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Modified Approach on Similarity Properties of Triangles and Trapezoids

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Abstract

The objective of this note is to provide presumably new technique to explain similarity properties in both triangles and trapezoids respectively. The author presented very short form of general conditions on similarity properties of triangles as well as trapezoids in a very elementary way.

Key words: similarity, triangles, trapezoids

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1. Introduction and preliminaries

Triangle is the simplest polygon with three edges and three vertices. It is one of the basic shapes in geometry. In Euclidean geometry any three points, when non- collinear, determine a unique triangle and a unique plane. The basic elements of any triangle are its sides and angles. Triangles are classified depending on relative sizes of their elements. In Euclidean geometry, a convex quadrilateral with at least one pair of parallel sides is referred to as a trapezoid in American and Canadian English but as a trapezium in English outside North America. A trapezium in Proclus' sense is a quadrilateral having one pair of its opposite sides parallel [1- 5].

Two geometrical objects are called similar if they both have the same shape, or one has the same shape as the mirror image of the other. Two figures that have the same shape but not necessarily the same size are similar. The ratio of the corresponding side lengths is the scale factor. More precisely, one can be obtained from the other by uniformly scaling (enlarging or reducing), possibly with additional translation, rotation and reflection. This means that either object can be rescaled, repositioned, and reflected, so as to coincide precisely with the other object. If two objects are similar, each is congruent to the result of a particular uniform scaling of the other. A modern and novel perspective of similarity is to consider geometrical objects similar if one appears congruent to the other when zoomed in or out at some level. For example, all circles are similar to each other, all squares are similar to each other, and all equilateral triangles are similar to each other. On the other hand, ellipses are not all similar to each other, rectangles are not all similar to each other, and isosceles triangles are not all similar to each other. If two angles of a triangle have measures equal to the measures of two angles of another triangle, then the triangles are similar. Corresponding sides of similar polygons are in proportion, and corresponding angles of similar polygons have the same measure [6-8]. Consider two similar triangles that is, ΔABE is similar to $\Delta A'B'E'$ ($\Delta ABE \sim \Delta A'B'E'$), as given in figure 1 below. Then by definition of similarity of triangles, we have the following relationships. ΔABE is similar to $\Delta A'B'E'$ if and only if $\angle BAE \cong \angle B'A'E'$, $\angle ABE \cong \angle A'B'E'$, $\angle AEB \cong \angle A'E'B'$ and $\frac{AB}{A'B'} = \frac{AE}{A'E'} = \frac{BE}{B'E'}$.

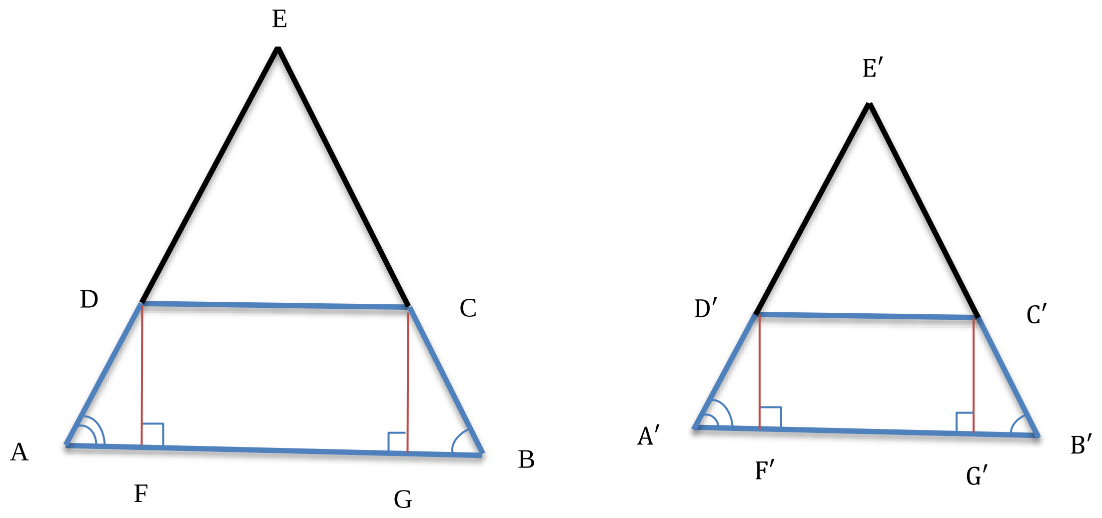

Figure 1: Two similar triangles

There are different properties on similarities of triangles. For example side-side-side similarity for triangles states that if corresponding sides of two triangles are proportional, then the triangles are similar. Side-angle-side similarity for triangles states that given two triangles, if two sides are proportional and the included angles are congruent, then the triangles are similar. Angle-angle similarity for triangles states that if two angles in one triangle are congruent, respectively, to two angles of a second triangle, then the triangles are similar. Thus from figure 1 above, $\Delta AFD \sim \Delta A'F'D'$ and $\Delta BGC \sim \Delta B'G'C'$.

Consider segments AB parallel to DC and $A'B'$ parallel to $D'C'$ in triangles ΔABE and $\Delta A'B'E'$ from figure 1 above. Definition of similarity of trapezoids states that trapezoid ABCD is similar to trapezoid $A'B'C'D'$ if and only if $\angle BAD \cong \angle B'A'D'$, $\angle ABC \cong \angle A'B'C'$, $\angle BCD \cong \angle B'C'D'$, $\angle ADC \cong \angle A'D'C'$, and $\frac{AB}{A'B'} = \frac{DC}{D'C'} = \frac{AD}{A'D'} = \frac{BC}{B'C'}$. By using the recent work done [6, 9, 10], we have $\Delta ABE \sim \Delta DCE$, $\Delta A'B'E' \sim \Delta D'C'E'$, and trapezoids $ABCD \sim A'B'C'D'$ if and only if $\angle BAD \cong \angle B'A'D'$, $\angle ABC \cong \angle A'B'C'$, $\angle BCD \cong \angle B'C'D'$, $\angle ADC \cong \angle A'D'C'$, and also $\frac{AB-DC}{A'B'-D'C'} = \frac{AD}{A'D'} = \frac{BC}{B'C'}$ which implies another properties on similarity of trapezoids. In this study, we presented general conditions that express similarity properties of triangles and also trapezoids. More general conditions on similarity properties of trapezoids have been examined by using known results [9] and on this note it has modified to very short forms.

2. Main Theorem

Theorem: $\Delta ABE \sim \Delta A'B'E' \Leftrightarrow \angle BAE \cong \angle B'A'E'$, $\angle ABE \cong \angle A'B'E'$, $\angle AEB \cong \angle A'E'B'$, *and*

$$\frac{AB+BE}{A'B'+B'E'} = \frac{AE}{A'E'} \quad \text{or} \quad \frac{AB+AE}{A'B'+A'E'} = \frac{BE}{B'E'} \quad \text{or} \quad \frac{AE+BE}{A'E'+B'E'} = \frac{AB}{A'B'}; \tag{2.1}$$

Trapezoids $ABCD \sim A'B'C'D' \Leftrightarrow \angle BAD \cong \angle B'A'D'$, $\angle ABC \cong \angle A'B'C'$, $\angle BCD \cong \angle B'C'D'$,

$$\angle ADC \cong \angle A'D'C' \text{ and } \frac{AB-DC}{A'B'-D'C'} = \frac{AD+BC}{A'D'+B'C'}. \tag{2.2}$$

Proof: First of all we have to prove our first identity (2.1). Let $\Delta ABE \sim \Delta A'B'E'$, then $\angle BAE \cong \angle B'A'E'$, $\angle ABE \cong \angle A'B'E'$, $\angle AEB \cong \angle A'E'B'$, and $\frac{AB}{A'B'} = \frac{AE}{A'E'} = \frac{BE}{B'E'} = k$, for $k \in \mathbb{R}^+$ by definition. From this we have:

$$\frac{AB}{A'B'} = \frac{AE}{A'E'} = \frac{BE}{B'E'} = k \tag{2.3}$$

$$\left.\begin{array}{l} AB = kA'B' \\ AE = kA'E' \\ BE = kB'E' \end{array}\right\} \tag{2.4}$$

Adding the first two equations of (2.4) and after simplifications with (2.3), we obtain:

$$\frac{AB+AE}{A'B'+A'E'} = \frac{BE}{B'E'} = k.$$

Or adding the first and third equations of (2.4) and after simplifications with (2.3), we have:

$$\frac{AB+BE}{A'B'+B'E'} = \frac{AE}{A'E'} = k.$$

Or adding the second and third equations of (2.4) and after simplifications with (2.3), we get:

$$\frac{AE+BE}{A'E'+B'E'} = \frac{AB}{A'B'} = k.$$

Let $\angle BAE \cong \angle B'A'E'$, $\angle ABE \cong \angle A'B'E'$, $\angle AEB \cong \angle A'E'B'$, and $\frac{AB+BE}{A'B'+B'E'} = \frac{AE}{A'E'}$ or $\frac{AB+AE}{A'B'+A'E'} = \frac{BE}{B'E'}$ or $\frac{AE+BE}{A'E'+B'E'} = \frac{AB}{A'B'}$. Let us take only one equation from these three alternative equations because others are also compute similarly. Thus we have:

$$\frac{AB+BE}{A'B'+B'E'} = \frac{AE}{A'E'} = k \text{ for } k \in \mathbb{R}^+.$$

$$\Rightarrow AB + BE = k(A'B' + B'E').$$

$$\Rightarrow AB = kA'B' \text{ and } BE = kB'E'; \text{ which can be proved by contradiction.}$$

$$\Rightarrow \frac{AB}{A'B'} = \frac{AE}{A'E'} = \frac{BE}{B'E'} = k.$$

$$\Rightarrow \Delta ABE \sim \Delta A'B'E'.$$

This completes the proof of (2.1).

Further, we have to attempt to establish our second identity (2.2). Let $ABCD \sim A'B'C'D'$. Using the recent work done [9], we have $\angle BAD \cong \angle B'A'D'$, $\angle ABC \cong \angle A'B'C'$, $\angle BCD \cong \angle B'C'D'$, $\angle ADC \cong \angle A'D'C'$, and $\frac{AB-DC}{A'B'-D'C'} = \frac{AD}{A'D'} = \frac{BC}{B'C'} = k$, for $k \in \mathbb{R}^+$. From these we have:

$$\frac{AB-DC}{A'B'-D'C'} = \frac{AD}{A'D'} = \frac{BC}{B'C'} = k \text{ for } k \in \mathbb{R}^+ \tag{2.5}$$

$$\left.\begin{array}{l} AD = kA'D' \\ BC = kB'C' \end{array}\right\} \tag{2.6}$$

Adding both equations of (2.6) and after simplifications with (2.5), we obtain:

$$\frac{AB-DC}{A'B'-D'C'}=\frac{AD+BC}{A'D'+B'C'}=k.$$

Let $\angle BAD\cong\angle B'A'D'$, $\angle ABC\cong\angle A'B'C'$, $\angle BCD\cong\angle B'C'D'$, $\angle ADC\cong\angle A'D'C'$, and $\frac{AB-DC}{A'B'-D'C'}=\frac{AD+BC}{A'D'+B'C'}$. From these we get:

$$\frac{AB-DC}{A'B'-D'C'}=\frac{AD+BC}{A'D'+B'C'}=k, \text{ for } k\in\mathbb{R}^+.$$

$$\Longrightarrow AD+BC=k(A'D'+B'C');\ \Longrightarrow AD+BC=kA'D'+kB'C'.$$

Since $\Delta\mathrm{AFD}\sim\Delta\mathrm{A'F'D'}$ and $\Delta\mathrm{BGC}\sim\Delta\mathrm{B'G'C'}$, we have:

$\Longrightarrow AD=kA'D'$ and $BC=kB'C'$. $\Longrightarrow \frac{AB-DC}{A'B'-D'C'}=\frac{\mathrm{AD}}{\mathrm{A'D'}}=\frac{\mathrm{BC}}{\mathrm{B'C'}}=k.$

$$\Longrightarrow \mathrm{ABCD}\sim\mathrm{A'B'C'D'}.$$

This completes the proof of (2.2).

Corollary 1: Trapezoids $\mathrm{ABCD}\sim\mathrm{A'B'C'D'}$ if and only if $\angle BAD\cong\angle B'A'D'$, $\angle ABC\cong\angle A'B'C'$, $\angle BCD\cong\angle B'C'D'$, $\angle ADC\cong\angle A'D'C'$ and $\frac{AB+DC}{A'B'+D'C'}=\frac{AD+BC}{A'D'+B'C'}$.

Proof. Let $\mathrm{ABCD}\sim\mathrm{A'B'C'D'}$. Then we have $\angle BAD\cong\angle B'A'D'$, $\angle ABC\cong\angle A'B'C'$, $\angle BCD\cong\angle B'C'D'$, $\angle ADC\cong\angle A'D'C'$, and $\frac{\mathrm{AB}}{\mathrm{A'B'}}=\frac{\mathrm{DC}}{\mathrm{D'C'}}=\frac{\mathrm{AD}}{\mathrm{A'D'}}=\frac{\mathrm{BC}}{\mathrm{B'C'}}=k$, for $k\in\mathbb{R}^+$ by definition. From these we have:

$$\left.\begin{array}{l}\mathrm{AB}=k\mathrm{A'B'}\\ \mathrm{DC}=k\mathrm{D'C'}\end{array}\right\} \tag{2.7}$$

$$\left.\begin{array}{l}\mathrm{AD}=k\mathrm{A'D'}\\ \mathrm{BC}=k\mathrm{B'C'}\end{array}\right\} \tag{2.8}$$

Using (2.7) - (2.8) and after simplifications with little algebra, we obtain:

$$\frac{AB+DC}{A'B'+D'C'}=\frac{AD+BC}{A'D'+B'C'}=k.$$

Let $\angle BAD\cong\angle B'A'D'$, $\angle ABC\cong\angle A'B'C'$, $\angle BCD\cong\angle B'C'D'$, $\angle ADC\cong\angle A'D'C'$, and $\frac{AB+DC}{A'B'+D'C'}=\frac{AD+BC}{A'D'+B'C'}$. From these we get:

$$\frac{AB+DC}{A'B'+D'C'}=\frac{AD+BC}{A'D'+B'C'}=k, \text{ for } k\in\mathbb{R}^+.$$

$$\Longrightarrow AD+BC=k(A'D'+B'C');\ \Longrightarrow AD+BC=kA'D'+kB'C'.$$

Since $\Delta\mathrm{AFD}\sim\Delta\mathrm{A'F'D'}$ and $\Delta\mathrm{BGC}\sim\Delta\mathrm{B'G'C'}$, we have:

$\Longrightarrow AD=kA'D'$ and $BC=kB'C'$. $\Longrightarrow \frac{AB+DC}{A'B'+D'C'}=\frac{\mathrm{AD}}{\mathrm{A'D'}}=\frac{\mathrm{BC}}{\mathrm{B'C'}}=k.$

Again we have:

$$\Longrightarrow AB+DC=k(A'B'+D'C');\ \Longrightarrow AB+DC=kA'B'+kD'C'.$$

$$\Longrightarrow AB=kA'B' \text{ and } DC=kD'C';\ \text{which can be proved by contradiction.}$$

$$\Longrightarrow \frac{AB}{A'B'} = \frac{DC}{D'C'} = k. \quad \Longrightarrow \frac{AB}{A'B'} = \frac{DC}{D'C'} = \frac{AD}{A'D'} = \frac{BC}{B'C'} = k, \text{ for } k \in \mathbb{R}^+.$$

$$\Longrightarrow \mathrm{ABCD} \sim \mathrm{A'B'C'D'}.$$

This completes the proof of corollary 1.

3. Conclusions

We found the following conditions from main result. Triangles $\Delta\mathrm{ABE} \sim \Delta\mathrm{A'B'E'} \Leftrightarrow \angle BAE \cong \angle B'A'E'$, $\angle ABE \cong \angle A'B'E'$, $\angle AEB \cong \angle A'E'B'$, and $\frac{AB+BE}{A'B'+B'E'} = \frac{AE}{A'E'}$ or $\frac{AB+AE}{A'B'+A'E'} = \frac{BE}{B'E'}$ or $\frac{AE+BE}{A'E'+B'E'} = \frac{AB}{A'B'}$; and trapezoids $\mathrm{ABCD} \sim \mathrm{A'B'C'D'} \Leftrightarrow \angle BAD \cong \angle B'A'D'$, $\angle ABC \cong \angle A'B'C'$, $\angle BCD \cong \angle B'C'D'$, $\angle ADC \cong \angle A'D'C'$, and $\frac{AB-DC}{A'B'-D'C'} = \frac{AD+BC}{A'D'+B'C'}$ or $\frac{AB+DC}{A'B'+D'C'} = \frac{AD+BC}{A'D'+B'C'}$. Significance of this study is to provide similarity properties of both triangles and trapezoids in short form.

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