Mathematical Theory and Modeling
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.7, No.3, 2017
www.iiste.org

On Rational Solutions for some systems

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ABSTRACT: In this work, Laurent series expansion compared with definition of rational function is used to find rational solutions with N simple poles for some systems.

Introduction

During the last three decades, there has been a great deal of interest in rational solution of integrable nonlinear evolution equations: This began with studies of the rational solutions of the Korteweg-de Vries (KdV) and Kadomtsev-Petviashvilli (KP) equations, but soon similar results were obtained concerning the Benjamin-Ono equation and the classical Boussinesq and AKNS systems, [2,3].

Rational solutions were found in the majority of the equations of mathematical physic that possess a rich algebraic structure: an infinite set of symmetries and related commuting flows. It was the rational solutions of such equations are special limiting forms of exponential "multisoliton" solutions and can be obtained from the latter once by long-wave degeneration, [6].

Further applications of rational solutions to soliton equations include the description of explode-decay waves and vortex solutions of the complex sine-Gordon equation, [3].

The direct methods for the construction of rational solutions have great importance in mathematical physics. For instance, such methods are based on the transformation of various forms of the τ function to the equation of interest, or on the group theory, or on the analytic properties of the Baker-Akhiezer function, [6].

The goal of this work is to present a method for finding rational solutions with N simple poles for some systems.

Description of the Method

Consider a given nonlinear PDE, say in two variables *x* and t

$$\frac{\partial u}{\partial t} = F(x,t,u,\frac{\partial u}{\partial x},\frac{\partial^2 u}{\partial x^2},\ldots,\frac{\partial^n u}{\partial x^n}) \qquad ,x\in C,t\in R \qquad \ldots(1.1)$$

In order to apply this method, we need the following steps:

Step 1: We assume that u(x,t) is a rational solution of the equation (1.1) with N simple poles, we use

$$u(x,t)=\frac{R_1(t)}{x-x_1(t)}+a_0(t)+a_1(t)(x-x_1(t)) \qquad \ldots(1.2)$$

where $R_1(t)$ is the residue of u(x,t) near $x_1(t)$ and

$$a_0(t)=\sum_{k=2}^{N}\frac{R_k(t)}{x_1(t)-x_k(t)}\ , \ x_1(t)\neq x_k(t)$$

$$a_1(t)=\sum_{k=2}^{N}\frac{-R_k(t)}{\left(x_1(t)-x_k(t)\right)^2}, \ x_1(t)\neq x_k(t).$$

Step 2: Letting $x - x_1$ to be ε, then Substitute the following derivatives of u(x,t) with respect the variables x and t in the equation

$$\frac{\partial u}{\partial t}=\frac{\dot{R}_1}{\varepsilon}+\frac{R_1\dot{x}_1}{\varepsilon^2}+\dot{a}_0+\dot{a}_1\varepsilon-a_1\dot{x}_1 \qquad \ldots(1.3a)$$

$$\frac{\partial^2 u}{\partial t^2}=\frac{\ddot{R}_1}{\varepsilon}+\frac{2\dot{R}_1\dot{x}_1+R_1\ddot{x}_1}{\varepsilon^2}+\frac{2R_1\dot{x}_1^2}{\varepsilon^3}+\ddot{a}_0a-2\dot{a}_1\dot{x}_1+\ddot{a}_1\varepsilon-a_1\ddot{x}_1 \qquad \ldots(1.3b)$$

$$\frac{\partial u}{\partial x}=\frac{-R_1(t)}{\varepsilon}+a_1(t)$$

$$\frac{\partial^n u}{\partial x^n}=\frac{(-1)^n n!R_1}{\varepsilon^{n+1}} \qquad n=2,3,\ldots \qquad \ldots(1.3c)$$

Step 3: Equating the coefficients of ε^i to zero, to have a nonlinear system of algebraic or differential equations and then solve the system.

Remarks*:*

(i) Our method is based on the comparison of the usual assumption for function with N simple poles

$$u(x,t)=\sum_{k=1}^{N}\frac{R_k}{x-x_k}\ , \ x\in £ \qquad \ldots(1.4)$$

and expand of functions by formal Laurent series near a specified pole (say x_1):

$$u(x,t)=\sum_{n=-1}^{\infty}a_n(x-x_1)^n \qquad \ldots(1.5)$$

Denote a_{-1} by R_1. We can see that

$$\sum_{n=0}^{\infty} a_n(t)(x - x_1(t))^n = \sum_{k=2}^{N} \frac{R_k(t)}{x - x_k(t)} \qquad \ldots(1.6)$$

(ii) The coefficients $a_0(t)$, $a_1(t)$, … in (1.6) carry information about the poles x_2, …, x_N.

(iii) The ansatz that we adopt, is:

$$u(x,t) = \frac{R_1(t)}{x - x_1(t)} + \sum_{n=0}^{\infty} a_n(t)\left(x - x_k(t)\right)^n \qquad \ldots(1.7)$$

and because of the difficulty of equating coefficients for $(x - x_1(t))$ of the nonlinear from PDE. Hence we will write the form (1.7) as (1.2) which we have directly in the first step of the method.

Rational Solution for Carleman Mode

In this section, an example illustrating that u and v may have a common pole has been given

We consider the system of PDEs

$$u_t + u_x = v^2 - u^2 \qquad \ldots(2.1a)$$

$$v_t - v_x = u^2 - v^2 \qquad \ldots(2.1b)$$

Which was introduced by T. Carleman as the two velocity model of the Boltzman equation,[7].

Let $x_1(t)$ be the common pole of u and v.

To apply the method of finding rational solution with N simple poles, the following ansatz will be used:

$$u(x,t) = \frac{R_1(t)}{x - x_1} + a_0(t) + a_1(t)(x - x_1) \qquad \ldots(2.2a)$$

$$v(x,t) = \frac{S_1(t)}{x - x_1} + b_0(t) + b_1(t)(x - x_1) \qquad \ldots(2.2b)$$

where

$$a_0(\mathrm{t})=\sum_{\mathrm{k=2}}^{\mathrm{N_1}}\frac{\mathrm{R_k(t)}}{x_1(\mathrm{t})-x_{\mathrm{k}}(\mathrm{t})},\ a_1(\mathrm{t})=\sum_{\mathrm{k=2}}^{\mathrm{N_1}}\frac{-\mathrm{R_k(t)}}{\left(x_1(\mathrm{t})-x_{\mathrm{k}}(\mathrm{t})\right)^2},\ b_0(\mathrm{t})=\sum_{\mathrm{k=2}}^{\mathrm{N_2}}\frac{\mathrm{S_k(t)}}{x_1(\mathrm{t})-x_{\mathrm{k}}(\mathrm{t})},\ \text{and}$$

$$b_1(\mathrm{t})=\sum_{\mathrm{k=2}}^{\mathrm{N_2}}\frac{-\mathrm{S_k(t)}}{\left(x_1(\mathrm{t})-x_{\mathrm{k}}(\mathrm{t})\right)^2}.$$

By substituting the functions (2.2) and their derivatives in equations (2.1) , the following identities have been obtained:

$$\begin{aligned}&R_1\overset{\bullet}{x_1}\varepsilon^{-2}+\overset{\bullet}{R_1}\varepsilon^{-1}+\overset{\bullet}{a_0}+\overset{\bullet}{a_1}\varepsilon-a_1\overset{\bullet}{x_1}-R_1\varepsilon^{-2}+a_1=S_1^2\varepsilon^{-2}+b_0^2+b_1^2\varepsilon^2+\\&2b_0S_1\varepsilon^{-1}+2b_1S_1+2b_0b_1\varepsilon-R_1^2\varepsilon^{-2}-a_0^2-a_1^2\varepsilon^2-2a_0R_1\varepsilon^{-1}-2a_1R_1-2a_0a_1\varepsilon\end{aligned}\qquad\ldots(2.3\text{a})$$

and

$$\begin{aligned}&S_1\overset{\bullet}{x_1}\varepsilon^{-2}+\overset{\bullet}{S_1}\varepsilon^{-1}+\overset{\bullet}{b_0}+\overset{\bullet}{b_1}\varepsilon-b_1\overset{\bullet}{x_1}+S_1\varepsilon^{-2}-b_1=R_1^2\varepsilon^{-2}+a_0^2+a_1^2\varepsilon^2+\\&2a_0R_1\varepsilon^{-1}+2a_1R_1+2a_0a_1\varepsilon-S_1^2\varepsilon^{-2}-b_0^2-b_1^2\varepsilon^2-2b_0S_1\varepsilon^{-1}-2b_1S_1-b_0b_1\varepsilon\end{aligned}\qquad\ldots(2.3\text{b})$$

From the coefficients of ε^2 in (2.3a) and (2.3b) we get:

$b_1^2-a_1^2=0$

$a_1^2-b_1^2=0$ then $a_1=\pm\, b_1$.

The coefficients of ε give:

$\overset{\bullet}{a_1}=2b_0b_1-2a_0a_1$

$\overset{\bullet}{b_1}=2a_0a_1-2b_0b_1$ then $\overset{\bullet}{a_1}=-\overset{\bullet}{b_1}$ which means $a_1=-\,b_1$.

From the coefficients of ε^0 we get:

$$a_0^{\bullet} - a_1 x_1^{\bullet} + a_1 = b_0^2 + 2b_1 S_1 - a_0^2 - 2a_1 R_1$$
$$b_0^{\bullet} - b_1 x_1^{\bullet} - b_1 = a_0^2 + 2a_1 R_1 - b_0^2 - 2b_1 S_1$$

Then

$$b_0^{\bullet} = -a_0^{\bullet} - 2a_1$$

and $\mathrm{b}_0 = -3a_0 + \mathrm{c}_1$ where c_1 arbitrary constants.

The coefficients of ε^{-1} give:

$$R_1^{\bullet} = 2b_0 S_1 - 2a_0 R_1$$
$$S_1^{\bullet} = 2a_0 R_1 - 2b_0 S_1$$

Then $\mathrm{R}_1(\mathrm{t}) = -\mathrm{S}_1(\mathrm{t}) + \mathrm{c}_2$, c_2 arbitrary constant.

From the coefficients of ε^{-2} we get:

$$R_1 x_1^{\bullet} - R_1 = S_1^2 - R_1^2$$
$$S_1 x_1^{\bullet} + S_1 = R_1^2 - S_1^2$$

Then $x_1^{\bullet} = -2S_1 + 1,$

and

$$x_1 = -2\mathrm{S}_1\mathrm{t} + \mathrm{t} + \mathrm{c}_3\ ,\ \ \mathrm{c}_3 \text{ arbitrary constant.}$$

Hence the solution will be:

$$\mathrm{u}(x,\mathrm{t}) = \frac{-\mathrm{S}_1 + \mathrm{c}_2}{x + 2\mathrm{S}_1\mathrm{t} - \mathrm{t} - \mathrm{c}_3} + a_0 + \mathrm{a}_1(x + 2\mathrm{S}_1\mathrm{t} - \mathrm{t} - \mathrm{c}_3)$$
$$\mathrm{v}(x,\mathrm{t}) = \frac{\mathrm{S}_1}{x + 2\mathrm{S}_1\mathrm{t} - \mathrm{t} - \mathrm{c}_3} - 3a_0 + \mathrm{c}_1 - \mathrm{a}_1(x + 2\mathrm{S}_1\mathrm{t} - \mathrm{t} - \mathrm{c}_3)$$

where c_1 and c_3 are arbitrary constants.

Dispersive Long Wave System and Rational Solution

Consider the following dispersive long wave system:

$$u_t = (u^2 - u_x + 2w)_x \qquad \ldots(2.4a)$$

$$w_t = (2uw + w_x)_x \qquad \ldots(2.4b)$$

The analytical solution of (2.4) with initial conditions:

$u(x,0)=2+2\tanh(x)$ and $w(x,0)=-1+2\,\mathrm{sech}(x)^2$ is given by Bai et al. as follows:[5]

$u(x,t)=2+2\tanh(x-2t)$ and $w(x,t)=-1+2\,\mathrm{sech}\,(x-2t)^2$.

The results on analytical and numerical solutions for system of dispersive long wave in (2 + 1) dimension can be found in [1] and [5].

Also, Zhang Jie and Fang Guo obtained a nonlinear transformation for simplifying a dispersive long wave equation by using the homogeneous method and found some new multi-Soliton solutions and exact traveling solution of the dispersive long wave equations from the linear partial equation, [4].

In this section, this example shows that one of the functions of a system may have rational solution while the other is not.

In order to obtain the rational solutions with N simple Poles for the system (2.4), we assume that the functions $u(x,t)$ and $w(x,t)$ have a common simple pole $x_1(t)$ and using the formulas:

$$u(x,t)=\frac{R_1(t)}{x-x_1}+a_0(t)+a_1(t)(x-x_1) \qquad \ldots(2.5a)$$

$$w(x,t)=\frac{S_1(t)}{x-x_1}+b_0(t)+b_1(t)(x-x_1) \qquad \ldots(2.5b)$$

Plugging (2.5) and their derivatives in equations (2.4) gives

$$R_1 \dot{x}_1 \varepsilon^{-2} + \dot{R}_1 \varepsilon^{-1} + \dot{a}_0 + \dot{a}_1 \varepsilon - a_1 \dot{x}_1 =$$
$$-2R_1^2 \varepsilon^{-3} - a_0 R_1 \varepsilon^{-2} + 2a_0 a_1 + 2a_1^2 \varepsilon - 2R_1 \varepsilon^{-3} - 2S_1 \varepsilon^{-2} + 2b_1$$

and

$$S_1 \dot{x}_1 \varepsilon^{-2} + \dot{S}_1 \varepsilon^{-1} + \dot{b}_0 + \dot{b}_1 \varepsilon - b_1 \dot{x}_1 =$$
$$2(R_1 \varepsilon^{-1} + a_0 + a_1 \varepsilon)(-S_1 \varepsilon^{-2} + b_1) + 2(-R_1 \varepsilon^{-2} + a_1)(S_1 \varepsilon^{-1} + b_0 + b_1 \varepsilon) + 2S_1 \varepsilon^{-2}$$

By equating the coefficients of ε^{-3} and ε^{-2} to zero we get:

$R_1(t) = -1$,

$S_2(t) = 0$,

$\dot{x}_1 = -2a_0$, and

$b_0(t) = 0$

The coefficients of ε^{-1} vanish identically, while the coefficient of ε^0 give:

$$\dot{a}_0 = 2b_1$$

From the coefficient of ε we get:

$$\dot{a}_1 = 2a_1^2$$
$$\dot{b}_1 = 4a_1 b_1$$

If $a_1 \neq 0$ then

$$a_1(t) = \frac{-1}{2t + c_1}$$

$$b_1(t) = \frac{c_2}{(2t + c_1)^2}$$

$$a_0(t) = \frac{-2c_2}{2t + c_1} + c_3$$

and $x_1(t) = 2c_2 \ln|2t + c_1| - 2c_3 t + c_4$

Where c_1, c_2, c_3 and c_4 are arbitrary constants.

Hence the solutions for equations (2.4) will be:

$$u(x,t)=\frac{-1}{x-2c_2\ln|2t+c_1|+2c_3t-c_4}-\frac{2c_2}{2t+c_1}+c_3-\frac{x-2c_2\ln|2t+c_1|+2c_3t-c_4}{2t+c_1}$$

…(2.6a)

$$w(x,t)=\frac{x-2c_2\ln|2t+c_1|+2c_3t-c_4}{(2t+c_1)^2} \qquad \ldots(2.6b)$$

The pole of (2.6a) is the zero of (2.6b).

When $a_1 = 0$ then u and w becomes:

$$u(x,t)=\frac{-1}{x+k_1t^2+2k_2t+k_3}+2k_1t+k_2$$

$$w(x,t)=k_1(x+2k_1t^2+2k_2t+k_3)$$

where k_1, k_2 and k_3 are an arbitrary constants.

Conclusion

In this method for obtaining rational solutions with simple poles using Laurent series expansion, the exact solution for nonlinear PDE using the whole Laurent series expansion may be difficult to be evaluated especially in nonlinear term . We expect there is a relation between the number of poles and the number of terms of using power series expansion.

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