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ON A GENERALIZATION OF LINEAR POSITIVE OPERATORS FOR FUNCTIONS OF GROUTH 2^{x+y} IN TWO DIMENSIONS (x, y)

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Abstract:

In this paper, we introduce a generalization of linear positive operators $L_n^{\sim}(f; x)$ which we studied in [1], generalized in two dimensions (x, y). We defined a sequence of linear positive operators to approximate unbounded functions in the domain $(\mathbb{R}_0 \times \mathbb{R}_0)$. We study some approximation properties for this operators $L_{n,m}^{\sim}(f; x, y)$ like Korovkin theorem and we proved Voronoviskaja – type asymptotic formula for the operators $L_{n,m}^{\sim}(f; x, y)$, whom we defined.

Key words: Linear positive operators, Korovkin Theorem, Voronoviskaja Theorem.

1.Introduction

In 1995 Lupas[9], defined and studied the identity;

$$\frac{1}{(1-a)^{\alpha}} = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} a^k \ , |a| < 1; \qquad \text{Where} (\alpha)_k = \begin{cases} \alpha(\alpha+1) \ \ldots (\alpha+k-1) & k \in N \coloneqq \{1,2,\ldots\} \\ 1 \ , & k = 0 \end{cases}. \quad (1.1)$$

for more details, see [9],[2],[4] and[5]) and he introduced the linear positive operators

$$L_n(f; x) = (1-a)^{nx} \sum_{k=0}^{\infty} \frac{(nx)_k}{k!} a^k f(\tfrac{k}{n}).$$

By letting $\alpha = nx$ and $x \geq 0$. Where $f: \mathbb{R}_0 \to \mathbb{R}$. In 1999 Agratini [2], Supposing that $L_n(1; x) = 1$, he found that $a = \frac{1}{2}$. Hence Lupas defined the linear positive operators;

$$L_n(f; x) = 2^{-nx} \sum_{k=0}^{\infty} \frac{(nx)_k}{2^k k!} f(\tfrac{k}{n}) \qquad (1.2)$$

In 2007 A. Erencyn and Fatma Tasdelen, [5], define a sequence of positive linear operators and study some approximation properties for it that he fine the generalization of the operators (1.2) above, so he define this operators:

$L_n(f; x) = 2^{-a_n x} \sum_{k=0}^{\infty} \frac{(a_n x)_k}{2^k k!} f(\frac{k}{b_n})$, where $x \in \mathbb{R}_0, n \in \mathbb{N}$, $\{a_n\}, \{b_n\}$ some increasing and unbounded sequence. In 2014 Mohammed and Sadiq [1] defined a new sequence of positive and linear operators as fellow:

$$L_n^{\sim}(f; x) = \frac{1}{G_x} \ 2^{-nx} \sum_{k=0}^{\infty} \frac{(nx)_{k+r}}{2^{k+r} \ (k+r)!} f(\tfrac{k+r}{n}) \qquad (1.3)$$

where $G_x = 2^{-nx} \sum_{k=0}^{r} \frac{(nx)_k}{2^k \ k!}$ or $G_x = 2^{-nx} \sum_{k=0}^{\infty} \frac{(nx)_{k+r}}{2^{k+r} \ (k+r)!}$

$G_x := \sum_{k=0}^{r} d_{n,k}(x)$ or $G_x := \sum_{k=0}^{\infty} d_{n,k+r}(x)$

Then, it can be written the operator (1.3) as follow

$L_n^{\sim}(f; x) = \frac{1}{G_x} \sum_{k=0}^{\infty} d_{n,k+r}(x) \ f(\frac{k+r}{n})$ where $x \in \mathbb{R}_0, n \in \mathbb{N}$, $\mathbb{R}_0 = [0, \infty)$, $\mathbb{N} \coloneqq \{1,2,\ldots\}$ and $r \in \mathbb{N}$.

In this paper, we define and study the operators $L_{n,m}^{\sim}(f; x, y)$ which represents the generalization of the operators (1.3) in two dimensions (x, y)([8], [6] and [12]) as follow:

$$L_{n,m}^{\sim}(f; x, y) = \frac{1}{G_x} \frac{1}{G_y} \sum_{k=0}^{\infty} d_{n,k+r}(x) \sum_{j=0}^{\infty} d_{m,j+s}(y) \ f(\tfrac{k+r}{n}, \tfrac{j+s}{m}) \qquad (1.4)$$

where $d_{n,k+r}(x) = 2^{-nx} \sum_{k=0}^{\infty} \frac{(nx)_{k+r}}{2^{k+r}\,(k+r)!}$

and $x, y \in \mathbb{R}_0$, $f \in C_{\rho,q}(\mathbb{R}_0 \times \mathbb{R}_0)$,

$C_{\rho,q}$: The space of all continuous and unbounded functions f on the area $\mathbb{R}_0 \times \mathbb{R}_0$, with growth $(O(2^{\rho+q}))$, for some $\rho, q > 0$,with the norm

$\|f(x,y)\|_{\rho,q} = Sup_{(x,y)\epsilon(\mathbb{R}_0\times\mathbb{R}_0)} \frac{|f(x,y)|}{\rho(x)q(y)}$, where $\rho(x) = 2^x$.

2. Main Results:

Firstly, we study the convergent conditions on the operators (1.4), and before that we need to offer some theorems (Korovkin theorem [3] and [11]) Mohammed and Sadiq proved in 2014 [1], on the operators (1.3):

Theorem(2.1) (Korovkin Theorem):[1]

For $x \in \mathbb{R}_0$, $f \in C_\rho$ and by applying Korovkin Theorem on the operator $L_n^{\sim}(f;x)$, we have:

1) $L_n^{\sim}(1;x) = 1$

2) $L_n^{\sim}(t;x) = x + \frac{2r}{n\ G_x} d_{n,r}(x)$.

3) $L_n^{\sim}(t^2;x) = x^2 + \frac{2x}{n} + r d_{n,r}(x)\left[\frac{6nx+2r+6}{3n^2 G_x}\right]$

4) $L_3^{\sim} = \frac{8}{7} r d_{n,r}(x)\left[r^2 + \frac{(nx+r)(3r+nx+5)+21+nx(33+7nx)+4r(nx+3)}{4}\right]$

5) $L_4^{\sim} = +n^4x^4G_x + 12n^3x^3G_x + 36n^2x^2G_x + 26nxG_x$

$$+\frac{16}{15} r d_{n,r}(x)\begin{bmatrix} 1.8214n^3x^3 + 15.3362n^2x^2 + 34.6249nx + 1.4285 + r^3 + 1.6071r^2 \\ +11.5892r + 0.4999r^2nx + 0.75rn^2x^2 + 5.9642rnx \end{bmatrix}.$$

Lemma (2.2): For the operator $L_n^{\sim}(f;x,y)$, where $f(t,u) \in C_{\rho,q}(\mathbb{R}_0 \times \mathbb{R}_0)$, we have:

$\lim_{n,m\to\infty}\left\|L_{n,m}^{\sim}(f;x,y) - f(x,y)\right\|_{\rho,q} = 0$ if and only if

1) $\lim_{n,m\to\infty}\left\|L_{n,m}^{\sim}(1;x,y) - 1\right\|_{\rho,q} = 0$; (2.1)

2) $\lim_{n,m\to\infty}\left\|L_{n,m}^{\sim}(t;x,y) - x\right\|_{\rho,q} = 0$; (2.2)

3) $\lim_{n,m\to\infty}\left\|L_{n,m}^{\sim}(u;x,y) - y\right\|_{\rho,q} = 0$; (2.3)

4) $\lim_{n,m\to\infty}\left\|L_{n,m}^{\sim}(t^2+u^2;x,y) - (x^2+y^2)\right\|_{\rho,q} = 0$; (2.4)

for $x, y \in \mathbb{R}_0$ and $f \in C_{\rho,q}(\mathbb{R}_0 \times \mathbb{R}_0)$.

Proof: Clearly,

1) $\lim_{n,m\to\infty}\left\|L_{n,m}^{\sim}(1;x,y) - 1\right\|_{\rho,q} =$

$$\left|\frac{1}{G_x}\frac{1}{G_y}\sum_{k=0}^{\infty} d_{n,k+r}(x)\sum_{j=0}^{\infty} d_{m,j+s}(y)\ \ ) - 1\right| = 0$$

2) $\lim_{n,m\to\infty}\left\|L_{n,m}^{\sim}(t;x,y) - x\right\|_{\rho,q} = \left|\frac{1}{G_x}\frac{1}{G_y}\sum_{k=0}^{\infty}\left(\frac{k+r}{n}\right) d_{n,k+r}(x)\sum_{j=0}^{\infty} d_{m,j+s}(y) - x\right| =$
$\left|\frac{1}{G_x}\sum_{k=0}^{\infty}(\frac{k+r}{n})\ d_{n,k+r}(x) - x\right|$

By using theorem (2.1) (2) we get;

$L_n^{\sim}(t;x) = x + \frac{2r}{n\ G_x} d_{n,r}(x)$

that means $\frac{1}{G_x}\sum_{k=0}^{\infty}\left(\frac{k+r}{n}\right) d_{n,k+r}(x) = x + \frac{2r}{n\ G_x} d_{n,r}(x)$ and this a broach to x as $n \to \infty$.

Then $\left|\frac{1}{G_x}\sum_{k=0}^{\infty}(\frac{k+r}{n})\ d_{n,k+r}(x) - x\right| \to 0$ as $n \to \infty$.

Therefore $\lim_{n,m\to\infty}\left\|L_{n,m}^{\sim}(t;x,y) - x\right\|_{\rho,q} = \frac{2r}{nG_x} d_{n,r}(x)$ which tend to 0 as $n \to \infty$.

We proved (2.1) and (2.2), by the same way we shall prove (2.3)

3) $\lim_{n,m\to\infty}\|L_n^{\sim}(u;x,y)-y\|_{\rho,q}=\left|\frac{1}{G_x}\frac{1}{G_y}\sum_{k=0}^{\infty}d_{n,k+r}(x)\sum_{j=0}^{\infty}\left(\frac{j+s}{m}\right)d_{m,j+s}(y)\ -y\right|=$

$\left|\frac{1}{G_y}\sum_{j=0}^{\infty}(\frac{j+s}{m})\ d_{m,j+s}(y)-y\right|$

By using theorem (2.1) (2) we have;

$L_m^{\sim}(u;y)=y+\frac{2s}{m\ G_y}d_{m,s}(y);$

Then $\left|\frac{1}{G_y}\sum_{j=0}^{\infty}(\frac{j+s}{m})\ d_{m,j+s}(y)-y\right|\to 0$ as $m\to\infty$.

Therefore $\lim_{n,m\to\infty}\left\|L_{n,m}^{\sim}(u;x,y)-y\right\|_{\rho,q}=\frac{2s}{mG_y}d_{m,s}(y)$ which tend to 0 as $m\to\infty$.

Now, we want to prove (2.4) bellow;

4) $\lim_{n,m\to\infty}\left\|L_{n,m}^{\sim}(t^2+u^2;x,y)-(x^2+y^2)\right\|_{\rho,q}$

$=\lim_{n,m\to\infty}\left\|L_{n,m}^{\sim}\left(\left(\frac{k+r}{n}\right)^2+\left(\frac{j+s}{m}\right)^2;x,y\right)-(x^2+y^2)\right\|_{\rho,q}$

$=\left|\frac{1}{G_x}\frac{1}{G_y}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\left(\left(\frac{k+r}{n}\right)^2+\left(\frac{j+s}{m}\right)^2\right)d_{n,k+r}(x)d_{m,j+s}(y)\ -(x^2+y^2)\right|$

$=\left(\frac{1}{G_x}\sum_{k=0}^{\infty}\left(\frac{k+r}{n}\right)^2 d_{n,k+r}(x)-x^2\right)\left(\frac{1}{G_y}\sum_{j=0}^{\infty}\left(\frac{j+s}{m}\right)^2 d_{m,j+s}(y)\ -y^2\right)$

By using theorem (2.1) (3) we have;

$L_n^{\sim}(t^2;x)=x^2+\frac{2x}{n}+rd_{n,r}(x)\left[\frac{6nx+2r+6}{3n^2G_x}\right],$

So, $\left(\frac{1}{G_x}\sum_{k=0}^{\infty}\left(\frac{k+r}{n}\right)^2 d_{n,k+r}(x)-x^2\right)=\frac{2x}{n}+rd_{n,r}(x)\left[\frac{6nx+2r+6}{3n^2G_x}\right]$

which tend to 0 when $n\to\infty$.

and $\frac{1}{G_y}\sum_{j=0}^{\infty}\left(\frac{j+s}{m}\right)^2 d_{m,j+s}(y)\ -y^2=\frac{2y}{m}+sd_{m,s}(y)\left[\frac{6my+2s+6}{3m^2G_y}\right]$

which tend to 0 as $m\to\infty$.

So the proof is complete. ∎

Later, we shall prove Voronoviskaja theorem on the operators (1.4) and to prove it we need the next lemma to get the moment to the operators (1.3) which proved by Mohammed and Sadiq in 2014 in [1].

Lemma (2.3):[1] Let $r\in\mathbb{N}$, then for all $x\in\mathbb{R}_0$ and $n\in\mathbb{N}$, we have

1) $T_{n,0}^{\sim}(x)=1.$
2) $T_{n,1}^{\sim}(x)=\frac{2r}{n\ G_x}d_{n,r}(x).$
3) $T_{n,2}^{\sim}(x)=rd_{n,r}(x)\left(\frac{6nx+2r+6}{3n^2G_x}+\frac{4x}{nG_x}\right)+\frac{2x}{n}.$
4) $T_{n,3}^{\sim}(x)=rd_{n,r}(x)\left(\frac{8r^2+2(nx+r)(3r+nx+5)+42+2nx(33+7nx)+8r(nx+3)}{7n^3G_x}-\frac{2rx+6}{n^2G_x}\right)+\frac{6x}{n^2}.$
5) $T_{n,4}^{\sim}(x)=rd_{n,r}(x)\Big[5.9428\frac{x^3}{nG_x}+28.358\frac{x^2}{n^2G_x}+36.9332\frac{x}{n^3G_x}+4.8\frac{rx^2}{n^2G_x}+0.3618\frac{rx}{n^3G_x}++\frac{1.5237}{n^4G_x}+$ $1.0666\frac{r^3}{n^4G_x}+1.7142\frac{r^2}{n^4G_x}-7.9999\frac{r^2}{n^3G_x}+12.3618\frac{r}{n^4G_x}+0.5332\frac{xr^2}{n^3G_x}-9.1428\frac{rx}{n^2G_x}-9.1428\frac{x^2}{nG_x}-$ $9.4285\frac{x}{n^2G_x}-19.4285\frac{r}{n^3G_x}-\frac{24}{n^3G_x}\Big]+\frac{12x^2}{n^2}+\frac{36x}{n^3}.$

Now, we ready to state and prove Voronovskaja theorem

Theorem (2.4):[11] *(Voronovskaja theorem)*

Suppose that $f\in C_{\rho,q}(\mathbb{R}_0\times\mathbb{R}_0)$, $\rho,q\epsilon\mathbb{N}^0$, and suppose that

$\frac{\partial^2 f(x,y)}{\partial x^2}$ and $\frac{\partial^2 f(x,y)}{\partial y^2}$ exist at a point $(x,y)\in\mathbb{R}_0\times\mathbb{R}_0$.

Then

$\lim_{n\to\infty}n\left(L_{n,n}^{\sim}(f(t,u);x,y)-f(x,y)\right)=x\frac{\partial^2 f(x,y)}{\partial x^2}+y\frac{\partial^2 f(x,y)}{\partial y^2}.$

Proof: By using Taylor's expansion for $f(t,u)$ about (x,y) ,we have; [10]

$$f(t,u) = f(x,y) + f_x'(x,y)(t-x) + f_y'(x,y)(z-y) + \frac{1}{2}\left(f_{xx}''(x,y)(t-x)^2 + 2f_{xy}''(x,y)(u-y)^2\right)$$
$$+\varphi(t,u;x,y)\sqrt{(t-x)^4) - (u-y)^4}$$

where $(t,u) \in (0,\infty) \times (0,\infty)$, and $\varphi(t,u) = \varphi(t,u;x,y)$is a function belonging to $C_{\rho,q} \in [0,\infty) \times [0,\infty)$ and $\varphi(t,u) \to 0$ when $(t,u) \to (x,y)$ for $n,m \in \mathbb{N}$.

So, by depending on $L_{n,n}^{\sim}(f;x)$ is a sequence of linear positive operators, we have;

$$L_{n,n}^{\sim}(f(t,u);x,y) = f(x,y) + \frac{\partial f(x,y)}{\partial x} L_n^{\sim}((t-x);x)$$
$$+\frac{\partial f(x,y)}{\partial y} L_n^{\sim}((u-y);y)$$
$$+\frac{1}{2}\left[\frac{\partial^2 f(x,y)}{\partial x^2} L_n^{\sim}((t-x)^2;x) + \frac{\partial^2 f(x,y)}{\partial x \partial y} L_n^{\sim}((t-x);x) L_n^{\sim}((u-y);y) + \frac{\partial^2 f(x,y)}{\partial y^2} L_n^{\sim}((u-y)^2;y)\right] +$$
$$L_{n,n}^{\sim}\left(\varphi(t,u)\sqrt{(t-x)^4 + (u-y)^4};x,y\right)$$

Here, using lemma (2.3), we have:

$$\lim_{n\to\infty} n\left(L_{n,n}^{\sim}(f(t,u);x,y) - f(x,y)\right) =$$
$$\frac{\partial f(x,y)}{\partial x}\frac{2r}{nG_x} d_{n,r}(x) + \frac{\partial f(x,y)}{\partial y}\frac{2r}{nG_x} d_{n,r}(x) +$$
$$\frac{1}{2}\left[\frac{\partial^2 f(x,y)}{\partial x^2}\left\{r d_{n,r}(x) + \left(\frac{6nx+2r+6}{3n^2 G_x} + \frac{4x}{nG_x}\right)\frac{2x}{n}\right\} + \frac{\partial^2 f(x,y)}{\partial x \partial y}\frac{2r}{nG_x} d_{n,r}(x)\frac{2r}{nG_y} d_{n,r}(y) + \frac{\partial^2 f(x,y)}{\partial y^2}\left\{r d_{n,r}(y) +\right.\right.$$
$$\left.\left.\left(\frac{6ny+2r+6}{3n^2 G_y} + \frac{4y}{nG_y}\right)\frac{2y}{n}\right\}\right] + nL_{n,n}^{\sim}\left(\varphi(t,u)\sqrt{(t-x)^4 + (u-y)^4};x,y\right)$$

$$\lim_{n\to\infty} n\left(L_{n,n}^{\sim}(f(t,u);x,y) - f(x,y)\right) =$$
$$x\frac{\partial^2 f(x,y)}{\partial x^2} + y\frac{\partial^2 f(x,y)}{\partial y^2} + \lim_{n\to\infty} n\,L_{n,n}^{\sim}\left(\varphi(t,u)\sqrt{(t-x)^4 + (u-y)^4};x,y\right).$$

By using the Cauchy - Schwartz inequality; we get:

$$\left|L_{n,n}^{\sim}\left(\varphi(t,u)\sqrt{(t-x)^4 + (u-y)^4};x,y\right)\right|$$
$$\leq \left(L_{n,n}^{\sim}(\varphi^2(t,u);x,y)\right)^{1/2} \cdot \left(L_n^{\sim}((t-x)^4;x) + L_n^{\sim}((u-y)^4;y)\right)^{1/2}$$
$$\leq \varphi^2\left(L_n^{\sim}((t-x)^4;x) + L_n^{\sim}((u-y)^4;y)\right)^{1/2}$$

Which tend to zero when n tend to ∞, (by applying lemma {(2.3) (5) }.

Hence, $\lim_{n\to\infty} n\,L_{n,n}^{\sim}\left(\varphi(t,u)\sqrt{(t-x)^4 + (u-y)^4};x,y\right) = 0$.

Therefore,

$$\lim_{n\to\infty} n\left(L_{n,n}^{\sim}(f(t,u);x,y) - f(x,y)\right) = x\frac{\partial^2 f(x,y)}{\partial x^2} + y\frac{\partial^2 f(x,y)}{\partial y^2}. \qquad \blacksquare$$

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