

Unique Invariant Point Theorems for Random operators In Hilbert Space

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Abstract

We find unique common random fixed point of two random operators in closed subset of a separable Hilbert space by considering a sequence of measurable functions satisfying Theorem 1.1 and Theorem 1.2.

Keywords: Separable Hilbert space, random operators, common random fixed point.

Definition 1.1: A function $f: \Omega \times C \rightarrow C$ is Said to be random operator, if

$F(\cdot, x) : \Omega \rightarrow C$ is measurable for all $x \in C$.

Definition 1.2: A function $f: \Omega \times C \rightarrow C$ is Said to be measurable, if

$f'(B \cap C) \in \Sigma$ for every Borel subset B of H.

Definition 1.3: A function $f: \Omega \times C \rightarrow C$ is Said to be continuous, if

For fixed $\epsilon \in \Omega$, $f(t, \cdot) : C \rightarrow C$ is Continuous.

Definition 1.4: A measurable function $g: \Omega \rightarrow C$ is Said to be random fixed point of the random operator $f: \Omega \times C \rightarrow C$, is Continuous.

Main Result

Theorem 1.1: Let C be a non empty subset of Hilbert Space H. Let R and S be continuous random operations defined on C such that for $\xi \in \Omega$,

$T(\xi): C \rightarrow C$ satisfying the condition :

$$\| Rx - Sy \|^2 \leq \alpha \max [\| x - y \|^2, \{ \| x - Rx \|^2 + \| y - Sy \|^2 \}, \frac{1}{2} \\ \{ \| x - Sy \|^2 + \| y - Rx \|^2, \frac{\| x - y \|^2 + \| x - Sy \|^2 + \| x - Rx \|^2}{1 + \| x - y \| \| x - Sy \| \| x - Rx \|} \}]$$

For $0 \leq \alpha \leq \frac{1}{2}$ and.

Then the sequence $\{g_n\}$ converges to the unique common random fixed point of R and S.

Proof: Let $\{g_n\}$ be a sequence of function defined as :

$$g_{2n+1}(\xi) = R(\xi, g_{2n}(\xi)), g_{2n+2}(\xi) = S(\xi, g_{2n+1}(\xi)),$$

For $\xi \in \Omega$ and $n = 0, 1, 2, 3, \dots$

$$\| g_{2n+1}(\xi) - g_{2n}(\xi) \|^2 = \| R(\xi, g_{2n}(\xi)) - S(\xi, g_{2n-1}(\xi)) \|^2 \\ \leq \alpha \max [\| g_{2n}(\xi) - g_{2n-1}(\xi) \|^2, \{ \| g_{2n}(\xi) - R(\xi, g_{2n}(\xi)) \|^2 + \| g_{2n-1}(\xi) - S(\xi, g_{2n-1}(\xi)) \|^2 \},$$

$$\frac{1}{2} \| g_{2n}(\xi) - S(\xi, g_{2n-1}(\xi)) \|^2 + \| g_{2n-1}(\xi) - R(\xi, g_{2n}(\xi)) \|^2 \},$$

$$\frac{\| g_{2n}(\xi) - g_{2n-1}(\xi) \|^2 + \| g_{2n}(\xi) - S(\xi, g_{2n-1}(\xi)) \|^2 + \| g_{2n}(\xi) - R(\xi, g_{2n}(\xi)) \|^2}{1 + \| g_{2n}(\xi) - g_{2n-1}(\xi) \| \| g_{2n}(\xi) - S(\xi, g_{2n-1}(\xi)) \| \| g_{2n}(\xi) - R(\xi, g_{2n}(\xi)) \|}]$$

$$= \alpha \max [\| g_{2n}(\xi) - g_{2n-1}(\xi) \|^2, \{ \| g_{2n}(\xi) - g_{2n+1}(\xi) \|^2$$

$$+ \| g_{2n-1}(\xi) - g_{2n}(\xi) \|^2 \}, \frac{1}{2} \{ \| g_{2n}(\xi) - g_{2n}(\xi) \|^2 \},$$

$$\{ \| g_{2n-1}(\xi) - g_{2n+1}(\xi) \|^2 \},$$

$$\frac{\| g_{2n}(\xi) - g_{2n-1}(\xi) \|^2 + \| g_{2n}(\xi) - g_{2n}(\xi) \|^2 + \| g_{2n}(\xi) - g_{2n+1}(\xi) \|^2}{1 + \| g_{2n}(\xi) - g_{2n-1}(\xi) \| \| g_{2n}(\xi) - g_{2n}(\xi) \| \| g_{2n}(\xi) - g_{2n+1}(\xi) \|}]$$

$$= \alpha \max [\| g_{2n}(\xi) - g_{2n-1}(\xi) \|^2, \{ \| g_{2n}(\xi) - g_{2n+1}(\xi) \|^2$$

$$+ \| g_{2n-1}(\xi) - g_{2n}(\xi) \|^2 \}, \frac{1}{2} \{ \| g_{2n-1}(\xi) - g_{2n+1}(\xi) \|^2 \},$$

$$\{ \| g_{2n}(\xi) - g_{2n-1}(\xi) \|^2 + \| g_{2n}(\xi) - g_{2n+1}(\xi) \|^2 \}]$$

$$\begin{aligned}
 &= \alpha \max \{ \|\ g_{2n}(\xi) - g_{2n-1}(\xi) \|^2, \{ \|\ g_{2n}(\xi) - g_{2n+1}(\xi) \|^2 \\
 &\quad + \|\ g_{2n-1}(\xi) - g_{2n}(\xi) \|^2 \}, \\
 &\quad \frac{1}{2} \{ \|\ g_{2n-1}(\xi) - g_{2n}(\xi) + g_{2n}(\xi) - g_{2n+1}(\xi) \|^2 \}, \\
 &\{ \|\ g_{2n}(\xi) - g_{2n-1}(\xi) \|^2 + \|\ g_{2n}(\xi) - g_{2n+1}(\xi) \|^2 \} \}
 \end{aligned}$$

$$\begin{aligned}
 &= \alpha \max \{ \|\ g_{2n}(\xi) - g_{2n-1}(\xi) \|^2, \{ \|\ g_{2n}(\xi) - g_{2n+1}(\xi) \|^2 \\
 &\quad + \|\ g_{2n-1}(\xi) - g_{2n}(\xi) \|^2 \}, \\
 &\frac{1}{2} \{ 2 \|\ g_{2n-1}(\xi) - g_{2n}(\xi) + 2 \|\ g_{2n}(\xi) - g_{2n+1}(\xi) \|^2 - \|\ g_{2n-1}(\xi) - \\
 &\ g_{2n}(\xi) - g_{2n}(\xi) + g_{2n+1}(\xi) \|^2 \}, \{ \|\ g_{2n}(\xi) - g_{2n-1}(\xi) \|^2 \\
 &\quad + \|\ g_{2n}(\xi) - g_{2n+1}(\xi) \|^2 \} \}
 \end{aligned}$$

$$\because (\| x + y \|^2 + \| x - y \|^2 = 2 \| x \|^2 + 2 \| y \|^2) \quad (\text{By parallelogram})$$

$$\begin{aligned}
 &\leq \alpha \{ \|\ g_{2n}(\xi) - g_{2n+1}(\xi) \|^2 + \|\ g_{2n-1}(\xi) - g_{2n}(\xi) \|^2 \} \\
 &\therefore \|\ g_{2n+1}(\xi) - g_{2n}(\xi) \|^2 \leq \frac{\alpha}{1-\alpha} \|\ g_{2n-1}(\xi) - g_{2n}(\xi) \|^2
 \end{aligned}$$

$$\text{Similarly, } \|\ g_{2n}(\xi) - g_{2n-1}(\xi) \|^2 \leq \frac{\alpha}{1-\alpha} \|\ g_{2n-2}(\xi) - g_{2n-1}(\xi) \|^2$$

Hence, Proceeding in the same manner , we get

$$\|\ g_n(\xi) - g_{n+1}(\xi) \|^2 \leq \frac{\alpha}{1-\alpha} \|\ g_{n-1}(\xi) - g_n(\xi) \|^2$$

Since $0 \leq \frac{\alpha}{1-\alpha} \leq 1$, therefore $\{g_n(\xi)\}$ be a Cauchy sequence and hence $\{g_n\}$ converges in H.

As $n \rightarrow \infty, g_n(\xi) \rightarrow g(\xi)$

Since C is closed and $g: C \rightarrow C$ be a function.

For all $\xi \in \Omega$.

$$\begin{aligned}
 & \|g(\xi) - S(\xi, g(\xi))\|^2 = \|g(\xi) - g_{2n+1}(\xi) + g_{2n+1}(\xi) - S(\xi, g(\xi))\|^2 \\
 & = 2\|g(\xi) - g_{2n+1}(\xi)\|^2 + 2\|g_{2n+1}(\xi) - S(\xi, g(\xi))\|^2 \\
 & \quad \|g(\xi) - g_{2n+1}(\xi) - g_{2n+1}(\xi) + S(\xi, g(\xi))\|^2 \\
 & \leq 2\|g(\xi) - g_{2n+1}(\xi)\|^2 + 2\|g_{2n+1}(\xi) - S(\xi, g(\xi))\|^2 \\
 & = 2\|g(\xi) - g_{2n+1}(\xi)\|^2 + 2\|R(\xi, g_{2n}(\xi)) - S(\xi, g(\xi))\|^2 \\
 & = 2\|g(\xi) - g_{2n+1}(\xi)\|^2 + 2\alpha \max\{\|g_{2n}(\xi) - g(\xi)\|^2, \\
 & \quad \{\|g_{2n}(\xi) - R(\xi, g_{2n}(\xi))\|^2 + \|g(\xi) - S(\xi, g(\xi))\|^2\}, \\
 & \quad \frac{1}{2}\{\|g_{2n}(\xi) - S(\xi, g(\xi))\|^2 + \|g(\xi) - R(\xi, g_{2n}(\xi))\|^2\}, \\
 & \quad \frac{\|g_{2n}(\xi) - g(\xi)\|^2 + \|g_{2n}(\xi) - S(\xi, g(\xi))\|^2 + \|g_{2n}(\xi) - R(\xi, g_{2n}(\xi))\|^2}{1 + \|g_{2n}(\xi) - g(\xi)\| \|g_{2n}(\xi) - S(\xi, g(\xi))\| \|g_{2n}(\xi) - R(\xi, g_{2n}(\xi))\|}\} \\
 & = 2\|g(\xi) - g_{2n+1}(\xi)\|^2 + 2\alpha \max\{\|g_{2n}(\xi) - g(\xi)\|^2, \\
 & \quad \|g_{2n}(\xi) - g_{2n+1}(\xi)\|^2 + \|g(\xi) - S(\xi, g(\xi))\|^2\} \\
 & \quad \frac{1}{2}\{\|g_{2n}(\xi) - S(\xi, g(\xi))\|^2 + \|g(\xi) - g_{2n+1}(\xi)\|^2\}, \\
 & \quad \frac{\|g_{2n}(\xi) - g(\xi)\|^2 + \|g_{2n}(\xi) - S(\xi, g(\xi))\|^2 + \|g_{2n}(\xi) - g_{2n+1}(\xi)\|^2}{1 + \|g_{2n}(\xi) - g(\xi)\| \|g_{2n}(\xi) - S(\xi, g(\xi))\| \|g_{2n}(\xi) - g_{2n+1}(\xi)\|}\}
 \end{aligned}$$

Making $n \rightarrow \infty$, we have

$$\|g(\xi) - S(\xi, g(\xi))\|^2 \leq 2\alpha \|g(\xi) - S(\xi, g(\xi))\|^2$$

Since $0 \leq \alpha \leq \frac{1}{2}$, so $\forall \xi \in \Omega$

We say that $S(\xi, g(\xi)) = g(\xi)$

Similarly, we can prove that $R(\xi, g(\xi)) = g(\xi)$.

Again, if $f: \Omega \times C \rightarrow C$ is a continuous random operation on a non empty subset C of a separated Hilbert space H , then for any measurable function $F: \Omega \rightarrow C$, the function $F(\xi, g(\xi)) = g(\xi)$ is also measurable .

Therefore the sequence of measurable function $\{g_n\}$ converges to measurable function with $R(\xi, g(\xi)) = g(\xi) = S(\xi, g(\xi))$ which shows that $g: \Omega \rightarrow C$ is common random fixed point of R and S.

Uniqueness: Let $h: \Omega \rightarrow C$ be another common random fixed point of R and S. Then:

$$\begin{aligned} \|g(\xi) - h(\xi)\|^2 &= \|R(\xi, g(\xi)) - S(\xi, h(\xi))\|^2 \\ &\leq \alpha \max[\|g(\xi) - h(\xi)\|^2, \|g(\xi) - R(\xi, g(\xi))\|^2 \\ &+ \|h(\xi) - S(\xi, h(\xi))\|^2], \frac{1}{2} \{ \|g(\xi) - S(\xi, h(\xi))\|^2 + \|h(\xi) - R(\xi, g(\xi))\|^2 \} \\ &\frac{\|g(\xi) - h(\xi)\|^2 + \|g(\xi) - S(\xi, h(\xi))\|^2 + \|g(\xi) - R(\xi, g(\xi))\|^2}{1 + \|g(\xi) - h(\xi)\| \|g(\xi) - S(\xi, h(\xi))\| \|g(\xi) - R(\xi, g(\xi))\|} \\ &= \alpha \max [\|g(\xi) - h(\xi)\|^2, \|g(\xi) - h(\xi)\|^2, 2 \|g(\xi) - h(\xi)\|^2] \\ &= 2 \alpha \|g(\xi) - h(\xi)\|^2 \end{aligned}$$

Which is a contradiction because :

$$\alpha \leq \frac{1}{2}$$

This shows that:

$$g(\xi) = h(\xi)$$

Hence R and S have a unique common random fixed Point.

Theorem 1.2: Let C be a non-empty subset of Hilbert space H. Let R and S be continuous random operations defined on C such that for

$\xi \in \Omega, T(\xi) : C \rightarrow C$ satisfying the condition.

$$\begin{aligned} \|Rx - Sy\|^2 &= \alpha \|x - y\|^2 + \beta \{ \|x - Rx\|^2 + \|y - Sy\|^2 \} + \\ &\frac{\gamma}{2} \|x - Sy\|^2 + \|y - Rx\|^2 \} + \end{aligned}$$

$$\delta \left\{ \frac{\|x - y\|^2 + \|x - Sy\|^2 + \|x - Rx\|^2}{1 + \|x - y\| \|x - Sy\| \|x - Rx\|} \right\} + \frac{\eta}{2} \left\{ \frac{\|x - Rx\|^2 + \|y - Sy\|^2 + \|x - Sy\|^2}{1 + \|y - Rx\| \|y - Sy\| \|x - Sy\|} \right\}$$

$$+ \rho \left\{ \frac{\|x - Rx\|^2 + \|y - Ry\|^2 + \|y - Sy\| \|x - Sy\|}{1 + \|x - Rx\| \|x - Sy\| + \|y - Ry\| \|x - Sy\|} \right\}$$

For

$$\alpha + 2(\beta + \gamma + \delta) + 5\left(\frac{\eta}{2} + \rho\right) < 1$$

And $\alpha, \beta, \gamma, \delta, \eta, \rho > 0$

Then $\{g_n\}$ converges to the unique common random fixed point R and S.

Proof: Let $\{g_n\}$ be a sequence of the function defined for $\xi \in \Omega$ and $n=0,1,2,3,\dots$

$$g_{2n+1}(\xi) = R(\xi, g_{2n}(\xi)) ,$$

$$g_{2n+2}(\xi) = S(\xi, g_{2n+1}(\xi))$$

$$\begin{aligned} & \|g_{2n+1}(\xi) - g_{2n}(\xi)\|^2 = \|R(\xi, g_{2n}(\xi)) - S(\xi, g_{2n-1}(\xi))\|^2 \\ & \leq \alpha [\|g_{2n}(\xi) - g_{2n-1}(\xi)\|^2 + \beta \|g_{2n}(\xi) - Rg_{2n}(\xi)\|^2 \\ & + \|g_{2n-1}(\xi) - Sg_{2n-1}(\xi)\|^2 + \frac{\gamma}{2} \{ \|g_{2n}(\xi) - Sg_{2n-1}(\xi)\|^2 \\ & + \|g_{2n-1}(\xi) - Rg_{2n}(\xi)\|^2 \}] \\ & + \delta \left\{ \frac{\|g_{2n}(\xi) - g_{2n-1}(\xi)\|^2 + \|g_{2n}(\xi) - Sg_{2n-1}(\xi)\|^2 + \|g_{2n}(\xi) - Rg_{2n}(\xi)\|^2}{1 + \|g_{2n}(\xi) - g_{2n-1}(\xi)\| + \|g_{2n}(\xi) - Sg_{2n-1}(\xi)\| \|g_{2n}(\xi) - Rg_{2n}(\xi)\|} \right\} \\ & + \frac{\eta}{2} \left\{ \frac{\|g_{2n-1}(\xi) - Rg_{2n}(\xi)\|^2 + \|g_{2n-1}(\xi) - Sg_{2n-1}(\xi)\|^2 + \|g_{2n}(\xi) - Sg_{2n-1}(\xi)\|^2}{1 + \|g_{2n-1}(\xi) - Rg_{2n}(\xi)\| \|g_{2n-1}(\xi) - Sg_{2n-1}(\xi)\| \|g_{2n}(\xi) - Sg_{2n-1}(\xi)\|} \right\} + \\ & \rho \left\{ \frac{\|g_{2n}(\xi) - Rg_{2n}(\xi)\|^2 + \|g_{2n-1}(\xi) - Rg_{2n}(\xi)\|^2 + \|g_{2n-1}(\xi) - Sg_{2n-1}(\xi)\|^2 \|g_{2n}(\xi) - Sg_{2n-1}(\xi)\|^2}{1 + \|g_{2n}(\xi) - Rg_{2n}(\xi)\| + \|g_{2n}(\xi) - Sg_{2n-1}(\xi)\| + \|g_{2n-1}(\xi) - Rg_{2n}(\xi)\| \|g_{2n}(\xi) - Sg_{2n-1}(\xi)\|} \right\} \\ & = \alpha \|g_{2n}(\xi) - g_{2n-1}(\xi)\|^2 + \beta \{ \|g_{2n}(\xi) - g_{2n+1}(\xi)\|^2 + \|g_{2n-1}(\xi) - \\ & g_{2n}(\xi)\|^2 \} + \frac{\gamma}{2} \{ \|g_{2n}(\xi) - g_{2n}(\xi)\|^2 + \|g_{2n-1}(\xi) - g_{2n+1}(\xi)\|^2 \\ & + \delta \left\{ \frac{\|g_{2n}(\xi) - g_{2n-1}(\xi)\|^2 + \|g_{2n}(\xi) - g_{2n}(\xi)\|^2 + \|g_{2n}(\xi) - g_{2n+1}(\xi)\|^2}{1 + \|g_{2n}(\xi) - g_{2n-1}(\xi)\| \|g_{2n}(\xi) - g_{2n}(\xi)\| \|g_{2n}(\xi) - g_{2n+1}(\xi)\|} \right\} \\ & + \frac{\eta}{2} \left\{ \frac{\|g_{2n-1}(\xi) - g_{2n+1}(\xi)\|^2 + \|g_{2n-1}(\xi) - g_{2n}(\xi)\|^2 + \|g_{2n}(\xi) - g_{2n}(\xi)\|^2}{1 + \|g_{2n-1}(\xi) - g_{2n+1}(\xi)\| + \|g_{2n-1}(\xi) - g_{2n}(\xi)\| \|g_{2n}(\xi) - g_{2n}(\xi)\|} \right\} \end{aligned}$$

$$\begin{aligned}
 & +\rho\left\{\frac{\|g_{2n}(\xi)-g_{2n+1}(\xi)\|^2+\|g_{2n-1}(\xi)-g_{2n+1}(\xi)\|^2+\|g_{2n-1}(\xi)-g_{2n}(\xi)\|^2\|g_{2n}(\xi)-g_{2n}(\xi)\|^2}{1+\|g_{2n}(\xi)-g_{2n+1}(\xi)\|\|g_{2n}(\xi)-g_{2n}(\xi)\|+\|g_{2n-1}(\xi)-g_{2n+1}(\xi)\|\|g_{2n}(\xi)-g_{2n}(\xi)\|}\right\} \\
 & =\alpha\|g_{2n}(\xi)-g_{2n-1}(\xi)\|^2+\beta\{\|g_{2n}(\xi)-g_{2n+1}(\xi)\|^2+\|g_{2n-1}(\xi)-g_{2n}(\xi)\|^2\}+\frac{\gamma}{2}\{\|g_{2n-1}(\xi)-g_{2n+1}(\xi)\|^2+\delta\|g_{2n}(\xi)-g_{2n-1}(\xi)\|^2 \\
 & +\|g_{2n}(\xi)-g_{2n+1}(\xi)\|^2\}+\frac{\eta}{2}\{\|g_{2n-1}(\xi)-g_{2n+1}(\xi)\|^2 \\
 & +\|g_{2n-1}(\xi)-g_{2n}(\xi)\|^2\}+\rho\|g_{2n}(\xi)-g_{2n+1}(\xi)\|^2+\|g_{2n-1}(\xi)-g_{2n+1}(\xi)\|^2\} \\
 & =(\alpha+\beta+\delta+\frac{\eta}{2})\|g_{2n}(\xi)-g_{2n-1}(\xi)\|^2+(\beta+\delta+\rho) \\
 & \quad \|g_{2n}(\xi)-g_{2n+1}(\xi)\|^2+(\frac{\gamma}{2}+\frac{\eta}{2}+\rho)\|g_{2n-1}(\xi)-g_{2n+1}(\xi)\|^2
 \end{aligned}$$

NOW,

$$\begin{aligned}
 \|g_{2n-1}(\xi)-g_{2n+1}(\xi)\|^2 & =[\|g_{2n-1}(\xi)-g_{2n}(\xi)\|^2]+ \\
 & [\|g_{2n}(\xi)-g_{2n+1}(\xi)\|^2] \dots\dots\dots(1.2.1)
 \end{aligned}$$

By using parallelogram, we have :

$$\begin{aligned}
 \|g_{2n-1}(\xi)-g_{2n+1}(\xi)\|^2 & =2\|g_{2n-1}(\xi)-g_{2n}(\xi)\|^2+ \\
 2\|g_{2n}(\xi)-g_{2n+1}(\xi)\|^2 & -\|[g_{2n-1}(\xi)-g_{2n}(\xi)]-[g_{2n}(\xi)-g_{2n+1}(\xi)]\|^2 \\
 & \leq 2\|g_{2n-1}(\xi)-g_{2n}(\xi)\|^2+2\|g_{2n}(\xi)-g_{2n+1}(\xi)\|^2
 \end{aligned}$$

Therefore by equation (1.2.1), we have:

$$\begin{aligned}
 \|g_{2n+1}(\xi)-g_{2n}(\xi)\|^2 & \leq(\alpha+\beta+\delta+\frac{\eta}{2})\|g_{2n}(\xi)-g_{2n-1}(\xi)\|^2+ \\
 & (\beta+\delta+\rho)\|g_{2n}(\xi)-g_{2n+1}(\xi)\|^2 \\
 +(\frac{\alpha}{2}+\frac{\eta}{2}+\rho)[2\|g_{2n-1}(\xi)-g_{2n}(\xi)\|^2 & +2\|g_{2n}(\xi)-g_{2n+1}(\xi)\|^2] \\
 \|g_{2n+1}(\xi)-g_{2n}(\xi)\|^2 & \leq K\|g_{2n}(\xi)-g_{2n-1}(\xi)\|^2
 \end{aligned}$$

Where:

$$K=\frac{\alpha+\beta+\gamma+\delta+3\frac{\eta}{2}+2\rho}{1-(\beta+\gamma+\delta+\eta+3\delta)} < 1,$$

Similarly, $\|g_{2n}(\xi) - g_{2n-1}(\xi)\|^2 \leq \|g_{2n-1}(\xi) - g_{2n-2}(\xi)\|^2$

Proceeding in the same manner, we have:

$$\|g_n(\xi) - g_{n+1}(\xi)\|^2 \leq K \|g_{n-1}(\xi) - g_n(\xi)\|^2, \forall \xi \in \Omega$$

Therefore $g_n(\xi)$ is a Cauchy sequence and hence it is convergent in the Hilbert spaces H.

As $n \rightarrow \infty$, $g_n(\xi) \rightarrow g(\xi)$. Since C is closed and $g:C \rightarrow C$ be a function for all $\xi \in \Omega$.

$$\begin{aligned} & \|g(\xi) - S(\xi, g(\xi))\|^2 = \|g(\xi) - g_{2n+1}(\xi)\|^2 + \|g_{2n+1}(\xi) - S(\xi, g(\xi))\|^2 \\ & \leq 2 \|g(\xi) - g_{2n+1}(\xi)\|^2 + 2 \|g_{2n+1}(\xi) - S(\xi, g(\xi))\|^2, \text{ By parallelogram} \\ & = 2 \|g(\xi) - g_{2n+1}(\xi)\|^2 + 2 \|R(\xi, g_{2n}(\xi)) - S(\xi, g(\xi))\|^2 \\ & = 2 \|g(\xi) - g_{2n+1}(\xi)\|^2 + 2[\alpha \|g_{2n}(\xi) - g(\xi)\|^2 + \\ & \beta\{\|g_{2n}(\xi) - Rg_{2n}(\xi)\|^2 + \|g(\xi) - Sg(\xi)\|^2\} + \frac{\gamma}{2}\{\|g_{2n}(\xi) - Sg(\xi)\|^2 \\ & + \|g(\xi) - Rg_{2n}(\xi)\|^2\} \\ & + \delta\left\{\frac{\|g_{2n}(\xi) - g(\xi)\|^2 + \|g_{2n}(\xi) - Sg(\xi)\|^2 + \|g_{2n}(\xi) - Rg_{2n}(\xi)\|^2}{1 + \|g_{2n}(\xi) - g(\xi)\| \|g_{2n}(\xi) - Sg(\xi)\| \|g_{2n}(\xi) - Rg_{2n}(\xi)\|}\right\} \\ & + \frac{\eta}{2}\left\{\frac{\|g(\xi) - Rg_{2n}(\xi)\|^2 + \|g(\xi) - Sg(\xi)\|^2 + \|g_{2n}(\xi) - Sg(\xi)\|^2}{1 + \|g(\xi) - Rg_{2n}(\xi)\| \|g(\xi) - Sg(\xi)\| \|g_{2n}(\xi) - Sg(\xi)\|}\right\} + \\ & \rho\left\{\frac{\|g_{2n}(\xi) - Rg_{2n}(\xi)\|^2 + \|g(\xi) - Rg_{2n}(\xi)\|^2 + \|g(\xi) - Sg(\xi)\|^2 + \|g_{2n}(\xi) - Sg(\xi)\|^2}{1 + \|g_{2n}(\xi) - Rg_{2n}(\xi)\| \|g_{2n}(\xi) - Sg(\xi)\| + \|g(\xi) - Rg_{2n}(\xi)\| \|g_{2n}(\xi) - Sg(\xi)\|}\right\}] \\ & = 2 \|g(\xi) - g_{2n+1}(\xi)\|^2 + 2[\alpha \|g_{2n}(\xi) - g(\xi)\|^2 + \\ & \beta\{\|g_{2n}(\xi) - g_{2n+1}(\xi)\|^2 + \|g(\xi) - Sg(\xi)\|^2\} + \frac{\gamma}{2}\{\|g_{2n}(\xi) - Sg(\xi)\|^2 + \\ & \|g(\xi) - g_{2n+1}(\xi)\|^2\} \\ & + \delta\left\{\frac{\|g_{2n}(\xi) - g(\xi)\|^2 + \|g_{2n}(\xi) - Sg(\xi)\|^2 + \|g_{2n}(\xi) - g_{2n+1}(\xi)\|^2}{1 + \|g_{2n}(\xi) - g(\xi)\| \|g_{2n}(\xi) - Sg(\xi)\| \|g_{2n}(\xi) - g_{2n+1}(\xi)\|}\right\} \\ & + \frac{\eta}{2}\left\{\frac{\|g(\xi) - g_{2n+1}(\xi)\|^2 + \|g(\xi) - Sg(\xi)\|^2 + \|g_{2n}(\xi) - Sg(\xi)\|^2}{1 + \|g(\xi) - g_{2n+1}(\xi)\| \|g(\xi) - Sg(\xi)\| \|g_{2n}(\xi) - Sg(\xi)\|}\right\} \end{aligned}$$

$$+ \rho \left\{ \frac{\|g_{2n}(\xi) - g_{2n+1}(\xi)\|^2 + \|g(\xi) - g_{2n+1}(\xi)\|^2 + \|g(\xi) - Sg(\xi)\|^2 \|g_{2n}(\xi) - Sg(\xi)\|^2}{1 + \|g_{2n}(\xi) - g_{2n+1}(\xi)\| \|g_{2n}(\xi) - Sg(\xi)\| + \|g(\xi) - g_{2n+1}(\xi)\| \|g_{2n}(\xi) - Sg(\xi)\|} \right\}$$

Making $n \rightarrow \infty$, we get

$$\|g(\xi) - S(\xi, g(\xi))\|^2 \leq (2\beta + \gamma + 2\delta + 2\eta + 4\rho) \|g(\xi) - S(\xi, g(\xi))\|^2$$

Since $(2\beta + \gamma + 2\delta + 2\eta + 4\rho) < 1$

Therefore for $\xi \in \Omega$, we say that $S(\xi, g(\xi)) = g(\xi)$.

Similarly we can prove that $R(\xi, g(\xi)) = g(\xi)$

Again if $F: \Omega \rightarrow C \rightarrow C$ is a continuous random operations on a non empty subset C of a separated Hilbert space H . Then for any measurable function

$F: \Omega \rightarrow C$ the function $F(\xi, g(\xi)) = g(\xi)$ is also measurable.

Therefore the sequence of measurable function $\{g_n\}$ converges to measurable

Function with $R(\xi, g(\xi)) = g(\xi) = S(\xi, g(\xi))$,

Which shows that $g: \Omega \rightarrow C$ is common random fixed point of R and S .

Uniqueness: Let $h: \Omega \rightarrow C$ be another common random fixed point of R and S . Then:

$$\begin{aligned} \|g(\xi) - h(\xi)\|^2 &= \|R(\xi, g(\xi)) - S(\xi, h(\xi))\|^2 \\ &\leq \alpha \|g(\xi) - h(\xi)\|^2 + \beta \{ \|g(\xi) - Rg(\xi)\|^2 + \|h(\xi) - Sh(\xi)\|^2 \} \end{aligned}$$

$$+ \frac{\gamma}{2} \{ \|g(\xi) - Sh(\xi)\|^2 + \|h(\xi) - Rg(\xi)\|^2 \}$$

$$+ \delta \left\{ \frac{\|g(\xi) - h(\xi)\|^2 + \|g(\xi) - Sh(\xi)\|^2 + \|g(\xi) - Rg(\xi)\|^2}{1 + \|g(\xi) - h(\xi)\| \|g(\xi) - Sh(\xi)\| \|g(\xi) - Rg(\xi)\|^2} \right\} +$$

$$\frac{\eta}{2} \left\{ \frac{\|h(\xi) - Rg(\xi)\|^2 + \|h(\xi) - Sh(\xi)\|^2 + \|g(\xi) - Sh(\xi)\|^2}{1 + \|h(\xi) - Rg(\xi)\| \|h(\xi) - Sh(\xi)\| \|g(\xi) - Sh(\xi)\|} \right\}$$

$$+ \rho \left\{ \frac{\|g(\xi) - Rg(\xi)\|^2 + \|h(\xi) - Rg(\xi)\|^2 + \|h(\xi) - Sh(\xi)\| \|g(\xi) - Sh(\xi)\|}{1 + \|g(\xi) - Rg(\xi)\| \|g(\xi) - Sh(\xi)\| + \|h(\xi) - Rg(\xi)\|^2 \|g(\xi) - Sh(\xi)\|} \right\}$$

$$= (\alpha + \gamma + 2\delta + \eta + \rho) \|g(\xi) - h(\xi)\|^2$$

Which is a contradiction because:

$$\alpha + \gamma + 2\delta + \eta + \rho < 1.$$

This shows that $g(\xi) = h(\xi)$

Hence R and S have a unique common random fixed point.

Corollary: Let C be a non empty subset of Hilbert Space H. Let R and S be continuous random operations defined on C such that for

$\xi \in \Omega$, $T(\xi): C \rightarrow C$ satisfying the condition.

$$\|x - Rx\|^2 + \|y - Sy\|^2 \leq \alpha \|x - y\|^2$$

For $1 \leq \alpha \leq 2$.

Then the sequence $\{g_n\}$ converges to the unique common random fixed point of R and S.

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