Mathematical Theory and Modeling
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.6, No.1, 2016
www.iiste.org

STABILITY ANALYSIS OF AN SIR EPIDEMIC MODEL WITH SPECIFIC NONLINER INCIDENCE RATE

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Abstract

We study an SIR epidemic model with a specific non linear incidence rate function. The stability of the disease-free equilibrium and the endemic equilibrium are found and an appropriate Dulac function was constructed for investigating the global stability of an endemic equilibrium. We illustrate the theoretical results by carrying numerical simulation.

Keywords: epidemic, nonlinear incidence, inhibitory effect, disease-free equilibrium, endemic equilibrium, global stability.
2010 AMS Subject Classification: 92D30, 93A30, 93D30, 34D23.

1. Introduction

Epidemiology is the study of the distribution and determinants of health related states or events in specified populations, and the application of this study to control health problems [12]. The term epidemiology is derived from Greek words that can be translated into the phrase "the study of that which is upon the people". In the study of epidemiological model incidence rate plays an important role. An incidence rate is defined as the number of new health related events or cases of a disease in a population exposed to the risk in a given time period. Epidemic models with nonlinear incidence rate have been studied and developed by many authors. In order to model this disease transmission process several authors employ the incidence functions: The earliest one is the bilinear incidence rate λSI used by Kermack and Mckendrick [11] in 1927, where λ, S and I denote the transmission rate, the number of susceptible population and the infectious population respectively. In 1978, Capasso and Serio [5] introduced a saturated incidence rate $\frac{\lambda I}{1+\alpha I}$ by research of the Cholera epidemic spread in Bari. Also in 1978, May and Anderson [1] proposed the saturated incidence rate $\frac{\lambda SI}{1+\alpha S}$. The general incidence rate $\frac{\lambda I^p S}{1+\alpha I^q}$ was proposed by Liu et. al. [13-14] in 1986-87, Derick and Ven Den Driessche [6] in 1993, etc. Ruan and Wang [10] studied an epidemic model with a specific nonlinear incidence rate $\frac{\lambda I^2 S}{1+\alpha I^2}$ and presented a detailed quantitative analysis and bifurcation analysis and Bogdanov-Takens bifurcation for the model in 2003. Kar and Batabyal [17] proposed an SIR model with non-monotonic incidence rate in 2010 suggested by Xiao and Ruan [22] incorporating with a treatment function proposed by Wang [21] and so on. In this paper, we drive a model which includes a specific nonlinear incidence rate with the psychological or inhibitory effect measuring parameters.

The paper is organized as follows: In the next section, we present the model diagram and formulation of mathematical model. In section 3 we drive the disease free equilibrium and the endemic equilibrium with stability conditions. Finally, we give some numerical simulations and concluding in section 4, closed paper.

2. The Basic Mathematical Model

The model we analyze in this chapter is considered under the framework of the following nonlinear ordinary differential equations:

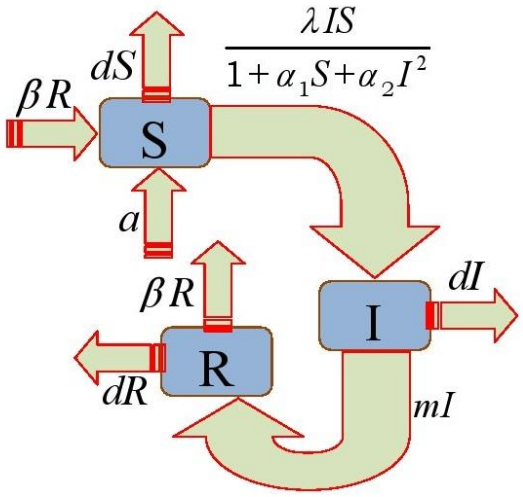

Figure 2.1: The Model Diagram

$$\left.\begin{aligned}\frac{dS}{dt} &= a - dS - \frac{\lambda SI}{1+\alpha_1 S+\alpha_2 I^2} + \beta R \\ \frac{dI}{dt} &= \frac{\lambda SI}{1+\alpha_1 S+\alpha_2 I^2} - (d+m)I \\ \frac{dR}{dt} &= mI - (d+\beta)R\end{aligned}\right\} \tag{2.1}$$

where, $S(t)$, $I(t)$, $R(t)$ denote the number of susceptible, infected, recovered individuals respectively; a is the recruitment rate of the population, d is the natural death rate of the population, λ is the proportionality constant, β is the rate at which recovered individuals lose immunity and return to susceptible class, m is the natural recovery rate of the infective individuals, α_1 and α_2 are the parameter measures of the psychological or inhibitory effect.

The incidence rate $\frac{\lambda SI}{1+\alpha_1 S+\alpha_2 I^2}$, where λI measures the infecting force of the disease and $\frac{1}{1+\alpha_1 S+\alpha_2 I^2}$ describes the psychological or inhibitory effect of the behavioral change to the susceptible individuals when the number of infective individuals is very large. Notice that if $\alpha_1 = 0$ the incidence rate becomes $\frac{\lambda SI}{1+\alpha_2 I^2}$ and if $\alpha_2 = 0$ it becomes $\frac{\lambda SI}{1+\alpha_1 S}$, when α_1 and α_2 are both equal to zero the incidence rate becomes the bilinear incidence rate.

3. Equilibrium Points and Stability

We first consider the existence of equilibria of system (2.1). For any values of parameters, model (2.1) always has a disease-free equilibrium $E_0 = (\frac{a}{d}, 0, 0)$. To find the positive equilibria, set $S^* = \frac{(d+m)(1+\alpha_2 I^2)}{\lambda - \alpha_1(d+m)}$, $R^* = \frac{mI}{(d+\beta)}$ and I is given as a root of the quadratic equation $aI^2 + bI + c = 0.$

where, $a = [\alpha_2 d(d+m)]$, $b = [\{\lambda - \alpha_1(d+m)\}\{(d+m) - \frac{\beta m}{d+\beta}\}]$, $c = [(\alpha_1 a + d)(d+m) - \lambda a]$.

Clearly, the above equation will have a positive root if $\Delta_1 > 0$ and $R_0 > 1$, where R_0 is the basic reproduction number given as follows: $R_0 = \frac{\lambda a}{(\alpha_1 a + d)(d+m)}$.

Now $I^* = \frac{-[\lambda\{1 - \frac{\alpha_1(d+m)}{\lambda}\}\{(d+m) - \frac{\beta m}{d+\beta}\}] + \sqrt{\Delta_1}}{2\alpha_2 d(d+m)}$

where, $\Delta_1 = [\lambda\{1 - \frac{\alpha_1(d+m)}{\lambda}\}\{(d+m) - \frac{\beta m}{d+\beta}\}]^2 - 4\alpha_2 d(d+m)^2(\alpha_1 a + d)[1 - R_0].$

Theorem 3.1 The plane $S + I + R = b/d$ is a manifold of system (2.1) which is attracting in the first octant.

Proof. Summing up the three equations in (2.1) and denoting $N(t) = S(t) + I(t) + R(t)$ we have

$$\frac{dN}{dt} = a - dN. \tag{3.1}$$

It is clear that $N(t) = a/d$ is a solution of equation (3.1) and for any $N(t_0) \geq 0,$ the general solution of the equation (3.1) is

$$N = \frac{1}{d}[a - (a - dN(t_0))e^{-d(t-t_0)}].$$

Thus

$$\lim_{t\to\infty} N(t) = \frac{a}{d}$$

This implies the conclusion.

It is clear that the limit set of system (2.1) is on the plane $S + I + R = a/d$. Thus, we focus on the reduced system

$$\left.\begin{aligned} \frac{dI}{dt} &= \frac{\lambda d I(\frac{a}{d} - I - R)}{(d + \alpha_1 a) + (\alpha_2 I - \alpha_1)dI - \alpha_1 dR} - (d+m)I \equiv F_1(I, R) \\ \frac{dR}{dt} &= mI - (d+\beta)R \equiv F_2(I, R) \end{aligned}\right\} \tag{3.2}$$

In order to study the properties of the disease-free equilibrium E_0 and the endemic equilibrium E^*, we rescale (3.2) by

$$\left.\begin{aligned} x &= \frac{\lambda}{d+\beta} I,\ y = \frac{\lambda}{d+\beta} R,\ T = (d+\beta)t \\ \frac{dx}{dT} &= \frac{px}{1 + qx^2 - r(x+y)}(A - x - y) - ux \\ \frac{dy}{dT} &= wx - y \end{aligned}\right\} \tag{3.3}$$

where

$p=\frac{d}{(d+\alpha_1 a)}$, $q=\frac{\alpha_2 d(d+\beta)^2}{\lambda^2(d+\alpha_1 a)}$, $r=\frac{\alpha_1 d(d+\beta)}{\lambda(d+\alpha_1 a)}$, $A=\frac{\lambda a}{d(d+\beta)}$, $u=\frac{(d+m)}{(d+\beta)}$, $w=\frac{m}{(d+\beta)}$. Here $E_0(0,0)$ is the disease free equilibrium and the unique positive equilibrium (x^*, y^*) of the system (3.3). (x^*, y^*) exists if $u-Ap<0$ and is given by

$$qux^{*2}+[(1+w)(p-ru)]x^*+(u-Ap)=0 \tag{3.4}$$

Therefore

$$x^*=\frac{[(1+w)(ru-p)]+\Delta_2}{2qu}\ ,\quad y^*=wx^*.$$

Where, $\Delta_2=\sqrt{[(1+w)(p-ru)]^2-4qu(u-Ap)}$

The Jacobian matrix corresponding to $E_0(0,0)$ is

$$M_0=\begin{pmatrix} Ap-u & 0 \\ w & -1 \end{pmatrix}$$

The Jacobian matrix corresponding to (x^*, y^*) is

$$M_1=\begin{pmatrix} \frac{px^*[qx^{*2}(2w+1)-A(2qx^*+r)-1]}{[1+qx^{*2}-x^*(1+w)r]^2} & \frac{-px^*[qx^{*2}+Ar+1]}{[1+qx^{*2}-x^*(1+w)r]^2} \\ w & -1 \end{pmatrix}$$

We have that

$$det(M_1)=\frac{px^*[-q(1+w)x^{*2}+2Aqx^*+(1+Ar)(1+w)]}{[1+qx^{*2}-x^*(1+w)r]^2}$$

The sign of $det(M_1)$ is the sign of the

$$P_1=-q(1+w)x^{*2}+2Aqx^*+(1+Ar)(1+w). \tag{3.5}$$

Now $u\times$ (3.5)+(1+w)$\times$ (3.4), we have

$$uP_1=\{2Aqu+(1+w)^2(p-ru)\}\{x^*+\frac{(1+w)\{u(1+Ar)+(u-Ap)\}}{2Apu+(1+w)^2(p-ru)}\}$$

Now substituting $x^*=\frac{-(1+w)(p-ru)+\Delta_2}{2qu}$

we get

$$uP_1=\frac{1}{2qu}[\{2Aqu+(1+w)^2(p-ru)\}\Delta_2-(1+w)\{\Delta_2^2-4Aqru^2\}]$$

$$\therefore\quad P_1=\frac{(1+w)\Delta_2}{2qu^2}[(1+w)(p-ru)+\frac{2Aqu}{1+w}-\Delta_2]+2Ar(1+w)$$

Since $\left((1+w)(p-ru)+\frac{2Aqu}{1+w}\right)^2-\Delta_2^2=\frac{4A^2q^2u^2}{(1+w)}+4qu^2(1-Ar)>0$, when $Ar<1$.

Therefore, $P_1>0$, and hence $det(M_1)$ is positive for any set of parameters.

$$Trac(M_1)=\frac{px^*[qx^{*2}(1+2w)-A(2qx^*+r)-1]}{[1+qx^{*2}-r(1+w)x^*]^2}-1$$

So the sign $Trac(M_1)$ is determined by

$$P_2=-q^2x^{*4}+\{p(1+2w)+2q(1+w)\}rx^{*3}-\{2q(1+Ap)+(1+w)^2r^2\}x^{*2}+\{2(1+w)r-p(1+2Ar)\}x^*-1($$

3.6)

After some algebraic calculation using (3.4) and (3.6) we get, $u^3vP_2=P_3x^*-P_4$, where

$$P_3 = [(1+w)(p-ru)][(1+w)^2(p-ru)^2(2ru+1)+2Apqu(u+1)+qu(1+w)(1+2w)(p-ru)$$
$$+qr^2u^3(1+w^2)]+pqu^2(1+2w)(Ap-u)+qu^2[2Apr(1-u)+p(2Arw-u)]$$
$$P_4 = [(1+w)^2(ru-p)^2+2pu^2(1+Ap)+r^2u^2(1+w)^2+qu(1+2w)(1+w)(p-ru)$$
$$+2ru(1+w)^2(p-ru)+qu(Ap-u)](Ap-u)+qu^3.$$

Therefore P_3 and P_4 are positive for any set of parameters with $Ap > u$. So when the point (x^*, y^*) exists, it is locally stable if $x^* < P_4/P_3$.

Thus we have the following theorem.

Theorem 3.2

(i) When the basic reproduction number $R_0 \le 1$, there exists no positive equilibrium of the system (3.3) and in that case the only disease free equilibrium (0, 0) is a stable node.

(ii) When $R_0 > 1$, there exists a unique positive equilibrium of the system (3.3) and in that case (0, 0) is an unstable saddle point. Also the condition for which the unique positive equilibrium will be locally stable if $x^* < P_4/P_3$.

Global Stability.

Theorem3.3 System (3.2) does not have non trivial periodic orbits.

Proof. Consider system (3.2) for $I > 0$ and $R > 0$: Take a Dulac function

$$D(I,R) = \frac{(d+\alpha_1 a)+(\alpha_2 I-\alpha_1)dI-\alpha_2 dR}{\lambda dI}$$

$$\frac{\partial}{\partial I}(DF_1)+\frac{\partial}{\partial R}(DF_2) = -1-\frac{(d+\beta)(d+\alpha_1 a)}{\lambda dI}-\frac{[(2\alpha_2 I-\alpha_1)(d+m)+(\alpha_2 I-\alpha_1)(d+\beta)-\alpha_1 m]}{\lambda} < 0.$$ The conclusion follows.

4. Numerical Simulation and Conclusion

Case I. When $a=3,\ d=0.2,\ \lambda=0.3,\ \alpha_1=0.1,\ \alpha_2=0.7,\ \beta=0.1,\ m=0.5$, then the basic reproduction number $R_0 = 2.57142857 > 1$, all three components $S(t),\ I(t)$ and $R(t)$, approach to their steady state values as $t \to \infty$, the disease become endemic (Figure 4.1).

Case II. Again, if we take $a=15,\ d=0.01,\ \lambda=0.5,\ \alpha_1=0.7,\ \alpha_2=0.1,\ \beta=0.5,\ m=0.9$, then the basic reproduction number $R_0 = 0.7841825 < 1$ component $S(t)$ approaches to its steady state value while $I(t)$ and $R(t)$ approach to zero as $t \to \infty$, the disease dies out (Figure 4.2).

By rescaling, the system (3.3) reduces to

$$\left.\begin{aligned}\frac{dx}{dT} &= \frac{0.4x}{1+0.28x^2-0.04(x+y)}(15-x-y)-2.34x\\ \frac{dy}{dT} &= 1.67x-y\end{aligned}\right\} \qquad (4.1)$$

By rescaling, the system (3.3) we see that $(u-pA) < 0$, and hence there exists the unique positive equilibrium $x^* = 1.73949302$ and $y^* = 2.90495334$. For the above choice of parameters $P_3 = 26.5371813$ and $P_4 = 222.752042$, $P_4/P_3 = 8.39396014$ and therefore the sufficient condition for local stability i.e. $x^* < P_4/P_3$ is satisfied here.

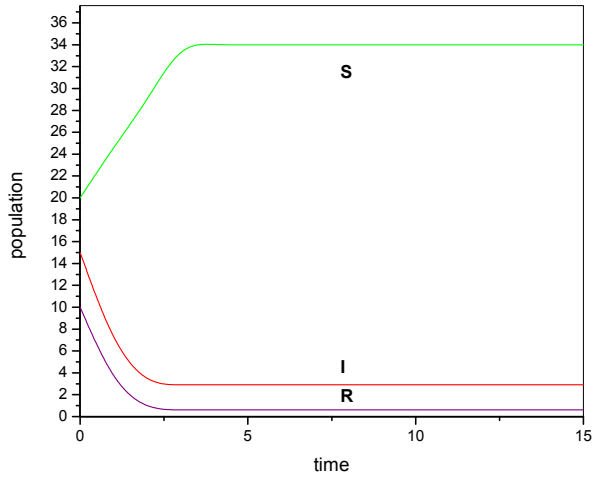

Figure 4.1: The plot represents that the disease endemic

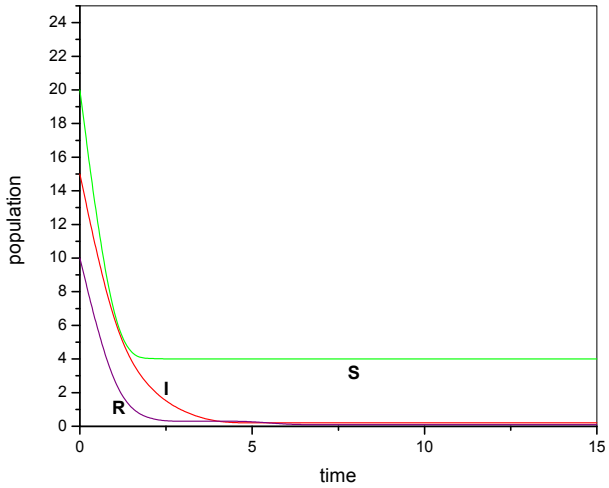

Figure 4.2: The plot represents that the disease dies out

Using the numerical simulations, we conclude that the basic reproduction number plays an important role to control the disease. When $R_0 \leq 1$ disease dies out and the disease free equilibrium is globally attractive. When $R_0 > 1$ the endemic equilibrium is globally stable, i.e. epidemic occurs. We also see that R_0 depend on the parameter α_1 explicit α_2 but α_1 and α_2 play important role in numerical simulations.

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