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On FIDEs System by Modified Sumudu Decomposition Method

Madiha Tahir, Mariam Tahir and Jamshad Ahmad[*]
Department of Mathematics, Faculty of Sciences, NCBA&E, Pakistan
[*]*Department of Mathematics, Faculty of Sciences, University of Gujrat, Pakistan*

Abstract: In this paper, the technique of modified Sumudu decomposition method has been employed to solve a system of Fredholm integro-differential equations with initial conditions. Two examples are discussed to show applicability, reliability and the performance of the modified sumudu decomposition method. This study showed the capability, simplicity and effectiveness of the modified approach.
Keywords: Modified Sumudu decomposition method; System of Fredholm integro-differential equations.

1. Introduction

Mathematical modeling of real-life phenomena usually results in functional equations like ordinary or partial differential equations, stochastic differential equations, and integral and integro-differential equations. Many mathematical formulations of physical phenomena contain integro-differential equations. The integro-differential equations play an important role in engineering, mechanics, physics, chemistry, biology, economics and electrostatic. Some important problems in science and engineering can usually be reduced to a system of integral and integro-differential equations. Integro-differential equations have attracted much attention and solving these equations has been one of the interesting tasks for mathematicians. In this research we try to introduce a solution of system of linear integro-differential equations the following form:

$$\begin{cases} U_1^{(n_1)}(x) = f_1(x) + \int_{\alpha_1}^{\beta_1} k_1\left(x,t,U_1(t),U_2(t),\ldots,U_p(t)\right)dt, \\ U_2^{(n_2)}(x) = f_2(x) + \int_{\alpha_2}^{\beta_2} k_2\left(x,t,U_1(t),U_2(t),\ldots,U_p(t)\right)dt, \\ \vdots \\ U_p^{(n_p)}(x) = f_p(x) + \int_{\alpha_p}^{\beta_p} k_p\left(x,t,U_1(t),U_2(t),\ldots,U_p(t)\right)dt, \end{cases} \tag{1}$$

with initial conditions:

$$U_i^{(j)}(x_0) = U_{ij}, i = 1,\ldots,p, j = 0,1,\ldots,n_i - 1.$$

In literature, many researchers paid attention in solving such types of integro-differential equations and system. Recently, Biazar in [4] used the Adomian decomposition method for solving the system of integro-differential equations. Also Davari applied the some method for solving linear Fredholm integro-differential equations. Maleknejad in [9] used the Galerkin methods with Hybrid Functions for solving linear integro-differential equation system. Biazar et al. in [5] by He's Homotopy perturbation method for systems of integro-differential equations, Arikoglu and Ozkol in [2], by using differential Transform method, Maleknejad et al. in [8] by using rationalized Haar functions method can be solved to integro-differential equations system. Pour-Mahmoud et al. in [10] presented the Tau method for the numerical solution of systems of Fredholm integro-differential equations.

The integral transform has been used to solve many different types of differential and integro differential equations. For similar problems, Sumudu transform was introduced and further applied to several ODEs as well as PDEs. For example, in [1], this transform was applied to the one-dimensional neutron transport equation. In [3, 6] some properties were studied. Recently, Kılıc ̧man et al. applied this transform to solve the system of differential equations [7], since there are some interesting properties that Sumudu transform [12] satisfies such as if

$$f(t) = \sum_{n=0}^{\infty} a_n t^n \text{ Then } F(u) = \sum_{n=0}^{\infty} n! a_n u^n, \tag{2}$$

In this paper, we use Sumudu transform which is defined over the set of the following functions:

$$A = \{ f(t) \mid \exists M, \tau_1, \tau_2 > 0, | f(t) | < Me^{|t|/\tau_j}, if\ t \in (-1)^j \times [0,\infty) \} \tag{3}$$

by the following formula:

$$G(u) = S[f(t)] = \int_0^{\infty} f(ut)e^{-t}dt, u \in (-\tau_1, \tau_2). \tag{4}$$

We introduce a sumudu transform with modified decomposition method for solving Eq. (1). The modified technique of Adomian decomposition method developed in [11, 14] will form a useful basis for studying the system of Fredholm integro-differential equations. The modified sumudu decomposition method has a constructive attraction in that it provides the exact solution by computing only two terms of the decomposition series.

2. Modified Sumudu Decomposition Method

Consider following PDE

$$LU(x,t)+RU(x,t)+NU(x,t)=h(x,t), U(x,0)=f(x), U_t(x,0)=g(x), \quad (5)$$

Implementing Sumudu transform to both sides of Eq. (5), we have

$$S[LU(x,t)]+S[RU(x,t)]+S[NU(x,t)]=S[h(x,t)],$$

By using the differentiation property of Sumudu transform, we obtain

$$\frac{1}{u^2}S[U(x,t)]-\frac{1}{u^2}f(x)-\frac{1}{u}g(x)+S[RU(x,t)]+S[NU(x,t)]=S[h(x,t)],$$

$$S[U(x,t)]=f(x)+ug(x)-u^2S[RU(x,t)]-u^2S[NU(x,t)]+u^2S[h(x,t)]. \quad (6)$$

According to ADM, the series solution $U(x)$ is

$$U(x)=\sum_{n=0}^{\infty}U_n(x), \quad (7)$$

The nonlinear term is decomposed as

$$NU(x,t)=\sum_{n=0}^{\infty}A_n(U), \quad (8)$$

where A_n is the Adomian's polynomials which can be determined by following formula

$$A_n=\frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\right]_{\lambda=0},$$

Using (6), (7) and (8) we get

$$\sum_{n=0}^{\infty}S[U_n(x,t)]=f(x)+ug(x)-u^2S[R\sum_{n=0}^{\infty}U_n(x,t)]-u^2S[\sum_{n=0}^{\infty}A_n(U)]+u^2S[h(x,t)],$$

(9)

Recursive relations for the modified sumudu decomposition method are formulated as:

$$S[U_0(x,t)]=k_1(x,u), \quad (10)$$

$$S[U_1(x,t)]=k_2(x,u)-u^2S[RU_0(x,t)]-u^2S[A_0(U)], \quad (11)$$

$$S[U_{n+1}(x,t)]=-u^2S[Ru_n(x,t)]-u^2S[A_n(U)], n\geq 1, \quad (12)$$

Where $k_1(x,u)$ and $k_2(x,u)$ are Sumudu transform of $k_1(x,t)$ and $k_2(x,t)$ respectively. Inverse Sumudu transform to Eqs. (10)- (12) gives our required recursive relation as follows

$$U_0(x,t)=k_1(x,t), \quad (13)$$

$$U_1(x,t)=k_2(x,t)-S^{-1}\left[u^2S[RU_0(x,t)]-u^2S[A_0(U)]\right], \quad (14)$$

$$U_{n+1}(x,t)=-S^{-1}\left[u^2S[RU_n(x,t)]-u^2S[A_n(U)]\right], n\geq 1. \quad (15)$$

3. Numerical Applications

Two examples for testing the accuracy of the proposed technique are given.

Example 3.1 We consider the following system [15]

$$\begin{cases} U_1''(x)=\dfrac{3x}{10}+6-\int_0^1 2xt(U_1(t)-3U_2(t))dt, \\ U_2''(x)=15x+\dfrac{4}{5}-\int_0^1 3(2x+t^2)(U_1(t)-2U_2(t))dt, \end{cases} \quad (16)$$

With initial conditions

$$U_1(0)=1, U_2(0)=-1, U_1^{'}(0)=0, U_2^{'}(0)=2, \quad (17)$$

With the exact solution

$$U_1(x)=3x^2+1,\ U_2(x)=x^3+2x-1.$$

Implementing the Sumudu transform on Eq. (16) and by using the initial conditions, we have

$$\begin{cases} S[U_1(x)]=1+\dfrac{3}{10}u^3+6u^2-u^3\int_0^1 2t(U_1(t)-3U_2(t))dt, \\ S[U_2(x)]=-1+2u+15u^3+\dfrac{4}{5}u^2-6u^3\int_0^1 (U_1(t)-2U_2(t))dt-3u^2\int_0^1 t^2(U_1(t)-2U_2(t))dt, \end{cases} \quad (18)$$

With inverse Sumudu transform, we get

$$\begin{cases} U_1(x)=1+\dfrac{1}{20}x^3+3x^2-\dfrac{1}{3}x^3\int_0^1 t(U_1(t)-3U_2(x)(t))dt, \\ U_2(x)=-1+2x+\dfrac{5}{2}x^3+\dfrac{2}{5}x^2-x^3\int_0^1 (U_1(t)-2U_2(x)(t))dt-\dfrac{3}{2}x^2\int_0^1 t^2(U_1(t)-2U_2(x)(t))dt, \end{cases} \quad (19)$$

Consequently, we obtain

$$\begin{cases} U_{10}(x)=3x^2+1, \\ U_{20}(x)=x^3+2x-1. \end{cases} \quad (20)$$

and

$$\begin{cases} U_{11}(x)=\dfrac{1}{20}x^3-\dfrac{1}{3}x^3\int_0^1 t(U_{10}(t)-3U_{20}(x)(t))dt, =\dfrac{1}{20}x^3-\dfrac{1}{3}x^3\left(\dfrac{3}{20}\right)=0 \\ U_{21}(x)=\dfrac{3}{2}x^3+\dfrac{2}{5}x^2-x^3\int_0^1 (U_{10}(t)-2U_{20}(x)(t))dt-\dfrac{3}{2}x^2\int_0^1 t^2(U_{10}(t)-2U_{20}(x)(t))dt, \\ \qquad =\dfrac{3}{2}x^3+\dfrac{2}{5}x^2-x^3\left(\dfrac{3}{2}\right)-\dfrac{3}{2}x^2\left(\dfrac{4}{15}\right)=0 \end{cases} \quad (21)$$

and also we take

$$\begin{cases} U_{1,n+1}(x)=0, \\ U_{2,n+1}(x)=0, \end{cases} n\geq 1. \quad (22)$$

The solution of given system is

$$\begin{cases} U_1(x)=3x^2+1, \\ U_2(x)=x^3+2x-1. \end{cases} \quad (23)$$

The graphical representation for the solution of given system (16) is

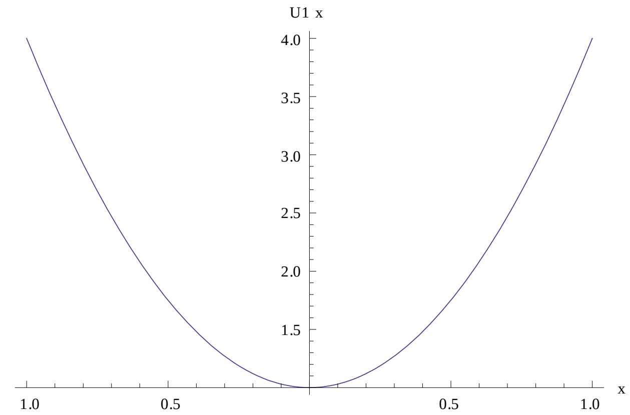

Fig: 3.1

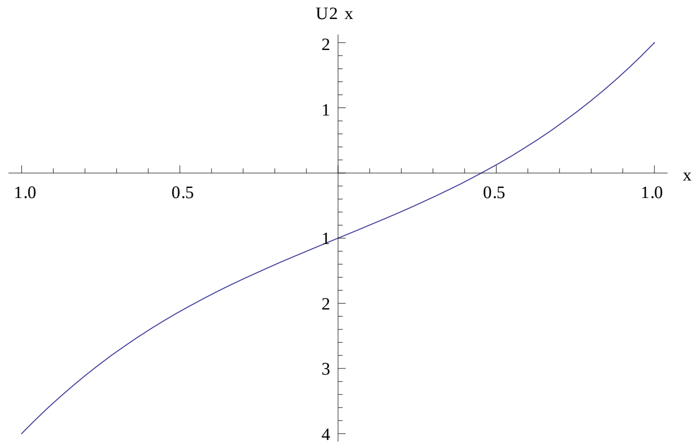

Fig: 3 2

Example 3.2 We consider the following system [15]

$$\begin{cases} U_1'(x) = -\frac{1}{4}x^2 + \int_0^1 \left[te^x U_1(t) + tx^2 U_2(t)\right]dt, \\ U_2'(x) = 2x + x\sin 1 - xe\sin 1 - \frac{1}{4}\cos x + \int_0^1 \left[x\sin t U_1(t) + t\cos x U_2(t)\right]dt, \end{cases} \quad (24)$$

with initial conditions

$$U_1(0) = 1, U_2(0) = 0, \quad (25)$$

With the exact solution

$$U_1(x) = e^x, U_2(x) = x^2.$$

Taking the Sumudu transform on Eq.(24) and by using the initial conditions we have

$$\begin{cases} S[U_1(x)] = 1 - \frac{u^3}{2} + \frac{u}{1-u}\int_0^1 tU_1(t)dt + 2u^3\int_0^1 tU_2(t)dt, \\ S[U_2(x)] = 2u^2 + u^2\sin 1 - u^2 e\sin 1 - \frac{u}{4(1+u^2)} + u^2\int_0^1 \sin 1 U_1(t)dt + \frac{u}{1+u^2}\int_0^1 tU_2(t)dt, \end{cases}$$

(26)

With the inverse Sumudu transform, we get

$$\begin{cases} U_1(x) = 1 - \frac{x^3}{12} + \left(e^x - 1\right)\int_0^1 tU_1(t)dt + \frac{x^3}{3}\int_0^1 tU_2(t)dt, \\ U_2(x) = x^2 + \frac{x^2}{2}\sin 1 - \frac{x^2}{2}e\sin 1 - \frac{\sin x}{4} + \frac{x^2}{2}\int_0^1 \sin 1 U_1(t)dt + \sin x\int_0^1 tU_2(t)dt, \end{cases} \tag{27}$$

Following the above procedure, if we assume solution of the form (7) and using the modified recursive relation, we obtain

$$\begin{cases} U_{10}(x) = e^x, \\ U_{20}(x) = x^2. \end{cases} \tag{28}$$

and

$$\begin{cases} U_1(x) = 1 - \frac{x^3}{12} - e^x + \left(e^x - 1\right)\int_0^1 tU_{10}(t)dt + \frac{x^3}{3}\int_0^1 tU_{20}(t)dt \\ \quad = 1 - \frac{x^3}{12} - e^x + \left(e^x - 1\right)(1) + \frac{x^3}{3}\left(\frac{1}{4}\right) = 0 \\ U_2(x) = \frac{x^2}{2}\sin 1 - \frac{x^2}{2}e\sin 1 - \frac{\sin x}{4} + \frac{x^2}{2}\int_0^1 \sin 1 U_{10}(t)dt + \sin x\int_0^1 tU_{20}(t)dt \\ \quad = \frac{x^2}{2}\sin 1 - \frac{x^2}{2}e\sin 1 - \frac{\sin x}{4} + \frac{x^2}{2}\sin 1(e-1) + \frac{\sin x}{4} = 0 \end{cases} \tag{29}$$

and also we take

$$\begin{cases} U_{1,n+1}(x) = 0, \\ U_{2,n+1}(x) = 0, \end{cases} n \geq 1. \tag{30}$$

The solution of given system is

$$\begin{cases} U_1(x) = e^x, \\ U_2(x) = x^2. \end{cases} \tag{31}$$

The graphical representation for the solution of given system (24)

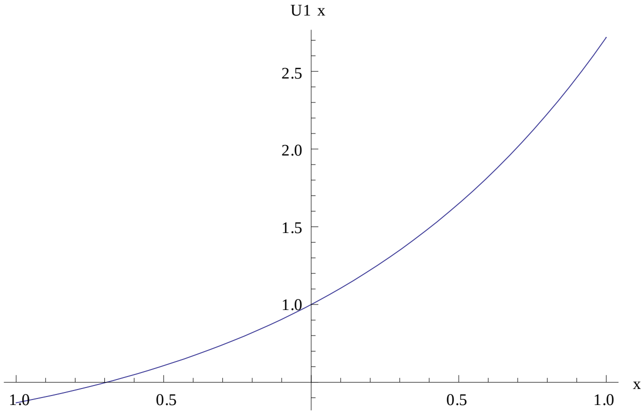

Fig: 3.3

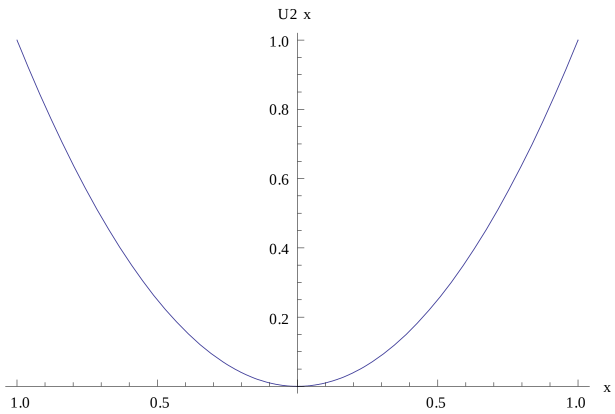

Fig: 3.4

4. Conclusion

In this paper, the modified Sumudu decomposition method is employed for solving system of Fredholm integro-differential equations. The proposed technique produces numerical results which are of highly reasonable accuracy. This study showed that the modified approach reduces the volume of the computational work. A conclusion may be draw from the obtained results that the modified Sumudu decomposition method provides with accurate approximate solutions to exact solutions.

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