepsilon+2)}z^{\varepsilon+3}\left(\mathfrak{D}_*^{t,\varepsilon}\left[\frac{(1+t)}{\varepsilon^2(\varepsilon+1)^2(\varepsilon+2)^2}z^3\left(\mathfrak{D}_*^{t,\varepsilon}(\mathfrak{f}_1''' * \mathfrak{f}_2''')(z)\right) + \frac{t}{\varepsilon^2(\varepsilon+1)^2(\varepsilon+2)^2(\varepsilon+3)^2}z^4\left(\mathfrak{D}_*^{t,\varepsilon}(\mathfrak{f}_1'''' * \mathfrak{f}_2'''')(z)\right)\right]\right)'''$$

$$+\frac{t}{\varepsilon(\varepsilon+1)(\varepsilon+2)(\varepsilon+3)}z^{\varepsilon+4}\left(\mathfrak{D}_*^{t,\varepsilon}\left[\frac{(1+t)}{\varepsilon^2(\varepsilon+1)^2(\varepsilon+2)^2}z^3\left(\mathfrak{D}_*^{t,\varepsilon}(\mathfrak{f}_1''' * f_2''')(z)\right) + \frac{t}{\varepsilon^2(\varepsilon+1)^2(\varepsilon+2)^2(\varepsilon+3)^2}z^4\left(\mathfrak{D}_*^{t,\varepsilon}(\mathfrak{f}_1'''' * \mathfrak{f}_2'''')(z)\right)\right]\right)''''$$

$$\prec \left(\left(\frac{1+\alpha_1 z}{1+\beta_1 z}\right) * \left(\frac{1+\alpha_2 z}{1+\beta_2 z}\right)\right).$$

Then

$$\mathfrak{y} \in \mu_\varepsilon\left(t, \mathcal{S}; \left(\frac{1+\alpha_1 z}{1+\beta_1 z}\right) * \left(\frac{1+\alpha_2 z}{1+\beta_2 z}\right)\right),$$

and

$$\mathbb{h} \in \mu_\varepsilon\left(t, \mathcal{S}; \left(\frac{1+\alpha_1 z}{1+\beta_1 z}\right) * \left(\frac{1+\alpha_2 z}{1+\beta_2 z}\right)\right).$$

The proof of (16) and (17) is complete, by same method we obtain

$$\frac{(1+t)}{\varepsilon(\varepsilon+1)(\varepsilon+2)}z^{\varepsilon+3}\left(\mathfrak{D}_*^{t,\varepsilon}\mathbb{h}(z)\right)''' + \frac{t}{\varepsilon(\varepsilon+1)(\varepsilon+2)(\varepsilon+3)}z^{\varepsilon+4}\left(\mathfrak{D}_*^{t,\varepsilon}\mathbb{h}(z)\right)'''' \prec \left(\left(\frac{1+\alpha_1 z}{1+\beta_1 z}\right) * \left(\frac{1+\alpha_2 z}{1+\beta_2 z}\right)\right). \quad (21)$$

Where $\mathbb{h}$ is given by (17). The proof of (18) we get by (21) and Theorem (1).

**Theorem 5:** let $\mathcal{A} \in \mu_\varepsilon$ and $\mathfrak{f} \in \mu_\varepsilon(t, \mathcal{S}: \mathbb{h})$ and

$$\text{Re}\,\{z^\varepsilon \mathcal{A}(z)\} \geq \frac{1}{2}. \quad (22)$$

Then $(\mathfrak{f} * \mathcal{A}) \in \mu_\varepsilon(t, \mathcal{S}: \mathbb{h})$

**Proof.** If $\mathfrak{f} \in M_\varepsilon(t, \mathcal{S}: \mathbb{h})$ given by (1) and $\mathcal{A} \in \mu_\varepsilon$ we have

$$\frac{(1+t)}{\varepsilon(\varepsilon+1)(\varepsilon+2)}z^{\varepsilon+3}\left(\mathfrak{D}_*^{t,\varepsilon}(\mathfrak{f} * \mathcal{A})(z)\right)''' + \frac{t}{\varepsilon(\varepsilon+1)(\varepsilon+2)(\varepsilon+3)}z^{\varepsilon+4}\left(\mathfrak{D}_*^{t,\varepsilon}(\mathfrak{f} * \mathcal{A})(z)\right)''''$$

$$=\frac{(1+t)}{\varepsilon(\varepsilon+1)(\varepsilon+2)}\left[z^{\varepsilon}\mathcal{A}(z)*\left(z^{\varepsilon+3}\left(\mathfrak{D}_{*}^{t,\varepsilon}(\mathfrak{f}(z))\right)^{'''}\right)\right]$$
$$+\frac{t}{\varepsilon(\varepsilon+1)(\varepsilon+2)(\varepsilon+3)}\left[z^{\varepsilon}\mathcal{A}(z)*\left(z^{\varepsilon+4}\left(\mathfrak{D}_{*}^{t,\varepsilon}(\mathfrak{f}(z))\right)^{''''}\right)\right]$$

$$=\{z^{\varepsilon}\mathcal{A}(z)\}*\Psi(z) \tag{23}$$

$\Psi$ (z)= $\frac{(1+t)}{\varepsilon(\varepsilon+1)(\varepsilon+2)}z^{\varepsilon+3}\left(\mathfrak{D}_{*}^{t,\varepsilon}\mathfrak{f}(z)\right)^{'''}+\frac{t}{\varepsilon(\varepsilon+1)(\varepsilon+2)(\varepsilon+3)}z^{\varepsilon+4}\left(\mathfrak{D}_{*}^{t,\varepsilon}\mathfrak{f}(z)\right)^{''''}$.

The function $z^{\varepsilon}\mathcal{A}(z)$ has the Herglotz representation [7], by (22).

$$z^{\varepsilon}\mathcal{A}(z)=\int_{|x|=1}\frac{d\varepsilon(x)}{1-xz} \quad (\text{z} \in ©) \tag{24}$$

The probability measure ( $\varepsilon$ ) defined on the unit circle $|x|=1$, and

$$\int_{|x|=1}d\varepsilon(x)=1.$$

Because $\mathbb{h}$ is convex univalent in ©.

By (21) and (24) give now

$$\frac{(1+t)}{\varepsilon(\varepsilon+1)(\varepsilon+2)}z^{\varepsilon+3}\left(\mathfrak{D}_{*}^{t,\varepsilon}(\mathfrak{f}*\mathcal{A})(z)\right)^{'''}+\frac{t}{\varepsilon(\varepsilon+1)(\varepsilon+2)(\varepsilon+3)}z^{\varepsilon+4}\left(\mathfrak{D}_{*}^{t,\varepsilon}(\mathfrak{f}*\mathcal{A})(z)\right)^{''''}$$

$=\int_{|x|=1}\Psi(xz)d\varepsilon(z)\prec\mathbb{h}(z)$.

Then $(\mathfrak{f}*\mathcal{A})\in\mu_{\varepsilon}(t,\mathcal{S}:\mathbb{h})$.

**Corollary:** Suppose $\mathfrak{f}\in\mu_{\varepsilon}(t,\mathcal{S}:\mathbb{h})$ be given by (1) and let

$$Re\left(1+\sum_{\mathcal{S}=1}^{\infty}\frac{\delta}{k+\delta}z^{\mathcal{S}}\right)>\frac{1}{2}.$$

Then

$$\Phi_{\varepsilon,\delta}(\mathfrak{f})=\frac{\delta}{z^{\varepsilon+\delta}}\int_{0}^{z}L^{\varepsilon+\delta-1}\mathfrak{f}(L)dL \;\; (\delta>-\varepsilon),$$

$$and\ \Phi_{\varepsilon,\delta}(\mathfrak{f})\in\mu_{\varepsilon}(t,\mathcal{S}:\mathbb{h}).$$

**Proof.** Let $\mathfrak{f}\in\mu_{\varepsilon}(t,\mathcal{S}:\mathbb{h})$ be defined in (1). Then

$$\Phi_{\varepsilon,\delta}\frac{\delta}{z^{\varepsilon+\delta}}\int_{0}^{z}L^{\varepsilon+\delta-1}\mathfrak{f}(L)dL=z^{-\varepsilon}+\sum_{\mathcal{S}=1}^{\infty}\frac{\delta}{\mathcal{S}+\delta}a_{\mathcal{S}-\varepsilon}z^{\mathcal{S}-\varepsilon},$$

$$= \left( z^{-\varepsilon} + \sum_{\mathcal{S}=1}^{\infty} a_{\mathcal{S}-\varepsilon} z^{\mathcal{S}-\varepsilon} \right) * \left( z^{-\varepsilon} + \sum_{\mathcal{S}=1}^{\infty} \frac{\delta}{\mathcal{S}+\delta} a_{\mathcal{S}-\varepsilon} z^{\mathcal{S}-\varepsilon} \right) = (\mathfrak{f} * \phi) * z, \quad (25)$$

where

$$\phi(z) = z^{-\varepsilon} + \sum_{\mathcal{S}=1}^{\infty} \frac{\delta}{\mathcal{S}+\delta} z^{\mathcal{S}-\varepsilon} (\delta > -\varepsilon),$$

and $\phi \in M_{\varepsilon}$. We give

$$Re\{z^{\varepsilon}\phi(z)\} = Re\left(1 + \sum_{\mathcal{S}=1}^{\infty} \frac{\delta}{\mathcal{S}+\delta} z^{\mathcal{S}}\right) > \frac{1}{2}, \quad (26)$$

Thus $\phi_{\varepsilon,\delta}(\mathfrak{f}) \in \mu_{\varepsilon}(t, \mathcal{S}: \mathbb{h})$, by (25) (26) and theorem (5).

## References

[1] Abdul Rahman S. J. Applications on differential subordination involving linear operator. Basrah Journal of Science (A)2016;34(2),85-93.

[2] Aisha H, Adolf B, Jeyaraman M.P. Certain third order differential subordination results of meromorphic multivalent functions , Asia Pacific Journal of Mathematics. 2015 ;2 (2). 76-87.

[3] Aouf M.K, Mostafay A.O. Meromorphic subordination results for p-valent functions associated with convolution, Hacettepe Journal of Mathematics and Statistics .2015; (4) (2): 255 – 260.

[4] Atshan W.G, Mohammed T. K. Some Interesting Properties of a Subclass of Meromorphic Univalent Functions Defined by Hadamard Product, (2014); 9 (3): 1184–1188, Google Scholar.

[5] Atshan W.G, Najah A. On a new class of meromorphic multivalent functions defined by fractional differ – integral operator, Gournal of kufa for mathematics and computer.2018;5:12-20.

[6] Maria A.ACU, Approximatian by certain positive linear operators. 2016; Habilitation thesis.

[7] Meiyan, C, Linliu J, A family of meromorphic functions involving generalized mittag –Leffler function, journal of mathmatical inequalities (2018); 12(4): 943–951. (Communicated by H. M. Srivastava).

[8] Miller S. S. and Mocanu P. T., Differetial subordinations and univalent functions, 7ichigan Math. J. 1981;28: 157-171.

[9] Raina R. K. and Srivastava H. M, Inclusion and neighborhood properties of some analytic and multivalnt functions, J. Ineqel. Pure Appl. Math., 2006; 7(1):A5,1-6.

ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online) DOI: 10.7176/MTM
Vol.9, No.2, 2019

[10] Ruscheweyh S, stankiewicz J. Subordination under convex univalent functions, Bull. Polish Acad. Sci. Math; (1981); 33:499-502.