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On 2- Self_adjoint Operators

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Abstract

In this paper we give a generalization of self-adjoint operators defined on a Hilbert space which we called 2-self-adjoint operators. In spite of that we established some characterizations and properties of such type of operators .Moreover we have found some of their relationships with the other classes of operators .Futhermore ,we study the spectrum of this type of operators

Keywords: Operators on Hilbert space ,Self-adjoint operator,Hyponormal operator,2-normal operators,n-power normal operator,Quasi-posinormal operator.

1. Introduction:

Let H be a complex Hilbert space and ,B(H) be the algebra of all bounded linear operator on H. In [3] Jibrill introduced the class of 2-normal operators, T$\in$ B(H) is 2-normal operator if $T^2T^*=T^*T^2$. Later he gave a generalization of the class of normal operators which he called a n-power normal operators [4,5]. T$\in$ B(H) is n -power normal operator if $T^nT^*=T^*T^n$. In this paper we give a generalization of the self-adjoint operators defined on a Hilbert space which we called 2-self-adjoint operator. We establish some characterization and properties of such type of operators .Moreover, we prove some general results about this class. An operator T is called hyponormal operator iff $\|T^*x\| \le \|Tx\|$ for all x in H . T is called dominant operator iff Range(T-λ)$\subseteq$Range(T-λ)* for each $\lambda \in \sigma(T)$. T is called posinormal if λ=0. In[1] Amelia studied the class of posinormal operators and she proved many interesting properties of such type of operator. One of the authors introduced the class of quasi-posinormal operators. T $\in$B(H) is qusai-posinormal operators iff Range (T^2) $\subseteq$Range(T^*)[1,2,6]. In this paper we also study the relationship of 2-self-adjoint with these classes above.

2. The class of 2-Self-Adjoint operators:

In this section we give a definition of 2-self-adjoint operators defined on H and its relation with the self-adjoint operator defined on H .

Definition 2.1: Let T$\in$B(H) we say that T is a 2-self-adjoint operator defined on H iff $T^2=T^{*2}$.

The class of a 2-self-adjoint defined on H is denoted by 2-Se(H).

Example 2.2: Let T: H$\rightarrow$H and H is any complex Hilbert space ,which defined as follows

Tx=5ix, for all x $\in$ H . Then T$\in$Se(H).

It is clear that if T is self-adjoint operator then T $\in$Se(H).However T in this example is not self-adjoint operator.

Remark 2.3: From definition we have T$\in$Se(H) if and only if T^* $\in$2-Se(H).

Proposition 2.4: i-If T$\in$Se(H) then T^n $\in$Se(H) for n is even interger number.

ii- If T and I-T $\in$Se(H) then T is self- adjoint operator .

iii- Let A and B are unitary equivalent then A $\in$2-Se(H) if and only if B $\in$2-Se(H) .

Proof : From definition one can prove (i)and (ii). For (iii) let $A=U^*BU$ where U is a unitary operator.

Now $A^2= U^*BU\,U^*BU= U^*B^2U= U^*B^{*2}U= (U^*B^*U)^2=A^{*2}$ hence A $\in$2-Se(H).

Proposition 2.5: Let T ,S$\in$B(H) , if T ,S $\in$2-Se(H) then the following statements are true.

i- If TS=ST then TS as well as ST $\in$2-Se(H).

ii- (T+S) $\in$2-Se(H) iff Im(ST)= - Im(ST).

Proof:we have $(TS)^2=T^2S^2=T^{*2}S^{*2}=T^{2*}S^{2*}=(S^2T^2)^*=(ST)^{2*}=(TS)^{*2}$

which implies that TS and ST are in 2-Se(H).

i) Suppose that T+S$\in$2-Se(H). then $(T+S)^2=(T^*+S^*)^2$

and $(T+S)^2= T^2+TS+ST+S^2$, $(T^*+S^*)^2 =T^{*2}+T^*S^*+S^*T^*+S^{*2}=T^{*2}+(ST)^*+(TS)^*+S^{*2}$ which implies that TS+ST $=(ST)^*+(TS)^*$, hence Im(ST)=-Im(TS).

Now if Im(ST)= - Im(TS) then $(ST)-(ST)^*= - (TS) +(TS)^*$.

Now $(T+S)^2=T^2+TS+ST+S^2=T^{*2}+(ST)^*+(TS)^*+S^{*2}=(T+S)^{*2}$ and T+S $\in$2-Se(H).

Corollary 2.6: Let T$\in$B(H) be a self-adjoint operator, if λ is real or pure imaginary number then λT$\in$2-Se(H).

3- Relationships with Other Operators

In this section we are going to investigate some of relationships of the class of 2-self-adjoint operators with other classes of operators these are the class of normal operators, the class of hyponormal operators , the class of dominant operators, the class of posinormal operators, the class of 2-normal operators as wellas the class of n-power normal operators , and quasi-posinormal operators.

It is well known that {T $\in$B(H) : T is normal operator} $\subseteq$ {T $\in$B(H) : T is hyponormal operator} $\subseteq$ {T $\in$B(H) : T is dominant operator}$\subseteq${T $\in$B(H) : T is posinormal operator}$\subseteq${T $\in$B(H) : T is quasi-posinormal operator} [6]. In[4] Jibril show that the class of 2-normal operators is belongs to the class of n-power normal operators.

Example 3.1: i- Let T: H$\rightarrow$H, where H is any complex Hilbert space, defined as follows

Tx=zx , for each x in H and for each complex number z . Then $TT^*=T^*T$ and $T^2\neq T^{*2}$ therefore T is normal operator and T is not 2- self-adjoint operator.

ii- Let H=$\ell_2(\mathbb{C})$, T:H$\rightarrow$H define as follows $T(x_1, x_2, x_3,\ldots) =(0,x_1,0,0,\ldots)$. It is easy to check that $T^*(x_1, x_2, x_3,\ldots)=(x_2,0, 0,0,\ldots)$.

Now $T^2(x_1, x_2, x_3,\ldots)= T(0,x_1,0,0,\ldots) =(0,0,0,0,\ldots)$ and $T^{*2}(x_1, x_2, x_3,\ldots)= T^*(x_2,0, 0,0,\ldots) =(0,0,0,0,\ldots)$.

If we take $x=(1,0,0,0,\ldots)$ then $TT^*x=T\,(0,0,0,0,\ldots) =(0,0,0,0,\ldots)$, $T^*Tx=T^*(0,1,0,0,0,\ldots)= (1,0,0,0,\ldots)$, hence T is 2- self-adjoint operator and T is not normal operator .Now if we take $x=(0,1,0,0,0,\ldots)$ then $\|T^*x\|^2 =\|(1,0,0,0,\ldots)\|^2 = 1$, but $\|Tx\|^2 =\|(0,0,0,0,\ldots)\|^2 =0$ then T is not hyponormal operator .

It is clear that the operator which defined in Example 3.1 belongs to the class of hyponormal operators , dominant operators, and posinormal operators which is not 2- self-adjoint operator.

It is enough to check that the operator $T(x_1, x_2, x_3,\ldots) =(0,x_1,0,0,\ldots)$ in Example 3.1(ii) is not dominant operator and not posinormal operator ,clearly that $0\in\sigma(T)$. Now let y= (0,1,0,0,0,…,0,…) then y$\in$Range(T) and $T^*(x)\neq y$ for all x in H . Hence y$\notin$ Range(T^*) , therefore T is not posinormal operator ,hence T is not dominant

Remark 3.2: The last example shows that the class of 2-self adjoint operators and the class of of normal operators, the class of hyponormal operators , the class of dominant operators as well as the class of posinormal operators are independent .

Proposition 3.3: Let T$\in$B(H) then

i- If T $\in$2-Se(H) then T is 2-normal operator therefore T is n-power normal operator.

ii-If T $\in$2-Se(H) and T is a partial isomerty operator then T is 3-normal oerator i.e $T^3T^*=T^{*3}T^*$.

Remark 3.4: Example 3.1(i) shows that T is 2-normal operator as wellas n-power normal operator which is not 2-self-adjoint operator.

Proposition 3.5: If T $\in$2-Se(H) then T is quasi-posinormal operator.

Proof:Since T $\in$2-S(H) we have $T^2=T^{*2}$. So Range (T^2)= Range(T^{*2}) $\subseteq$Range(T^*) which implies that T is a quasi-posinormal operators.

Note that The converse of the Is not true the following example illustrates this .

Example 3.6: Let H= $\ell_2(\mathbb{C})$, the unilateral shift operator on H is defined by $T(x_1, x_2, x_3,) =(0,x_1, x_2, x_3,\ldots)$.It is known that $T^*(x_1, x_2, x_3,\ldots) =(x_2,x_3, x_4,\ldots\ldots)$ and $T^2 (x_1, x_2, x_3,\ldots) =T(0,x_1, x_2, x_3,\ldots) =(0,0,x_1, x_2, x_3,\ldots)$.Now let y$\in$Range$(T^2)$ then $y=(0,0,x_1, x_2, x_3,\ldots)$ for some x in H .If we assume $x=(0,0,0,x_1, x_2, x_3,\ldots)$ then $T^*(x)=(0,0,x_1, x_2, x_3,\ldots)$ = y, and y $\in$Range(T^*) . Hence T is quasi-posinormal operator.Now if we take $x=(1,1,1,x_4,x_5, \ldots..)$ then $T^2(x)=T(0,1,1,1,x_4, x_5, \ldots..) =(0,0,1,1,1,x_4, x_5, \ldots..)$,but $T^{*2}(x)=T^*(1,1,x_4,x_5, \ldots..)=(1,x_4,x_5, \ldots..)$, therefore T is not 2- self-adjoint operator.

4- The spectrum of 2- self-adjoint Operators

The spectrum of self-adjoint operator defined on Hilbert space has been studied in the literature .In this section we study spectrum of 2-self-adjoint operator defined on Hilbert space .It is known that the spectrum of self-adjoint operator, $\sigma(T)$ is a subset of R.

Theorem 4.1: Let T $\in$2-S(H) then $\sigma(T)\subseteq$R or $\sigma(T)\subseteq$ iR ,where iR ={ix: x$\in$R}.

Proof :Suppose $\lambda\in\sigma(T)$ and λ=a+ib, where a ,b are real numbers , then $\lambda^2\in\sigma(T^2)$ by spectral mapping theorem, therefore $\lambda^2 = a^2-b^2+2iab$ is real number which implies that ab=0 hence $\lambda\in$R or $\lambda \in$ iR.

Proposition 4.2: Let T $\in$2-Se(H) then if $\lambda\in\sigma(T^2)$ then λ is real number .

Proof:Let $\lambda\in\sigma(T^2)$ then there exist $0\neq x \in$H, such that $T^2x=\lambda x$, therefore $<\lambda x,x>=<T^2x,x>=<x, T^{*2}x>=<x, T^2x>=<x, \lambda x>$, hence $(\lambda-\bar{\lambda})<x,x>=0$ and $\lambda=\bar{\lambda}$.

Proposition 4.3: Let T $\in$2-S(H) then $E_{T^2}(\lambda)$ reduces T.

Proof:Let $x\in E_{T^2}(\lambda)$ then $T^2x=\lambda x$ for some $0\neq x \in$H . Now $T^2(Tx)=T(\lambda x)= \lambda(Tx)$ and $Tx\in E_{T^2}(\lambda)$ hence $E_{T^2}(\lambda)$ is invariant under T.And $T^2(T^*x)=T^{*2}(T^*x)= T^*(T^{*2}x)= T^* (T^2x)= T^* (\lambda x)= \lambda(T^*x)$ and $T^*x\in E_{T^2}(\lambda)$, hence $E_{T^2}(\lambda)$ is invariant under T^*.

Theorem 4.4: Let T$\in$2-Se(H),if T is invertible operator then $T^{-1} \in$2-Se(H)

Proof::$(T^{-1})^2=(T^2)^{-1}=(T^{*2})^{-1}=((T^*)^{-1})^2=((T^{-1})^*)^2$ then $T^{-1}\in$2-Se(H)

Corollary 4.5:T-$\lambda\in$2-Se(H) for all $\lambda \notin \sigma(T)$, then $(T-\lambda)^{-1}\in$2-Se(H) .

Proposition 4.6: If T$\in$2-Se(H) and T^2 or T^{*2} is onto then

i- Range (T)=Range(T*).
ii- T and T^* are invertible operators

Proof:i-Suppose that T^2 is onto then Range(T^2) =H=Range(T^{*2}) $\subseteq$Range(T^*) then Range(T^*) =H, since Range(T^2) $\subseteq$Range(T) therefore Range (T)=Range(T*). Similarly if T^{*2} is onto .

ii) From (i) we have Range (T)=Range(T*)=H, hence hence $R(T^{*})^{\perp}=N(T)=H^{\perp}=\{0\}$ then T is one to one so T is invertible operator. Similarly if T^{*2} is onto.

Example 3.1(i) shows that the converse of (i) is not true

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