# On the Global Existence of Solutions to a System of Second – Order Nonlinear Differential Equations

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**Abstract**

Solutions of second – order nonlinear differential system is investigated. A sufficient condition for every solution of the system to exist globally is obtained. Sufficient conditions are placed on the functions in the system that guarantee global existence of solutions to the system and these conditions are put into a theorem which is proved.

## 1.0 Introduction

Consider the second – order nonlinear differential system:

$$x' = \frac{1}{a(x)}[c(y) - b(x)]$$

$$y' = -a(x)[h(x) - e(t)] \qquad (1)$$

where a: **R** $\longrightarrow$ $(0,\infty)$, b, c, h: **R** $\longrightarrow$ R and e: **R** $\longrightarrow$ **R** are continuous. Our aim in this paper is to establish criteria under which solutions to the second – order nonlinear differential system in Eq.(1) exist globally.

System (1) can be regarded as a mathematical model for many phenomena in applied sciences. It has been investigated by several authors. Constantin Adrian[2] considered sufficient conditions for the continuability of solutions for perturbed differential equations and some results for the global existence of solutions were obtained. Mustafa and Rogovchenko[6] proved global existence of solutions for a general class of nonlinear second-order differential equations that includes, in particular, Van der Pol, Rayleigh and Lienard equations and relevant examples were discussed. Then Cemil Tunc and Timur Ayhan[3] dealt with the global existence and boundedness of solutions for a certain nonlinear integro-differential equation of second order with multiple constant delays and obtained some new sufficient conditions which guarantee the global existence and boundedness of solutions to the considered equation. Takasi Kusano and William F. Trench[7] established sufficient conditions that ensure that the solutions to nonlinear differential equations exist on a given interval and have the prescribed asymptotic behaviour. Our result is closely related to the one obtained in the paper[1].

**Definition 1.1**: Consider the following Cauchy problem for a nonlinear system:

$$\begin{cases} y''(t) = f(t, y(t)), \\ y'(t_0) = x_0, \end{cases} \tag{1.1}$$

where $(t_o, x_o) \in A \dot{\subseteq} \mathbf{R} \times \mathbf{R}^n = \mathbf{R}^{n+1}$ is fixed, with A, a given open set and $f: A \longrightarrow \mathbf{R}^n$ continuous. We say that a function $y: I \longrightarrow \mathbf{R}^n$ is a solution of (1.1) on I if $I \dot{\subseteq} \mathbf{R}$ is an open interval, $t_o \in I$, $y \in c^1(I, \mathbf{R}^n)$, $y'(t) = f(t, y(t))$ for all $t \in I$ and $y(t_o) = x_o$

**Definition 1.2**: We say that the problem (1.1) admits global solutions or is globally solvable if for every open interval $I \dot{\subseteq} \mathbf{R}$ such that

$\{x \in \mathbf{R}^n | (t, x) \in A\} \neq \emptyset \quad \forall t \in I$, there exists a function $y: I \longrightarrow \mathbf{R}^n$ which is solution of (1.1) on I.

**Main Result**

Main result will be presented on the global existence of solutions to system (1.1) under general conditions on the nonlinearities.

Define $\int_0^y c(s)ds$ , $H(x) = \int_0^x a^2(s)\, h(s)ds$

Then we have the following:

**Theorem 2.1**: Assume that

(i) there exists some $k \geq 0$ such that

$$sgn(x)H(x) + k \geq 0, \quad x \in \mathbf{R} \tag{2.2}$$

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$$sgn\,(y)C(y)+k\geq 0, \qquad y\in \mathbf{R}$$

(ii) there exists some $N\geq 0$ and $Q>0$ such that

$$\left| H(x) \right| < Q,\ |x| > N, \qquad (2.3)$$

$$\left| C(y) \right| < Q,\ |y| > N,$$

(iii) $\lim_{|y|\to\infty} sgn(y)\,C(y) = \mathrm{Q}\lim_{|x|\to\infty}[\left(\frac{1}{Q-sgn(x)H(x)}\right)+sgn(x)b(x)] = \infty$

(iv) there exists two positive functions $\mu, w \in c([0,\ K+Q),(0,\ \infty))$ such that $a(x)\,|c(y)| \leq \min\begin{bmatrix} \mu(sgn(x)H(x)+K)+w(sgn(x)H(x)+K), \\ \mu(sgn(y)C(y)+K)+w(sgn(y)C(y)+K) \end{bmatrix}$

$$|x| > N, |y| > N \qquad (2.4)$$

(v) $sgn(x)a(x)b(x)h(x) \geq -[\mu(sgn(x)H(x)+K)+w(sgn(x)H(x)+K)]$, $x > N$ *and* $|h(x)| \leq M < \infty, x \in \mathrm{R}$

If

$$\int_0^{K+Q}\frac{ds}{\mu(s)+w(s)} = \infty,$$

then every solution of (1.1) exists globally.

**Proof**: Due that $a{:}\,\mathbf{R}\to(o,\infty), b, c, h : \mathbf{R}\to \mathrm{R}$ and $e{:}\,\mathbf{R}\to\mathbf{R}$ are continuous, by Peano's Existence Theorem [6 ], we have that the system (1.1) with any initial data $(x_0\,,\ y_0)$ possesses a solution$(x(t), y(t))$ on $[O\,,T]$ for some maximal $T>0$. If $T<\infty$, one has

$$\lim_{t\to T}(|x(t)| + |y(t)|) = \infty$$

First, assume that $\lim_{t\to T}|y(t)| = \infty$

Since $\mathrm{y(t)}$ is continuous, there exists $0\leq T_0 < T$ such that

$$y(t) > N, \qquad t\in\ [T_0, T] \qquad (2.7)$$

Take $V_1(t,x,y) = sgn(y)C(y)+K,\ \ t\in\mathbf{R}_+,\ \ x,y\in\mathbf{R}.$ Differentiating $V_1(t,x,y)$ with respect to t along solution $(x(t),y(t))$ of (1.1), we have

$$dV_1 = \frac{\partial V_1}{\partial t}dt + \frac{\partial V_1}{\partial x}dx + \frac{\partial V_1}{\partial y}dy$$

 

Dividing both sides by $dt$, we have

$$\frac{dV_1}{dt} = \frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x}\frac{dx}{dt} + \frac{\partial V_1}{\partial y}\frac{dy}{dt} = 0 + 0 + \frac{\partial C(y)}{\partial y}y'$$

But

$$C(y) = \int_0^y c(s)ds \text{ and } y' = -a(x)[h(x) - e(t)] = -a(x)h(x) + a(x)e(t)$$

$$\Rightarrow \frac{\partial c(y)}{\partial y} = c(y)$$

$$\frac{dV_1}{dt} = sgn(y)c(y)[-a(x)h(x) + a(x)e(t)]$$

$$\frac{dV_1}{dt} = sgn(y)[-a(x)\, c(y)\, h(x) + a(x)\, c(y)\, e(t)]$$

Using triangle inequality, we have

$$\frac{dV_1}{dt} \leq |a(x)c(y)|\,|h(x) + e(t)|$$

$$\leq |a(x)c(y)|\,(|h(x)| + |e(t)|)$$

$$\leq (|h(x)| + |e(t)|)\ a(x)\,|c(y)|$$

Recall that $|h(x)| \leq M$ and also by assumption (2.4), We have

$$\frac{dV_1}{dt} \leq (M + |e(t)|)\ [\mu\,(sgn(y)\,c(y) + K) + w(sgn(y)\,c(y) + K)],$$

$$t \in [T_{0,}T] \tag{2.8}$$

Since $o \leq sgn(y(t))\ C(y(t)) + K < Q < Q + K,\ t \in [\,T_0, T], we\ obtain$

$$dV_1 \leq (M + |e(t)|)\ [\mu(sgn(y)\,c(y) + K) + w(sgn(y)\,c(y) + K]dt$$

$$dV_1(t) \leq (M + |e(t)|)\ [(sgn(y)\,c(y) + K)(\mu + w)]dt$$

Dividing both sides by $(V_1(t)) + w(V_1(t))$ , we have

$$\frac{dV_1(t)}{\mu(V_1(t)) + w(V_1(t))} \leq \frac{(M + \ |e(t)|\ )[V_1(t)(\mu + w)]dt}{V_1(t)(\mu + w)}$$

$$\leq (M + \ |e(t)|\ )dt, \qquad t \in [T_{0,}T] \tag{2.9}$$

Denote $V_1(t) = V_1(t, x(t), y(t))$

Since

$$\lim_{|y|\to\infty} sgn(y)c(y) = Q, \qquad \int_0^{K+Q} {\partial s}/{(\mu(s) + w(s)))} = \infty,$$

$y(t), c(y)$ are continuous, there exists $T_0 \leq t_1 < t_2 < T$ such that

$$\int_{V_1(t_1)}^{V_{1(t_2)}} \frac{ds}{\mu(s) + w(s)}$$

$$\int_{V_1(t_1)}^{V_{1(t_2)}} \frac{ds}{\mu(s) + w(s)} > M\int_0^T dt + \int_0^T |e(t)|\,dt \tag{2.10}$$

Integrating **(2.9)** with respect to **t**, we obtain

$$M\int_0^T dt + \int_0^T |e(t)|\,dt < \int_{V_1(t_1)}^{V_{1(t_2)}} \frac{ds}{\mu(s) + w(s)} = \int_{t_1}^{t_2} \frac{dV_1(t)}{\mu\big(V_1(t)\big) + w\big(V_1(t)\big)}$$
$$\leq \int_{t_1}^{t_2} (M + \ |e(t)|\ )\,dt \leq M\int_0^T dt + \int_0^T |e(t)|\,dt \qquad 2.11$$

$\exists\ an\ M > 0\ \ni$

$|y(t)| \leq M \qquad t \in [0,T]$ 2.12

By the result above, we have $\lim_{t\to T} |x(t)| = \infty$

Setting $V_2(t,x,y) = sgn(x)\,H(x) + K \qquad t \in \mathbf{R}_+,\ x,y \in \mathbf{R}$ 2.13

Since $x(t)$ is continuous, $\exists\ 0 \leq T_1 < T \ \ni$

$|x(t)| > N,\ t \in [T_1, T]$ 2.14

We differentiate $V_2(t,x,y)$ with respect to $t$ along solution $(x(t), y(t))$ of (1.1), we obtain $dV_2 = \frac{\partial V_2}{\partial t}\,dt + \frac{\partial V_2}{\partial t}\,dx + \frac{\partial V_2}{\partial y}dy$

$$\frac{dV_2}{dt} = \frac{\partial V_2}{\partial t} + \frac{\partial V_2}{\partial x}\,\frac{dx}{dt} + \frac{\partial V_2}{\partial y}\,\frac{dy}{dt}$$

Following the same process as in (2.8), we obtain

$$\frac{dV_2}{dt} = sgn(x)[a(x)c(y)h(x) - a(x)b(x)h(x)]$$

$$\leq (M+1)[\mu(sgn(x)H(x)+k) + w(sgn(x)H(x)+k)] \quad (2.15)$$

$$\frac{dV_1(t)}{\mu(V_1(t)) + w(V_1(t))} \leq (M+1)dt, \qquad t \in [T_1, T] \qquad (2.16)$$

Denote $V_2(t) = V_2(t, x(t), y(t))$

Since $\lim_{|x|\to\infty} sgn(x)H(x) = Q, \quad \int_0^{K+Q} \left(\frac{ds}{(\mu(s)+w(s))}\right) = \infty, x(t), H(x)$

are continuous, there exists $T_1 \leq t_3 < t_4 < T$ *such that*

$$\int_{V_2(t_3)}^{V_2(t_4)} \frac{ds}{\mu(s)+w(s)} > (M+1)\int_0^T \partial t \qquad (2.17)$$

Integrating (2.16) on $[t_3, t_4]$ with respect to **t** and using the above inequality, we obtain:

$$(M+1)\int_0^T \partial t \; < \; \int_{V_2(t_3)}^{V_2(t_4)} \frac{\partial s}{\mu(s)+w(s)} = \int_{t_3}^{t_4} \frac{\partial V_2(t)}{\mu\big(V_2(t)\big)+w\big(V_2(t)\big)} \quad (2.18)$$

$$\leq \int_{t_3}^{t_4} (M+1)dt \leq (M+1)\int_0^T dt$$

which is a contradiction.

Considering $\lim_{|x|\to\infty}\left(\frac{1}{\big(Q - sgn(x)H(x)\big)}\right)$

By (ii), we have

$$\lim_{|x|\to\infty} sgn(x)b(x) = \infty$$

Set

$$W(t,x,y) = x, \qquad t \in \mathbf{R}_+, x, y \in \mathbf{R} \quad (2.19)$$

Then, along solutions to (1.1), we have

$$\frac{dW}{dt} = \frac{1}{a(x)}[c(y) - b(x)] \quad (2.20)$$

If $\lim_{t\to T} x(t) = \infty$, we deduce that there exists $x_1 and\ x_2$ such that

$$x_0 < x_1 < x_2 \text{ and } \frac{dW}{\partial t} < 0, \qquad x_1 \leq x < x_2, \qquad |\,y\,| \leq M \quad (2.21)$$

Then, by the continuity of the solution, there exist $0 < t_1 < t_2 < T \quad such\ that\ x(t_1) = x_1$

$x(t_2) = x_2$ . Integrating (2.21) on $[t_1, t_2]$, we have

$$W\big(t_1, x(t_1), y(t_1)\big) = x_1 > x_2 = W\big(t_2, x(t_2), y(t_2)\big) \quad (2.22)$$

This contradicts $x_1 < x_2$ . Hence $x(t)$ is bounded from above.

Similarly, if $\lim_{t \to T} x(t) = -\infty$, we can obtain a contradiction by setting W(t,x,y) = -x. It follows that x(t) is also bounded from above. This forces T = ∞ and hence the proof is complete.

## Conclusion

The main usefulness of this paper is the establishment of the sufficient conditions that guarantee global existence of solutions to a system of second-order nonlinear differential equations and the investigation that if solutions to a system of second-order nonlinear differential equations exist globally, that is, the solutions are defined throughout the entire real axis, then the solutions are also bounded from above and below.

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## References

1. Chuanxi Qian, "Boundedness and asymptotic behaviour of solutions of a second-order nonlinear system, Journal of Topology, Bull.London Math.Soc.(1992) 24(3): 281-288 doi:1112//ms/24.3.281.
2. Constantin A, "Global existence of solutions for perturbed differential equations", Annali di Matematica Pura ed Applicata, vol. 168, pp.237-299, 1995.
3. Cemil Tunc, Timur Ayhan, "Global existence and boundedness of solutions of a certain nonlinear integro-differential equation of second order with multiple deviating arguments", Journal of Inequalities and Applications, DO1 10.1186/s 13660-016-0987-2, 2016.
4. Changjian Wu, Shushuang Hao, Changwei Xu, "Global existence and boundedness of solutions to a second-order nonlinear differential system", Journal of Applied Mathematics, vol.2012, Article ID 603783, 12 pages.
5. Kato J., "On a boundedness condition for solutions of a generalized Lienard equations", vol. 65, no.2, pp. 269-286, 1986.

6. Mustafa O.G., Rogovchenko Y.V., "Global existence of solutions for a class of nonlinear differential equations", Applied Mathematics Letters 16, pp.753-758, 2003.
7. Takasi Kusano, William F., "Global existence for solutions of nonlinear differential equations with prescribed asymptotic behaviour", Annali di Matematica pura ed applicata(IV), vol CXLII, pp. 329-381.
8. Lakshmikantham, Leela S., "Differential and Integral Inequalities", vol. I: Ordinary Differential Equation Academic Press, London, Uk, 1969.
9. Yin Z., "Global existence and boundedness of solutions to a second order nonlinear differential system", Studia Scientiarum Mathematicarum Hungarica, vol.41, no.4, pp.365-378, 2004.