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Blow-up Time for a Semilinear Heat Equation with a Gradient Term

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Abstract

We consider the semilinear Heat equation with a gradient term, which takes the form $u_t = \Delta u + e^u - |\nabla u|^2$, defined on a ball in R^n. It has been proved before that for a special initial function, the solution of this equation blows-up in finite time. In this paper, we use a certain change of variables to transform the equation to a linear equation and we use the explicit Euler method, to compute the approximate blow-up solutions and times for a numerical experiment in one dimensional space.

Keywords: Semilinear heat equation, Blow-up, Radial function

1. Introduction

In this paper, we consider the following initial-boundary value problem:

$$\begin{aligned} &u_t = \Delta u + e^u - |\nabla u|^2, && (x,t) \in B_R \times (0,T), \\ &u(x,t) = 0, && (x,t) \in \partial B_R \times (0,T) \qquad \ldots\ldots\ldots \quad (1) \\ &u(x,0) = u_0(x), && x \in B_R \end{aligned}$$

where B_R is a ball in R^n, with radius $R > 0$, and center $x = 0$, u_0 is positive in B_R, and non increasing radial function vanish on the boundary of B_R.

From these properties of initial function, any solution of problem (1) has to be positive and radially decreasing, see [6 ,7]. This problem has been studied firstly by J. Bebernes and D. Eberly, [2], they showed for a certain initial function, that the local existence and regularity of classical solution of this problem can be guaranteed for some time $0 < T < \infty$, which means the solution blows up in the blow-up time T. In other words, the solution becomes unbounded in finite time, moreover, it has been proved that, the only possible blow-up point is $x = 0$.

i.e. $$u(0,t) \xrightarrow{t \to T} \infty$$

Later, in [7], it has been shown, that the upper point wise estimate takes the following form:

$$u(x,t) \le \frac{1}{2\alpha}[\log C - m\log(|x|)], \quad (x,t) \in B_R/\{0\} \times (0,T),$$

where $C > 0, m > 0, \ \alpha \in (0,1/2]$

While, the upper and lower blow-up rate estimate take the following form

$$\log C_1 - \log(T-t) \le u(x,t) \le \log C_2 - \log(T-t), \quad (x,t) \in B_R \times (0,T),$$

Where C_1, C_2 are positive constant.

In fact, these estimates are the same as the estimates of the problem (1) without gradient term, see [3,7], this means, the gradient term has no effect on these estimates. Moreover, it is well known that, for any solution of this problem, the gradient still bounded as long as the solution is bounded.

From above, we see that, this problem has been studied very well theoretically, while its numerical study still in early stage. However, the blow-up solutions and times for problem (1), where the gradient term is absent, have been studied in [9].

In this paper, we will compute the blow-up solutions and times for problem (1), in one dimensional space with a special initial function. And that will be done, by using explicit Euler method, to compute the numerical solutions to an equivalent problem to (1).

2. Equivalent linear problem

Problem (1) can be transformed to a linear initial value problem, which is easier to deal with, and that can be done by using the following change of variable

$v = 1 - exp(-u),$

so, problem (1), can be rewritten as follows:

$$v_t = \Delta v + 1, \quad (x,t) \in B_R \times (0,T),$$

$$v(x,t) = 0 \qquad (x,t) \in \partial B_R \times (0,T) \qquad \ldots\ldots\ldots\ldots(2)$$

$$u(x,0) = v_0(x), \qquad x \in B_R$$

where, v_0 is a continuous function on B_R such that $v_0(\partial B_R) = 0,$ and

$v_0(x) = 1 - \exp(-u_0(x)) < 1$.

It is clear that, if the solution of problem (1), u, blows up at the blow up time T, and at the blow-up point

x_0, then

$$\lim_{t\to T} v = \lim_{t\to T}(1 - e^{-u}) = 1 \quad .$$

Which means, u blows-up at (x_0, T), if and only if $v(x_0,t) \underset{t\to T}{\longrightarrow} 1$

It is well known that, nonhomogeneous heat equation with the initial-boundary conditions, problem (2), has a unique global stable classical solution. That means when v attends the value 1 at the first time, T, then we can consider this time is the blow-up time for problem (1).
i.e.

$$\lim_{t\to T} u(x_0,t) = \lim_{t\to T}[-ln(1 - v(x_0,t))] = \infty$$

3. Finite difference scheme

The numerical blow-up solutions and times for problem (1) in one dimensional space, where the gradient term is absent, were studied by some authors, where the reaction term f of power or exponential types, see [1,8,9]. While, as far as we know, the numerical study for problem (1) still in the new stage.
It well known that Euler explicit method is stable and convergent, when it is used to compute the approximate solution of non homogenous heat equation. Therefore, in order to compute the numerical blow-up times and solution of problem (1) in one dimensional space, we will compute the numerical solution of the equivalent linear problem (2), by using Euler explicit finite difference algorithm
It is clear that, problem (1), in one dimensional space can be written as follows:

$$u_t = u_{xx} + e^u - (u_x)^2 , \quad (x,t) \in (-R,R) \times (0,T),$$

$$u(x,t) = 0, \qquad x = \pm R,\ t \in (0,T) \qquad \ldots\ldots\ldots\ldots\ldots(3)$$

$$u(x,0) = u_0(x), \qquad x \in (-R,R)$$

While the equivalent problem (2) becomes

$$v_t = v_{xx} + 1 , \quad (x,t) \in (-R,R) \times (0,T),$$

$$v(x,t) = 0 \qquad x = \pm R,\ t \in (0,T) \qquad \ldots\ldots\ldots\ldots\ldots(4)$$

$$v(x,0) = v_0(x), \qquad x \in B_R$$

We can use finite difference operators to get the discrete problem for the last problem as follows:

For J a positive integer, we set $J = 2R/h$, and we defined the grids:

$x_0 = -R, \quad x_j = x_0 + ih, \quad 0 \le j \le J, \quad x_J = R$

and let $t_0 = 0, \; t_{n+1} = t_n + k$, *n=0,1,….*

And we denote to the approximate value of *v*, *u* at the point (x_j, t_n) by u_j^n, v_j^n respectively. We approximate the time derivative u_t by the forward finite difference operator, while the second order derivatives is approximated by the standard second order center finite difference operators. Thus, problem (4) can be written in the discrete form as follows:

$$\frac{v_j^{n+1} - v_j^n}{k} = \frac{v_j^n - 2v_j^n + v_{j-1}^n}{h^2} + 1$$

So, we get

$$v_j^{n+1} = (1-2r)\,v_j^n + r\left(v_{j+1}^n - v_{j-1}^n\right) + k\,, \qquad r = \frac{k}{h^2} \quad ,$$

$$v_0^n = v_J^n = 0, \quad \forall n \qquad \text{……….} (5)$$

$$v_j^0 = v_0(x_j), \quad j = 1,2, \ldots .J$$

Where *k* and *h* should be chosen such that the last formula is stable

i.e. $r \le 1/2$

Form the definition of *v*, we have

$$u_j^n = -\ln\left(1 - v_j^n\right), \quad j = 1,2,3, \ldots .J \qquad \ldots\ ..(6)$$

4. Blow-up time of the discrete problem

Definition:- Let $\{u^n\} = \left(u_0^n, u_1^n, \ldots .. u_J^n\right)$, and $\{v^n\} = \left(v_0^n, v_1^n, \ldots .. v_J^n\right)$ $n \ge 0$ are the numerical solutions of problem (3) and (4), respectively, we say that $\{u^n\}$ achieves blow-up in the finite time T_J^m , if

$$\lim_{n \to m} \|u^n\|_\infty = \infty, \quad where \quad T_J^m = mk < \infty$$

In fact, we will consider that T_J^m is the blow-up time, when, $\|v^m\|_\infty \ge 1$

For some $0 < m \in N$,

where $\|u^n\|_\infty = \sqrt{u_1^n + u_2^n + \cdots u_J^n}$, $\|v^n\|_\infty = \sqrt{v_1^n + v_2^n + \cdots v_J^n}$

5. Numerical Experiment with Matlab program

In this section, we present some numerical approximation to the blow-up solutions and times of problem (3), where *R=1*, with a special initial function:

$$u_0 = 5(1 - x^2) , \qquad -1 \le x \le 1 ,$$

Thus, $v_0 = 1 - e^{-u_0} = 1 - e^{5(x^2-1)} , \quad -1 \le x \le 1$

Is it clear that, u_0, v_0 are positive and symmetric for $-1 < x < 1$,

while they vanish at $x = -1 , x = 1$, moreover, they non increasing in (0,1) and each of them takes its

maximum at $x = 0$. So, form maximum principles, we can show that for a small time T, the solutions *u* and *v* of

problem (3) and (4), respectively, have the same above properties of u_0, v_0 .

Therefore, according to the known blow-up results for the semilinear heat equation with gradient term, (see [2]), the blow-up in problem (3), occurs only at a single point, which is $x = 0$

While, the solution of problem (4) may become equal or more than one, at more than one point or in a subinterval of (-1,1).

Our aim is to compute numerically, the first time T, when $v \ge 1$, which will be considered finally the blow-up time of the main problem (3).

This problem will be solved numerically by using Euler explicit method, (5), (6), which was suggested in section 4, for different values of the mesh size *h*, to exam experimentally, if there exist any rate of convergence for the numerical blow-up times with using the following Matlab program:

```
1- numx = 101;    %number of grid points in x

2- numt = 50;  %number of time steps to be iterated over

3- dx = 2/(numx-1);

4- t(1)=0;
```

```
v = zeros([numx,numt]);
x = -1:dx:1;    %vector of x values, to be used for plotting
u0=5*(1-x.^2);
y=1-exp(-u0);
v(:,1)=y';
v(1,:)=0 ;v(numx,:)=0;
dt=dx^2/2;
for j=1:numt-1
r=dt/(dx^2);
 t(j+1)=t(j)+dt;
for i=2:numx-1
v(i,j+1)=(1-2*r)*v(i,j)+r*v(i+1,j)+r*v(i-1,j)+dt*(1);
 end
 end
for j=1:numt
if max(v(:,j))>=1
 cbut=t(j)
 m=j
display('the solution blows up')
 break
 end
```



```
end
u=zeros([numx,m-1]);
for j=1:m-1
s=-1*log(1-v(:,j));
u(:,j)=s';
end
for i=1:m
plot3(t(i)*ones(numx),x,v(:,i),'b');
hold on
end
grid
xlabel('t');
ylabel('x');
zlabel('v');
for i=1:m-1
plot3(t(i)*ones(numx),x,u(:,i),'b');
hold on
end
grid
xlabel('t');
ylabel('x');
```

```
47- zlabel('u');
```

In the next tables (1) and (2), we show the numerical blow-up times with respect to the meshes J=10-100, and the iterative errors those can be got form using the error form $E_J = |T_{2J}^m - T_J^m|$, where *m* is referred to the number of iteration, when numerical blow-up occurs.

Table 1,
Computed blow-up times

J	T_J^m	m	J	T_J^m	m
10	0.020000000000000	2	60	0.007777777777778	15
20	0.010000000000000	3	70	0.007346938775510	19
30	0.008888888888889	5	80	0.007500000000000	25
40	0.007500000000000	7	90	0.007407407407407	31
50	0.008000000000000	11	100	0.007400000000000	38

Table2,
Errors in the numerical blow-up times of u, using $E_J = |T_{2J}^m - T_J^m|$

J	E_J
10	0.010000000000000
20	0.002500000000000
30	0.001111111111111
40	0.000000000000000
50	0.000600000000000

The next figures (1) and (2) show the evolutions of the numerical bow-up solutions *u & v*,of problems (1) & (2) respectively, which arise from using Euler explicit method, before blow-up occurs, for different values to J .

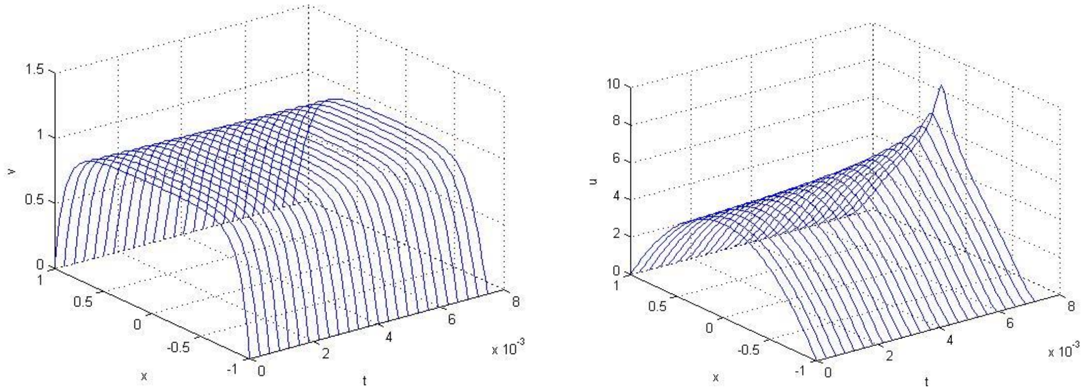

Figure 1,

Numerical solutions of u & v before blow-up occurs , $J = 80$, $t \in [0, T_J^m)$

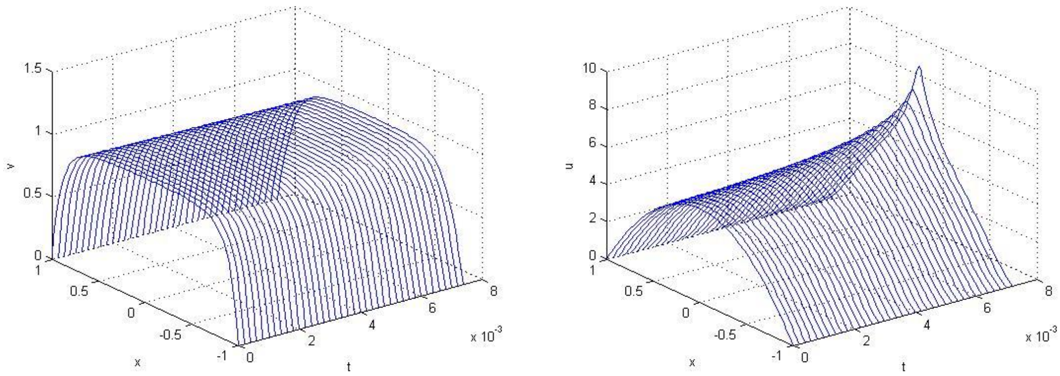

Figure 2,

Numerical solutions of u & v before blow-up occurs , $J = 100$, $t \in [0, T_J^m)$

6. Conclusions

In this paper, we have used a finite difference scheme with a Matlab program, to find the approximate blow-up time for a semilinear heat equation with a gradient term and a certain initial function., Since, it well known that, the solution of this problem, has to blow up in finite time, it is difficult to compute directly, the numerical solutions of this problem, therefore, we have computed rather the numerical solution of the equivalent linear problem, which has been suggested in section 2. When the computed numerical solution of the equivalent problem becomes equal or more than the value one, at a finite time, we can consider this time is the numerical blow-up time of the main problem. Moreover, from the numerical results, in section 5, we can point out the following conclusions

- Increasing the values of J, leads almost to increase the number of iterations, m, when the numerical blow- up occurs, and decreasing the errors E_J .
- The table of error (2), in the computed blow-up times, shows that, there is no error can be get where $J = 40$.
- The figures (1) & (2) show that, the blow-up in problem (1) occurs only at a single point and that confirms the theoretical results see [2,7].

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Maan A. Rasheed was born in Baghdad, 1980. He graduated from university of Baghdad in 2002 and got BSc in Mathematics , in 2005 he got MSc from university of Baghdad in Applied Mathematics (Numerical analysis of parabolic partial differential equations), and then in 2006 he became a lecturer in Mathematics at Al Mustansirya university until 2009, and he got a scholarship from Iraqi government to study Ph.D. degree in united kingdom, then he got his Ph.D. in 2012 from university of Sussex in England. The field of Ph.D. was on blow-up phenomena in partial differential equations. Since 2013 and so far he is working as a lecturer in Mathematics at Al Mustansirya university, and he has taught many subjects for BSc and MSc students and he has done many publications in the field of partial differential equations.