# One , Two and Multi-Component Gompertz Stress- strength Reliability Estimation

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**Abstract**

The reliability function for a component which has strength independently exposed three stresses ; $R_1$, two component parallel which are subjected to a common stress; $R_2$, and for a multi-component; $R(s,k)$ using Gompertz distribution with unknown location and known shape parameters, are parameter derived from a stress-strength models. Estimate the reliability $R_1$, $R_2$ and $R(s,k)$ by three methods (MLE, LSE and WLSE) and also in the numerical simulation study a comparison between the three estimates by MES is introduced.

**Keywords:** Gompertz distribution, components stress-strength, reliability, estimation, ML, LS and WLS estimation.

## 1. Introduction

The reliability of stress- strength context is used by Church and Harris 1970 [Church & Harris(1970)], it defined as $R = p(x < y)$ , where $X$ represent the stress random variable and $Y$ represent the strength random variable. [see: Asgharzadch et al(2013) Beg (1980) Karam & Ghanim (2015), Kotz S. Lumelskii et al(2003), Kundu & Gupta(2005)]

The reliability of component (or a system) can be represented in various forms depending on the structure of the system , where the component (or the system) would fail, if the stress exceeds its strength.

The Gompertz distribution plays an important role in modeling survival times, human mortality and actuarial tables, it also used as a survival model in reliability. The Gompertz force of mortality in demography (Gompertz 1825) at age x, is:

$$G(x) = ae^{bx} \quad x \geq 0 \ , a, b > 0$$

where a denotes the level of the force of mortality at age o , and b the rate of aging. This distribution has a continuous pdf with location parameter a and shape parameter b , given as: $f(x) = a\, e^{bx - \frac{a}{b}(e^{bx}-1)}$, $I_x[0,\infty)$ and cdf given as: $F(x) = 1 - e^{-\frac{a}{b}(e^{bx}-1)}$, while the Mgf of this distribution is given by its Laplace transform. [see: Lenart (2012), Saracoglu B., Kaya et al(2009), Wu et al(2004)].

The present study considering stress – strength Gompertz reliability of two models, first; for one component waving strength put up with three stresses , and the second models; for multi-component models. Related expressions with definitions and notations are described in section 2 for the one component reliability, for two component in section 3 and in section 4 for the multi component reliability when the stress and strength are Gompertz two parameter random variables. The estimation of the parameters and the three reliabilities are made by ML , LS and WLS estimation methods in section 5. Finally, the simulation study for comparing between them by MSE are explained in section 6.

## 2. Three Stress- one Strength Component Reliability

In this article the stress – strength reliability of one component subjected to three stress is examined , where the reliability of a component being exposed two exponential stresses and having Gamma strength have been obtained by karaday et al[Karaday et al(2011)].

Now Let the strength random variable of the component represented by Y as a Gompertz r.v. with two parameters $(\beta, \alpha)$ , and the component subjected stress random variables are represented by $X_i$ , $i = 1,2,3$ following Gompertz distribution with the parameters $(\beta_i, \alpha)$; $i = 1,2,3$.

Probability density functions (pdf) and cumulative distribution functions cdf of the r. variables are given as:

$$F_i(x_i) = 1 - e^{-\frac{\beta_i}{\alpha}(e^{\alpha x_i - 1})} \qquad x_i \geq 0 \;\; ; \;\; \beta_i, \alpha > 0, \; i = 1,2,3$$

and

$$F_Y(y) = 1 - e^{-\frac{\beta}{\alpha}(e^{\alpha y} - 1)} \qquad y \geq 0 \; , \; \beta, \alpha > 0$$

$$f(y) = \beta \, e^{\alpha y} \quad e^{-\frac{\beta}{\alpha}(e^{\alpha y} - 1)} \qquad y \geq 0 \; , \; \beta, \alpha > 0 \qquad \ldots.. \qquad (1)$$

Hence , the model reliability of such a component, $R_1$, is given by:

$$R_1 = P(\max(X_1, X_2, X_3) < Y)$$

$$R_1 = \int_0^\infty \int_0^y \int_0^y \int_0^y f(x_1, x_2, x_3, y) \; dx_3 \; dx_2 \; dx_1 \; dy$$

Since the r. variables are non- identical independently distributed, then:

$$R_1 = \int_0^\infty F_1(y) \;\; F_2(y) \;\; F_3(y) \;\; f(y) \, dy$$

Then for Gompertz densities in (1), we derive the reliability R as:

$$R_1 = \int_0^\infty [1 - \; e^{-\frac{\beta_1}{\alpha}(e^{\alpha y} - 1)}] \; [1 - \; e^{-\frac{\beta_2}{\alpha}(e^{\alpha y} - 1)}]$$

$$[1 - \; e^{-\frac{\beta_3}{\alpha}(e^{\alpha y} - 1)}] \; \beta e^{\alpha y} \, e^{-\frac{\beta}{\alpha}(e^{\alpha y} - 1)} \, dy$$

$$R_1 = \int_0^\infty [1 - \; e^{-\frac{\beta_1}{\alpha}(e^{\alpha y} - 1)} - \; e^{-\frac{\beta_2}{\alpha}(e^{\alpha y} - 1)} - e^{-\frac{\beta_3}{\alpha}(e^{\alpha y} - 1)} + e^{-\frac{1}{\alpha}(e^{\alpha y} - 1)(\beta_1 + \beta_2)} + e^{-\frac{1}{\alpha}(e^{\alpha y} - 1)(\beta_1 + \beta_3)} + e^{-\frac{1}{\alpha}(e^{\alpha y} - 1)(\beta_2 + \beta_3)} - e^{-\frac{1}{\alpha}(e^{\alpha y} - 1)(\beta_1 + \beta_2 + \beta_3)}] \, \beta \, e^{\alpha y} e^{-\frac{\beta_1}{\alpha}(e^{\alpha y} - 1)} dy$$

By transformation , we get the final Expression of $R_1$ as:

$$R_1 = 1 - \frac{\beta}{\beta + \beta_1} - \frac{\beta}{\beta + \beta_2} - \frac{\beta}{\beta + \beta_3} + \frac{\beta}{\beta + \beta_1 + \beta_2} + \frac{\beta}{\beta + \beta_1 + \beta_3} + \frac{\beta}{\beta + \beta_2 + \beta_3} - \frac{\beta}{\beta + \beta_1 + \beta_2 + \beta_3} \qquad \ldots.(2)$$

**3. Two Component Reliability**

The system reliability $R_2$ of two parallel components having Gompertz strength r. variables $Y_i; i = 1,2$ with parameters $(\theta_i, \alpha); i = 1,2$ subject to a common Gompertz random stress x with parameters $(\theta, \alpha)$ , is the probability of maximum two strengths under one stress. [Hanagal (1998)]

$$R_2 = P[\, X < \max(Y_1, Y_2)]$$

$$R_2 = \int_x \left[1 - F_{y_1}(x)\ F_{y_2}(x)\right] f_x(x)\ dx$$

$$R_2 = \int_0^\infty \left[1 - (1 - e^{-\frac{\theta_1}{\alpha}(e^{\alpha z}-1)}\ (1 - e^{-\frac{\theta_2}{\alpha}(e^{\alpha z}-1)})\right] f_x(x)\ dx$$

$$R_2 = \int_0^\infty \left(e^{-\frac{\theta_1}{\alpha}(e^{\alpha x}-1)} + e^{-\frac{\theta_2}{\alpha}(e^{\alpha x}-1)} - e^{-\frac{\theta_1+\theta_2}{\alpha}(e^{\alpha x}-1)}\right)\ \theta e^{-\frac{\theta}{\alpha}(e^{\alpha x}-1)}\ e^{\alpha x}\ dx$$

then by transformation, we get the reliability $R_2$as:

$$\therefore R_2 = \frac{\theta}{\theta+\theta_1} + \frac{\theta}{\theta+\theta_2} - \frac{\theta}{\theta+\theta_1+\theta_2} \qquad \text{….. (3)}$$

**4. Multi-component Reliability**

The s-out of- k system functions when at least s $(1 \le s \le k)$ of components survive a common random stress x developed by Bhattachoryya and Johnson (1974) under the assumption that all components were subjected to a common stress [Bhattachryya & Johnson (1974)]. Where

R(s,k)= p(at least s of k variables $Y_1, Y_2, \dots, Y_n > X$)

$$= \sum_{i=s}^{k} C_i^k \int_x \left[1 - F_y(x)\right]^i F_y(x)^{k-i}\ \ dG(x)$$

Let $Y_i;\ i = 1, \dots k$ be Gompertz strength r. variables with parameters $(\lambda, \alpha)$ and X be Gompertz stress r. variable with parameter $(\gamma, \alpha)$, then $R(s,k)$ , can be derived as:

$$R(s,k) = \sum_{i=s}^{k} C_i^k\ \int_0^\infty [\, e^{-\frac{\lambda}{\alpha}(e^{\alpha x}-1)}]^i\ [\, e^{-\frac{\lambda}{\alpha}(e^{\alpha x}-1)}]^{k-i}\ \gamma\ e^{\alpha x}\ e^{-\frac{\gamma}{\alpha}(e^{\alpha x}-1)}\ dx$$

Let u = $e^{-\frac{\lambda}{\alpha}(e^{\alpha x}-1)}$ , and by transformation, we get:

$$R(s,k) = \sum_{i=s}^{k} C_i^k\ \frac{\gamma}{\lambda} \int_0^1 u^{i+\frac{\gamma}{\lambda}-1}\ \ (1-u)^{k-i}\, du$$

and then

$$R(s,k) = \frac{\gamma}{\lambda} \sum_{i=s}^{k} C_i^k\ \ \beta\left(i + \frac{\gamma}{\lambda}\ ,\ k - i + 1\right) \qquad \dots \qquad \dots(4)$$

**5. The Maximum Likelihood Estimators**

Let $y_1, y_2, \dots, y_m$ be a random Sample from Gompertz distribution , then the Likelihood function will be

$$L\left(\beta, \alpha, \underline{y}\right) = \prod_{i=1}^{m} f(y_i, \beta, \alpha)\ \ = \beta^n\ \ e^{\alpha \sum_{i=1}^{m} y_i}\ \ e^{-\frac{\beta}{\alpha}\sum_{i=1}^{m}(e^{\alpha y_i}-1)}$$

Then by maximizing the Log-likelihood function with respect to $\beta$ and we get the ML estimator for $\beta$ , say $\hat{\beta}_{ML}$ , as

$$\hat{\beta}_{ML} = \frac{\alpha m}{\sum_{i=1}^{m}(e^{\alpha y_i - 1})}$$

by the same way; let $x_{11}, \dots, x_{1n1};\ x_{21}, \dots, x_{2n2}$ and $x_{31}, \dots, x_{3n3}$ be a random samples from Gompertz of $n_1$ , $n_2$ and $n_3$ sample sizes. Then the ML estimator of $\beta_i$ ; $i = 1,2,3$ are given as:

$$\hat{\beta}_{IML} = \frac{\alpha n_I}{\sum_{i=1}^{n_I}(e^{\alpha x_{Ii} - 1})} \qquad I = 1,2,3$$

then the ML estimator of R, say $\hat{\beta}_{ML}$, by the invariant property of this method given as:

$$\hat{R}_{1ML} = 1 - \frac{\hat{\beta}_{ML}}{\hat{\beta}_{ML} + \hat{\beta}_{1ML}} - \frac{\hat{\beta}_{ML}}{\hat{\beta}_{ML} + \hat{\beta}_{2ML}} - \frac{\hat{\beta}_{ML}}{\hat{\beta}_{ML} + \hat{\beta}_{3ML}}$$

$$+ \frac{\hat{\beta}_{ML}}{\hat{\beta}_{ML}+\hat{\beta}_{1ML}+\hat{\beta}_{2ML}} + \frac{\hat{\beta}_{ML}}{\hat{\beta}_{ML}+\hat{\beta}_{1ML}+\hat{\beta}_{2ML}}$$

$$+ \frac{\hat{\beta}_{ML}}{\hat{\beta}_{ML}+\hat{\beta}_{2ML}+\hat{\beta}_{3ML}} - \frac{\hat{\beta}_{ML}}{\hat{\beta}_{ML}+\hat{\beta}_{1ML}+\hat{\beta}_{2ML}+\hat{\beta}_{3ML}}$$

Now let $x_1, x_2, \dots\ x_n$ ; $y_{11}, y_{12}, \dots\ y_{1m_1}; y_{21}, y_{22}, \dots, y_{2m_2}$ be three random sample from Gompertz distribution, then the ML estimator of $\theta$ , $\theta_1, \theta_2$ ; say $\hat{\theta}_{ML}$ , $\hat{\theta}_{1ML}$ and $\hat{\theta}_{2ML}$ , are given as:

$$\hat{\theta}_{ML} = \frac{\alpha n}{\sum_{i=1}^{n}(e^{\alpha x_i - 1})}, \quad \hat{\theta}_{IML} = \frac{\alpha m_I}{\sum_{i=1}^{m_I}(e^{\alpha y_{Ii} - 1})} \qquad I = 1,2$$

Then the ML estimator of $R_2$ ; say $\hat{R}_{2ML}$ , is by

$$\hat{R}_{2ML} = \frac{\hat{\theta}_{ML}}{\hat{\theta}_{ML}+\hat{\theta}_{1ML}} + \frac{\hat{\theta}_{ML}}{\hat{\theta}_{ML}+\hat{\theta}_{2ML}} - \frac{\hat{\theta}_{ML}}{\hat{\theta}_{ML}+\hat{\theta}_{1ML}+\hat{\theta}_{2ML}}$$

Finally , the ML estimator for $R_{(s,k)}$ , say $R_{ML(s,k)}$ , by the same property is given by:

$$\hat{R}_{ML(s,k)} = \frac{\hat{\gamma}_{ML}}{\hat{\lambda}_{ML}} \sum_{i=s}^{k} C_i^k \ \ B\left(k - i + 1\ , \frac{\hat{\gamma}_{ML}}{\hat{\lambda}_{ML}} + i\right)$$

where the random samples $y_{1i}, y_{2i}, \dots, y_{m_i}$ ; $i = 1, .., k$ and $x_1 , x_2, \dots, x_n$ are taken from Gompertz distribution , and the ML estimators of $\lambda$ and $\gamma$ are:

$$\hat{\lambda}_{ML} = \frac{\alpha m}{\sum_{i=1}^{m}(e^{\alpha y_i - 1})} \quad \text{and} \quad \hat{\gamma}_{ML} = \frac{\alpha n}{\sum_{i=1}^{n}(e^{\alpha x_i - 1})}$$

**6. The Last Square Estimator**

The LS estimator can be obtain by minimizing ; $\sum_{i=1}^{n}\left(F\left(y_{(i)} - p_i\right)\right)^2$ with respect to the unknown parameters.

Now suppose $y_1, y_2, \dots, y_n$ is a random Sample of size n from Gompertz dist. With cdf $F(x)$ in eq. (1), and $y(i)$ , $i = 1,2, \dots, m$ is the order sample of this r. variable, where

$$E\left(F\left(y_{(i)}\right)\right) = \frac{i}{m+1}\ , \ \text{var}\left(F\left(y_{(i)}\right)\right) = \frac{i(m-i+1)}{(m+1)^2(m+2)}\ i = 1,2, \dots, m$$

 

Then the LS estimator of $\beta$ , say $\hat{\beta}_{LS}$ , can be obtained by minimizing $\sum_{i=1}^{m}\left[\ln(1-p_i)+\frac{\beta}{\alpha}(e^{\alpha y_{(i)}}-1)\right]^2$ with respect to $\beta$ as:

$$\sum_{i=1}^{m}(e^{\alpha y_{(i)}}-1)\ln(1-p_i)+\frac{\beta}{\alpha}\sum_{i=1}^{n}(e^{\alpha y_{(i)}}-1)^2=0$$

so

$$\hat{\beta}_{LS}=\frac{-\alpha\sum_{i=1}^{m}\left(e^{\alpha y_{(i)}}-1\right)\ln(1-p_i)}{\sum_{i=1}^{m}\left(e^{\alpha y_{(i)}}-1\right)^2}\quad \text{where } p_i=\frac{i}{m+1}, \text{ i=1,2,…,m.}$$

By the same way we get the LS estimator of $\beta_1$ , $\beta_2$ and $\beta_3$, say $\hat{\beta}_{1LS}$ $\hat{\beta}_{2LS}$ and $\hat{\beta}_{3LS}$ as:

$$\hat{\beta}_{ILS}=\frac{-\alpha\sum_{i=1}^{n_I}\left(e^{\alpha x_{I(i)}}-1\right)\ln(1-p_i)}{\sum_{i=1}^{n_I}\left(e^{\alpha x_{I(i)}}-1\right)^2}\qquad I=1,2,3$$

and the approximated LS estimator of $R_1$, say $\hat{R}_{1LS}$ , is given by replacing the LS parameter estimators instead of the parameter in eq. (2) the LS estimators of $\theta,\theta_1$ and $\theta_2$; say $\hat{\theta}_{LS}$ ,$\hat{\theta}_{1LS}$ ,$\hat{\theta}_{2LS}$ , are replaced in eq.(3) to get the LS estimator for $R_2$, $\hat{R}_{2LS}$ , where:

$$\hat{\theta}_{LS}=\frac{-\alpha\sum_{i=1}^{n}\left(e^{\alpha x_{(i)}}-1\right)\ln(1-p_i)}{\sum_{i=1}^{n}\left(e^{\alpha x_{(i)}}-1\right)^2},\ \hat{\theta}_{ILS}=\frac{-\alpha\sum_{i=1}^{m_I}\left(e^{\alpha y_{I(i)}}-1\right)\ln(1-p_i)}{\sum_{i=1}^{m_I}\left(e^{\alpha y_{I(i)}}-1\right)^2}\qquad I=1,2$$

Finally the LS estimators of $\lambda$ and $\beta$ , say $\hat{\lambda}_{LS}$ and $\hat{\beta}_{LS}$ , instead of $\lambda$ and $\beta$ in equ. (4) to give the approximately LS estimator of $R_{(s,k)}$ , say $\hat{R}_{LS(s,k)}$ where

$$\hat{\lambda}_{LS}=\frac{-\alpha\sum_{i=1}^{m}\left(e^{\alpha y_{(i)}}-1\right)\ln(1-p_i)}{\sum_{i=1}^{m}\left(e^{\alpha y_{(i)}}-1\right)^2}\text{ and }\hat{\gamma}_{LS}=\frac{-\alpha\sum_{i=1}^{n}\left(e^{\alpha x_{(i)}}-1\right)\ln(1-p_i)}{\sum_{i=1}^{n}\left(e^{\alpha x_{(i)}}-1\right)^2}$$

**7. The Weighted Least Square Estimators**

It can be obtained by the same sample assumptions of LS estimation using the weight $w_i$ , which given as:

$$w_i=\frac{(m+1)^2\,(m+2)}{i(m-i+1)}\qquad i=1,2,\dots,m$$

then minimizing $\sum_{i=1}^{m}w_i\left[\ln(1-p_i)+\frac{\beta}{\alpha}(e^{\alpha y_{(i)}}-1)\right]^2$ with respect to the unknown parameter $\beta$ .

The WLS estimator for $\beta$ , say $\beta_{WLS}$ , is finally given as:

$$\hat{\beta}_{WLS}=\frac{\sum_{i=1}^{m}w_i\left(e^{\alpha y_{(i)}}-1\right)\ln(1-p_i)}{\sum_{i=1}^{m}w_i\left(e^{\alpha y_{(i)}}-1\right)^2}$$

and then the WLS reliability estimators, $\hat{R}_{1WLS}$, $\hat{R}_{2WLS}$ and $\hat{R}_{WLS(s,k)}$ are given by replacing the following WLS parameters estimators.

$$\hat{\beta}_{IWLS}=\frac{\sum_{i=1}^{n_I}w_i\left(e^{\alpha x_{I(i)}}-1\right)\ln(1-p_i)}{\sum_{i=1}^{n_I}w_i\left(e^{\alpha x_{I(i)}}-1\right)^2},\ I=1,2,3\ \text{ in eq.(2)}$$

$$\hat{\theta}_{WLS} = \frac{\sum_{i=1}^{n} w_i \left(e^{\alpha x_{(i)}}-1\right) \ln(1-p_i)}{\sum_{i=1}^{n} w_i \left(e^{\alpha x_{(i)}}-1\right)^2}$$

$$\hat{\theta}_{IWLS} = \frac{\sum_{i=1}^{m_I} w_i \left(e^{\alpha y_{I(i)}}-1\right) \ln(1-p_i)}{\sum_{i=1}^{m_I} w_i \left(e^{\alpha y_{I(i)}}-1\right)^2}, \quad I = 1,2 \text{ in eq.(3)}$$

$$\hat{\lambda}_{WLS} = \frac{\sum_{i=1}^{m} w_i \left(e^{\alpha y_{(i)}}-1\right) \ln(1-p_i)}{\sum_{i=1}^{m} w_i \left(e^{\alpha y_{(i)}}-1\right)^2}, \hat{\gamma}_{WLS} = \frac{\sum_{i=1}^{n} w_i \left(e^{\alpha x_{(i)}}-1\right) \ln(1-p_i)}{\sum_{i=1}^{n} w_i \left(e^{\alpha x_{(i)}}-1\right)^2} \text{ in eq.(4)}$$

## 6. Simulation study

A Mont Carlo Simulation is conducted, in this section, to compare the performances of the ML, LS and WLS estimators of $R_1, R_2$ and $R_{(s,k)}$ (based on 10000 replication).

It made by assuming three cases of $R_1$ , say $[(\beta = 3 , \beta_1 = 2 , \beta_2 = 5 , \beta_3 = 1), (\beta = 2 , \beta_1 = 1.5 , \beta_2 = 1.2 , \beta_3 = 1.8) and\ (\beta = 0.5 , \beta_1 = 0.4 , \beta_2 = 0.3 , \beta_3 = 0.2)]$ , three cases of $R_2$, say $[(\theta = 3 , \theta_1 = 2 ,\ \theta_2 = 5) , (\theta = 0.5 , \theta_1 = 0.4 ,\ \theta_2 = 0.3 )\ and\ (\theta = 2 , \theta_1 = 1.5 ,\ \theta_2 = 1.2)]$ and four cases for $R_{(s,k)}$ , say $[(\gamma = 0.5 , \lambda = 0.3 , s = 1 , k = 3), (\gamma = 3 , \lambda = 4 , s = 1 , k = 3), (\gamma = 3, \lambda = 4 , s = 2, k = 4),$ and $(\gamma = 0.5 , \lambda = 0.3 , s = 2 , k = 4)]$ for different sample sizes assuming 10 for the small, 30 for moderate and 50 for large sample sizes.

The real values of $R_1$ , $R_2$ and $R_{(s,k)}$ and the MSE values for these cases are recorded in tables from(1) to (7).

Form table (1) the ML estimator gives the best performance, accept for three cases of sample sizes, first for $\alpha = 2$ with moderate $m = 30$ ; second, fir $\alpha = 4$ with moderate $m = 50$ . And in table(2) , it gives the best for large $m = 50$ for $\alpha = 2$ and the same for $\alpha = 4$. While it gives the best performance for all cases and sample sizes in table(3).

For $R_2$ the ML estimator gives the best performance for all cases and sample sizes as in tables (4) , (5) and (6).

the best performance estimator for the multi component $R(s,k)$ was in orders ML, WLS and LS estimator for different cases and sample size when $s = 1$ and $k = 3$ for table(7) and (8). But it be in orders of WLS, ML and LS estimator for $s = 1$ and $k = 4$ of different cases and sample sizes in table(9) and (10).

## 7. Conclusion

The reliability, $R_1 = P[\max(X_1, X_2, X_3) < Y]$ , of a component being exposed to $X_1, X_2$ and $X_3$ Gompertz stresses and having $Y$ Gompertz strength with unknown location parameter and common known shape parameters is obtained. Then driving the two component $R_2$, and the multi-component reliability $R(s,k)$.

MSE criteria used to make a comparison between three different estimators for each reliability function, where ML give the best performance for $R_1$ and $R_2$ , while WLS estimators was the best for $R(s,k)$ .

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**Table(1):**MSE values of $(\beta = 3, \beta_1 = 2, \beta_2 = 5, \beta_3 = 1)$, $R_1$=0.1356

| α=2 | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| **Samples** | | | | **Mean** | | | **MSE** | | | |
| **n1** | **n2** | **n3** | **m** | **MLE** | **LSE** | **WLSE** | **MLE** | **LSE** | **WLSE** | **Best** |
| 10 | 10 | 10 | 10 | 0.1420 | 0.1430 | 0.1439 | 0.0043 | 0.0051 | 0.0057 | MLE |
| 10 | 10 | 10 | 30 | 0.1431 | 0.1290 | 0.1330 | 0.0026 | 0.0027 | 0.0035 | MLE |
| 10 | 10 | 10 | 50 | 0.1439 | 0.1249 | 0.1304 | 0.0021 | 0.0021 | 0.0029 | MLE,LSE |
| 10 | 10 | 30 | 10 | 0.1391 | 0.1477 | 0.1470 | 0.0037 | 0.0049 | 0.0055 | MLE |
| 10 | 10 | 50 | 10 | 0.1374 | 0.1483 | 0.1466 | 0.0034 | 0.0047 | 0.0052 | MLE |
| 10 | 30 | 10 | 10 | 0.1432 | 0.1464 | 0.1468 | 0.0044 | 0.0054 | 0.0060 | MLE |
| 10 | 50 | 10 | 10 | 0.1419 | 0.1452 | 0.1455 | 0.0045 | 0.0053 | 0.0059 | MLE |
| 30 | 10 | 10 | 10 | 0.1419 | 0.1480 | 0.1477 | 0.0043 | 0.0053 | 0.0059 | MLE |
| 50 | 10 | 10 | 10 | 0.1399 | 0.1483 | 0.1477 | 0.0041 | 0.0053 | 0.0059 | MLE |
| α=4 | | | | | | | | | | |
| 10 | 10 | 10 | 10 | 0.1434 | 0.1439 | 0.1447 | 0.0046 | 0.0053 | 0.0059 | MLE |
| 10 | 10 | 10 | 30 | 0.1433 | 0.1288 | 0.1330 | 0.0026 | 0.0026 | 0.0034 | MLE,LSE |
| 10 | 10 | 10 | 50 | 0.1437 | 0.1244 | 0.1297 | 0.0022 | 0.0022 | 0.0029 | MLE,LSE |
| 10 | 10 | 30 | 10 | 0.1389 | 0.1476 | 0.1469 | 0.0037 | 0.0050 | 0.0056 | MLE |
| 10 | 10 | 50 | 10 | 0.1381 | 0.1499 | 0.1485 | 0.0036 | 0.0050 | 0.0056 | MLE |

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 10 | 30 | 10 | 10 | 0.1422 | 0.1447 | 0.1449 | 0.0044 | 0.0053 | 0.0059 | MLE |
| 10 | 50 | 10 | 10 | 0.1427 | 0.1466 | 0.1470 | 0.0045 | 0.0055 | 0.0062 | MLE |
| 30 | 10 | 10 | 10 | 0.1409 | 0.1470 | 0.1466 | 0.0042 | 0.0052 | 0.0058 | MLE |
| 50 | 10 | 10 | 10 | 0.1411 | 0.1493 | 0.1486 | 0.0042 | 0.0053 | 0.0059 | MLE |

**Table(2):** MSE values of $(\beta = 2, \beta_1 = 1.5, \beta_2 = 1.2, \beta_3 = 1.8)$ $R_1$=0.1725

| α=2 | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| **Samples** | | | | **Mean** | | | **MSE** | | | |
| **n1** | **n2** | **n3** | **m** | **MLE** | **LSE** | **WLSE** | **MLE** | **LSE** | **WLSE** | **Best** |
| 10 | 10 | 10 | 10 | 0.1771 | 0.1778 | 0.1784 | 0.0061 | 0.0070 | 0.0079 | MLE |
| 10 | 10 | 10 | 30 | 0.1790 | 0.1613 | 0.1660 | 0.0034 | 0.0037 | 0.0047 | MLE |
| 10 | 10 | 10 | 50 | 0.1792 | 0.1558 | 0.1621 | 0.0028 | 0.0031 | 0.0040 | MLE |
| 10 | 10 | 30 | 10 | 0.1755 | 0.1816 | 0.1811 | 0.0058 | 0.0071 | 0.0079 | MLE |
| 10 | 10 | 50 | 10 | 0.1753 | 0.1829 | 0.1820 | 0.0058 | 0.0073 | 0.0082 | MLE |
| 10 | 30 | 10 | 10 | 0.1748 | 0.1825 | 0.1816 | 0.0055 | 0.0069 | 0.0077 | MLE |
| 10 | 50 | 10 | 10 | 0.1753 | 0.1856 | 0.1840 | 0.0054 | 0.0068 | 0.0076 | MLE |
| 30 | 10 | 10 | 10 | 0.1751 | 0.1818 | 0.1810 | 0.0057 | 0.0071 | 0.0079 | MLE |
| 50 | 10 | 10 | 10 | 0.1757 | 0.1846 | 0.1833 | 0.0056 | 0.0071 | 0.0079 | MLE |
| α=4 | | | | | | | | | | |
| 10 | 10 | 10 | 10 | 0.1781 | 0.1786 | 0.1791 | 0.0062 | 0.0072 | 0.0082 | MLE |
| 10 | 10 | 10 | 30 | 0.1791 | 0.1614 | 0.1660 | 0.0034 | 0.0036 | 0.0047 | MLE |
| 10 | 10 | 10 | 50 | 0.1791 | 0.1557 | 0.1619 | 0.0027 | 0.0030 | 0.0040 | MLE |
| 10 | 10 | 30 | 10 | 0.1763 | 0.1822 | 0.1817 | 0.0058 | 0.0072 | 0.0081 | MLE |
| 10 | 10 | 50 | 10 | 0.1756 | 0.1825 | 0.1814 | 0.0057 | 0.0070 | 0.0077 | MLE |
| 10 | 30 | 10 | 10 | 0.1756 | 0.1832 | 0.1822 | 0.0056 | 0.0071 | 0.0079 | MLE |
| 10 | 50 | 10 | 10 | 0.1752 | 0.1849 | 0.1831 | 0.0055 | 0.0071 | 0.0078 | MLE |
| 30 | 10 | 10 | 10 | 0.1760 | 0.1827 | 0.1819 | 0.0058 | 0.0070 | 0.0078 | MLE |
| 50 | 10 | 10 | 10 | 0.1749 | 0.1838 | 0.1825 | 0.0057 | 0.0072 | 0.0080 | MLE |

 

**Table(3):**MSE values of ($\beta = 0.5$ , $\beta_1 = 0.4$ , $\beta_2 = 0.3$ , $\beta_3 = 0.2$), $R_1$=0.1192

| α=2 | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| **Samples** | | | | **Mean** | | | **MSE** | | | |
| **n1** | **n2** | **n3** | **m** | **MLE** | **LSE** | **WLSE** | **MLE** | **LSE** | **WLSE** | **Best** |
| 10 | 10 | 10 | 10 | 0.1250 | 0.1263 | 0.1272 | 0.0040 | 0.0048 | 0.0055 | MLE |
| 10 | 10 | 10 | 30 | 0.1267 | 0.1130 | 0.1169 | 0.0022 | 0.0022 | 0.0029 | MLE,LSE |
| 10 | 10 | 10 | 50 | 0.1261 | 0.1075 | 0.1126 | 0.0018 | 0.0018 | 0.0024 | MLE,LSE |
| 10 | 10 | 30 | 10 | 0.1235 | 0.1307 | 0.1303 | 0.0036 | 0.0048 | 0.0054 | MLE |
| 10 | 10 | 50 | 10 | 0.1232 | 0.1317 | 0.1304 | 0.0035 | 0.0046 | 0.0051 | MLE |
| 10 | 30 | 10 | 10 | 0.1246 | 0.1305 | 0.1304 | 0.0038 | 0.0048 | 0.0054 | MLE |
| 10 | 50 | 10 | 10 | 0.1251 | 0.1327 | 0.1322 | 0.0038 | 0.0050 | 0.0055 | MLE |
| 30 | 10 | 10 | 10 | 0.1256 | 0.1304 | 0.1305 | 0.0039 | 0.0048 | 0.0054 | MLE |
| 50 | 10 | 10 | 10 | 0.1251 | 0.1308 | 0.1306 | 0.0038 | 0.0048 | 0.0054 | MLE |
| α=4 | | | | | | | | | | |
| 10 | 10 | 10 | 10 | 0.1273 | 0.1288 | 0.1298 | 0.0042 | 0.0050 | 0.0057 | MLE |
| 10 | 10 | 10 | 30 | 0.1271 | 0.1133 | 0.1174 | 0.0023 | 0.0023 | 0.0030 | MLE,LSE |
| 10 | 10 | 10 | 50 | 0.1269 | 0.1083 | 0.1135 | 0.0018 | 0.0018 | 0.0025 | MLE,LSE |
| 10 | 10 | 30 | 10 | 0.1233 | 0.1306 | 0.1303 | 0.0035 | 0.0047 | 0.0053 | MLE |
| 10 | 10 | 50 | 10 | 0.1236 | 0.1327 | 0.1317 | 0.0035 | 0.0047 | 0.0053 | MLE |
| 10 | 30 | 10 | 10 | 0.1260 | 0.1316 | 0.1314 | 0.0038 | 0.0048 | 0.0054 | MLE |
| 10 | 50 | 10 | 10 | 0.1254 | 0.1330 | 0.1324 | 0.0037 | 0.0047 | 0.0053 | MLE |
| 30 | 10 | 10 | 10 | 0.1249 | 0.1297 | 0.1297 | 0.0039 | 0.0047 | 0.0053 | MLE |
| 50 | 10 | 10 | 10 | 0.1247 | 0.1304 | 0.1299 | 0.0038 | 0.0047 | 0.0053 | MLE |

**Table(4):**MSE values of ($\theta = 3$ , $\theta_1 = 2$ , $\theta_2 = 5$), $R_2$=0.6750

| α=2 | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **Samples** | | | **Mean** | | | **MSE** | | | |
| **n1** | **n2** | **m** | **MLE** | **LSE** | **WLSE** | **MLE** | **LSE** | **WLSE** | **Best** |
| 10 | 10 | 10 | 0.6698 | 0.6692 | 0.6689 | 0.0102 | 0.0116 | 0.0129 | MLE |
| 10 | 10 | 30 | 0.6665 | 0.6880 | 0.6825 | 0.0062 | 0.0070 | 0.0086 | MLE |
| 10 | 10 | 50 | 0.6658 | 0.6944 | 0.6875 | 0.0054 | 0.0063 | 0.0077 | MLE |
| 10 | 30 | 10 | 0.6748 | 0.6680 | 0.6696 | 0.0096 | 0.0111 | 0.0123 | MLE |
| 10 | 50 | 10 | 0.6746 | 0.6651 | 0.6672 | 0.0096 | 0.0114 | 0.0127 | MLE |
| 30 | 10 | 10 | 0.6805 | 0.6659 | 0.6688 | 0.0083 | 0.0101 | 0.0115 | MLE |
| 50 | 10 | 10 | 0.6815 | 0.6615 | 0.6654 | 0.0078 | 0.0097 | 0.0109 | MLE |
| α=4 | | | | | | | | | |
| 10 | 10 | 10 | 0.6721 | 0.6726 | 0.6726 | 0.0103 | 0.0117 | 0.0129 | MLE |
| 10 | 10 | 30 | 0.6673 | 0.6891 | 0.6844 | 0.0062 | 0.0070 | 0.0086 | MLE |
| 10 | 10 | 50 | 0.6686 | 0.6973 | 0.6907 | 0.0054 | 0.0063 | 0.0077 | MLE |
| 10 | 30 | 10 | 0.6741 | 0.6671 | 0.6687 | 0.0098 | 0.0114 | 0.0126 | MLE |
| 10 | 50 | 10 | 0.6745 | 0.6646 | 0.6665 | 0.0098 | 0.0115 | 0.0127 | MLE |
| 30 | 10 | 10 | 0.6783 | 0.6632 | 0.6659 | 0.0082 | 0.0101 | 0.0114 | MLE |
| 50 | 10 | 10 | 0.6807 | 0.6610 | 0.6652 | 0.0077 | 0.0097 | 0.0110 | MLE |

**Table(5):**MSE values of $(\theta = 0.5, \theta_1 = 0.4, \theta_2 = 0.3)$, $R_2$=0.7639

| α=2 | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **Samples** | | | **Mean** | | | **MSE** | | | |
| **n1** | **n2** | **m** | **MLE** | **LSE** | **WLSE** | **MLE** | **LSE** | **WLSE** | **Best** |
| 10 | 10 | 10 | 0.7606 | 0.7599 | 0.7592 | 0.0080 | 0.0091 | 0.0101 | MLE |
| 10 | 10 | 30 | 0.7556 | 0.7749 | 0.7705 | 0.0048 | 0.0052 | 0.0064 | MLE |
| 10 | 10 | 50 | 0.7535 | 0.7793 | 0.7725 | 0.0042 | 0.0046 | 0.0058 | MLE |
| 10 | 30 | 10 | 0.7639 | 0.7524 | 0.7540 | 0.0066 | 0.0082 | 0.0093 | MLE |
| 10 | 50 | 10 | 0.7633 | 0.7483 | 0.7510 | 0.0066 | 0.0085 | 0.0095 | MLE |
| 30 | 10 | 10 | 0.7624 | 0.7524 | 0.7536 | 0.0074 | 0.0091 | 0.0101 | MLE |
| 50 | 10 | 10 | 0.7620 | 0.7490 | 0.7508 | 0.0071 | 0.0089 | 0.0099 | MLE |
| α=4 | | | | | | | | | |
| 10 | 10 | 10 | 0.7589 | 0.7582 | 0.7575 | 0.0081 | 0.0093 | 0.0104 | MLE |
| 10 | 10 | 30 | 0.7548 | 0.7739 | 0.7688 | 0.0047 | 0.0051 | 0.0065 | MLE |
| 10 | 10 | 50 | 0.7538 | 0.7798 | 0.7733 | 0.0042 | 0.0045 | 0.0057 | MLE |
| 10 | 30 | 10 | 0.7633 | 0.7519 | 0.7537 | 0.0067 | 0.0084 | 0.0094 | MLE |
| 10 | 50 | 10 | 0.7626 | 0.7475 | 0.7501 | 0.0066 | 0.0084 | 0.0094 | MLE |
| 30 | 10 | 10 | 0.7622 | 0.7529 | 0.7543 | 0.0072 | 0.0088 | 0.0097 | MLE |
| 50 | 10 | 10 | 0.7609 | 0.7480 | 0.7499 | 0.0072 | 0.0091 | 0.0101 | MLE |

**Table(6):**MSE values of ($\theta = 2, \theta_1 = 1.5, \theta_2 = 1.2$), $R_2$=0.7709

| α=2 | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **Samples** | | | **Mean** | | | **MSE** | | | |
| **n1** | **n2** | **m** | **MLE** | **LSE** | **WLSE** | **MLE** | **LSE** | **WLSE** | **Best** |
| 10 | 10 | 10 | 0.7658 | 0.7651 | 0.7645 | 0.0079 | 0.0090 | 0.0101 | MLE |
| 10 | 10 | 30 | 0.7610 | 0.7801 | 0.7754 | 0.0046 | 0.0050 | 0.0063 | MLE |
| 10 | 10 | 50 | 0.7609 | 0.7863 | 0.7800 | 0.0040 | 0.0043 | 0.0055 | MLE |
| 10 | 30 | 10 | 0.7691 | 0.7578 | 0.7593 | 0.0067 | 0.0083 | 0.0093 | MLE |
| 10 | 50 | 10 | 0.7705 | 0.7561 | 0.7585 | 0.0065 | 0.0083 | 0.0093 | MLE |
| 30 | 10 | 10 | 0.7688 | 0.7594 | 0.7609 | 0.0072 | 0.0088 | 0.0098 | MLE |
| 50 | 10 | 10 | 0.7692 | 0.7562 | 0.7584 | 0.0068 | 0.0086 | 0.0095 | MLE |
| α=4 | | | | | | | | | |
| 10 | 10 | 10 | 0.7643 | 0.7643 | 0.7639 | 0.0078 | 0.0089 | 0.0099 | MLE |
| 10 | 10 | 30 | 0.7616 | 0.7808 | 0.7760 | 0.0048 | 0.0052 | 0.0064 | MLE |
| 10 | 10 | 50 | 0.7613 | 0.7865 | 0.7801 | 0.0040 | 0.0043 | 0.0055 | MLE |
| 10 | 30 | 10 | 0.7706 | 0.7591 | 0.7603 | 0.0065 | 0.0082 | 0.0092 | MLE |
| 10 | 50 | 10 | 0.7697 | 0.7552 | 0.7576 | 0.0064 | 0.0082 | 0.0091 | MLE |
| 30 | 10 | 10 | 0.7677 | 0.7578 | 0.7589 | 0.0070 | 0.0086 | 0.0096 | MLE |
| 50 | 10 | 10 | 0.7686 | 0.7561 | 0.7584 | 0.0069 | 0.0085 | 0.0095 | MLE |

**Table(7):**MSE values of ($\gamma = 0.5$ , $\lambda = 0.3$ , $s = 1$ , $k = 3$), $R_{(s,k)}$=0.8685

| α=2 | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| **Sample** | | **Mean** | | | **MSE** | | | |
| **n** | **m** | **MLE** | **LSE** | **WLSE** | **MLE** | **LSE** | **WLSE** | **Best** |
| 10 | 10 | 0.8974 | 0.8973 | 0.8859 | 8.37e-08 | 8.31e-08 | 3.02e-08 | WLSE |
| 10 | 30 | 0.8659 | 0.8472 | 0.8297 | 6.88e-10 | 4.52e-08 | 1.50e-07 | MLE |
| 10 | 50 | 0.8985 | 0.9180 | 0.9224 | 8.99e-08 | 2.45e-07 | 2.90e-07 | MLE |
| 30 | 10 | 0.8402 | 0.7823 | 0.7785 | 7.99e-08 | 7.42e-07 | 8.10e-07 | MLE |
| 50 | 10 | 0.8968 | 0.8999 | 0.9109 | 8.00e-08 | 9.88e-08 | 1.79e-07 | MLE |
| 30 | 30 | 0.8496 | 0.8542 | 0.8694 | 3.55e-08 | 2.03e-08 | 8.10e-11 | WLSE |
| 30 | 50 | 0.8495 | 0.8504 | 0.8637 | 3.60e-08 | 3.27e-08 | 2.32e-09 | WLSE |
| 50 | 30 | 0.8768 | 0.8678 | 0.8579 | 6.82e-09 | 5.38e-11 | 1.12e-08 | LSE |
| 50 | 50 | 0.8746 | 0.8760 | 0.8660 | 3.76e-09 | 5.58e-09 | 6.36e-10 | WLSE |
| α=4 | | | | | | | | |
| 10 | 10 | 0.8196 | 0.8452 | 0.8609 | 2.39e-07 | 5.42e-08 | 5.83e-09 | WLSE |
| 10 | 30 | 0.9048 | 0.8682 | 0.8264 | 1.31e-07 | 1.11e-11 | 1.77e-07 | LSE |
| 10 | 50 | 0.8138 | 0.8638 | 0.8619 | 2.99e-07 | 2.24e-09 | 4.30e-09 | LSE |
| 30 | 10 | 0.8577 | 0.8616 | 0.8855 | 1.16e-08 | 4.81e-09 | 2.89e-08 | LSE |
| 50 | 10 | 0.8823 | 0.8652 | 0.8804 | 1.89e-08 | 1.07e-09 | 1.41e-08 | LSE |
| 30 | 30 | 0.8604 | 0.8167 | 0.7831 | 6.53e-09 | 2.68e-07 | 7.30e-07 | MLE |
| 30 | 50 | 0.8654 | 0.8561 | 0.8378 | 9.63e-10 | 1.54e-08 | 9.45e-08 | MLE |
| 50 | 30 | 0.8622 | 0.8471 | 0.8240 | 3.92e-09 | 4.59e-08 | 1.98e-07 | MLE |
| 50 | 50 | 0.8332 | 0.8441 | 0.8606 | 1.24e-07 | 5.96e-08 | 6.29e-09 | WLSE |

**Table(8):**MSE values of $(\gamma = 3, \lambda = 4, s = 1, k = 3)$, $R_{(s,k)}$=0.6675

| α=2 | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| **Sample** | | **Mean** | | | **MSE** | | | |
| **n** | **m** | **MLE** | **LSE** | **WLSE** | **MLE** | **LSE** | **WLSE** | **Best** |
| 10 | 10 | 0.6239 | 0.6554 | 0.6833 | 1.90e-07 | 1.48e-08 | 2.48e-08 | LSE |
| 10 | 30 | 0.5554 | 0.6508 | 0.6694 | 1.25e-06 | 2.79e-08 | 3.38e-10 | WLSE |
| 10 | 50 | 0.6573 | 0.6735 | 0.6667 | 1.05e-08 | 3.51e-09 | 6.76e-11 | WLSE |
| 30 | 10 | 0.7577 | 0.6787 | 0.6144 | 8.13e-07 | 1.24e-08 | 2.82e-07 | LSE |
| 50 | 10 | 0.6773 | 0.6347 | 0.6283 | 9.48e-09 | 1.07e-07 | 1.53e-07 | MLE |
| 30 | 30 | 0.6733 | 0.6718 | 0.6331 | 3.31e-09 | 1.85e-09 | 1.18e-07 | LSE |
| 30 | 50 | 0.5743 | 0.6373 | 0.6755 | 8.68e-07 | 9.11e-08 | 6.38e-09 | WLSE |
| 50 | 30 | 0.6455 | 0.6328 | 0.6603 | 4.87e-08 | 1.20e-07 | 5.18e-09 | WLSE |
| 50 | 50 | 0.6621 | 0.6550 | 0.6453 | 2.93e-09 | 1.57e-08 | 4.93e-08 | MLE |
| α=4 | | | | | | | | |
| 10 | 10 | 0.6856 | 0.6126 | 0.6028 | 3.25e-08 | 3.01e-07 | 4.18e-07 | MLE |
| 10 | 30 | 0.6183 | 0.6857 | 0.6743 | 2.42e-07 | 3.28e-08 | 4.51e-09 | WLSE |
| 10 | 50 | 0.6947 | 0.7107 | 0.6679 | 7.38e-08 | 1.85e-07 | 1.39e-11 | WLSE |
| 30 | 10 | 0.7104 | 0.6790 | 0.7262 | 1.83e-07 | 1.31e-08 | 3.44e-07 | LSE |
| 50 | 10 | 0.6887 | 0.5767 | 0.5996 | 4.47e-08 | 8.24e-07 | 4.61e-07 | MLE |
| 30 | 30 | 0.6610 | 0.6015 | 0.5492 | 4.28e-09 | 4.36e-07 | 1.39e-06 | MLE |
| 30 | 50 | 0.6633 | 0.6959 | 0.7313 | 1.79e-09 | 8.07e-08 | 4.06e-07 | MLE |
| 50 | 30 | 0.6307 | 0.6476 | 0.6996 | 1.35e-07 | 3.96e-08 | 1.02e-07 | LSE |
| 50 | 50 | 0.6709 | 0.6322 | 0.5691 | 1.16e-09 | 1.25e-07 | 9.68e-07 | MLE |

**Table(9):**MSE values of $(\gamma = 3, \lambda = 4, s = 2, k = 4)$, $R_{(s,k)}$=0.8000

| α=2 | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| **Sample** | | **Mean** | | | **MSE** | | | |
| **n** | **m** | **MLE** | **LSE** | **WLSE** | **MLE** | **LSE** | **WLSE** | **Best** |
| 10 | 10 | 0.5907 | 0.6237 | 0.6485 | 4.38e-06 | 3.10e-06 | 2.29e-06 | WLSE |
| 10 | 30 | 0.8765 | 0.9311 | 0.9502 | 5.85e-07 | 1.71e-06 | 2.25e-06 | MLE |
| 10 | 50 | 0.7829 | 0.8598 | 0.8594 | 2.93e-08 | 3.57e-07 | 3.53e-07 | MLE |
| 30 | 10 | 0.7098 | 0.7192 | 0.7605 | 8.14e-07 | 6.52e-07 | 1.55e-07 | WLSE |
| 50 | 10 | 0.7990 | 0.7854 | 0.7713 | 1.09e-10 | 2.12e-08 | 8.25e-08 | MLE |
| 30 | 30 | 0.8201 | 0.8227 | 0.8011 | 4.02e-08 | 5.13e-08 | 1.29e-10 | WLSE |
| 30 | 50 | 0.7990 | 0.7831 | 0.7405 | 9.37e-11 | 2.86e-08 | 3.53e-07 | MLE |
| 50 | 30 | 0.8183 | 0.8235 | 0.8129 | 3.36e-08 | 5.50e-08 | 1.66e-08 | WLSE |
| 50 | 50 | 0.7756 | 0.7990 | 0.8356 | 5.95e-08 | 9.39e-11 | 1.26e-07 | LSE |
| α=4 | | | | | | | | |
| 10 | 10 | 0.7364 | 0.7861 | 0.7932 | 4.03e-07 | 1.94e-08 | 4.61e-09 | WLSE |
| 10 | 30 | 0.7865 | 0.8200 | 0.8271 | 1.81e-08 | 3.99e-08 | 7.32e-08 | MLE |
| 10 | 50 | 0.7841 | 0.8138 | 0.8093 | 2.52e-08 | 1.90e-08 | 8.61e-09 | WLSE |
| 30 | 10 | 0.8231 | 0.8199 | 0.8250 | 5.33e-08 | 3.96e-08 | 6.26e-08 | LSE |
| 50 | 10 | 0.8712 | 0.8221 | 0.8155 | 5.07e-07 | 4.89e-08 | 2.40e-08 | WLSE |
| 30 | 30 | 0.7666 | 0.7907 | 0.8005 | 1.11e-07 | 8.59e-09 | 2.19e-11 | WLSE |
| 30 | 50 | 0.8267 | 0.8254 | 0.7884 | 7.12e-08 | 6.47e-08 | 1.34e-08 | WLSE |
| 50 | 30 | 0.7830 | 0.8251 | 0.8617 | 2.90e-08 | 6.31e-08 | 3.81e-07 | MLE |
| 50 | 50 | 0.8145 | 0.8057 | 0.7738 | 2.11e-08 | 3.29e-09 | 6.85e-08 | LSE |

**Table(10):**MSE values of ($\gamma = 0.5, \lambda = 0.3 , s = 2, k = 4$), $R_{(s,k)}$= 0.7525

| α=2 | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| **Sample** | | **Mean** | | | **MSE** | | | |
| **n** | **m** | **MLE** | **LSE** | **WLSE** | **MLE** | **LSE** | **WLSE** | **Best** |
| 10 | 10 | 0.7978 | 0.8029 | 0.7888 | 2.05e-07 | 2.54e-07 | 1.32e-07 | WLSE |
| 10 | 30 | 0.7710 | 0.7887 | 0.7987 | 3.43e-08 | 1.31e-07 | 2.13e-07 | MLE |
| 10 | 50 | 0.6453 | 0.6621 | 0.6521 | 1.14e-06 | 8.17e-07 | 1.01e-06 | WLSE |
| 30 | 10 | 0.6799 | 0.6675 | 0.6893 | 5.26e-07 | 7.22e-07 | 3.99e-07 | WLSE |
| 50 | 10 | 0.6791 | 0.6663 | 0.6876 | 5.38e-07 | 7.43e-07 | 4.20e-07 | WLSE |
| 30 | 30 | 0.8108 | 0.7679 | 0.7537 | 3.39e-07 | 2.38e-08 | 1.55e-10 | WLSE |
| 30 | 50 | 0.7431 | 0.7629 | 0.7611 | 8.71e-09 | 1.09e-08 | 7.48e-09 | WLSE |
| 50 | 30 | 0.7379 | 0.7029 | 0.6322 | 2.12e-08 | 2.46e-07 | 1.44e-06 | MLE |
| 50 | 50 | 0.7444 | 0.7764 | 0.7984 | 6.61e-09 | 5.70e-08 | 2.11e-07 | MLE |
| α=4 | | | | | | | | |
| 10 | 10 | 0.6666 | 0.6308 | 0.6203 | 7.37e-07 | 1.48e-06 | 1.74e-06 | MLE |
| 10 | 30 | 0.7252 | 0.7069 | 0.6762 | 7.41e-08 | 2.07e-07 | 5.82e-07 | LSE |
| 10 | 50 | 0.7905 | 0.8069 | 0.7758 | 1.44e-07 | 2.96e-07 | 5.42e-08 | WLSE |
| 30 | 10 | 0.7614 | 0.7114 | 0.7204 | 7.88e-09 | 1.68e-07 | 1.03e-07 | MLE |
| 50 | 10 | 0.9195 | 0.8982 | 0.8937 | 2.79e-06 | 2.12e-06 | 1.99e-06 | WLSE |
| 30 | 30 | 0.7978 | 0.7680 | 0.7480 | 2.05e-07 | 2.41e-08 | 2.04e-09 | WLSE |
| 30 | 50 | 0.8642 | 0.8389 | 0.8073 | 1.24e-06 | 7.46e-07 | 3.00e-07 | WLSE |
| 50 | 30 | 0.7836 | 0.7492 | 0.7488 | 9.70e-08 | 1.05e-09 | 1.38e-09 | LSE |
| 50 | 50 | 0.7413 | 0.7448 | 0.7822 | 1.25e-08 | 5.97e-09 | 8.83e-08 | LSE |