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# On Some Linear and Non-linear Fuzzy Integral Equations by Homotopy Perturbation Method

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**Abstract**
Many mathematical models are contributed to give rise to of linear and nonlinear integral equations. In this paper, we study the performance of recently developed technique homotopy perturbation method by implement on various types of linear and non-linear fuzzy Volterra integral equations of second kind, mixed fuzzy volterra fredholm integral equation and singular fuzzy integral equations. Obtained results show that technique is reliable, efficient and easy to use through recursive relations that involve integrals. Moreover, these particular examples show the reliability and the performance of proposed modifications.
**Keywords:** Homotopy perturbation method, linear fuzzy integral equations, non-linear fuzzy integral equations.

## 1. Introduction

Integral equation plays a vital role with in many disciplines of sciences, engineering and mathematics. Using of integral equations with exact parameter within many modeling physical problems is not quite easy or better to say impossible in real problems. To overcome this difficulty one of the most recent approach is to use fuzzy concept. Basic concept of fuzzy was first introduced by professor Zadeh in 1965 after his publication on fuzzy set theory [1, 2]. Thus in 1978 Dubois and Prade introduced the concept of arithmetic operations on fuzzy numbers or can say they presented the fuzzy calculus [3, 4], then as well as time pass many different fields of mathematics use this concept of fuzzy set theory and introduced fuzzy functions, relations, groups, subgroups etc. Recently twenty years ago in Japan a person name M. Sugeno introduced the concept of fuzzy integrals [5, 6], then it's becoming a research oriented topic. Hence mathematicians preferred to use fuzzy integral equation instead of using deterministic models of integral equations. Homotopy perturbation method is a coupling of perturbation method and homotopy technique was firstly introduced by He JH in 1999 [7, 8], then it was farther developed by him [9, 10]. This method successively proved accurate and fastly convergent. HPM is one of the most reliable and affective method to find the solution of linear and non-linear fuzzy integral equations. In this paper we shall discuss the analysis of HPM for fuzzy integral equation and concluded their advantage in application form on linear and non-linear problems of fuzzy integral equations.

## 2. Preliminaries

Let's introduced the notation needed in this paper. The bar under or over symbols denoted the fuzzy functions. Fuzzy number $u(\alpha)$ is basically the parametric representation of pair of functions $(\overline{u}(\alpha), \underline{u}(\alpha))$, $0 \le \alpha \le 1$ which satisfied the following properties [11]:

I. $\overline{u}(\alpha)$ Is bounded monotonically increasing function or can say is a bounded left continuous non-decreasing function over [0, 1].
II. $\underline{u}(\alpha)$ Is bounded monotonically decreasing function or can say is a bounded left continuous non-increasing function over [0, 1].
III. $\overline{u}(\alpha) \le \underline{u}(\alpha), 0 \le \alpha \le 1.$

If $\overline{u}(\alpha) = \underline{u}(\alpha), 0 \le \alpha \le 1$ then it's becomes Crisp number.

## 3. Definition: Fuzzy Integral Equation

An integral equation

$$u(x,\alpha) = f(x,\alpha) + \lambda \int_{a(x)}^{b(x)} k(x,t)u(t,\alpha)dt \tag{1}$$

is called fuzzy integral equation of second kind. Where $u(x,\alpha)$ and $f(x,\alpha)$ are fuzzy functions, $\alpha$ is the fuzzy parameter whose value lies between $[0,1]$ i.e. $0 \le \alpha \le 1$, $\lambda$ is constant parameter, $k(x,t)$ is known function of two variables x and t called kernel of fuzzy integral equation, $a(x)$ and $b(x)$ are limits of fuzzy integral equation, if both of limits $a(x)$ and $b(x)$ are constant, then integral equation is known as fredholm

fuzzy integral equation, if one of limit can say $a(x)$ is constant and one of limit say $b(x)$ is variable then equation is called fuzzy volterra integral equation.
The parametric representation of Eq. (1) is as follows,

$$\begin{cases} \underline{u}(x,\alpha) = \underline{f}(x,\alpha) + \lambda \int_{a(x)}^{b(x)} \underline{k(x,t)u(t,\alpha)}dt \\ \overline{u}(x,\alpha) = \overline{f}(x,\alpha) + \lambda \int_{a(x)}^{b(x)} \overline{k(x,t)u(t,\alpha)}dt \end{cases}$$

For $0 \le \alpha \le 1$
Where $u(x,\alpha) = (\underline{u}(x,\alpha), \overline{u}(x,\alpha))$ , $f(x,\alpha) = (\underline{f}(x,\alpha), \overline{f}(x,\alpha))$ and

$$\begin{cases} \underline{k(x,t)u(t,\alpha)} = k(x,t)\underline{u}(t,\alpha) \qquad k(x,t) \ge 0 \\ \overline{k(x,t)u(t,\alpha)} = k(x,t)\overline{u}(t,\alpha) \qquad k(x,t) \ge 0 \end{cases}$$

**4. Analysis of HPM to Fuzzy Integral Equations**
To solve Eq. (1) by HPM 1st we construct following homotopy,

$$\begin{cases} H(\underline{v},p,\alpha) = (1-p)\left[\underline{v}(x,\alpha) - \underline{u}_0(x,\alpha)\right] + p\left[\underline{v}(x,\alpha) - \underline{f}(x,\alpha) - \int_{a(x)}^{b(x)} k(x,t)\underline{v}(t,\alpha)dt\right] = 0 \\ H(\overline{v},p,\alpha) = (1-p)\left[\overline{v}(x,\alpha) - \overline{u}_0(x,\alpha)\right] + p\left[\overline{v}(x,\alpha) - \overline{f}(x,\alpha) - \int_{a(x)}^{b(x)} k(x,t)\overline{v}(t,\alpha)dt\right] = 0 \end{cases} \tag{2}$$

Thus the initial approximation is taken a

$$\begin{cases} \underline{u}_0(x,\alpha) = \underline{f}(x,\alpha) \\ \overline{u}_0(x,\alpha) = \overline{f}(x,\alpha) \end{cases} \tag{3}$$

Substituting Eq. (3) in Eq. (2) reduces to

$$\begin{cases} \underline{v}(x,\alpha) = \underline{f}(x,\alpha) + p\int_{a(x)}^{b(x)} k(x,t)\underline{v}(t,\alpha)dt \\ \overline{v}(x,\alpha) = \overline{f}(x,\alpha) + p\int_{a(x)}^{b(x)} k(x,t)\overline{v}(t,\alpha)dt \end{cases} \tag{4}$$

The solution of Eq. (2) is assumed as

$$\begin{cases} \underline{v}(x,\alpha) = \sum_{i=0}^{\infty} p^i \underline{v}_i(x,\alpha) \\ \overline{v}(x,\alpha) = \sum_{i=0}^{\infty} p^i \overline{v}_i(x,\alpha) \end{cases} \tag{5}$$

Where $(\underline{v}_i, \overline{v}_i)$ are unknown to determined.
Now by putting Eq. (5) in Eq. (4) and by comparing coefficient like power of p we get
The following iterations

$$p^0 : \begin{cases} \underline{v}_0(x,\alpha) = \underline{f}(x,\alpha) \\ \overline{v}_0(x,\alpha) = \overline{f}(x,\alpha) \end{cases} \tag{6}$$

$$p^1 : \begin{cases} \underline{v}_1(x,\alpha) = \int_{a(x)}^{b(x)} k(x,t)\underline{v}_0(t,\alpha)dt \\ \overline{v}_1(x,\alpha) = \int_{a(x)}^{b(x)} k(x,t)\overline{v}_0(t,\alpha)dt \end{cases} \tag{7}$$

$$p^2:\begin{cases}\underline{v}_2(x,\alpha)=\int_{a(x)}^{b(x)} k(x,t)\underline{v}_1(t,\alpha)dt\\ \overline{v}_2(x,\alpha)=\int_{a(x)}^{b(x)} k(x,t)\overline{v}_1(t,\alpha)dt\end{cases} \quad (8)$$

and so on…

Thus the solution of FIE-2 is given as

$$\begin{cases}\underline{u}(x,\alpha)=\lim_{p\to 1}\underline{v}(x,\alpha)=\sum_{i=0}^{\infty}\underline{v}_i(x,\alpha)\\ \overline{u}(x,\alpha)=\lim_{p\to 1}\overline{v}(x,\alpha)=\sum_{i=0}^{\infty}\overline{v}_i(x,\alpha)\end{cases} \quad (9)$$

## 5. Numerical Examples

**Example 5.1** Consider the fuzzy volterra integral equation of 2nd kind

$$u(x,\alpha)=f(x,\alpha)+\lambda\int_0^x \sinh x u(x,\alpha)dt \quad (10)$$

Where

$\lambda=1,\ 0\le x\le 1,\ 0\le t\le x,\ 0\le\alpha\le 1,\ k(x,t)=\sinh x$ *and* $f(x,\alpha)=(\underline{f}(x,\alpha),\overline{f}(x,\alpha))$ *i.e.*

$$f(x,\alpha)=((\cosh x+1-\cosh^2 x)(\alpha^2+\alpha),(\cosh x+1-\cosh^2 x)(4-\alpha^3-\alpha))$$

To solve Eq. (10) by homotopy perturbation method we construct following homotopy,

$$\begin{cases}H(\underline{v},p,\alpha)=\underline{v}(x,\alpha)-(\cosh x+1-\cosh^2 x)(\alpha^2+\alpha)-p\int_0^x \sinh x\underline{v}(t,\alpha)dt=0\\ H(\overline{v},p,\alpha)=\overline{v}(x,\alpha)-(\cosh x+1-\cosh^2 x)(4-\alpha^3-\alpha)-p\int_0^x \sinh x\overline{v}(t,\alpha)dt=0\end{cases} \quad (11)$$

Assume the solution of Eq. (11) can be assume as power series in p

$$\begin{cases}\underline{v}(x,\alpha)=\sum_{i=0}^{\infty}p^i\underline{v}_i(x,\alpha)\\ \overline{v}(x,\alpha)=\sum_{i=0}^{\infty}p^i\overline{v}_i(x,\alpha)\end{cases} \quad (12)$$

Utilizing Eq. (12) in Eq. (11) and by comparing coefficients like power of p we get

$$p^0:\begin{cases}\underline{v}_0(x,\alpha)=\underline{f}(x,\alpha)=(\cosh x+1-\cosh^2 x)(\alpha^2+\alpha)\\ \overline{v}_0(x,\alpha)=\overline{f}(x,\alpha)=(\cosh x+1-\cosh^2 x)(4-\alpha^3-\alpha)\end{cases} \quad (13)$$

$$p^1:\begin{cases}\underline{v}_1(x,\alpha)=\frac{-1}{4}(\alpha^2+\alpha)\sinh x(\sinh 2x-4\sinh x-2x)\\ \overline{v}_1(x,\alpha)=\frac{-1}{4}(4-\alpha^3-\alpha)\sinh x(\sinh 2x-4\sinh x-2x)\end{cases} \quad (14)$$

$$p^2:\begin{cases}\underline{v}_2(x,\alpha)=\frac{-1}{4}(\alpha^2+\alpha)\sinh x(\frac{2}{3}\sinh^3 x-2x\cosh x+2\sinh x-\sinh 2x+2x)\\ \overline{v}_2(x,\alpha)=\frac{-1}{4}(4-\alpha^3-\alpha)\sinh x(\frac{2}{3}\sinh^3 x-2x\cosh x+2\sinh x-\sinh 2x+2x)\end{cases} \quad (15)$$

$$\vdots$$

 

As we know the solution is given as

$$\begin{cases} \underline{u}(x,\alpha) = \sum_{i=0}^{\infty} \underline{v}_i(x,\alpha) \\ \overline{u}(x,\alpha) = \sum_{i=0}^{\infty} \overline{v}_i(x,\alpha) \end{cases} \tag{16}$$

Thus by utilizing above iterative results the series solution is given as

$$\begin{cases} \underline{u}(x,\alpha) = (\alpha^2+\alpha)(\cosh x - \sinh^2 x + \sinh^2 x - \frac{1}{2}x\sinh x + \frac{1}{2}x\sinh x - \frac{1}{4}\sinh x \sinh 2x + ...) \\ \overline{u}(x,\alpha) = (4-\alpha^3-\alpha)(\cosh x - \sinh^2 x + \sinh^2 x - \frac{1}{2}x\sinh x + \frac{1}{2}x\sinh x - \frac{1}{4}\sinh x \sinh 2x + ...) \end{cases} \tag{17}$$

And the exact solution is given as

$$\begin{cases} \underline{u}(x,\alpha) = \cosh x(\alpha^2+\alpha) \\ \overline{u}(x,\alpha) = \cosh x(4-\alpha^3-\alpha) \end{cases}. \tag{18}$$

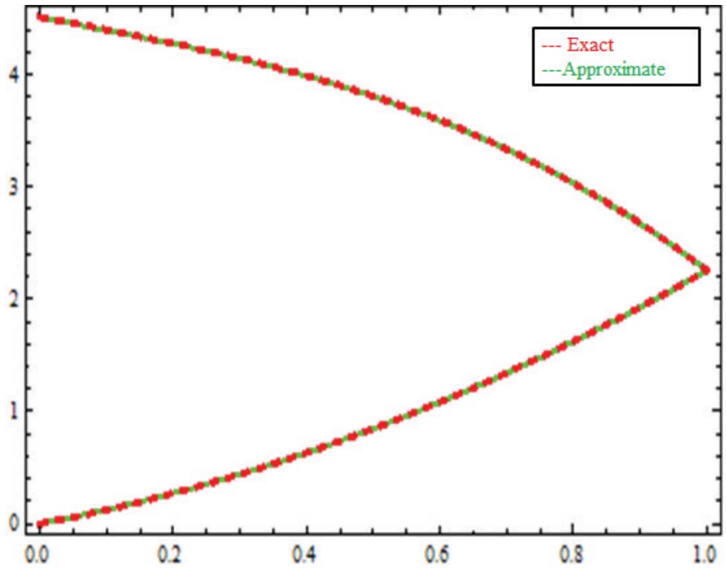


**Fig.1.** Plot of approximate and exact solution

**Example 5.2** Consider fuzzy fredholm integral equation of 2nd kind

$$u(x,\alpha) = f(x,\alpha) + \int_0^1 xtu(t,\alpha)dt \tag{19}$$

Where

$$0 \le x,t \le 1,\ 0 \le \alpha \le 1,\ \ k(x,t) = xt \ \ and \ \ f(x,\alpha) = (\underline{f}(x,\alpha), \overline{f}(x,\alpha)) \ \ i.e.$$

$$f(x,\alpha) = (\frac{2}{3}x\alpha, \frac{2}{3}x(3-\alpha))$$

To solve Eq. (19) by HPM 1st we construct homotopy as follows,

 

$$\begin{cases} H(\underline{v}, p, \alpha) = \underline{v}(x, \alpha) - \frac{2}{3}x\alpha - p\int_0^1 xt\underline{v}(x, \alpha)dt = 0 \\ H(\overline{v}, p, \alpha) = \overline{v}(x, \alpha) - \frac{2}{3}x\alpha - p\int_0^1 xt\overline{v}(x, \alpha)dt = 0 \end{cases} \tag{20}$$

The solution of Eq. (20) can be assumed as power series in p

$$\begin{cases} \underline{v}(x, \alpha) = \sum_{i=0}^{\infty} p^i \underline{v}_i(x, \alpha) \\ \overline{v}(x, \alpha) = \sum_{i=0}^{\infty} p^i \overline{v}_i(x, \alpha) \end{cases} \tag{21}$$

Now by putting Eq. (21) in Eq. (20) and by comparing coefficients like power of p we get

$$p^0 : \begin{cases} \underline{v}_0(x, \alpha) = \underline{f}(x, \alpha) = \frac{2}{3}x\alpha \\ \overline{v}_0(x, \alpha) = \overline{f}(x, \alpha) = \frac{2}{3}x(3-\alpha) \end{cases} \tag{22}$$

$$p^1 : \begin{cases} \underline{v}_1(x, \alpha) = \frac{2}{9}x\alpha \\ \overline{v}_1(x, \alpha) = \frac{2}{9}x(3-\alpha) \end{cases} \tag{23}$$

$$p^2 : \begin{cases} \underline{v}_2(x, \alpha) = \frac{2}{27}x\alpha \\ \overline{v}_2(x, \alpha) = \frac{2}{27}x(3-\alpha) \end{cases} \tag{24}$$

$$p^3 : \begin{cases} \underline{v}_3(x, \alpha) = \frac{2}{81}x\alpha \\ \overline{v}_3(x, \alpha) = \frac{2}{81}x(3-\alpha) \end{cases} \tag{25}$$

And so on…
As we know the solution is given as

$$\begin{cases} \underline{u}(x, \alpha) = \sum_{i=0}^{\infty} \underline{v}_i(x, \alpha) \\ \overline{u}(x, \alpha) = \sum_{i=0}^{\infty} \overline{v}_i(x, \alpha) \end{cases} \tag{26}$$

Thus by utilizing above iterative results the series form solution is given as

$$\begin{cases} \underline{u}(x, \alpha) = \frac{2}{3}x\alpha + \frac{2}{9}x\alpha + \frac{2}{27}x\alpha + \frac{2}{81}x\alpha + ... \\ \overline{u}(x, \alpha) = \frac{2}{3}x(3-\alpha) + \frac{2}{9}x(3-\alpha) + \frac{2}{27}x(3-\alpha) + \frac{2}{81}x(3-\alpha) + ... \end{cases} \tag{27}$$

And the exact form solution is given as

$$\begin{cases} \underline{u}(x,\alpha) = x\alpha \\ \overline{u}(x,\alpha) = x(3-\alpha) \end{cases}. \tag{28}$$

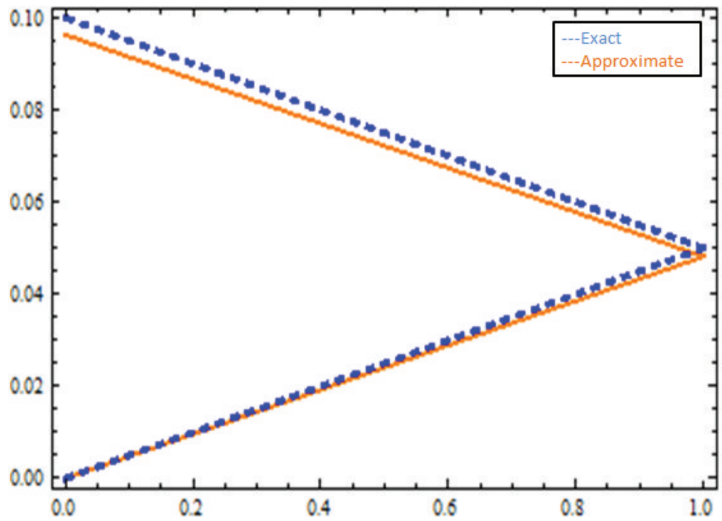


**Fig.2.** Plot of approximate and exact solution

**Example 5.3** Consider the mixed fuzzy volterra fredholm integral equation of 2nd kind

$$u(x,\alpha) = f(x,\alpha) + \int_0^x \int_0^{\frac{\pi}{2}} r u(t,\alpha)\,dt\,dr \tag{29}$$

Where

$$0 \le t \le \frac{\pi}{2}, 0 \le r \le x, 0 \le x \le 1, 0 \le \alpha \le 1,\ k(x,r,t) = r \ \ and \ \ f(x,\alpha) = (\underline{f}(x,\alpha), \overline{f}(x,\alpha)) \ \ i.e.$$

$$f(x,\alpha) = ((\sin x - \frac{1}{2}x^2)(\alpha^2+1), (\sin x - \frac{1}{2}x^2)(3-\alpha^3-\alpha))$$

To solve Eq. (29) by HPM 1st we construct homotopy as follows,

$$\begin{cases} H(\underline{v},p,\alpha) = \underline{v}(x,\alpha) - (\sin x - \frac{1}{2}x^2)(\alpha^2+1) - p\int_0^x \int_0^{\frac{\pi}{2}} r\underline{v}(t,\alpha)\,dt\,dr = 0 \\ H(\overline{v},p,\alpha) = \overline{v}(x,\alpha) - (\sin x - \frac{1}{2}x^2)(3-\alpha^3-\alpha) - p\int_0^x \int_0^{\frac{\pi}{2}} r\overline{v}(t,\alpha)\,dt\,dr = 0 \end{cases} \tag{30}$$

The solution of Eq. (30) can be assumed as a power series in p

$$\begin{cases} \underline{v}(x,\alpha) = \sum_{i=0}^{\infty} p^i \underline{v}_i(x,\alpha) \\ \overline{v}(x,\alpha) = \sum_{i=0}^{\infty} p^i \overline{v}_i(x,\alpha) \end{cases} \tag{31}$$

By putting Eq. (31) in Eq. (30) and by comparing coefficients like power of p we get

$$p^0 : \begin{cases} \underline{v}_0(x,\alpha) = \underline{f}(x,\alpha) = (\sin x - \frac{1}{2}x^2)(\alpha^2+1) \\ \overline{v}_0(x,\alpha) = \overline{f}(x,\alpha) = (\sin x - \frac{1}{2}x^2)(3-\alpha^3-\alpha) \end{cases} \tag{32}$$

$$p^1:\begin{cases}\underline{v}_1(x,\alpha)=(\frac{1}{2}x^2-\frac{\pi^3}{96}x^2)(\alpha^2+1)\\ \overline{v}_1(x,\alpha)=(\frac{1}{2}x^2-\frac{\pi^3}{96}x^2)(3-\alpha^3-\alpha)\end{cases} \tag{33}$$

$$p^2:\begin{cases}\underline{v}_2(x,\alpha)=(\frac{\pi^3}{96}x^2-\frac{\pi^6}{4608}x^2)(\alpha^2+1)\\ \overline{v}_2(x,\alpha)=(\frac{\pi^3}{96}x^2-\frac{\pi^6}{4608}x^2)(3-\alpha^3-\alpha)\end{cases} \tag{34}$$

and so on…

As we know the solution is given as

$$\begin{cases}\underline{u}(x,\alpha)=\sum_{i=0}^{\infty}\underline{v}_i(x,\alpha)\\ \overline{u}(x,\alpha)=\sum_{i=0}^{\infty}\overline{v}_i(x,\alpha)\end{cases} \tag{35}$$

By utilizing the above iterative results the series from solution is given as

$$\begin{cases}\underline{u}(x,\alpha)=(\alpha^2+1)(\sin x-\frac{1}{2}x^2+\frac{1}{2}x^2-\frac{\pi^3}{96}x^2+\frac{\pi^3}{96}x^2-\frac{\pi^6}{4608}x^2+...)\\ \overline{u}(x,\alpha)=(3-\alpha^3-\alpha)(\sin x-\frac{1}{2}x^2+\frac{1}{2}x^2-\frac{\pi^3}{96}x^2+\frac{\pi^3}{96}x^2-\frac{\pi^6}{4608}x^2+...)\end{cases} \tag{36}$$

And the exact form solution is given as

$$\begin{cases}\underline{u}(x,\alpha)=\sin x(\alpha^2+1)\\ \overline{u}(x,\alpha)=\sin x(3-\alpha^3-\alpha)\end{cases} \tag{37}$$

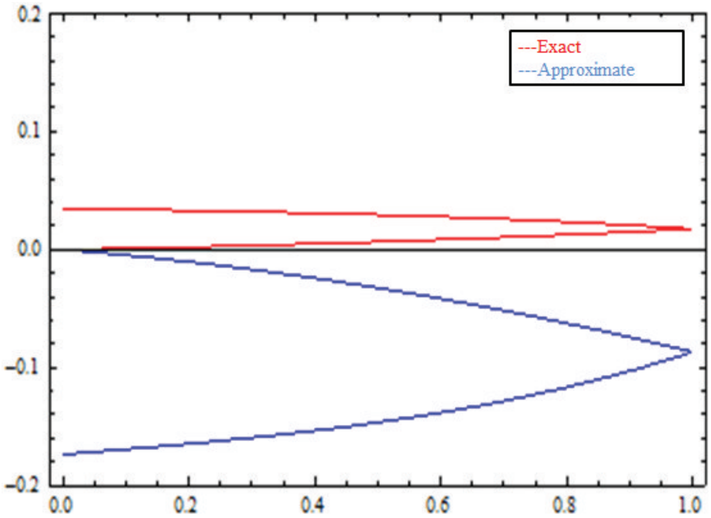


Fig.3.Plot of approximate and exact solution

**Example 5.4** Consider the singular fuzzy integral equation of 2nd kind

$$f(x,\alpha)=\int_0^x\frac{u(t,\alpha)}{\sqrt{x-t}}dt \tag{38}$$

Where

$\lambda = 1,\ 0 \le x \le 1,\ 0 \le t < x, 0 \le \alpha \le 1,\ k(x,t) = \dfrac{1}{\sqrt{x-t}}$ *and* $f(x,\alpha) = (\underline{f}(x,\alpha), \overline{f}(x,\alpha))$ *i.e.*

$$f(x,\alpha) = (x\alpha, x(2-\alpha))$$

Equivalent to

$$f(x,\alpha) = \int_0^x \frac{u(x,\alpha)}{\sqrt{x-t}} + \int_0^x \frac{u(t,\alpha) - u(x,\alpha)}{\sqrt{x-t}} dt \tag{39}$$

$$f(x,\alpha) = 2\sqrt{x}u(x,\alpha) + \int_0^x \frac{u(t,\alpha) - u(x,\alpha)}{\sqrt{x-t}} dt \tag{40}$$

$$u(x,\alpha) = \frac{f(x,\alpha)}{2\sqrt{x}} - \frac{1}{2\sqrt{x}} \int_0^x \frac{u(t,\alpha) - u(x,\alpha)}{\sqrt{x-t}} dt \tag{41}$$

To solve Eq. (41) by HPM 1st we construct homotopy as follows,

$$\begin{cases} H(\underline{v}, p, \alpha) = \underline{v}(x,\alpha) - \dfrac{\alpha\sqrt{x}}{2} - \dfrac{p}{2\sqrt{x}} \displaystyle\int_0^x \frac{\underline{v}(t,\alpha) - \underline{v}(x,\alpha)}{\sqrt{x-t}} dt = 0 \\ H(\overline{v}, p, \alpha) = \overline{v}(x,\alpha) - \dfrac{(2-\alpha)\sqrt{x}}{2} - \dfrac{p}{2\sqrt{x}} \displaystyle\int_0^x \frac{\overline{v}(t,\alpha) - \overline{v}(x,\alpha)}{\sqrt{x-t}} dt = 0 \end{cases} \tag{42}$$

Assume the solution of Eq. (42) can be written as power series in p

$$\begin{cases} \underline{v}(x,\alpha) = \displaystyle\sum_{i=0}^{\infty} p^i \underline{v}_i(x,\alpha) \\ \overline{v}(x,\alpha) = \displaystyle\sum_{i=0}^{\infty} p^i \overline{v}_i(x,\alpha) \end{cases} \tag{43}$$

Utilizing Eq. (43) in Eq. (42) and by comparing coefficients like power of p we get

$$p^0 : \begin{cases} \underline{v}_0(x,\alpha) = \dfrac{\alpha\sqrt{x}}{2} \\ \overline{v}_0(x,\alpha) = \dfrac{(2-\alpha)\sqrt{x}}{2} \end{cases} \tag{44}$$

$$p^1 : \begin{cases} \underline{v}_1(x,\alpha) = \dfrac{\alpha\sqrt{x}}{2}(1 - \dfrac{\pi}{4}) \\ \overline{v}_1(x,\alpha) = \dfrac{(2-\alpha)\sqrt{x}}{2}(1 - \dfrac{\pi}{4}) \end{cases} \tag{45}$$

$$p^2 : \begin{cases} \underline{v}_2(x,\alpha) = \dfrac{\alpha\sqrt{x}}{2}(1 - \dfrac{\pi}{4})^2 \\ \overline{v}_2(x,\alpha) = \dfrac{(2-\alpha)\sqrt{x}}{2}(1 - \dfrac{\pi}{4})^2 \end{cases} \tag{46}$$

and so on…

As we know the solution is given as

$$\begin{cases} \underline{u}(x,\alpha) = \displaystyle\sum_{i=0}^{\infty} \underline{v}_i(x,\alpha) \\ \overline{u}(x,\alpha) = \displaystyle\sum_{i=0}^{\infty} \overline{v}_i(x,\alpha) \end{cases} \tag{47}$$

Thus by utilizing above iterative results the series from solution is given as

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$$\begin{cases} \underline{u}(x,\alpha) = \dfrac{\alpha\sqrt{x}}{2}(1+(1-\dfrac{\pi}{4})+(1-\dfrac{\pi}{4})^2+...) \\ \overline{u}(x,\alpha) = \dfrac{(2-\alpha)\sqrt{x}}{2}(1+(1-\dfrac{\pi}{4})+(1-\dfrac{\pi}{4})^2+...) \end{cases} \quad (48)$$

And the exact solution is given as

$$\begin{cases} \underline{u}(x,\alpha) = \dfrac{2\sqrt{x}}{\pi}\alpha \\ \overline{u}(x,\alpha) = \dfrac{2\sqrt{x}}{\pi}(2-\alpha) \end{cases} . \quad (49)$$

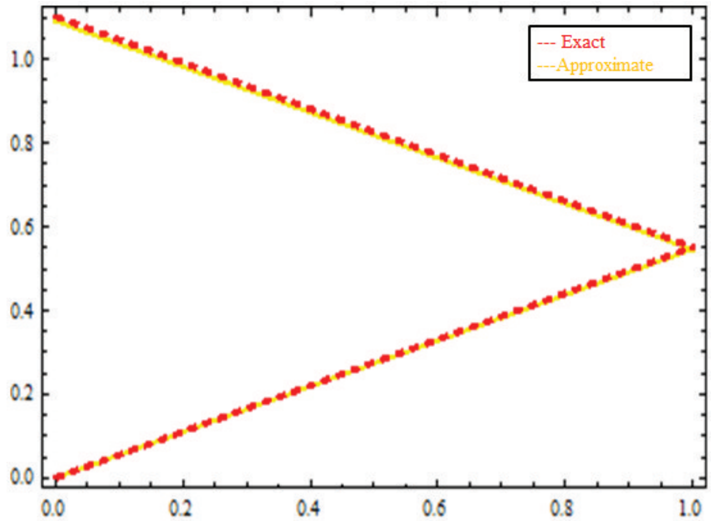


**Fig.4.** Plot of approximate and exact solution

**Example 5.5** Consider the non-linear fuzzy volterra integral equation of 2nd kind

$$u(x,\alpha) = f(x,\alpha) - \int_0^x u^2(t,\alpha)dt \quad (50)$$

Where

$\lambda = 1, 0 \le x \le 1, 0 \le t \le x, 0 \le \alpha \le 1,\ k(x,t) = 1\ \ and\ \ f(x,\alpha) = (\underline{f}(x,\alpha), \overline{f}(x,\alpha))\ \ i.e.$

$$f(x,\alpha) = (\alpha,(1-\alpha))$$

To solve Eq. (50) by HPM 1st we construct homotopy as follows,

$$\begin{cases} H(\underline{v},p,\alpha) = \underline{v}(x,\alpha) - \alpha + p\int_0^x \underline{v}^2(t,\alpha)dt = 0 \\ H(\overline{v},p,\alpha) = \overline{v}(x,\alpha) - (1-\alpha) + p\int_0^x \overline{v}^2(t,\alpha)dt = 0 \end{cases} \quad (51)$$

Assume the solution of Eq. (51) can be written as power series in p

$$\begin{cases} \underline{v}(x,\alpha) = \sum_{i=0}^{\infty} p^i \underline{v}_i(x,\alpha) \\ \overline{v}(x,\alpha) = \sum_{i=0}^{\infty} p^i \overline{v}_i(x,\alpha) \end{cases} \quad (52)$$

Now by putting Eq. (52) in Eq. (51) and by comparing coefficients like power of p we get

$$p^0: \begin{cases} \underline{v}_0(x,\alpha) = \alpha \\ \overline{v}_0(x,\alpha) = (1-\alpha) \end{cases} \tag{53}$$

$$p^1: \begin{cases} \underline{v}_1(x,\alpha) = -\alpha^2 x \\ \overline{v}_1(x,\alpha) = -(1-\alpha)^2 x \end{cases} \tag{54}$$

$$p^2: \begin{cases} \underline{v}_2(x,\alpha) = \alpha^3 x^2 \\ \overline{v}_2(x,\alpha) = (1-\alpha)^3 x^2 \end{cases} \tag{55}$$

$$p^3: \begin{cases} \underline{v}_3(x,\alpha) = -\alpha^4 x^3 \\ \overline{v}_3(x,\alpha) = -(1-\alpha)^4 x^3 \end{cases} \tag{56}$$

and so on …
As we know the solution is given as

$$\begin{cases} \underline{u}(x,\alpha) = \sum_{i=0}^{\infty} \underline{v}_i(x,\alpha) \\ \overline{u}(x,\alpha) = \sum_{i=0}^{\infty} \overline{v}_i(x,\alpha) \end{cases} \tag{57}$$

By utilizing the above iterative results the series form solution is given as

$$\begin{cases} \underline{u}(x,\alpha) = \alpha - \alpha^2 x + \alpha^3 x^2 - \alpha^4 x^3 + \ldots \\ \overline{u}(x,\alpha) = (1-\alpha) - (1-\alpha)^2 x + (1-\alpha)^3 x^2 - (1-\alpha)^4 x^3 + \ldots \end{cases} \tag{58}$$

And the exact form solution is given as

$$\begin{cases} \underline{u}(x,\alpha) = \dfrac{\alpha}{1+\alpha x} \\ \overline{u}(x,\alpha) = \dfrac{(1-\alpha)}{1+(1-\alpha)x} \end{cases}. \tag{59}$$

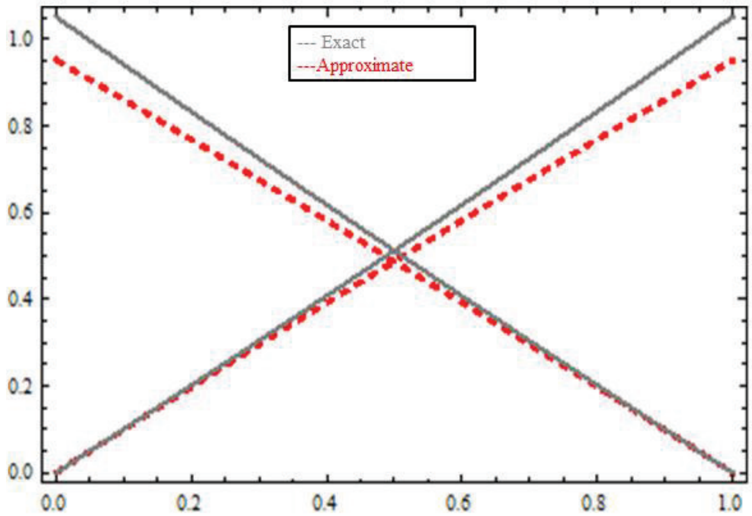


**Fig.5.** Plot of approximate and exact solution

**Example 5.6** Consider the non-linear fuzzy fredholm integral equation of 2nd kind

$$u(x,\alpha) = f(x,\alpha) + \int_0^1 \frac{xt}{10} u^2(t,\alpha)\,dt \tag{60}$$

Where

$$\lambda = 1,\ 0 \le x,t \le 1,\ 0 \le \alpha \le 1,\ k(x,t) = \frac{xt}{10} \ \ and \ \ f(x,\alpha) = (\underline{f}(x,\alpha), \overline{f}(x,\alpha)) \ \ i.e.$$

$$f(x,\alpha)=((\alpha-\frac{\alpha^2}{40})x,(\frac{76-36\alpha-\alpha^2}{40})x)$$

To solve Eq. (60) by HPM 1st we construct homotopy as follows,

$$\begin{cases} H(\underline{v},p,\alpha)=\underline{v}(x,\alpha)-(\alpha-\frac{\alpha^2}{40})x-p\int_0^1\frac{xt}{10}\underline{v}^2(t,\alpha)dt=0 \\ H(\overline{v},p,\alpha)=\overline{v}(x,\alpha)-(\frac{76-36\alpha-\alpha^2}{40})x-p\int_0^1\frac{xt}{10}\overline{v}^2(t,\alpha)dt=0 \end{cases} \quad (61)$$

The solution of Eq. (61) can be assumed as power series in p

$$\begin{cases} \underline{v}(x,\alpha)=\sum_{i=0}^{\infty}p^i\underline{v}_i(x,\alpha) \\ \overline{v}(x,\alpha)=\sum_{i=0}^{\infty}p^i\overline{v}_i(x,\alpha) \end{cases} \quad (62)$$

Utilizing Eq. (62) in Eq. (61) and by comparing coefficients like power of p we get

$$p^0:\begin{cases} \underline{v}_0(x,\alpha)=(\alpha-\frac{\alpha^2}{40})x \\ \overline{v}_0(x,\alpha)=(\frac{76-36\alpha-\alpha^2}{40})x \end{cases} \quad (63)$$

$$p^1:\begin{cases} \underline{v}_1(x,\alpha)=(\alpha-\frac{\alpha^2}{40})^2\frac{x}{40} \\ \overline{v}_1(x,\alpha)=(\frac{76-36\alpha-\alpha^2}{40})^2\frac{x}{40} \end{cases} \quad (64)$$

$$p^2:\begin{cases} \underline{v}_2(x,\alpha)=(\alpha-\frac{\alpha^2}{40})^3\frac{x}{800} \\ \overline{v}_2(x,\alpha)=(\frac{76-36\alpha-\alpha^2}{40})^3\frac{x}{800} \end{cases} \quad (65)$$

and so on…

Thus we know the solution is given as

$$\begin{cases} \underline{u}(x,\alpha)=\sum_{i=0}^{\infty}\underline{v}_i(x,\alpha) \\ \overline{u}(x,\alpha)=\sum_{i=0}^{\infty}\overline{v}_i(x,\alpha) \end{cases} \quad (66)$$

Thus by utilizing above iterative results the series form solution is given as

$$\begin{cases} \underline{u}(x,\alpha)=(\alpha-\frac{\alpha^2}{40})x+(\alpha-\frac{\alpha^2}{40})^2\frac{x}{40}+(\alpha-\frac{\alpha^2}{40})^3\frac{x}{800}+\ldots \\ \overline{u}(x,\alpha)=(\frac{76-36\alpha-\alpha^2}{40})x+(\frac{76-36\alpha-\alpha^2}{40})^2\frac{x}{40}+(\frac{76-36\alpha-\alpha^2}{40})^3\frac{x}{800}+\ldots \end{cases} \quad (67)$$

And the exact form solution is given as

$$\begin{cases} \underline{u}(x,\alpha) = \alpha x \\ \overline{u}(x,\alpha) = (2-\alpha)x \end{cases}. \tag{68}$$

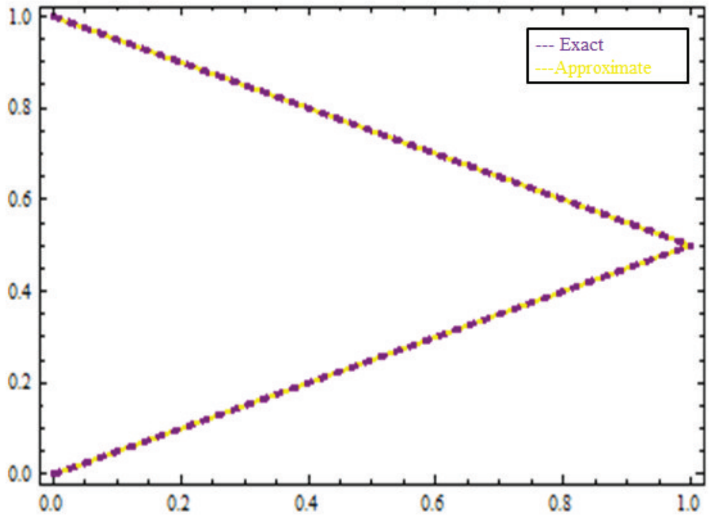


**Fig.6.** Plot of approximate and exact solution

**Example 5.7** Consider the non-linear singular fuzzy integral equation 2nd kind

$$f(x,\alpha) = \int_0^x \frac{u^3(t,\alpha)}{\sqrt{x-t}}dt \tag{69}$$

Where

$$\lambda = 1,\ 0 \le x \le 1,\ 0 \le t < x, 0 \le \alpha \le 1,\ k(x,t) = \frac{1}{\sqrt{x-t}}\ and\ f(x,\alpha) = (\underline{f}(x,\alpha), \overline{f}(x,\alpha))\ i.e.$$

$$f(x,\alpha) = (\frac{2048}{3003}(\alpha^2 + 2\alpha)^3 x^{\frac{13}{2}}, \frac{2048}{3003}(6-3\alpha^3)^3 x^{\frac{13}{2}})$$

Consider the transformation

$$\begin{aligned} &w(x,\alpha) = u^3(x,\alpha) \\ &u(x,\alpha) = \sqrt[3]{w(x,\alpha)} \end{aligned} \tag{70}$$

Carries equation (69) into

$$f(x,\alpha) = \int_0^x \frac{w(t,\alpha)}{\sqrt{x-t}}dt$$

Equivalent to

$$f(x,\alpha) = \int_0^x \frac{w(x,\alpha)}{\sqrt{x-t}}dt + \int_0^x \frac{w(t,\alpha) - w(x,\alpha)}{\sqrt{x-t}}dt$$

$$w(x,\alpha) = \frac{f(x,\alpha)}{2\sqrt{x}} - \frac{1}{2\sqrt{x}}\int_0^x \frac{w(t,\alpha) - w(x,\alpha)}{\sqrt{x-t}}dt \tag{71}$$

To solve Eq. (71) by HPM first we construct convex homotopy as follows,

$$\begin{cases} H(\underline{v},p,\alpha) = \underline{v}(x,\alpha) - \frac{2048}{3003}(\frac{(\alpha^2+2\alpha)^3}{2\sqrt{x}})x^{\frac{13}{2}} + \frac{p}{2\sqrt{x}}\int_0^x \frac{\underline{v}(t,\alpha) - \underline{v}(x,\alpha)}{\sqrt{x-t}}dt = 0 \\ H(\overline{v},p,\alpha) = \overline{v}(x,\alpha) - \frac{2048}{3003}(\frac{(6-3\alpha^3)^3}{2\sqrt{x}})x^{\frac{13}{2}} + \frac{p}{2\sqrt{x}}\int_0^x \frac{\overline{v}(t,\alpha) - \overline{v}(x,\alpha)}{\sqrt{x-t}}dt = 0 \end{cases} \tag{72}$$

Assume the solution of Eq. (72) an be written as power series in p

$$\begin{cases} \underline{v}(x,\alpha) = \sum_{i=0}^{\infty} p^i \underline{v}_i(x,\alpha) \\ \overline{v}(x,\alpha) = \sum_{i=0}^{\infty} p^i \overline{v}_i(x,\alpha) \end{cases} \tag{73}$$

Now by putting Eq. (73) in Eq. (72) and by comparing coefficients like power of p we get

$$p^0: \begin{cases} \underline{v}_0(x,\alpha) = \frac{1024}{3003}(\alpha^2+2\alpha)^3 x^6 \\ \overline{v}_0(x,\alpha) = \frac{1024}{3003}(6-3\alpha^3)^3 x^6 \end{cases} \tag{74}$$

$$p^1: \begin{cases} \underline{v}_1(x,\alpha) = \frac{1024}{3003}(\alpha^2+2\alpha)^3(1-\frac{1024}{3003})x^6 \\ \overline{v}_1(x,\alpha) = \frac{1024}{3003}(6-3\alpha^3)^3(1-\frac{1024}{3003})x^6 \end{cases} \tag{75}$$

$$p^2: \begin{cases} \underline{v}_2(x,\alpha) = \frac{1024}{3003}(\alpha^2+2\alpha)^3(1-\frac{1024}{3003})^2 x^6 \\ \overline{v}_2(x,\alpha) = \frac{1024}{3003}(6-3\alpha^3)^3(1-\frac{1024}{3003})^2 x^6 \end{cases} \tag{76}$$

$$\vdots$$

As we know the solution is given as

$$\begin{cases} \underline{w}(x,\alpha) = \sum_{i=0}^{\infty} \underline{v}_i(x,\alpha) \\ \overline{w}(x,\alpha) = \sum_{i=0}^{\infty} \overline{v}_i(x,\alpha) \end{cases} \tag{77}$$

Thus by utilizing the above iterative results the series form solution is given as

$$\begin{cases} \underline{w}(x,\alpha) = \frac{1024}{3003}(\alpha^2+2\alpha)^3 x^6 + \frac{1024}{3003}(\alpha^2+2\alpha)^3(1-\frac{1024}{3003})x^6 + \frac{1024}{3003}(\alpha^2+2\alpha)^3(1-\frac{1024}{3003})^2 x^6 + \dots \\ \overline{w}(x,\alpha) = \frac{1024}{3003}(6-3\alpha^3)^3 x^6 + \frac{1024}{3003}(6-3\alpha^3)^3(1-\frac{1024}{3003})x^6 + \frac{1024}{3003}(6-3\alpha^3)^3(1-\frac{1024}{3003})^2 x^6 + \dots \end{cases} \tag{78}$$

And the exact form solution is given as

$$\begin{cases} \underline{w}(x,\alpha) = (\alpha^2+2\alpha)^3 x^6 \\ \overline{w}(x,\alpha) = (6-3\alpha^3)^3 x^6 \end{cases} \tag{79}$$

Now by doing back substitution from Eq. (78) the exact solution is given as

$$\begin{cases} \underline{u}(x,\alpha) = \sqrt[3]{\underline{w}(x,\alpha)} = (\alpha^2+2\alpha)x^2 \\ \overline{u}(x,\alpha) = \sqrt[3]{\overline{w}(x,\alpha)} = (6-3\alpha^3)x^2 \end{cases} \tag{80}$$

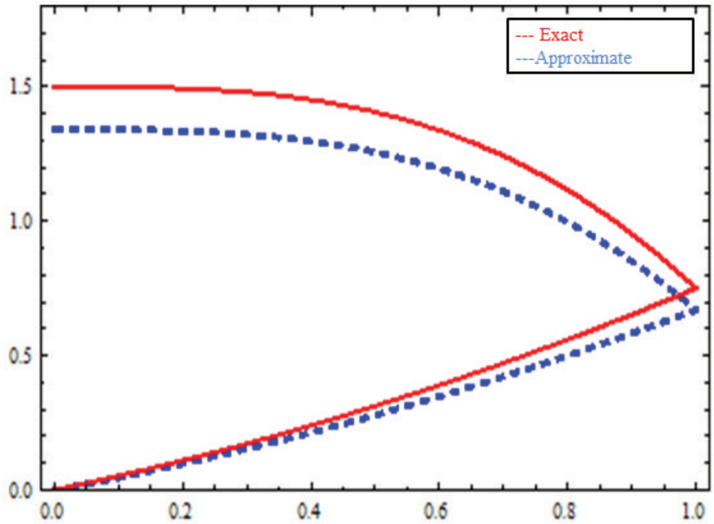


**Fig.7.** Plot of approximate and exact solution

## 6. Conclusion

In this paper we successfully implemented the homotopy perturbation method for finding the analytical solution of some linear and non-linear fuzzy integral equations. This technique proved really reliable and affective from achieved results. It gives fast convergence because by utilized less number of iterations we get approximate as well as exact solution.

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