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Analytical Solution of Multi-Pantograph Delay Differential Equations Via Sumudu Decomposition Method

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Abstract

In this paper, we apply Sumudu Decomposition Method (SDM) to solve the multi-pantograph delay differential equations with constant coefficients. Three problems are resolved to show the effectiveness and consistency of the SDM. The obtained results by this method provide solutions in a series form and in few terms. This technique successfully determines the convergence of the solution.

Keywords: Sumudu Decomposition Method (SDM), Multi-Pantograph Delay Differential Equations

1. Introduction

Pantograph is a kind of Delay Differential Equations. Pantograph is a device located on the electric locomotive. First time in 1851 electric locomotive was made. In 1895 electric locomotive was commissioned. First Time in 1971 Taylor and Ockendon developed the Mathematical model of pantograph [1]. Pantograph have a wide range of uses in different fields of sciences like, nonlinear dynamical systems ,subdivision of astronomy “astrophysics”, electro dynamics, probability concept on algebraic arrangements, cell growing , number theory .quantum mechanics, , collaborating problems, populace models, etc. Many authors have studied Pantograph equations which is kind of Delay Differential equations by recently. For instances, Liu and Li solved multi-pantograph delay equation by Runge-Kutta methods [2]. Evans and Raslan solved the delay differential equation by Adomian decomposition method [3]. Keskin et al. got the approximate solution by using differential transform method [4]. Sezer and Dascioglu [5] established Taylor method and advanced case or retarded case of generalized pantograph equations solved by this method. Yu solved Multi-pantograph delay equation by VIM [6]. Sezer et al. get approximate solution of multi-pantograph equation by using variable coefficients [7]. Singular Perturbed Multi-Pantograph Delay Equations were solved by Geng, F. Z. and Qian, S. P. with Method of Reprociding Kernel [8]. Cherruault, Y., Adomian, G., Abbaoui, K. and Rach, R. controlled Convergence of Decomposition Method [9]. Delay dynamic system was studied by El-Safty et al. with the spline function of 3-h step for getting approximate solution [11].

Several numerical schemes have been developed for solving advanced and retarded pantograph equations. The utmost significant one are method of collocation, methods of spline, Runge-Kutta methods, θ-methods, Homotopy perturbation method (HPM),Adomian decomposition method (ADM), and Taylor method, variational iteration method (VIM).

First time Watugala introduced Sumudu transform in his effort of work (Watugala, 1993). Many people then further developed it and used it to get solution of many problems. Belgacem et al recognized its fundamentals properties in (2003, 2006). Its Properties are very different and valuable that can help in science and engineering for solving many complicated applications.

On view of the above approaches we are intent to solve Pantograph equations by Sumudu Decomposition method, because this method yields an approximate and exact solution in a small number of terms and is easy to compute.

This paper is organized as follows. In section 2, the analysis of the method is introduced. Subsequently, in section 3, some examples are provided. Finally, in section 4, conclusion is presented.

2. Analysis of Sumudu Decomposition Method

A be a space of functions as follows

$$A = \{ f(t) : \tau_1, \tau_2 > 0, |f(t)| < Me^{t/\tau_j}, if\ t \in (-1)^j \times [0, \infty)\}, \tag{1}$$

Then the well-defined Sumudu transform of function is

$$G(u) = S[f(t)] := \int_0^\infty f(ut)e^{-t}dt, \qquad u \in (-\tau_1, \tau_2). \tag{2}$$

To demonstrate the basic idea of the SDM for DDEs, we consider a general Multi-Pantograph Eq. of the form

$$\begin{cases} u'(t) = p(t)u(t) + q_1(t)u(q_1 t) + q_2(t)u(q_2 t) + ... + q_j(t)u(q_j t) + h(t) \\ u(0) = u_0,\ 0 < t \le T, \end{cases} \tag{3}$$

Where $0 < q_1 < q_2 < ...q_j \le 1$ and h is given function.

By applying Sumudu transform on Eq.3, it becomes

$$S[u'(t)] = S\left[p(t)u(t) + \sum_{i=1}^{j} q_i(t)u(t) + h(t) \right], \tag{4}$$

By using Sumudu Transform Differentiation properties with initial condition, Eq.4 becomes

$$S[U(t)] = u_0 + uS[p(t)u(t)] + uS\left[\sum_{i=1}^{j} q_i(t)u(t)\right] + uS[h(t)], \tag{5}$$

Then by applying Inverse Sumudu Transform on Eq.5, it becomes

$$u(t) = S^{-1}\{u_0 uS[uS[h(t)]]\} + S^{-1}\left\{uS[p(t)u(t)] + uS\left[\sum_{i=1}^{j} q_i(t)u(t)\right]\right\}, \tag{6}$$

Now Represent the solution $u(t)$ as infinite series as

$$u(t) = \sum_{n=0}^{\infty} u_n(t), \tag{7}$$

By putting Eq.7 into Eq.6, we get

$$\sum_{n=0}^{\infty} u_n(t) = S^{-1}\{u_0 uS[uS[h(t)]]\} + S^{-1}\left\{uS\left[p(t)\sum_{n=0}^{\infty} u_n(t)\right] + uS\left[\sum_{i=1}^{j} q_i(t)\sum_{n=0}^{\infty} u_n(t)\right]\right\}, \tag{8}$$

Then by comparing the components of $u(t)$, we get the exact solution.

3. Explanatory Examples

Example 3.1. Consider linear 1st order Multi-Pantograph Equation

$$\begin{cases} u'(t) = 4u(\frac{t}{2}) - \frac{5}{6}u(t) + 9u(\frac{t}{3}) + t^2 - 1, \qquad 0 < t \le 1 \\ u(0) = 1, \end{cases} \tag{9}$$

Applying Sumudu Transform on both sides of Eq.9, we get

$$S[u'(t)] = S\left[4u(\frac{t}{2})\right] - S\left[\frac{5}{6}u(t)\right] + S\left[9u(\frac{t}{3})\right] + S[t^2 - 1], \tag{10}$$

By using Differentiation Property,

$$\frac{1}{u}S[U(t)] - \frac{1}{u}U(0) = S\left[4u(\frac{t}{2})\right] - S\left[\frac{5}{6}u(t)\right] + S\left[9u(\frac{t}{3})\right] + S[t^2 - 1], \tag{11}$$

Then using Initial Condition,

$$S[U(t)] = 1 + uS\left[4u(\frac{t}{2})\right] - uS\left[\frac{5}{6}u(t)\right] + uS\left[9u(\frac{t}{3})\right] + uS[t^2 - 1], \tag{12}$$

Taking Inverse Sumudu Transform,

$$u(t) = 1 + S^{-1}\left[uS\left[4u(\frac{t}{2})\right]\right] - S^{-1}\left[uS\left[\frac{5}{6}u(t)\right]\right] + S^{-1}\left[uS\left[9u(\frac{t}{3})\right]\right] + S^{-1}\left[uS[t^2 - 1]\right], \tag{13}$$

Represent $u(t)$ as infinite series,

$$u(t) = \sum_{n=0}^{\infty} u_n(t), \tag{14}$$

By putting Eq.14 into Eq.13, we get

$$\sum_{n=0}^{\infty} u_n(t) = 1 - t + \frac{t^3}{3} + S^{-1}\left[uS\left[4\sum_{n=0}^{\infty} u_n(\frac{t}{2})\right]\right] - S^{-1}\left[uS\left[\frac{5}{6}\sum_{n=0}^{\infty} u_n(t)\right]\right] + S^{-1}\left[uS\left[9\sum_{n=0}^{\infty} u_n(\frac{t}{3})\right]\right], \tag{15}$$

By comparing the coefficients of $u(t)$,

$$u_0(t) = 1 - t + \frac{t^3}{3},$$

$$u_1(t) = \frac{73}{6}t - \frac{25}{12}t^2,$$

$$u_2(t) = \frac{1825}{72}t^2 - \frac{175}{216}t^3,$$

$$u_3(t) = \frac{12775}{1296}t^3,$$

$$u_4(t) = 0,$$

$$\vdots,$$

Therefore, the solution which is exact is

$$u(t) = u_0(t) + u_1(t) + u_2(t) + u_3(t) + u_4(t) + \ldots,$$
$$= 1 + \frac{67}{6}t + \frac{1675}{72}t^2 + \frac{12157}{1296}t^3 + 0 + 0 + 0\ldots, \quad (16)$$
$$= 1 + \frac{67}{6}t + \frac{1675}{72}t^2 + \frac{12157}{1296}t^3.$$

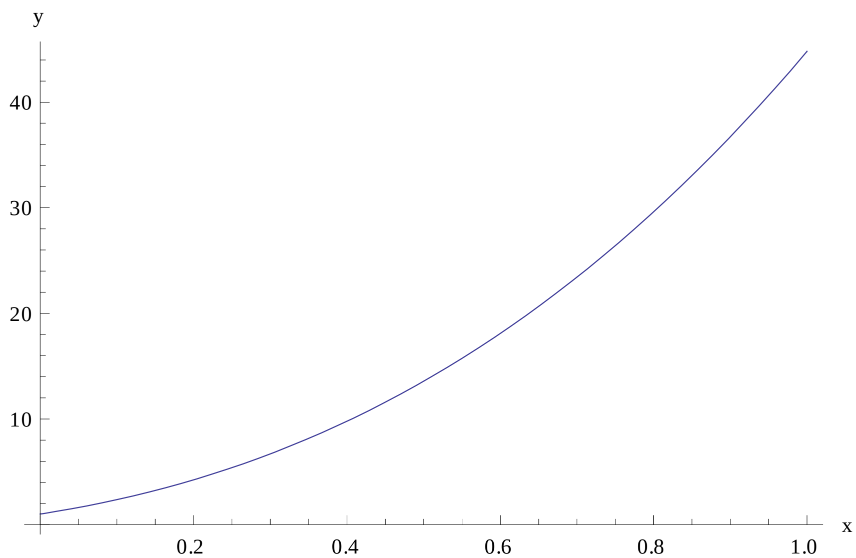

Fig 1:Graphical representation of obtained u(t) at different values of t

Example 3.2. Consider linear 2nd order Multi-Pantograph Equation

$$\begin{cases} u''(t) = u(\frac{t}{2}) + \frac{3}{4}u(t) - t^2 + 2, & 0 < t \le 1 \\ u(0) = 1, \quad u'(0) = 0. \end{cases} \quad (17)$$

Applying Sumudu Transform on both sides of Eq.17, we get

$$S[u''(t)] = S\left[u(\frac{t}{2})\right] + S\left[\frac{3}{4}u(t)\right] + S[-t^2 + 2], \quad (18)$$

By using Differentiation Property,

$$\frac{1}{u^2}S[U(t)] - \frac{1}{u^2}U(0) - \frac{1}{u}U'(0) = S\left[u(\frac{t}{2})\right] + S\left[\frac{3}{4}u(t)\right] - 2!u^2 + 2, \quad (19)$$

Then using Initial Condition,

$$S[U(t)] = -2!u^4 + 2u^2 + u^2 S\left[u(\frac{t}{2})\right] + u^2 S\left[\frac{3}{4}u(t)\right], \quad (20)$$

Taking Inverse Sumudu Transform,

$$u(t) = t^2 - \frac{t^4}{12} + S^{-1}\left[u^2 S\left[u(\frac{t}{2})\right]\right] + S^{-1}\left[u^2 S\left[\frac{3}{4}u(t)\right]\right], \tag{21}$$

Represent $u(t)$ as infinite series,

$$u(t) = \sum_{n=0}^{\infty} u_n(t), \tag{22}$$

By putting Eq.22 into Eq.21, we get

$$\sum_{n=0}^{\infty} u_n(t) = t^2 - \frac{t^4}{12} + S^{-1}\left[u^2 S\left[\sum_{n=0}^{\infty} u_n(\frac{t}{2})\right]\right] + S^{-1}\left[u^2 S\left[\frac{3}{4}\sum_{n=0}^{\infty} u_n(t)\right]\right], \tag{23}$$

By comparing the coefficients of $u(t)$,

$$u_0(t) = t^2 - \frac{t^4}{12},$$

$$u_1(t) = \frac{t^4}{12} - \frac{13t^6}{5760},$$

$$u_2(t) = \frac{13t^6}{5760} - \frac{91}{2949120}t^8,$$

$$\vdots,$$

Therefore, by noise term phenomenon the exact solution is

$$\begin{aligned} u(t) &= u_0(t) + u_1(t) + u_2(t) + u_3(t) + u_4(t) + \ldots, \\ &= t^2 \end{aligned} \tag{24}$$

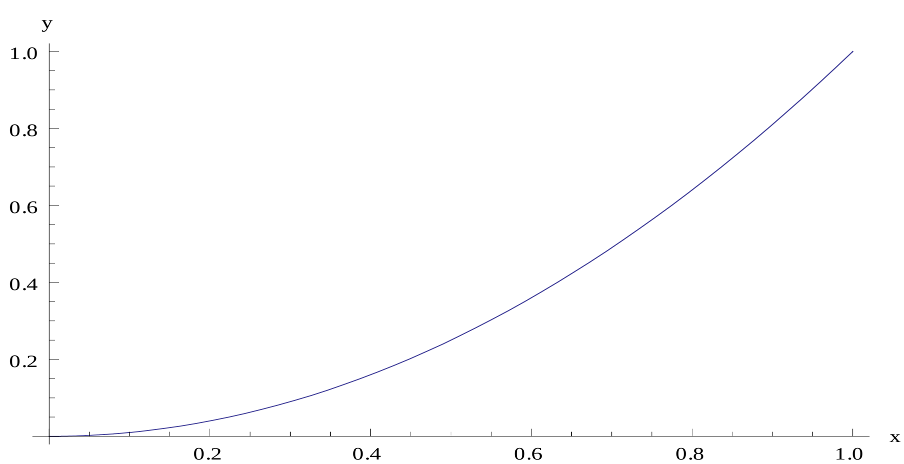

Fig 2: Graphical representation of obtained u (t)

Example 3.3. Consider linear 3rd order Multi-Pantograph Equation

$$\begin{cases} u'''(t) = e^{0.3-t} - u(t) - u(t-0.3), & 0 \le t \le 1 \\ u(0) = 1, \quad u'(0) = -1, \quad u''(0) = 1. & . \end{cases} \tag{25}$$

Applying Sumudu Transform on both sides of Eq.25, it becomes

$$S[u''(t)] = S[e^{0.3-t}] - S[u(t)] - S[u(t-0.3)], \tag{26}$$

By using Differentiation Property,

$$\frac{1}{u^3} S[U(t)] - \frac{1}{u^3} U(0) - \frac{1}{u^2} U''(0) - \frac{1}{u} U'(0) = \frac{e^{0.3}}{1+u} - S[u(t)] - S[u(t-0.3)], \tag{27}$$

Then using Initial Condition,

$$S[U(t)] = 1 - u + u^2 + \frac{e^{0.3}u^3}{1+u} - u^3 S[u(t)] - u^3 S[u(t-0.3)], \tag{28}$$

Taking Inverse Sumudu Transform,

$$u(t) = 1 - t + \frac{t^2}{2!} + e^{0.3}\left[1 - t + \frac{t^2}{2!} - e^{-t}\right] - S^{-1}[u^3 S[u(t)]] - S^{-1}[u^3 S[u(t-0.3)]], \tag{29}$$

Represent $u(t)$ as infinite series,

$$u(t) = \sum_{n=0}^{\infty} u_n(t), \tag{30}$$

By putting Eq.30 into Eq.29, we get

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(t) = 1 - t + \frac{t^2}{2!} + e^{0.3}\left[1 - t + \frac{t^2}{2!} - e^{-t}\right] - S^{-1}\left[u^3 S\left[\sum_{n=0}^{\infty} u_n(t)\right]\right] \\ - S^{-1}\left[u^3 S\left[\sum_{n=0}^{\infty} u_n(t-0.3)\right]\right], \end{aligned} \tag{31}$$

By comparing the coefficients of $u(t)$,

$$u_0(t) = 1 - t + \frac{t^2}{2!} + e^{0.3}\left[1 - t + \frac{t^2}{2!} - e^{-t}\right],$$

$$u_1(t) = -\frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!} + e^{0.3}\left[1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!} - e^{-t}\right],$$

$$\vdots,$$

Therefore, by cancelling the terms the exact solution is

$$u(t) = u_0(t) + u_1(t) + u_2(t) + u_3(t) + u_4(t) + \dots,$$
$$= 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \dots, \quad (32)$$
$$= e^{-t}.$$

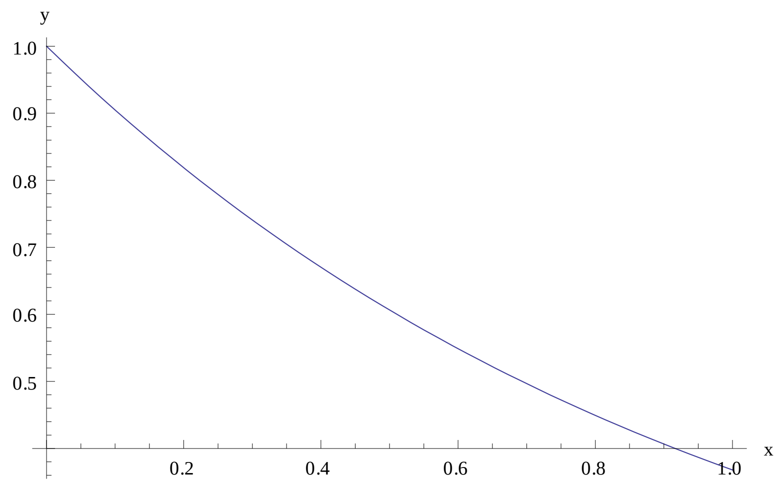

Fig 3: Graphical representation of obtained solution u (t)

4. Conclusion

In the current paper Sumudu Decomposition Method (SDM) successfully applied to linear Multi Pantograph Equations of different orders. Graphical representation of solutions of Pantograph equations is also presented. From the Numerical results of problems, it can be simply understood that this method obtains results as exact as possible.

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